The Truncated Moment Problem on $\mathbb{N}_0$

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Abstract

We find necessary and sufficient conditions for the existence of a probability measure on $\mathbb{N}_0$, the nonnegative integers, whose first n moments are a given n-tuple of nonnegative real numbers. The results, based on finding an optimal polynomial of degree n which is nonnegative on $\mathbb{N}_0$ (and which depends on the moments), and requiring that its expectation be nonnegative, generalize previous results known for $n = 1$, $n = 2$ (the Percus-Yamada condition), and partially for $n = 3$. The conditions for realizability are given explicitly for $n \leq 5$ and in a finitely computable form for $n \geq 6$. We also find, for all n, explicit bounds, in terms of the moments, whose satisfaction is enough to guarantee realizability. Analogous results are given for the truncated moment problem on an infinite discrete semi-bounded subset of $\mathbb{R}$.

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1 Introduction

In this paper we address the following question: given a positive integer n and an n-tuple $m = (m_1, m_2, \ldots, m_n)$ of real numbers, does there exist a

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probability measure μ on $\mathbb{N}_0$, the set of nonnegative integers, with the m_k's as moments: $E_\mu[X^k] = m_k$, where X is the identity random variable $X(i) = i$ on $\mathbb{N}_0$? When this is true we say that m is *realizable on* $\mathbb{N}_0$ or, when no confusion can arise, simply *realizable*. Specifically, we wish to give necessary and sufficient conditions on m for this realizability, in as simple a form as possible. For $j \leq n$ we write $m^{(j)} := (m_1, \ldots, m_j)$; thus m may be written as $m^{(n)}$ and we will use this notation when we want to emphasize the number of moments to be realized. We write $m_0 = 1$, so that $E_\mu[X^k] = m_k$ for $k = 0$ as well as $k = 1, \ldots n$.

This problem is a special case of the truncated (power) moment problem, in which one asks whether or not k given numbers (or vectors, or functions) can be realized as the first k moments of some random variable (or random vector, or random process) X whose support lies in a specified set or space $\mathcal{X}$. We then speak of the given data as *realizable on* $\mathcal{X}$ (for more details and references about this problem see e.g. [7], [16], [17, Chap. III] for the finite dimensional case $\mathcal{X} \subseteq \mathbb{R}^d$ and see e.g. [3], [13] for the infinite dimensional one). The main challenge in this area is to identify relevant and practically checkable conditions for realizability. Our results answer this question for the case $\mathcal{X} = \mathbb{N}_0$. This problem is very natural in many situations; for example, X could count the number of atoms in a container or the number of snakes in a pit.

We note here that in some cases one might, instead of specifying the numerical values of the m_j's, specify some relations $m_j \geq f_j(m_1, \ldots, m_{j-1})$, and ask whether such relations are compatible with the realizability of $m^{(j)}$ on $\mathbb{N}_0$. This occurs, for example, in a more general form in the classical theory of fluids. There the pair correlation function is given by some approximation schemes (Percus-Yevick, hyper-netted chain, etc.) as a function of the density, or the three-body correlation function is given as a function of the one and the two particle correlations (e.g., in the superposition approximation). Motivated by this, in the previous works [4, 12, 13] the truncated moment problem for X a point process on a subset Λ of $\mathbb{Z}^d$ or $\mathbb{R}^d$ was considered. If, instead of the full point process X, one takes the random variable given by the number of points in a fixed volume of $\mathbb{Z}^d$ or $\mathbb{R}^d$ then the problem reduces to the truncated moment problem on $\mathbb{N}_0$ considered in the current work. The sufficiency bounds given in Section 7 may in this way be useful for establishing realizability for point processes.

The case in which the support $\mathcal{X}$ is a discrete subset of $\mathbb{R}$ and X is the identity random variable is often called the *discrete moment problem*. For finite $\mathcal{X}$, e.g. $\mathcal{X} = \{0, 1, \dots, N\} \subseteq \mathbb{N}_0$, the problem has been extensively studied in connection with the problem of computing bounds for the probability that a certain number of events occurs in systems where only a few moments are known (see e.g. [14, 15, 20, 21, 22]). When $\mathcal{X}$ is an infinite discrete set the problem has been considered by Karlin and Studden [10, Chapter VII]. In particular, they characterize the cone of all n-tuples realizable on $\mathcal{X} = \mathbb{N}_0 \cup \{+\infty\}$ for the generalized moment problem (Tchebycheff systems), using techniques from convex geometry which since have become standard in moment theory (see also [9, 11]). The specific choice $\mathcal{X} = \mathbb{N}_0$ is also considered in [10, Sect. 8, Chap. VII], but the technique used by the authors characterizes the cone of all n-tuples realizable on $\mathbb{N}_0$ only up to an unknown parameter and hence does not provide a collection of necessary and sufficient conditions for realizability. To our knowledge the present paper contains the first computable necessary and sufficient realizability conditions for the truncated power moment problem, with arbitrary degree n, on $\mathbb{N}_0$. These results are here extended also to any infinite discrete semi-bounded subset of $\mathbb{R}$.

For the related *truncated Stieltjes moment problem*, in which $\mathcal{X} = \mathbb{R}_+$, explicit necessary and sufficient conditions for realizability are known [5]. Earlier works (see e.g. [1], [9], [10, Chapter V], [11], [23, p. 28 ff.]) did not provide such explicit conditions, due to the same technical restrictions present, e.g., in the work of Karlin and Studden for the truncated moment problem on $\mathbb{N}_0$. Let us reinterpret now the results of [5] in an inductive form (obtained in Appendix A; see in particular Corollary A.4) which is parallel to our treatment of the discrete case: for each $j = 1, \dots, n$ we state conditions which are necessary and sufficient for the realizability of $m^{(j)} := (m_1, \dots, m_j)$, *given* the realizability of $m^{(j-1)}$ (which is in itself clearly a necessary condition). At each stage we distinguish two types of realizability: *I-realizability*, in which $m^{(j)}$ lies in the interior of the set of realizable moment vectors, and *B-realizability*, in which $m^{(j)}$ lies on the boundary of this set. If $m^{(j-1)}$ is B-realizable, then $m^{(j)}$ is realizable if and only if m_j takes a certain unique value, computable from $m^{(j-1)}$, and then must be B-realizable; $m^{(j)}$ cannot be I-realizable. If $m^{(j-1)}$ is I-realizable, then realizability of $m^{(j)}$ is determined by the *Hankel matrix* C_j, where $C_j = A(k)$ if $j = 2k$ and $C_j = B(k)$ if $j = 2k+1$; here for $k \geq 0$ the $(k+1) \times (k+1)$ Hankel matrices are

$$A(k) := (m_{i+j})_{i,j=0}^{k}, \qquad B(k) := (m_{i+j+1})_{i,j=0}^{k}. \tag{1.1}$$

Specifically, $m^{(j)}$ is I-realizable if C_j is positive definite ($C > 0$) and is B-realizable if C_j is positive semidefinite ($C \geq 0$) but not positive definite.

We note that it is easy to see the relevance of the Hankel matrix for the Stieltjes problem, and in fact to see that realizability of $m^{(n)}$ requires that $C_j \geq 0$ for $j = 1, \ldots, n$. For if $2k \leq n$, $X_k = (1, X, \ldots, X^k)$, and ν realizes $m^{(n)}$ on $\mathbb{R}_+$, then for any $Q_k = (q_0, q_1, \ldots, q_k) \in \mathbb{R}^{k+1}$,

$$E_\nu[(Q_k \cdot X_k)^2] = E_\nu \left[\left(\sum_{i=0}^{k} q_i X^i \right)^2 \right] = Q_k^T A(k) Q_k = Q_k^T C_{2k} Q_k \tag{1.2}$$

and, if $2k + 1 \leq n$,

$$E_\nu[X(Q_k \cdot X_k)^2] = Q_k^T B(k) Q_k = Q_k^T C_{2k+1} Q_k, \tag{1.3}$$

must both be nonnegative. Obtaining sufficient conditions [5] is considerably more complicated. We remark that [5] gives explicitly computable necessary and sufficient conditions for the existence of a *unique* representing measure on $\mathbb{R}_+$; in this case the support of the measure is necessarily a finite set which is also explicitly computable (c.f. Proposition A.2 in Appendix A). When these conditions are satisfied the results of the current paper are not needed; one may simply check whether or not this support is contained in $\mathbb{N}_0$.

The approach in [5] can be extended to give necessary and sufficient conditions for the truncated moment problem on $\mathcal{X} \subset \mathbb{R}$ where $\mathcal{X}$ is defined by a finite number of polynomial inequalities, but the technique becomes more complex as the number of polynomials defining $\mathcal{X}$ increases. Since defining $\mathbb{N}_0$ in this way requires an infinite number of polynomial constraints, it is not clear how to apply the method to this case; a non-trivial modification seems to be necessary. In the present paper we introduce a new technique to get realizability conditions for the case $\mathcal{X} = \mathbb{N}_0$, based on an infinite family of polynomials which are different from the squares of polynomials used in [5] for the case $\mathcal{X} = \mathbb{R}_+$ (see (1.2) and (1.3)).

On the other hand, in structure our approach to the $\mathcal{X} = \mathbb{N}_0$ problem is strictly parallel to our reinterpretation of the results about the truncated Stieltjes moment problem given above. We use the same inductive procedure, and introduce the same notion of I- and B-realizability on $\mathbb{N}_0$. Again, if $m^{(j-1)}$ is B-realizable then $m^{(j)}$ is B-realizable if and only m_j takes a specific value, and B-realizability is the only possibility. The new element enters when $m^{(j-1)}$ is I-realizable. In this case we prove the existence of a polynomial

$P_j^{(m)}(x) = \sum_{i=0}^{j} p_i x^i$, which we take to be monic ($p_j = 1$), such that for any μ that realizes $m^{(n)}$ on $\mathbb{N}_0$, $E_\mu[P_j(X)]$ is minimized over all monic polynomials $P_j(x)$ of degree j, nonnegative on $\mathbb{N}_0$, by $P_j(x) = P_j^{(m)}(x)$. We then show that $m^{(j)}$ is realizable if and only if

$$E_\mu[P_j^{(m)}(X)] = \sum_{i=0}^{j} p_i m_i \geq 0, \tag{1.4}$$

with strict inequality (respectively equality) in (1.4) corresponding to I-realizability (respectively B-realizability). Condition (1.4) thus plays somewhat the role of positive semidefiniteness of the Hankel matrices in the Stieltjes theory.

The remaining problem is to find explicitly the polynomials $P_n^{(m)}$. Here the $n = 1$ case is trivial and the solution for $n = 2$ goes back to the work of Percus and Yamada [18, 19, 24] in statistical mechanics:

$$\begin{aligned} P_1^{(m)}(x) &:= x, \\ P_2^{(m)}(x) &:= (x-k)(x-(k+1)), \quad \text{with} \quad k = \lfloor m_1 \rfloor. \end{aligned}$$

The condition for realizability when $n = 1$ is thus $m_1 \geq 0$, and when $n = 2$ is

$$m_2 - m_1^2 \geq \theta(1-\theta), \qquad \theta = m_1 - \lfloor m_1 \rfloor, \tag{1.5}$$

which is known as the (Percus)-Yamada condition in the statistical mechanics literature. The similar condition ((5.6) below) in the $n = 3$ case can be derived from [15, 20]. The above conditions are necessary. In the current work we additionally show that they are also sufficient for realizability on $\mathbb{N}_0$.

The construction of $P_n^{(m)}$ is considerably more complicated for $n \geq 4$. In the current work we give explicit constructions in the cases $n = 4$ and $n = 5$, and a reasonably efficient recursive procedure for larger values of n.

The remainder of the paper is organized as follows. In Section 2 we establish necessary and sufficient conditions on $m^{(n)}$ for realizability on the set $\mathbb{N}_N = \{0, 1, 2, \ldots, N\}$; we are interested only in large N and always assume that $N \geq n$. The conditions will consist of the nonnegativity of a certain set of $O(N^n)$ affine functions of m. In Section 3 we give necessary and sufficient conditions for realizability on $\mathbb{N}_0$; these are nonnegativity conditions as for $\mathbb{N}_N$, but there are now an infinite number. In Section 4 we describe the classification of realizable moment vectors as I- or B-realizable and introduce the key polynomials $P_n^{(m)}$. Sections 5 and 6 are devoted to obtaining these polynomials: for $n = 1$, 2, and 3 in Section 5 and for $n \geq 4$ in Section 6, with a recursive procedure for general n described in Section 6.1 and the explicit formulas for $n = 4$ and $n = 5$ in Section 6.2. In Section 7 we discuss a sufficient condition for realizability on $\mathbb{N}_0$. In Section 8 we consider the problem of realizing given moments on an arbitrary infinite discrete subset of $\mathbb{R}_+$. Certain technical discussions are relegated to two appendices.

2 Realizability on $\mathbb{N}_N$

In this section we establish necessary and sufficient conditions for realizability of a moment vector $m^{(n)}$ by a probability measure on $\mathbb{N}_N = \{0, 1, \ldots, N\}$, where $N \geq n$, using similar techniques to the ones in [10]. Note that such techniques were used also in [19] for the realizability problem for point processes in the case $n = 2$. We begin with a geometrical lemma.

Lemma 2.1 *Let S be a finite subset of $\mathbb{R}^n$ which is not contained in any $n-1$ dimensional hyperplane. Then the convex hull of S has the form $\bigcap_{H \in \mathcal{H}} H$, where $\mathcal{H}$ is the family of all closed half spaces H containing S whose bounding hyperplane ∂H contains (at least) n points of S which do not belong to any $n-$ 2 dimensional affine subset of $\mathbb{R}^n$. Moreover, this representation is minimal: no half space may be omitted from the intersection.*

Proof: This is a consequence of Theorem 3.1.1 of [8]. ■

Now let $\mathcal{P}_n$ denote the set of monic polynomials of degree n in a single variable which have n distinct roots in and are nonnegative on $\mathbb{N}_0$, $\mathcal{P}_{n,N}$ denote the set of monic polynomials of degree n which have n distinct roots in and are nonnegative on $\mathbb{N}_N$, and $\mathcal{Q}_{n,N}$ denote the set of polynomials of degree n, with leading term $-x^n$, which have n distinct roots in and are nonnegative on $\mathbb{N}_N$. To describe these sets of polynomials more precisely we let $\mathcal{A}_n$ denote the set of n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers for which $\alpha_1 < \alpha_2 \cdots < \alpha_n$ and in addition:

- If n is even then $\alpha_{2k} = \alpha_{2k-1} + 1$ for $k = 1, \ldots, n/2$;
- If n is odd then $\alpha_1 = 0$ and $\alpha_{2k+1} = \alpha_{2k} + 1$ for $k = 1, \ldots, (n-1)/2$.

For $\alpha \in \mathcal{A}_n$ we define

$$P_\alpha(x) = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n). \tag{2.1}$$

Finally, let $R_N(x) = N-x$. It follows immediately from these definitions that the set $\mathcal{P}_n$ consists of all polynomials P_α with $\alpha \in \mathcal{A}_n$, that (using $N \geq n$) $\mathcal{P}_{n,N}$ consists of all polynomials P_α with $\alpha \in \mathcal{A}_n$ and $\alpha_n \leq N$, and that $\mathcal{Q}_{n,N}$ consists of all polynomials $R_N P_\alpha$ with $\alpha \in \mathcal{A}_{n-1}$ and $\alpha_{n-1} \leq N-1$.

Now to any polynomial $P(x) = \sum_{k=0}^{n} p_k x^k$ of degree at most n we associate the affine function L_P on $\mathbb{R}^n$ defined by

$$L_P(m) = \sum_{k=0}^{n} p_k m_k \tag{2.2}$$

(L_P is an affine rather than linear because m_0 takes the fixed value 1). Clearly (2.2) sets up a bijective correspondence between polynomials of degree n and affine functions on $\mathbb{R}^n$.

Theorem 2.2 *Suppose that $N \geq n$. Then the moment vector $m = m^{(n)}$ is realizable on $\mathbb{N}_N$ if and only if*

$$L_P(m) \geq 0 \quad \textit{and} \quad L_Q(m) \geq 0 \quad \textit{for every } P \in \mathcal{P}_{n,N} \textit{ and } Q \in \mathcal{Q}_{n,N}. \tag{2.3}$$

Moreover, none of the conditions in (2.3) *may be omitted.*

Proof: Since each $P \in \mathcal{P}_{n,N}$ and $Q \in \mathcal{Q}_{n,N}$ is nonnegative on $\mathbb{N}_N$, (2.3) is certainly necessary for realizability. Conversely, the set of probability measures on $\mathbb{N}_N$ is the set of all convex combinations of the point masses δ_k, $k = 0, 1, \ldots, N$, so the set S_N of all moment vectors realizable on $\mathbb{N}_N$ is the convex hull of the set S of all the corresponding vectors $v^{(k)}$, where $v_j^{(k)} = k^j$ for $j = 1, \ldots, n$ and $k = 0, 1, \ldots, N$. Note that for any n distinct positive indices $k_1, \ldots, k_n$ the set of moment vectors $v^{(k_i)}$ is linearly independent, since the matrix $\left(v_j^{(k_i)}\right)_{i,j=1,\ldots,n}$ is, up to a (nonzero) factor k_i in row i, a Vandermonde matrix with nonzero determinant. In particular, since $N \geq n$, S is not contained in any hyperplane of dimension $n-1$ and S_N may thus be characterized by Lemma 2.1. With the correspondence noted above between affine functions and polynomials, this characterization becomes

$$S_N = \bigcap_{H \in \mathcal{H}} H = \bigcap_{P \in \mathcal{R}} \{m \mid L_P(m) \geq 0\}, \tag{2.4}$$

with $\mathcal{R}$ the set of polynomials of degree n, normalized to have leading coefficient ± 1, which are nonnegative on $\mathbb{N}_N$ and have n distinct zeros $k_1, \dots, k_n$ in $\mathbb{N}_N$ (that the corresponding points $v^{(k_i)}$ do not belong to any $n-2$ dimensional affine subset follows from the linear independence pointed out above). This yields (2.3). ■

3 Realizability on $\mathbb{N}_0$

We now turn to necessary and sufficient conditions for realizability of $m = m^{(n)}$ on $\mathbb{N}_0$. Since any $P \in \mathcal{P}_n$ is nonnegative on $\mathbb{N}_0$ the condition

$$L_P(m) \geq 0 \quad \text{for every } P \in \mathcal{P}_n \tag{3.1}$$

is certainly necessary. By the results of [2], a moment vector m is realizable on $\mathbb{N}_0$ if and only if it is realizable on $\mathbb{N}_N$ for some N, so that m will be realizable if and only if (3.1) holds and in addition there exists an N such that $L_Q(m) \geq 0$ for every $Q \in \mathcal{Q}_{n,N}$. We want to replace the latter condition by one which does not refer explicitly to N.

Consider then a polynomial $P = P_\alpha \in \mathcal{P}_{n-1}$ and an integer N with $N > \alpha_{n-1}$, and let $\widehat{P}$ denote the polynomial $\widehat{P}(x) = xP(x)$. Both P and $\widehat{P}$ are nonnegative on $\mathbb{N}_0$ and thus realizability on $\mathbb{N}_0$ requires that

$$L_{\widehat{P}}(m) \geq 0 \quad \text{and} \quad L_P(m) \geq 0 \quad \text{for every } P \in \mathcal{P}_{n-1}. \tag{3.2}$$

(Note that the first condition here does not follow from (3.1), since $\widehat{P} = \widehat{P}_\alpha$ belongs to $\mathcal{P}_n$ if and only if $\alpha_1 > 0$, which is possible only if n is odd.) Let $Q_N = R_N P \in \mathcal{Q}_{n,N}$; since realizability on $\mathbb{N}_N$ for some N implies such realizability for all sufficiently large N, a necessary condition for realizability on some $\mathbb{N}_N$ is that for all $P \in \mathcal{P}_{n-1}$ and all sufficiently large N, $L_{Q_N}(m) = L_{R_N P}(m) \geq 0$, i.e.,

$$N L_P(m) \geq L_{\widehat{P}}(m). \tag{3.3}$$

But (3.2), with (3.3), requires in turn that

$$L_P(m) \geq 0 \quad \text{and if} \quad L_P(m) = 0 \quad \text{then} \quad L_{\widehat{P}}(m) = 0, \quad P \in \mathcal{P}_{n-1}. \tag{3.4}$$

We can now state the main result of this section.

Theorem 3.1 *The conditions* (3.1) *and* (3.4) *are necessary and (collectively) sufficient for realizability of* m *on* $\mathbb{N}_0$*.*

Theorem 3.1 will follow easily from part (b) of Theorem 3.2, given immediately below. Part (a) of that theorem will be used in Section 4. The proof of Theorem 3.2 is rather lengthy and we defer it to Section 3.1.

Theorem 3.2 *(a) If $L_P(m) > 0$ for all $P \in \mathcal{P}_k$, $1 \leq k \leq n-2$, then there exists a polynomial $P_n^{(m)} \in \mathcal{P}_n$ such that for all $P \in \mathcal{P}_n$,*

$$L_{P_n^{(m)}}(m) \leq L_P(m).$$

(b) If $L_P(m) \geq 0$ for all $P \in \mathcal{P}_{n-1}$ then there exists a finite set $\tilde{\mathcal{P}}_{n-1}^{(m)} \subset \mathcal{P}_{n-1}$ such that for all $P = P_\alpha \in \mathcal{P}_{n-1}$ and all $N > \alpha_{n-1}$ there is a $\widetilde{P} \in \tilde{\mathcal{P}}_{n-1}^{(m)}$ such that

$$L_{R_N \widetilde{P}}(m) \leq L_{R_N P}(m).$$

Proof of Theorem 3.1: Necessity of (3.1) and (3.4) is established above, so we must prove that these conditions imply the existence of some $N^{(m)}$ such that $L_Q(m) \geq 0$ for every $Q \in \mathcal{Q}_{n,N^{(m)}}$. Now by (3.4) and Theorem 3.2(b), $\tilde{\mathcal{P}}_{n-1}^{(m)}$ is defined. For any $\widetilde{P} \in \tilde{\mathcal{P}}_{n-1}^{(m)}$, (3.4) further implies that there exists an integer N' depending on $\widetilde{P}$ such that $L_{R_N\widetilde{P}}(m) = NL_{\widetilde{P}}(m) - L_{x\widetilde{P}}(m) \geq 0$ for $N \geq N'$, and since $\tilde{\mathcal{P}}_{n-1}^{(m)}$ is finite, there exists an integer $N^{(m)}$ such that for any $\widetilde{P} \in \tilde{\mathcal{P}}_{n-1}^{(m)}$, $L_{R_N\widetilde{P}}(m) \geq 0$ for $N \geq N^{(m)}$. But this suffices, since if $Q \in \mathcal{Q}_{n,N^{(m)}}$ then $Q = R_{N^{(m)}}P$ for some $P = P_\alpha \in \mathcal{P}_{n-1}$ with $\alpha_{n-1} < N^{(m)}$, and then for $\widetilde{P}$ as in Theorem 3.2(b), $L_Q(m) \geq L_{R_{N^{(m)}}\widetilde{P}}(m) \geq 0$. ■

We finally show that none the conditions (3.1) and (3.4) can be omitted.

Lemma 3.3 *Fix $n \geq 2$. Then:*

(a) For any $P_\alpha \in \mathcal{P}_n$ there exists a moment vector $m^{(n)}$ which is not realizable but which satisfies all conditions (3.1) *and* (3.4)*, except that $L_{P_\alpha}(m) < 0$;*

(b) For any $P_\alpha \in \mathcal{P}_{n-1}$ there exists a moment vector $m^{(n)}$ which is not realizable but which satisfies all conditions (3.1) *and* (3.4)*, except that $L_{P_\alpha}(m) < 0$;*

(c) For any $P_\alpha \in \mathcal{P}_{n-1}$ there exists a moment vector $m^{(n)}$ which is not realizable but which satisfies all conditions (3.1) *and* (3.4)*, except that $L_{P_\alpha}(m) = 0$ and $L_{\widehat{P}_\alpha}(m) > 0$.*

Proof: In the proof we will use the notation that if $P_\beta \in \mathcal{P}_k$ for some k then μ_β is the probability measure $\mu_\beta = k^{-1}\sum_{j=1}^k \delta_{\beta_j}$ and $v_\beta = v_\beta^{(n)}$ is the corresponding moment vector: $v_{\beta,i} = E_{\mu_\beta}[X^i]$, $i = 0, \dots, n$. Note that for any polynomial $P_\gamma \in \mathcal{P}_l$, $E_{\mu_\beta}[P_\gamma(X)] = 0$ if $\{\gamma_1, \dots, \gamma_l\} \supseteq \{\beta_1, \dots, \beta_k\}$ and otherwise $E_{\mu_\beta}[P_\gamma(X)] \geq 1/k$, since P_γ takes nonnegative integer values on $\mathbb{N}_0$. Then:

(a) For $P_\alpha \in \mathcal{P}_n$ define $m^{(n)}$ by $m^{(n-1)} = v_\alpha^{(n-1)}$ and $m_n = v_{\alpha,n} - 1/(2n)$. Then for $P_\gamma \in \mathcal{P}_{n-1}$, $L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] > 0$, and for $P_\gamma \in \mathcal{P}_n$ with $\gamma \neq \alpha$, $L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] - 1/(2n) \geq 1/(2n)$. On the other hand, $L_{P_\alpha}(m) = -1/(2n)$. Thus $m^{(n)}$ satisfies condition (a).

(b) For $P_\alpha \in \mathcal{P}_{n-1}$ define $m^{(n)}$ by $m^{(n-2)} = v_\alpha^{(n-2)}$, $m_{n-1} = v_{\alpha,n-1} - 1/(2(n-1))$, and $m_n = v_{\alpha,n}$. Then for $P_\gamma \in \mathcal{P}_{n-1}$, $L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] - 1/(2(n-1)) \geq 1/(2(n-1))$ for $\gamma \neq \alpha$ but $L_{P_\alpha}(m) = -1/(2(n-1))$. On the other hand, for $P_\gamma \in \mathcal{P}_n$,

$$L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] + \frac{1}{2(n-1)} \sum_{i=1}^{n} \gamma_i \geq E_{\mu_\alpha}[P_\gamma(X)] \geq 0. \tag{3.5}$$

Thus $m^{(n)}$ satisfies condition (b).

(c) Finally, for $P_\alpha \in \mathcal{P}_{n-1}$ define $m^{(n)}$ by $m^{(n-1)} = v_\alpha^{(n-1)}$ and $m_n = v_{\alpha,n} + c$ for some $c > 0$. Then for $P_\gamma \in \mathcal{P}_{n-1}$, $L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] \geq 0$, and in particular $L_{P_\alpha}(m) = 0$, while for $P_\gamma \in \mathcal{P}_n$, $L_{P_\gamma}(m) = E_{\mu_\alpha}[P_\gamma(X)] + c > 0$. Thus m satisfies condition (c). But since $L_{P_\alpha}(m) = 0$, if there is a measure ν realizing m then it be supported on $\{\alpha_1, \dots, \alpha_{n-1}\}$ and so, by the invertibility of the Vandermonde matrix, must in fact be μ_α. But then $E_\nu[X^n] = v_{\alpha,n} < m_n$, a contradiction. ■

Remark 3.4 In [6] R. E. Curto and L. A. Fialkow study the truncated moment problem on subsets of $\mathbb{R}^d$ using conditions based on extensions of moment sequences. The analogous condition in the case considered here would be the following:

$$L_P(m) \geq 0 \text{ for all } P \in \mathcal{P}_{n-1} \cup \mathcal{P}_n, \text{ and there exists a real number } \lambda \text{ such that if } m' = (m_1, m_2, \dots, m_n, \lambda) \text{ then } L_Q(m') \geq 0 \text{ for all } Q \in \mathcal{P}_{n+1}. \tag{3.6}$$

It is clear that conditions (3.1) and (3.4) are jointly equivalent to (3.6), since both of these give necessary and sufficient conditions for realizability. It would be interesting to see this equivalence directly. It is easy to see that (3.6) implies (3.1) and (3.4); for details see Proposition B.3 in Appendix B.

It is in general an open question whether or not the converse implication can also be proven directly, but this result may be obtained from Theorem 3.2 when the hypotheses of that theorem, with n replaced by $n+1$, are fulfilled (that is, when $L_P(m) > 0$ for $P \in \mathcal{P}_k$, $1 \le k \le n-1$). We omit details. We finally point out that in (3.6) we require an extension of $L_P(m)$ to a functional on polynomials of degree $n+1$, whereas [6] requires an extension to degree $n+2$. This is actually a direct consequence of the proof of [6, Theorem 2.2], if one takes into account that we are restricting the support a priori to be $\mathbb{N}_0$ and so a subset of $\mathbb{R}^+$.

3.1 Proof of Theorem 3.2

In this subsection we prove Theorem 3.2. For convenience we will in fact prove part (b) of the theorem with n replaced by $n+1$, that is, we will prove that if $L_P(m) \ge 0$ for all $P \in \mathcal{P}_n$ then there exists a finite set $\widetilde{\mathcal{P}}_n^{(m)} \subset \mathcal{P}_n$ such that for all $P = P_\alpha \in \mathcal{P}_n$ and all $N > \alpha_n$ there is a $\widetilde{P} \in \widetilde{\mathcal{P}}_n^{(m)}$ such that $L_{R_N\widetilde{P}}(m) \le L_{R_NP}(m)$. In particular, this means that in discussing Theorem 3.2 (b), and similarly in discussing Lemma 3.6 (b), we will assume that we are given a moment vector $m = m^{(n+1)} = (m_1, \dots, m_n, m_{n+1})$, so that $L_{R_NP}(m)$ is defined for $P \in \mathcal{P}_n$.

We begin by introducing some notation. Fix n, let $q = \lfloor n/2 \rfloor$, and write $i_0 = n - 2q$; if n is even then $i_0 = 0$ while if n is odd then $i_0 = 1$, so that α_{i_0+1} is the smallest α_i which is part of a pair $(\alpha_i, \alpha_{i+1}) = (j, j+1)$. If $\mathcal{J} = (J_1, J_2, \dots, J_q)$ is a strictly increasing q-tuple of positive integers and l is an integer satisfying $0 \le l \le q$ then we write $\mathcal{P}_{n,l,\mathcal{J}}$ for the set of polynomials $P_\alpha \in \mathcal{P}_n$ such that

$$\alpha_{i_0+2l} \le J_l, \quad \text{if } l > 0, \quad \text{and} \quad J_{l+1} < \alpha_{i_0+2l+2}, \quad \text{if } l < q. \tag{3.7}$$

We will speak of $\alpha_1, \dots, \alpha_{i_0+2l}$ as the *small* roots and $\alpha_{i_0+2l+1}, \dots, \alpha_n$ as the *large* roots at scale l, where the scale will not be mentioned if it is clear from context. If $l = 0$ then there are no small roots if n is even and one small root ($\alpha_1 = 0$) if n is odd; if $l = q$ there are no large roots and $\mathcal{P}_{n,l,\mathcal{J}}$ is in fact finite. Finally, for $0 \le l < q$ and $\gamma \in \mathcal{A}_{i_0+2l}$, with $\gamma_{i_0+2l} \le J_l$ when $l \ge 1$, we define $\beta = \beta(\gamma, l, \mathcal{J}) \in \mathcal{A}_n$ by

$$\beta_i = \gamma_i, \quad i \le i_0 + 2l; \qquad \beta_i = J_{l+1} + i - (i_0 + 2l) - 1, \quad i_0 + 2l < i \le n. \tag{3.8}$$

Note that $P_\beta \in \mathcal{P}_{n,l,\mathcal{J}}$ and that, given that $\beta_i = \gamma_i$ for $i = 1, \ldots, i_0 + 2l$, the values of β_i for $i_0 + 2l < i \le n$ are the smallest possible values consistent with this fact.

Proof of Theorem 3.2: The theorem is an immediate consequence of Lemmas 3.5 and 3.6 below; we summarize the argument here. We first show, in Lemma 3.5, that the sets $\mathcal{P}_{n,l,\mathcal{J}}$ for $0 \le l \le q$, which are clearly disjoint by (3.7), in fact partition $\mathcal{P}_n$. It then follows from Lemma 3.6 that for an appropriate choice of $\mathcal{J}$ and for any $N \in \mathbb{N}$ the infima over $P \in \mathcal{P}_n$ of $L_P(m)$ and $L_{R_N P}(m)$, when taken over each $\mathcal{P}_{n,l,\mathcal{J}}$ separately, are achieved on a finite subset $\mathcal{P}^*_{n,l,\mathcal{J}}$ of $\mathcal{P}_{n,l,\mathcal{J}}$. For $0 < l < q$ this is

$$\mathcal{P}^*_{n,l,\mathcal{J}} = \{P_{\beta(\gamma,l,\mathcal{J})} \mid \gamma \in \mathcal{A}_{i_0+2l}, \gamma_{i_0+2l} \le J_l\} \subset \mathcal{P}_{n,l,\mathcal{J}}, \tag{3.9}$$

for $l = 0$ we take $\mathcal{P}^*_{n,l,\mathcal{J}} = \{P_{\beta(\gamma,l,\mathcal{J})} \mid \gamma \in \mathcal{A}_{i_0+2l}\}$, and for $l = q$ we take $\mathcal{P}^*_{n,l,\mathcal{J}} = \mathcal{P}_{n,q,\mathcal{J}}$. Then part (b) of the theorem follows immediately, with $\tilde{\mathcal{P}}_n^{(m)} = \bigcup_{l=0}^{q} \mathcal{P}^*_{n,l,\mathcal{J}}$, and for part (a) we may simply take $P_n^{(m)}$ to be a polynomial on which $\inf_{P \in \tilde{\mathcal{P}}_n^{(m)}} L_P(m)$ is achieved. We note during the proof of Lemma 3.6 that $\mathcal{J}$, and so the sets $\mathcal{P}^*_{n,l,\mathcal{J}}$, can be chosen uniformly for m in bounded subsets of $\mathbb{R}^n$. ∎

Lemma 3.5 *Every $P_\alpha \in \mathcal{P}_n$ belongs to $\mathcal{P}_{n,l,\mathcal{J}}$ for some l, $0 \le l \le q$.*

Proof: Suppose that for some P_α we have $P_\alpha \notin \mathcal{P}_{n,l,\mathcal{J}}$ for $l = 0, \ldots, q-1$. Then we claim that $\alpha_{i_0+2l} \le J_l$ for $l = 1, \ldots, q$, which implies that $P_\alpha \in \mathcal{P}_{n,q,\mathcal{J}}$. The claim is proved by induction on l. For from $P_\alpha \notin \mathcal{P}_{n,0,\mathcal{J}}$ it follows that $\alpha_{i_0+2} \le J_1$. Similarly, from $\alpha_{i_0+2l} \le J_l$ and $P_\alpha \notin \mathcal{P}_{n,l+1,\mathcal{J}}$ it follows that $\alpha_{i_0+2(l+1)} \le J_{l+1}$. ∎

Lemma 3.6 *Let B be a bounded subset of $\mathbb{R}^n$. Then there exist $J_1, J_2, \ldots, J_q$ as above such that for $k = 0, \ldots, q-1$:*
(a) If $m \in B$ and $L_P(m) > 0$ for all $P \in \bigcup_{j=0}^{n-2} \mathcal{P}_j$ then for $P_\alpha \in \mathcal{P}_{n,k,\mathcal{J}}$,

$$L_{P_\alpha}(m) \ge L_{P_\beta}(m), \tag{3.10}$$

with $\beta = \beta(\alpha_1, \ldots, \alpha_{i_0+2k}, k, \mathcal{J})$ as given in (3.8).
(b) If $L_P(m) \ge 0$ for all $P \in \mathcal{P}_n$ then for $P_\alpha \in \mathcal{P}_{n,k,\mathcal{J}}$ and $N > \alpha_n$,

$$L_{R_N P_\alpha}(m) \ge L_{R_N P_\beta}(m), \tag{3.11}$$

with $\beta = \beta(\alpha_1, \ldots, \alpha_{i_0+2k}, k, \mathcal{J})$ given by (3.8).

Proof: Note that $P_{n,k,\mathcal{J}}$ and the index β of (3.10)–(3.11) are well defined once $J_1, \ldots, J_{k+1}$ have been specified. Thus we may proceed by induction, assuming that $J_1, \ldots, J_l$ have been constructed so that (a) and (b) are satisfied for $k < l$ and proving the existence of J_{l+1} so that they are satisfied for $k = l$. The case $l = 0$ is similar to other cases and we treat all values of l together.

Suppose now that we fix $\gamma \in \mathcal{A}_{i_0+2l}$ with $\gamma_{i_0+2l} \le J_l$. (More precisely, this holds if $l \ge 1$; if $l = 0$ then γ must be (0) if $i_0 = 1$ and an empty 0-tuple of indices if $i_0 = 0$). We will show that there then exists a number $J^{(\gamma)} > J_l$ such that if $J_{l+1} \ge J^{(\gamma)}$ then whenever $P_\alpha \in \mathcal{P}_{n,l,\mathcal{J}}$ has small roots (on scale l) given by γ, i.e., $\alpha_i = \gamma_i$ for $1 \le i \le i_0 + 2l$, and large roots such that for some j with $l < j \le q$,

$$\alpha_{i_0+2j-1} > \max\{\alpha_{i_0+2j-2} + 1, \beta_{i_0+2j-1}(\gamma, l, \mathcal{J})\}, \tag{3.12}$$

then under the hypotheses of (a),

$$L_{P_\alpha}(m) > L_{P_{\alpha'}}(m), \tag{3.13}$$

and under the hypotheses of (b),

$$L_{R_N P_\alpha}(m) > L_{R_N P_{\alpha'}}(m) \text{ if } N > \alpha_n. \tag{3.14}$$

Here $P_{\alpha'}$ is obtained from P_α by decreasing by 1 the values of a pair of zeros; specifically, $\alpha'_i = \alpha_i - 1$ if $i \in \{i_0 + 2j - 1, i_0 + 2j\}$ and $\alpha'_i = \alpha_i$ otherwise, so that by (3.12), $\alpha' \in \mathcal{A}_n$ and $P_{\alpha'} \in \mathcal{P}_{n,l,\mathcal{J}}$.

Once the existence of $J^{(\gamma)}$ is established, the induction step follows easily. For since there are only a finite number of $\gamma \in \mathcal{A}_{i_0+2l}$ with $\gamma_{i_0+2l} \le J_l$, we may define $J_{l+1} = \sup_\gamma J^{(\gamma)}$, so that (3.13) and (3.14) hold for all γ and all $P_\alpha \in \mathcal{P}_{n,l,\mathcal{J}}$, and from this obtain (3.10) and (3.11) by decreasing the large values of α, one pair at a time, generating a sequence $\alpha \to \alpha' \to \alpha'' \to \cdots \to \alpha^{(M)}$ with $\alpha^{(M)} = \beta(\alpha_1, \ldots, \alpha_{i_0+2l}, l, \mathcal{J})$.

We will separately find tentative values of $J^{(\gamma)}$ which lead to (3.13) under the hypotheses of (a) and to (3.14) under the hypotheses of (b); the larger of these two values is then the actual $J^{(\gamma)}$. In each case we write $P_\alpha \in \mathcal{P}_{n,l,\mathcal{J}}$ as $P_\alpha(x) = T_\alpha(x) Q_\gamma(x)$, where Q_γ contains the factors $x - \gamma_i = x - \alpha_i$ for small α_i (if $l = i_0 = 0$ then $Q_\alpha(x) = 1$) and T_α the corresponding factors for large α_i, and use

$$(x - \alpha)(x - \alpha - 1) - (x - \alpha + 1)(x - \alpha) = -2(x - \alpha), \tag{3.15}$$

to simplify the difference of the two sides of (3.13) or (3.14).

In case (a), we have from (3.15) that

$$T_\alpha(x) - T_{\alpha'(x)} = -2 \prod_{\substack{i>i_0+2l \\ i\neq i_0+2j}} (x-\alpha_i) = \sum_{p=0}^{2(q-l)-1} (-1)^p B_p x^p, \tag{3.16}$$

where B_p is twice the $(2(q-l)-p-1)^{\text{th}}$ symmetric function of the large roots, omitting α_{i_0+2j}. An easy computation shows that when $0 \le p \le 2(q-l)-2$ and all large roots satisfy $\alpha_i \ge J^{(\gamma)}$,

$$B_p \ge J^{(\gamma)} \frac{p+1}{2(q-l)-p-1} B_{p+1} \ge \frac{J^{(\gamma)}}{2(q-l)-1} B_{p+1}. \tag{3.17}$$

Thus by choosing $J^{(\gamma)}$ appropriately we may ensure that, for $J_{l+1} \ge J^{(\gamma)}$ and $P \in \mathcal{P}_{n,l,\mathcal{J}}$, all the ratios B_p/B_{p+1} are as large as we wish. As $Q_\alpha \in \bigcup_{i=0}^{n-2} \mathcal{P}_i$ we have by the hypothesis of (a) that $L_{Q_\alpha}(m) > 0$ and so the sum

$$L_{(T_\alpha - T_{\alpha'})Q_\alpha}(m) = \sum_{p=0}^{2(q-l)-1} (-1)^p B_p L_{x^p Q_\alpha}(m) \tag{3.18}$$

will be dominated, for sufficiently large $J^{(\gamma)}$, by the summand for $p = 0$. Hence, for such $J^{(\gamma)}$, (3.18) is positive, that is, (3.13) holds. How large one must choose $J^{(\gamma)}$ depends only on the $L_{x^p Q_\alpha}(m)$ and hence $J^{(\gamma)}$ can be chosen uniformly for $m \in B$.

To construct $J^{(\gamma)}$ in case (b) we proceed similarly. In parallel to (3.16) we now have

$$R_N(x)\big[T_\alpha(x) - T_{\alpha'(x)}\big] = \sum_{p=0}^{2(q-l)} (-1)^p C_p x^p, \tag{3.19}$$

where, again using (3.15), we see that C_p is twice the $(2(q-l)-p)^{\text{th}}$ symmetric function of the large roots, but with α_{i_0+2j} replaced by N. Hence by choosing $J^{(\gamma)}$ appropriately and requiring that all large roots satisfy $\alpha_i \ge J^{(\gamma)}$ and that $N \ge J^{(\gamma)}$ we may make all the ratios C_p/C_{p+1} arbitrarily large. The analogue of (3.18) is

$$L_{R_N(T_\alpha - T_{\alpha'})Q_\alpha}(m) = \sum_{p=0}^{2(q-l)} (-1)^p C_p L_{x^p Q \alpha}(m). \tag{3.20}$$

In this case we do not know that $L_{Q_\alpha}(m) > 0$, but certainly for sufficiently large $J^{(\gamma)}$ this sum is dominated by the term corresponding to $p = p_0$, where p_0 is the smallest index p such that $L_{x^pQ\alpha}(m) \neq 0$. But by Lemma 3.7 below, $(-1)^{p_0} L_{x^{p_0}Q\alpha}(m) > 0$, so that (3.20) is positive. As in case (a) the choice of $J^{(\gamma)}$ is uniform for $m \in B$. ■

Lemma 3.7 *Suppose that* $L_P(m) \geq 0$ *for all* $P \in \mathcal{P}_n$*, and that* l *and* p_0 *satisfy* $0 \leq l < q$ *and* $0 \leq p_0 \leq 2(q-l)$*. If* $Q \in \mathcal{P}_{i_0+2l}$ *satisfies* $L_{x^pQ}(m) = 0$ *for* $0 \leq p < p_0$ *then* $(-1)^{p_0} L_{x^{p_0}Q}(m) \geq 0$*.*

Proof: Let $k = 2(q-l)$. Given $Q \in \mathcal{P}_{i_0+2l}$ we choose $T \in \mathcal{P}_k$ so that its zeros are all greater than the largest zero of Q and hence $P = QT \in \mathcal{P}_n$. Then

$$T(x) = \sum_{p=0}^{k} (-1)^p A_p x^p, \tag{3.21}$$

with A_p the $(n-p)^{\text{th}}$ symmetric function of the roots of T, and as in (3.17) we may make all the ratios A_p/A_{p+1} arbitrarily large by choosing the roots of T large. Now we are given that

$$L_{TQ}(m) = \sum_{p=0}^{k} (-1)^p A_p L_{x^pQ}(m) \geq 0; \tag{3.22}$$

if $L_{x^pQ}(m) = 0$ for all $p < p_0$ then the p_0^{th} summand in (3.22) must be nonnegative, that is $(-1)^{p_0} L_{x^{p_0}Q}(m) \geq 0$. ■

4 A finite set of realizability conditions

Theorem 3.1 gives a characterization of realizability in terms of the values of infinitely many affine forms $L_P(m)$. The aim of this section is to give a procedure for determining realizability which involves evaluating only a small number of these forms. We begin with a definition which partitions the set of realizable moment vectors into two disjoint subsets, termed I-realizable and B-realizable. The terminology reflects the fact that I-realizable moment vectors lie in the interior of the set of realizable moment vectors and B-realizable ones on the boundary of that set (see Remark 4.5(b)).

Definition 4.1 A moment vector $m = m^{(n)}$ which is realizable on $\mathbb{N}_0$ is *I-realizable* if strict positivity holds in (3.1) and (3.4), that is, if $L_P(m) > 0$ for all $P \in \mathcal{P}_n \cup \mathcal{P}_{n-1}$; otherwise, that is if $L_P(m) = 0$ for some $P \in \mathcal{P}_n \cup \mathcal{P}_{n-1}$, it is *B-realizable.*

Lemma 4.2 *Suppose that $m = m^{(n)}$ is such that $m^{(n-1)}$ is B-realizable, and let $P \in \mathcal{P}_{n-i}$, with $i = 1$ or $i = 2$, be such that $L_P(m) = 0$. Then*
(a) $m^{(n-1)}$ is realized by a unique measure whose support is contained in the zeros of P, and
(b) $m^{(n)}$ is realizable if and only if $L_{x^iP}(m) = 0$, and the latter condition uniquely determines m_n.

Proof: Let μ be a measure on $\mathbb{N}_0$ realizing $m^{(n-1)}$, so that $0 = L_P(m) = E_\mu[P(X)]$, where $X : \mathbb{N}_0 \to \mathbb{N}_0$ is the identity. Since P is nonnegative on $\mathbb{N}_0$, the support of μ must be a subset of the $n-i$ distinct zeros $\alpha_1, \ldots, \alpha_{n-i}$ of P, so that $\mu = \sum_{j=1}^{n-i} c_j \delta_{\alpha_j}$ for some $c_1, \ldots, c_{n-i}$, and $m_k = E_\mu[X^k] = \sum_j c_j \alpha_j^k$ for $0 \le k \le n-i$. As the vectors $(\alpha_j, \alpha_j^2, \ldots, \alpha_j^{n-i})$, $1 \le j \le n-i$, are linearly independent, $c_1, \ldots, c_{n-i}$ and hence μ are uniquely determined by $m_1, \ldots, m_{n-i}$, which proves (a). If $m^{(n)}$ is realizable then the realizing measure must be μ, so that $L_{x^iP}(m) = E_\mu[X^i P(X)] = 0$, and conversely if $L_{x^iP}(m) = 0$ then μ realizes $m^{(n)}$; this proves the first statement of (b). The second statement of (b) follows from the fact that $x^i P(x) = x^n + \text{lower order terms}$. ■

Lemma 4.3 *If $m^{(n-1)}$ is I-realizable then $L_Q(m) > 0$ for all $Q \in \bigcup_{k=1}^{n-1} \mathcal{P}_k$.*

Proof: By definition, $L_Q(m) > 0$ if $Q \in \mathcal{P}_{n-2} \cup \mathcal{P}_{n-1}$. Assume then that for some $k \in \{1, \ldots, n-3\}$ and $Q \in \mathcal{P}_k$, $L_Q(m) = 0$. As $m^{(n-1)}$ is realizable, there exists a realizing measure μ on $\mathbb{N}_0$ and, as in the proof of Lemma 4.2, the support of μ must be contained in the zero set of Q. If $(n-1)-k$ is even, respectively odd, choose a polynomial T from $\mathcal{P}_{n-1-k}$, respectively $\mathcal{P}_{n-2-k}$, all the zeros of which are distinct from those of Q, so that TQ belongs to $\mathcal{P}_{n-1}$, respectively $\mathcal{P}_{n-2}$. Because the support of μ is contained in the zero set of TQ, $L_{TQ}(m) = 0$, a contradiction. ■

The next theorem gives an inductive procedure to determine whether a given n-tuple m is realizable and, in addition, whether I-realizable or B-realizable.

Theorem 4.4 *Suppose $n \in \mathbb{N}$ and $m = m^{(n)} = (m_1, \ldots, m_n) \in \mathbb{R}^n$. Then:*
(a) If $n = 1$ then $m = (m_1)$ is I-realizable if $m_1 > 0$, B-realizable if $m_1 = 0$, and not realizable if $m_1 < 0$.
(b) If $n \geq 2$ and $m^{(n-1)}$ is not realizable then $m^{(n)}$ is not realizable.
(c) If $n \geq 2$ and $m^{(n-1)}$ is I-realizable then a minimizing polynomial $P_n^{(m)}$, as in Theorem 3.2(a), exists, and $m^{(n)}$ is realizable if and only if

$$L_{P_n^{(m)}}(m) \geq 0. \tag{4.1}$$

In this case, $m^{(n)}$ is I-realizable if the inequality (4.1) *is strict and B-realizable if equality holds.*
(d) If $n \geq 2$ and $m^{(n-1)}$ is B-realizable, so that $L_P(m) = 0$ for some $P \in \mathcal{P}_{n-i}$ with $i \in \{1, 2\}$, then $m^{(n)}$ is realizable, and in particular B-realizable, if and only if $L_{x^i P}(m) = 0$.

Proof: (a) and (b) are trivial. To verify (c) we note that because $m^{(n-1)}$ is I-realizable, $P_n^{(m)} \in \mathcal{P}_n$ exists by Theorem 3.2(a) and Lemma 4.3, and for any $P \in \mathcal{P}_n$, $L_P(m) \geq L_{P_n^{(m)}}(m)$. The conclusion then follows immediately from Definition 4.1 and Theorem 3.1. Finally, for (d), if $m^{(n-1)}$ is B-realizable then Lemma 4.2 immediately gives the stated criterion for realizability, and this must be B-realizability, since by definition $L_P(m) = 0$ for some $P \in \mathcal{P}_{n-1} \cup \mathcal{P}_{n-2}$ and this, with Lemma 4.3, would contradict I-realizability of $m^{(n)}$. ■

Remark 4.5 (a) To make the inductive procedure of Theorem 4.4 effective we must be able to determine explicitly the polynomials $P_n^{(m)}$ for $n \geq 2$. In the next sections we do this explicitly for $n = 2, \ldots, 5$ and give a recursive construction for $n \geq 6$.
(b) As remarked above, I-realizable moment vectors lie in the interior of the set of all realizable moments, and B-realizable ones on the boundary. The first statement follows from the fact that, as explained in the proof of Theorem 3.2, the infimum of $L_P(m)$ over $P \in \mathcal{P}_n$ is in fact a minimum over a finite set of polynomials, and this set can be chosen uniformly in m on compact sets; thus if this minimum is strictly positive for m it will be positive also for nearby moment vectors. On the other hand, if m is B-realizable then $L_P(m) = 0$ for some $P \in \mathcal{P}_{n-i}$ with $i = 0$ or 1, and decreasing m_{n-i} by an

arbitrarily small amount gives a moment vector m' with $L_P(m') < 0$, so that m' is not realizable.

(c) If $m = m^{(n)}$ is B-realizable then there is a minimum index k such that $L_Q(m) = 0$ for some $Q \in \mathcal{P}_k$, so that $m^{(j)}$ is I-realizable if $j < k$ and B-realizable if $j \geq k$. Then following the ideas of the proof of Lemma 4.2 one sees immediately that the support of the unique measure realizing m is contained in the set of zeros of Q. Moreover, this support cannot be a subset of the zero set of any polynomial $\widetilde{Q} \in \mathcal{P}_j$ with $j < k$, by the minimality of k, so that any realizing measure μ satisfies

$$\left\lfloor \frac{k+1}{2} \right\rfloor \leq |\operatorname{supp} \mu| \leq k. \tag{4.2}$$

Any value for $|\operatorname{supp} \mu|$ in the range (4.2) is possible. For example, the upper bound is achieved if $m^{(n)}$ is the moment vector of a measure with support $K_k = \{1, 2, \ldots, k\}$ if k is even or $K_k = \{0, 1, 2, \ldots, k-1\}$ if k is odd, and other values are achieved when $m^{(n)}$ is the moment vector of a measure with support obtained from K_k by omitting an arbitrary set of odd integers.

(d) One may ask similarly about possible values of $|\operatorname{supp} \mu|$ when the moment vector $m = m^{(n)}$ is I-realizable and μ is a measure realizing m. In this case the realizing measure is not uniquely determined by m, so it is natural to ask for the minimum possible size of the support (which will of course depend on m). A result of Carathéodory (see, for example, [8]) implies that there is a measure μ realizing m supported on at most $n+1$ points of $\mathbb{N}_0$; then because the support of μ cannot be contained in the zero set of any $Q \in \mathcal{P}_j$ with $j \leq n$, since otherwise m would be B-realizable,

$$\left\lfloor \frac{n+2}{2} \right\rfloor \leq \min_{\mu} |\operatorname{supp} \mu| \leq n+1, \tag{4.3}$$

where the minimum is over all the measures realizing m. In the opposite direction one may show that there exists a set $S \subset \mathbb{N}_0$ with $n+1 \leq |S| < \infty$ such that there exists a measure realizing m with support S, and moreover that for any such S and any $S' \subset \mathbb{N}_0$ with $S \subset S'$ there exists also a realizing measure with support S'. In particular, one can find a measure with support equal to $\mathbb{N}_0$. We do not have a general answer to how small $|S|$ may be.

The restriction $|S| \geq n+1$ above arises as follows. Let us denote the minimum in (4.3) by j_m and the support of a corresponding realizing measure by S_m; if $j_m = |S_m| = n+1$ then certainly $|S| \geq n+1$, so we may suppose that

$j_m \le n$. Note first that for any j with $\lfloor (n+2)/2 \rfloor \le j \le n$ the set of moment vectors $\tilde{m}$ with $j_{\tilde{m}} = j$ is a countable family of simplices of dimension $j-1$ (and hence in particular has Lebesgue measure zero). Now S_m itself cannot play the role of S above, for any $S' \subset \mathbb{N}_0$ with $S' \supset S_m$ and $|S'| = n+1$ could not be the support of a realizing measure for m, since m would lie on the boundary of the open simplex of moment vectors realized by measures with support S'. On the other hand, if $j_m \le n-1$ then generically there will not exist a realizing measure with support S satisfying $j_m < |S| \le n$, since the set of moment vectors with such a realizing measure has Lebesgue measure zero within the simplex of moment vectors which have a realizing measure with support S_m.

(e) There may be several minimizing polynomials for $m^{(n)}$, i.e., polynomials which satisfy the conclusion of Theorem 3.2(a), but the set of such polynomials does not depend on m_n. For if $\widetilde{m} = (m_1, \ldots, m_{n-1}, \widetilde{m}_n)$ then for $Q \in \mathcal{P}_n$, $L_Q(m) - m_n = L_Q(\widetilde{m}) - \widetilde{m}_n$, from which it follows that $L_P(m) = \inf_{Q \in \mathcal{P}_n} L_Q(m)$ if and only if $L_P(\widetilde{m}) = \inf_{Q \in \mathcal{P}_n} L_Q(\widetilde{m})$.

(f) We will discuss the nonuniqueness of the minimizing polynomial for $m^{(n)}$ when $m^{(n-1)}$ is I-realizable, the only case for which this polynomial is needed in Theorem 4.4. Let $\widetilde{m}_n \in \mathbb{R}$ be defined by the condition $L_{P_n^{(m)}}(\widetilde{m}) = 0$, where $\widetilde{m}^{(n)} = (m_1, \ldots, m_{n-1}, \widetilde{m}_n)$ and $P_n^{(m)}$ is some minimizing polynomial for $m^{(n)}$; note that by (e) this condition is independent of the choice of $P_n^{(m)}$. Then $\widetilde{m}^{(n)}$ is B-realizable, so that by Lemma 4.2 there is a unique realizing measure μ for $\widetilde{m}^{(n)}$ with support in the zero set of $P_n^{(m)}$. When the support of μ contains fewer than n points there will be several minimizing polynomials, specifically, those polynomials in $\mathcal{P}_n$ whose zero set contains this support.

(g) Some of the results which have been obtained above by reference to a realizing measure for m may also be obtained or strengthened by purely algebraic means. See Appendix B.

5 Realizability for $n = 2, 3$

Theorem 4.4 gives simple and explicit realizability conditions when $n = 1$. In this section we obtain the polynomials $P_n^{(m)}$ when $n = 2$ and 3 and thus give simple conditions for these values of n. We first define

$$k_1 := \lfloor m_1 \rfloor \quad \text{and} \quad \theta_1 := m_1 - k_1, \tag{5.1}$$

and, if $m_1 > 0$,

$$k_2 := \left\lfloor \frac{m_2}{m_1} \right\rfloor \quad \text{and} \quad \theta_2 := \frac{m_2}{m_1} - k_2. \tag{5.2}$$

Theorem 5.1 *Suppose that n is 2 or 3 and that we are given the moment vector $m = m^{(n)} \in \mathbb{R}^n$. Then*
(a) If $n = 2$ and $m^{(1)}$ is I-realizable (that is, $m_1 > 0$) then

$$P_2^{(m)}(x) = (x - k_1)(x - k_1 - 1), \tag{5.3}$$

and $m^{(2)}$ is realizable if and only if

$$m_2 - m_1^2 \geq \theta_1(1 - \theta_1). \tag{5.4}$$

In particular, $m^{(2)}$ is I-realizable if the inequality in (5.4) *is strict, and B-realizable if equality holds there.*
(b) If $n = 3$ and $m^{(2)}$ is I-realizable (that is, by (a), if $m_1 > 0$ and $m_2 - m_1^2 > \theta_1(1 - \theta_1)$) then

$$P_3^{(m)}(x) = x(x - k_2)(x - k_2 - 1), \tag{5.5}$$

and $m^{(3)}$ is realizable iff

$$\frac{m_3}{m_1} - \left(\frac{m_2}{m_1}\right)^2 \geq \theta_2(1 - \theta_2). \tag{5.6}$$

In particular, $m^{(3)}$ is I-realizable if the inequality in (5.6) *is strict, and B-realizable otherwise.*

Proof: (a) Recall that $\mathcal{P}_2$ is the set of all polynomials of the form $T_k(x) = (x - k)(x - k - 1)$ with $k \in \mathbb{N}_0$. But for any $k \in \mathbb{N}_0$ with $k \neq k_1$, a simple computation shows that

$$L_{T_k}(m) - L_{T_{k_1}}(m) = (k - k_1)^2 \left(1 + \frac{1 - 2\theta_1}{k - k_1}\right) \geq 0, \tag{5.7}$$

where the inequality follows from $|k - k_1| \geq 1 \geq |1 - 2\theta_1|$. Thus, $P_2^{(m)} = T_{k_1}$. Moreover,

$$L_{T_{k_1}}(m) = m_2 - (2k_1 + 1)m_1 + k_1(k_1 + 1) = m_2 - m_1^2 - \theta_1(1 - \theta_1), \tag{5.8}$$

and (5.4) follows from Theorem 4.4.

(b) $\mathcal{P}_3$ is the set of all polynomials $S_k(x) = x(x-k)(x-k-1)$ with $k \in \mathbb{N}$. In parallel to (5.7) and (5.8), we have that for any $k \geq 1$ with $k \neq k_2$,

$$L_{S_k}(m) - L_{S_{k_2}}(m) = (k-k_2)^2 \left(1 + \frac{1-2\theta_2}{k-k_2}\right) m_1 \geq 0, \tag{5.9}$$

and

$$\begin{aligned} L_{S_{k_2}}(m) &= m_3 - (2k_2+1)m_2 + k_2(k_2+1)m_1 \\ &= m_3 - \left(\frac{m_2^2}{m_1}\right) - \theta_2(1-\theta_2)m_1. \end{aligned} \tag{5.10}$$

Thus $P_3^{(m)} = S_{k_2}$ and (5.10), with Theorem 4.4, yields (5.6). ■

We note that (5.4) was given by Percus and Yamada [18, 19, 24] as a necessary condition for realizability.

Remark 5.2 Theorem 5.1 covers, for $n = 2$ and 3, the determination of realizability when $m^{(n-1)}$ is I-realizable. The method of making this determination when $m^{(n-1)}$ is B-realizable is implicit in Theorem 4.4, but for clarity we discuss this briefly here.
(a) When $n = 2$, $m^{(1)} = (m_1)$ is B-realizable iff $m_1 = 0$, that is, iff $L_x(m) = 0$. Then Theorem 4.4(d) tells us that $m^{(2)} = (m_1, m_2)$ is realizable (and necessarily B-realizable) iff $L_{x^2}(m) = m_2 = 0$.
(b) When $n = 3$, $m^{(2)} = (m_1, m_2)$ is B-realizable if either (i) $m_1 = m_2 = 0$ (here we have used (a)) or (ii) $m_1 > 0$ and $L_{P_2^{(m)}}(m) = 0$, i.e., by Theorem 5.1(a), $m_2 - m_1^2 = \theta_1(1-\theta_1)$. Theorem 4.4(d) tells us that in case (i), $m^{(3)} = (m_1, m_2, m_3)$ is realizable iff $L_{x^3}(m) = m_3 = 0$, and in case (ii), $m^{(3)}$ is realizable iff $L_{xP_2^{(m)}}(m) = L_{x(x-k_1)(x-k_1-1)}(m) = 0$, i.e., if

$$m_3 = (2k_1+1)m_2 - k_1(k_1+1)m_1. \tag{5.11}$$

It can in fact be shown that in case (ii), $k_2 = k_1$ unless $0 < m_1 < 1$, when $k_1 = 0$, $k_2 = 1$, $\theta_1 = m_1$, and $\theta_2 = 0$; in any case (5.11) holds iff (5.6) holds with equality.

6 Realizability for $n \geq 4$

In this section we first obtain more detailed properties of the polynomials $P_n^{(m)}$ for general n, and then derive from these, in Section 6.1, an iterative procedure which reduces the computation of $P_n^{(m)}$ to the solution of a moment problem of degree $n-2$. In Section 6.2 we specialize to the cases $n=4$ and $n=5$, in which we can give very explicit conditions for realizability. Throughout we assume that we are given a realizable moment vector $m^{(n-1)} = (m_1, \ldots, m_{n-1})$ (recall from Remark 4.5(e) that $P_n^{(m)}$ does not depend on m_n). Theorem 3.2 implies that $P_n^{(m)}$ is defined as long as $m^{(n-2)}$ is I-realizable, but we will assume throughout this section that in fact $m^{(n-1)}$ is I-realizable; this assumption, which simplifies the construction of $P_n^{(m)}$, is justified by the fact that it is only this case which is needed for the inductive scheme outlined in Theorem 4.4.

We first consider the well-understood Stieltjes problem, that is, the problem of realizing a moment vector on $\mathbb{R}_+$, and for this purpose recall from Section 1 and Appendix A the definitions of I-realizability and B-realizability for this problem, and of the Hankel matrices (see (1.1)). In particular, for $\widehat{m}_n \in \mathbb{R}$ we let $\widehat{A}(k)$, $\widehat{B}(k)$, and $\widehat{C}_n$ be the Hankel matrices formed from the moment vector $\widehat{m}^{(n)} := (m_1, \ldots, m_{n-1}, \widehat{m}_n)$.

Theorem 6.1 *Suppose that $m^{(n-1)}$ is I-realizable for the Stieltjes problem and that $\widehat{m}_n$ is the smallest value such that $\widehat{C}_n \geq 0$. Then $\widehat{m}^{(n)}$ is realizable for the Stieltjes problem by a unique measure ν; moreover, if n is even with $n = 2k$ then $|\operatorname{supp} \nu| = k$ and $0 \notin \operatorname{supp} \nu$, while if n is odd with $n = 2k+1$ then $|\operatorname{supp} \nu| = k+1$ and $0 \in \operatorname{supp} \nu$.*

Proof: This is a fairly immediate consequence of the results in [5]; for details see Proposition A.2(c). ■

In the remainder of this section we will let $\widehat{m}^{(n)}$ and ν be as in Theorem 6.1. We will write $\operatorname{supp} \nu = \{y_1, \ldots, y_k\}$ when $n = 2k$ and $\operatorname{supp} \nu = \{0, y_1, \ldots, y_k\}$ when $n = 2k+1$, where $y_1 < y_2 < \cdots < y_k$ and, if n is odd, $0 < y_1$. It is important to note that $\operatorname{supp} \nu$ can be computed explicitly as the set of roots of a certain polynomial determined by $m^{(n-1)}$; see Remark A.3.

Our approach to the realization problem on $\mathbb{N}_0$ is quite parallel to the above. Suppose that $P_n^{(m)}(x) = P_\alpha(x)$, with $\alpha = (\alpha_1, \ldots, \alpha_n)$, and let $\widetilde{m}_n \in \mathbb{R}$ be the unique number for which, with $\widetilde{m}^{(n)} = (m_1, \ldots, m_{n-1}, \widetilde{m}_n)$, $L_{P_n^{(m)}}(\widetilde{m}) = 0$. Then $\widetilde{m}_n$ is the smallest value for which $\widetilde{m}^{(n)}$ is realizable; moreover (see Remark 4.5(f)), $\widetilde{m}^{(n)}$ is B-realizable and by Lemma 4.2 there is a unique realizing measure μ for $\widetilde{m}^{(n)}$, with $\operatorname{supp} \mu \subset \{\alpha_1, \ldots, \alpha_n\}$.

Remark 6.2 (a) If $\operatorname{supp}\nu \subset \mathbb{N}_0$ then $\widetilde{m}_n = \widehat{m}_n$ and $m^{(n)}$ is realizable on $\mathbb{N}_0$ if and only if it is realizable for the Stieltjes problem. We will therefore typically assume below that $\operatorname{supp}\nu \not\subset \mathbb{N}_0$.
(b) When $\operatorname{supp}\nu \not\subset \mathbb{N}_0$ there must be a point $t \in \mathbb{N}_0$ with $t \in \operatorname{supp}\mu$ and $t \notin \operatorname{supp}\nu$, since $|\operatorname{supp}\mu| \geq |\operatorname{supp}\nu|$ by Remark 4.5(c).
(c) If n is odd then the I-realizability of $m^{(n-1)}$, together with Remark 4.5(c), implies that $0 \in \operatorname{supp}\mu$.

Now one may easily check when $n = 2$, $\nu = \delta_{m_1}$, and when $n = 3$, $\nu = (1 - m_1^2/m_2)\delta_0 + (m_1/m_2^2)\delta_{m_1}$, and one may determine the corresponding values of α (where again $P_n^{(m)}(x) = P_\alpha(x)$) from Theorem 5.1. This leads to:

$$\begin{array}{lll} n = 2: & \operatorname{supp}\nu = \{m_1\}, & \alpha = (\lfloor m_1 \rfloor, \lfloor m_1 \rfloor + 1); \\ n = 3: & \operatorname{supp}\nu = \{0, m_2/m_1\}, & \alpha = (0, \lfloor m_2/m_1 \rfloor, \lfloor m_2/m_1 \rfloor + 1). \end{array} \tag{6.1}$$

These two examples thus suggest a close connection between $\operatorname{supp}\nu$ and the values of α (which are the possible points of $\operatorname{supp}\mu$); one might hope, for example, that when $n = 4$, $\alpha = \{\lfloor y_1 \rfloor, \lfloor y_1 \rfloor + 1, \lfloor y_2 \rfloor, \lfloor y_2 \rfloor + 1\}$. Consideration of explicit examples shows that this is not always the case, but, as we now show, knowledge of $\operatorname{supp}\nu$ does give some information about $\operatorname{supp}\mu$.

We first prove an interleaving property. Here and below we will use the fact that if $Q(x)$ is a polynomial of degree at most $n-1$ then $E_\mu[Q(X)] = E_\nu[Q(X)] = L_Q(m)$. From this it follows that if $Q(x) \geq 0$ either on $\operatorname{supp}\nu$ or on $\operatorname{supp}\mu$ then $L_Q(m) \geq 0$, and if also $Q(x_0) > 0$ at some point x_0 of $\operatorname{supp}\nu$ or of $\operatorname{supp}\mu$ then $L_Q(m) > 0$. Similarly, if $Q(x) = 0$ on $\operatorname{supp}\nu$ or on $\operatorname{supp}\mu$ then $L_Q(m) = 0$.

Proposition 6.3 *Suppose that* $\operatorname{supp}\nu \not\subset \mathbb{N}_0$. *Then if* $n = 2k$ *or* $n = 2k+1$ *there exist points* $\eta_0, , \ldots, \eta_k$ *in* $\operatorname{supp}\mu$ *such that*

$$\eta_0 < y_1 < \eta_1 < y_2 < \cdots < y_k < \eta_k. \tag{6.2}$$

and if n *is odd,* $0 < \eta_0$. *In particular,* $|\operatorname{supp}\mu| > |\operatorname{supp}\nu|$.

Proof: For $j = 1, \ldots, k-1$ we choose points $z_j, z_j' \in (y_j, y_{j+1})$ such that no point of $\operatorname{supp}\mu$ lies any of the intervals $(y_j, z_j]$ and $[z_j', y_{j+1})$. Suppose first that n is even, with $n = 2k$. To show that there must be an η_j with $\eta_j \in (y_j, y_{j+1})$, $j = 1, \ldots, k-1$, we suppose not and consider the polynomial

$$Q_j(x) = \prod_{l=1}^{k}(x - y_l) \prod_{l=1}^{j-1}(x - z_l) \prod_{l=j+1}^{k-1}(x - z_l'). \tag{6.3}$$

Then $Q_j(x) = 0$ for $x \in \operatorname{supp}\nu$, so $L_{Q_j}(m) = 0$, but $Q_j(x) \geq 0$ on $\operatorname{supp}\mu$, with $Q_j(t) > 0$ for $t \in \operatorname{supp}\mu \setminus \operatorname{supp}\nu$ (see Remark 6.2), so $L_{Q_j}(m) > 0$, a contradiction. For existence of η_0 and η_k we argue similarly from

$$Q_0(x) = \prod_{l=1}^{k}(x-y_l)\prod_{l=1}^{k-1}(x-z'_l) \quad \text{and} \quad Q_k(x) = -\prod_{l=1}^{k}(x-y_l)\prod_{l=1}^{k-1}(x-z_l). \tag{6.4}$$

For $n = 2k+1$ we consider similarly $\widehat{Q}_j(x) = xQ_j(x)$, $j = 0, \ldots, k$. ■

The next result shows that knowledge of $y_1, \ldots, y_k$ tells us about at least one of the pairs $(\alpha_i, \alpha_{i+1}) = (\alpha_i, \alpha_i + 1)$ in the fashion suggested by (6.1). To state it we write $Y_j := \lfloor y_j \rfloor$ for $j = 1, \ldots, k$.

Corollary 6.4 *Suppose that* $\operatorname{supp}\nu \not\subset \mathbb{N}_0$. *Then:*
(a) If $n = 2k$ *then for some* j, $1 \leq j \leq k$:

$$\alpha_{2j-1} = Y_j,\ \alpha_{2j} = Y_j + 1,\ \textit{and}\ \{\alpha_{2j-1}, \alpha_{2j}\} \subset \operatorname{supp}\mu; \tag{6.5}$$

(b) If $n = 2k+1$ *then for some* j, $1 \leq j \leq k$:

$$\alpha_{2j} = Y_j,\ \alpha_{2j+1} = Y_j + 1,\ \textit{and}\ \{\alpha_{2j}, \alpha_{2j+1}\} \subset \operatorname{supp}\mu. \tag{6.6}$$

Proof: We treat the even case $n = 2k$, using the notation of Proposition 6.3; the odd case is similar. Define

$$S = \big\{ j \in \{1, 2, \ldots, k\} \,\big|\, \alpha_{2j-1} \leq \eta_{j-1} \big\}. \tag{6.7}$$

Certainly $1 \in S$, since $\eta_0 \in \operatorname{supp}\mu \subset \{\alpha_1, \ldots, \alpha_{2k}\}$. Let $j_0 = \max S$. If $j_0 = k$ then necessarily $\alpha_{2k-1} = \eta_{k-1}$ and $\alpha_{2k} = \eta_k$, so that, since $\alpha_{2k} = \alpha_{2k-1} + 1$, (6.5) holds with $j = k$. Suppose then that $j_0 < k$. Because $\alpha_{2j_0+1} > \eta_{j_0}$ we must have $\alpha_{2j_0-1} = \eta_{j_0-1}$, $\alpha_{2j_0} = \eta_{j_0}$, so that (6.5) holds with $j = j_0$. ■

6.1 General inductive procedure

We can now give a general procedure for the reduction of the truncated moment problem of degree n, $n \geq 4$, to several truncated moment problems of degree $n-2$. For the moment we suppose that $\operatorname{supp}\nu \not\subset \mathbb{N}_0$ and fix l with $1 \leq l \leq k = \lfloor n/2 \rfloor$. From $n \geq 4$ and our assumption that $m^{(n-1)}$ is I-realizable it follows that

$$m_2 - (2Y_l + 1)m_1 + Y_l(Y_l + 1) > 0. \tag{6.8}$$

We may thus define $c_l(m) := (m_2 - (2Y_l + 1)m_1 + Y_l(Y_l + 1))^{-1}$ and so the new moment vector $M_l^{(n-2)}(m) := (M_{l,0}(m), \ldots, M_{l,n-2}(m))$ by

$$M_{l,i}(m) := c_l(m)(m_{i+2} - (2Y_l + 1)m_{i+1} + Y_l(Y_l + 1)m_i), \tag{6.9}$$

where the factor $c(m)$ insures that $M_{l,0}(m) = 1$. For the moment we suppress the dependence of $M(m)$ on l. The definition is chosen so that for any polynomial Q of degree at most $n - 2$,

$$L_Q(M(m)) = c(m)L_{(x-Y_l)(x-Y_l-1)Q}(m).$$

Lemma 6.5 *(a) If the probability measure σ realizes $m^{(n)}$ on $\mathbb{N}_0$ then the probability measure σ' with $d\sigma'(x) = c(m)(x - Y_l)(x - Y_l - 1)d\sigma(x)$ realizes $M^{(n-2)}(m)$ on $\mathbb{N}_0$.*
(b) $M^{(n-3)}(m)$ is I-realizable.

Proof: (a) This follows from $E_{\sigma'}[X^k] = c(m)E_\sigma[(X - Y_1)(X - Y_1 - 1)X^k]$.
(b) We use the characterization of I-realizability given in Remark 4.5(b). Let U be a neighborhood of $m^{(n-1)}$ such that if $\bar{m}^{(n-1)} \in U$ then $\bar{m}^{(n-1)}$ is realizable. Since the matrix $\left(\partial M_p/\partial m_{q+2}\right)_{p,q=1}^{n-3}$ obtained from (6.9) is triangular, with nonzero diagonal elements $c(m)$, we may apply the inverse function theorem to the map (6.9), at fixed m_1 and m_2, to conclude that there is a neighborhood V of $M^{(n-3)}(m)$ such that if $\bar{M}^{(n-3)} \in V$ then $\bar{M}^{(n-3)} = M^{(n-3)}(\bar{m})$ for some $\bar{m} \in U$. But then (a) implies that $\bar{M}^{(n-3)}$ is also realizable. ■

Recall that $\widetilde{m}_n$ is the minimal value for which $\widetilde{m}^{(n)} = (m_1, \ldots, m_{n-1}, \widetilde{m}_n)$ is realizable on $\mathbb{N}_0$ and that μ then denotes the unique probability measure realizing $\widetilde{m}^{(n)}$; note that then Lemma 6.5(a) implies that $M^{(n-2)}(\widetilde{m})$ is also realizable. We define $\mathcal{N}_n^m := \operatorname{supp}\mu$; $\mathcal{N}_n^m$ contains at most n points.

We give below a procedure to compute $\mathcal{N}_n^m$ by induction on n. This solves the realizability problem, for knowledge of $\mathcal{N}_n^m$ determines realizability: one must simply choose a polynomial $P \in \mathcal{P}_n$ whose zeros contain N_n^m (see Remark 4.5(f)); then $m^{(n)}$ is I-realizable if and only if $L_P(m^{(n)}) > 0$, and is B-realizable if and only if $L_P(m^{(n)}) = 0$.

Theorem 6.6 *If $l = j$, where j has the property that $\{Y_j, Y_{j+1}\} \subset \operatorname{supp}\mu$, then $M^{(n-2)}(\widetilde{m})$ is B-realizable by a unique probability measure μ'. The support of μ' is disjoint from $\{Y_j, Y_j + 1\}$ and $\operatorname{supp}\mu = (\operatorname{supp}\mu') \cup \{Y_j, Y_j + 1\}$.*

Proof: Note that a j with $\{Y_j, Y_{j+1}\} \subset \operatorname{supp}\mu$ exists by Corollary 6.4. By Lemma 6.5(b) and Theorem 3.2 there exists a minimizing polynomial $P_{n-2}^{(M)}$ for $M^{(n-2)}(m)$. Let $P_n^{(m)}$ be a minimizing polynomial for $m^{(n)}$; minimality of $\widetilde{m}_n$ implies that $L_{P_n^{(m)}}(\widetilde{m}) = 0$. By Corollary 6.4, $P_n^{(m)} = (x - Y_j)(x - Y_j - 1)Q_{n-2}^{(m)}$ for some $Q_{n-2}^{(m)} \in \mathcal{P}_{n-2}$. But then $L_{Q_{n-2}^{(m)}}(M(\widetilde{m})) = c(m)L_{P_n^{(m)}}(\widetilde{m}) = 0$, and as $M^{(n-2)}(\widetilde{m})$ is realizable it must be B-realizable and $Q_{n-2}^{(m)}$ must be a minimizing polynomial for it. By Lemma 4.2, $M^{(n-2)}(\widetilde{m})$ is realized by a unique probability measure μ' which must be $d\mu'(x) = c(m)(x - Y_j)(x - Y_j - 1)d\mu(x)$. The support properties of μ' follow. ■

We can now describe the inductive procedure for computing $\mathcal{N}_n^m$. The key difficulty is that we do not know *a priori* a "correct" index j arising from Corollary 6.4, and must carry our the recursion for each possible index (see step 4 below).

1. The base cases are $n = 2$ and $n = 3$. For these it follows from (6.1) and the accompanying discussion that $\mathcal{N}_2^m = \{m_1\}$ if $m_1 \in \mathbb{N}_0$ and otherwise $\mathcal{N}_2^m = \{\lfloor m_1 \rfloor, \lfloor m_1 \rfloor + 1\}$. Similarly $\mathcal{N}_3^m = \{0, m_2/m_1\}$ if $m_2/m_1 \in \mathbb{N}_0$, and otherwise $\mathcal{N}_3^m = \{0, \lfloor m_2/m_1 \rfloor, \lfloor m_2/m_1 \rfloor + 1\}$.

The induction for $n > 2$ proceeds as follows:

2. Determine $\operatorname{supp}\nu$, that is, $\{y_1, \ldots, y_k\}$ if $n = 2k$ or $\{0, y_1, \ldots, y_k\}$ if $n = 2k + 1$. The procedure is given in Remark A.3; in summary:

   • If $n = 2k$ then $y_1, \ldots, y_k$ are the roots of $x^k - \sum_{i=0}^{k-1} \varphi_i x^i = 0$, where $(\varphi_0, \ldots, \varphi_{k-1}) = (m_k, \ldots, m_{n-1})A(k-1)^{-1}$.

   • If $n = 2k+1$ then $0, y_1, \ldots, y_k$ are the roots of $x^{k+1} - \sum_{i=1}^{k} \varphi_i x^i = 0$, where $(\varphi_1, \ldots, \varphi_k) = (m_{k+1}, \ldots, m_{n-1})B(k-1)^{-1}$.

3. If $\operatorname{supp}\nu \subset \mathbb{N}_0$ then $\mathcal{N}_n^m = \operatorname{supp}\nu$.

4. If $\operatorname{supp}\nu \not\subset \mathbb{N}_0$ then for each l, $l = 1, \ldots, k$, define $M_l(m)$ by (6.9). By Lemma 6.5(b), $M_l^{(n-3)}(m)$ is I-realizable. Find recursively the corresponding support $\mathcal{N}_{l,n-2}^{M(m)}$. If $\mathcal{N}_{l,n-2}^{M(m)} \cap \{Y_l, Y_l + 1\} \neq \emptyset$ then reject this value of l.

5. Choose for each l a polynomial $Q_l \in \mathcal{P}_n$ whose set of roots contains $\mathcal{N}_{l,n-2}^{M(m)} \cup \{Y_l, Y_l + 1\}$.

6. Find i such that $L_{Q_i}(m)$ is minimal among all $L_{Q_l}(m)$, $l = 1, \dots, k$. Then Q_i is a minimizing polynomial for $m^{(n)}$. There is a unique realizing measure for $m^{(n-1)}$ with support in the zero set of Q_i (it is also the unique realizing measure for $\widetilde{m}^{(n)}$) which may be calculated by the procedure outlined in Lemma 4.2(a). $\mathcal{N}_n^m$ is the support of this measure.

Theorem 6.7 *The set $\mathcal{N}_n^m$ produced in step 1, 3, or 6 of the above algorithm is the support of μ.*

Proof: In the case in which $\mathcal{N}_n^m$ arises at step 1 or step 3 this follows from Section 5 or Remark 6.2(a), respectively. Suppose then that $\mathcal{N}_n^m$ arises at step 6. Theorem 6.6 implies that if $l = j$, with j as in Corollary 6.4, then Q_l is a minimizing polynomial for $m^{(n)}$; note that the procedure will not terminate at step 4 in this case. Thus if i is as in step 6, $L_{Q_i}(m) \le L_{Q_j}(m)$ implies that Q_i is also a minimizing polynomial. The characterization of $\mathcal{N}_n^m$ then follows from Remark 4.5(f). ∎

Because there are $k = \lfloor n/2 \rfloor$ choices for l at step 4, the algorithm can require $\lfloor n/2 \rfloor!$ stages. For moderate size of n this should not be a real restriction. The time might be shortened by the fact that the procedure can terminate at step 4, but we have no estimate for how often this may occur.

6.2 Explicit formulas for $n = 4$ and 5

We now specialize to the cases $n = 4$ and $n = 5$. Of course, the recursive procedure of Section 6.1 could be used to reduce these to the $n = 2$ and $n = 3$ cases of Section 5, but there is a simpler answer: we can obtain explicit formulas for $\operatorname{supp} \nu$ and hence for $P_n^{(m)}$. In stating the relevant theorems we assume, by Remark 6.2(c), that $\operatorname{supp} \nu \not\subset \mathbb{N}_0$. When $n = 4$ we define

$$t_1 = \frac{m_3 - (2Y_2+1)m_2 + Y_2(Y_2+1)m_1}{m_2 - (2Y_2+1)m_1 + Y_2(Y_2+1)m_0}, \qquad T_1 = \lfloor t_1 \rfloor; \tag{6.10}$$

$$t_2 = \frac{m_3 - (2Y_1+1)m_2 + Y_1(Y_1+1)m_1}{m_2 - (2Y_1+1)m_1 + Y_1(Y_1+1)m_0}, \qquad T_2 = \lfloor t_2 \rfloor. \tag{6.11}$$

Theorem 6.8 *Suppose that $n = 4$, that ν and μ are as above and that $\operatorname{supp} \nu \not\subset \mathbb{N}_0$. Then $\operatorname{supp} \mu \subset \{T_1, T_1+1, T_2, T_2+1\}$ with $|\operatorname{supp} \mu| \ge 3$; moreover, $T_2 \ge T_1 + 1$, so that if $T_2 > T_1 + 1$ one may take*

$$P_4^{(m)}(x) = (x - T_1)(x - T_1 - 1)(x - T_2)(x - T_2 - 1), \tag{6.12}$$

and if $T_2 = T_1 + 1$ *one may take for example*

$$P_4^{(m)}(x) = (x - T_1)(x - T_2)(x - T_2 - 1)(x - T_2 - 2). \qquad (6.13)$$

Proof: From Corollary 6.4 it follows that either $\{Y_1, Y_1 + 1\} \subset \operatorname{supp} \mu$ or $\{Y_2, Y_2 + 1\} \subset \operatorname{supp} \mu$. Consider the first case, in which $\operatorname{supp} \mu$ is either $\{Y_1, Y_1+1, k\}$ or $\{Y_1, Y_1+1, k, k+1\}$, with $k > Y_1+1$ an integer. Let $F_\tau(x) = (x - Y_1)(x - Y_1 - 1)(x - \tau)$, so that the linear equation $L_{F_\tau}(m) = 0$ has root $\tau = t_2$. Now if $\operatorname{supp} \mu = \{Y_1, Y_1 + 1, k\}$ then $L_{F_k}(m) = 0$ so that $t_2 = T_2 = k$, while if $\operatorname{supp} \mu = \{Y_1, Y_1 + 1, k, k + 1\}$ then $L_{F_k}(m) > 0$ and $L_{F_{k+1}}(m) < 0$, so that $k < t_2 < k + 1$, $k = T_2$ and $\operatorname{supp} \mu = \{Y_1, Y_1 + 1, T_2, T_2 + 1\}$. Note that Proposition 6.3 implies that $T_2 + 1 > y_2 > Y_1 + 1$, so that $T_2 \geq Y_2$ and $T_2 > Y_1 + 1$.

To complete the proof in the case under consideration we need only show that $T_1 = Y_1$, i.e., that $Y_1 \leq t_1 < Y_1 + 1$. Let $G_\tau(x) = (x - \tau)(x - Y_2)(x - Y_2 - 1)$, so that the equation $L_{G_\tau}(m) = 0$ has root $\tau = t_1$. Now $G_{Y_1}(x)$ is nonnegative for $x \in \operatorname{supp} \mu$, so that $L_{G_{Y_1}}(m) \geq 0$. On the other hand, $G_{y_1}(y_1) = 0$ and $G_{y_1}(y_2) \leq 0$, since $y_1 < Y_2 \leq y_2 < Y_2 + 1$, so that since ν, which realizes $m^{(3)}$, has support $\{y_1, y_2\}$, $L_{G_{y_1}}(m) = E_\nu[G_{y_1}(X)] \leq 0$. Since $L_{G_{Y_1}}(m) \geq 0$ and $L_{G_{y_1}}(m) \leq 0$, $Y_1 \leq t_1 \leq y_1 < Y_1 + 1$. This completes the proof when $\{Y_1, Y_1 + 1\} \subset \operatorname{supp} \mu$.

The case $\{Y_2, Y_2 + 1\} \subset \operatorname{supp} \mu$ is handled similarly. Now $\operatorname{supp} \mu$ is either $\{k, Y_2, Y_2 + 1\}$, with $k < Y_2$, or $\{k, k + 1, Y_2, Y_2 + 1\}$, with $k + 1 < Y_2$, and in either case $k = T_1$. Thus $T_2 \geq T_1 + 1$ in all cases. ■

Remark 6.9 (a) We summarize here the procedure to determine realizability for the case $n = 4$. If $m^{(3)}$ is I-realizable (see Theorem 5.1 for the conditions to be checked) then:

- Find the solutions y_1, y_2 of the equation (A.4). If y_1 and y_2 are integers then m is realizable if and only if $\det A(2) \geq 0$ and I-realizable if $\det A(2) > 0$.

- Otherwise, define $Y_1 := \lfloor y_1 \rfloor$ and $Y_2 := \lfloor y_2 \rfloor$, and compute T_1 and T_2 from (6.10) and (6.11).

- If $T_2 > T_1 + 1$ then define $P_4^{(m)}$ by (6.12); otherwise, i.e. if $T_2 = T_1 + 1$, define $P_4^{(m)}$ by (6.13). Then $m^{(4)}$ is realizable if and only if $L_{P_4^{(m)}}(m) \geq 0$; it is I-realizable if the inequality is strict and B-realizable otherwise.

(b) We may also relate Theorem 6.8 to the general inductive procedure introduced in Section 6.1. The index j of Corollary 6.4 must be either 1 or 2. If $j = 1$, so that $\{Y_1, Y_1 + 1\} \subset \operatorname{supp}\mu$, then step 4 of the procedure gives $M_{1,1}(m) = t_2$, so that a recursive application of step 1 gives $\mathcal{N}_{2,1}^{M(m)} = \{T_2\}$ if $t_2 \in \mathbb{N}_0$ and $\mathcal{N}_{2,1}^{M(m)} = \{T_2, T_2+1\}$ otherwise. By what appears to be a lucky accident (which does not seem to generalize to $n \geq 6$), however, (6.10) gives a value of T_1 which coincides with Y_1. The analysis when $j = 2$ is similar, so that (6.12) or (6.13) holds whatever the value of j.

The case $n = 5$ is similar to that of $n = 4$. Here we we define

$$\widetilde{t}_1 = \frac{m_4 - (2Y_2+1)m_3 + Y_2(Y_2+1)m_2}{m_3 - (2Y_2+1)m_2 + Y_2(Y_2+1)m_1}, \qquad \widetilde{T}_1 = \lfloor \widetilde{t}_1 \rfloor; \tag{6.14}$$

$$\widetilde{t}_2 = \frac{m_4 - (2Y_1+1)m_3 + Y_1(Y_1+1)m_2}{m_3 - (2Y_1+1)m_2 + Y_1(Y_1+1)m_1}, \qquad \widetilde{T}_2 = \lfloor \widetilde{t}_2 \rfloor. \tag{6.15}$$

Theorem 6.10 *Suppose that* $n = 5$, *that* ν *and* μ *are as above and that* $\operatorname{supp}\nu \not\subset \mathbb{N}_0$. *Then* $\operatorname{supp}\mu \subset \{0, \widetilde{T}_1, \widetilde{T}_1 + 1, \widetilde{T}_2, \widetilde{T}_2 + 1\}$ *with* $0 \in \operatorname{supp}\mu$ *and* $|\operatorname{supp}\mu| \geq 4$*; moreover,* $\widetilde{T}_2 \geq \widetilde{T}_1 + 1$, *so that if* $\widetilde{T}_2 > \widetilde{T}_1 + 1$ *one may take*

$$P_5^{(m)}(x) = x(x - \widetilde{T}_1)(x - \widetilde{T}_1 - 1)(x - \widetilde{T}_2)(x - \widetilde{T}_2 - 1), \tag{6.16}$$

and if $\widetilde{T}_2 = \widetilde{T}_1 + 1$ *one may take for example*

$$P_5^{(m)}(x) = x(x - \widetilde{T}_1)(x - \widetilde{T}_2)(x - \widetilde{T}_2 - 1)(x - \widetilde{T}_2 - 2). \tag{6.17}$$

Proof: The proof is completely parallel to that of Theorem 6.8, with the replacement of the polynomials $F_\tau(x)$ and $G_\tau(x)$ by $\widetilde{F}_\tau(x) = xF_\tau(x)$ and $\widetilde{G}_\tau(x) = xG_\tau(x)$, respectively. ■

7 Sufficient condition for realizability on $\mathbb{N}_0$

In this section we obtain a sufficient condition for realizability, which will be I-realizability, of a moment vector $m^{(n)}$ on $\mathbb{N}_0$. To do so we introduce a new class of polynomials $\mathcal{W}_n$: when $n = 2k$, $\mathcal{W}_n$ consists of all polynomials of the form $W_\beta(x) = V_\beta(x)V_\beta(x-1)$, where $V_\beta(x) = (x - \beta_1)\cdots(x - \beta_k)$ with $\beta_1, \ldots, \beta_k$ real numbers, and when $n = 2k+1$, $\mathcal{W}_n$ consists of all polynomials of the form $xW_\beta(x)$ with $W_\beta \in \mathcal{W}_{n-1}$. Note that $\mathcal{W}_n \supset \mathcal{P}_n$, so that by

Theorem 3.1, $m^{(n)}$ is realizable if $L_W(m) > 0$ for all $W \in \mathcal{W}_j$, $j = 1, \dots, n$. This is our sufficient condition; note that the strict positivity of all $L_W(m)$ and hence of all $L_P(m)$, $P \in \bigcup_{j=1}^n \mathcal{P}_j$, implies that this is I-realizability.

To obtain this condition in a useful form we first consider $j = 2k$ and note that if $V_\beta(x) = \sum_{i=0}^k c_i x^i$ then $V_\beta(x-1) = \sum_{i=0}^k b_i x^i$ where $c = (c_0, \dots, c_k)^T$ and $b = (b_0, \dots, b_k)^T$ are related by $b = H(k)c$, with $H(k)$ the $(k+1)\times(k+1)$ matrix defined by

$$H(k)_{il} = \begin{cases} 0, & \text{if } i > l, \\ (-1)^{l-i}\binom{l}{i}, & \text{if } i \le l. \end{cases} \tag{7.1}$$

If $m^{(n)}$ is realizable and μ is a realizing measure then

$$L_{W_\beta}(m) = E_\mu[W_\beta(X)] = c^T A(k)H(k)c = \frac{1}{2}c^T\big(H(k)^T A(k) + A(k)H(k)\big)c. \tag{7.2}$$

Thus if the matrix $D_j = D_{2k} := \big(H(k)^T A(k) + A(k)H(k)\big)/2$ is positive definite then $L_W(m) > 0$ for all $W \in \mathcal{W}_j$. For $j = 2k+1$ one argues similarly that a sufficient condition for $L_W(m) > 0$, $W \in \mathcal{W}_j$, is that D_j be positive definite, where $D_j = D_{2k+1} := \big(H(k)^T B(k) + B(k)H(k)\big)/2$. We have proved:

Theorem 7.1 *$m^{(n)}$ is realizable on $\mathbb{N}_0$, and in fact I-realizable, if all the matrices D_j, $j = 1, \dots, n$, are positive definite.*

The first few of the matrices D_j are

$$D_1 = \begin{pmatrix} m_1 \end{pmatrix}, \qquad D_2 = \begin{pmatrix} 1 & m_1 - 1/2 \\ m_1 - 1/2 & m_2 - m_1 \end{pmatrix},$$

$$D_3 = \begin{pmatrix} m_1 & m_2 - m_1/2 \\ m_2 - m_1/2 & m_3 - m_2 \end{pmatrix},$$

$$D_4 = \begin{pmatrix} 1 & m_1 - 1/2 & m_2 - m_1 + 1/2 \\ m_1 - 1/2 & m_2 - m_1 & m_3 - 3m_2/2 + m_1/2 \\ m_2 - m_1 + 1/2 & m_3 - 3m_2/2 + m_1/2 & m_4 - 2m_3 + m_2 \end{pmatrix}.$$

The p, q entry of D_j, $p, q = 0, \dots, \lfloor j/2 \rfloor$, is just the corresponding entry of the Hankel matrix C_j, that is, m_{p+q}, modified by the addition of a linear combination of lower moments. Positive definiteness of all the C_j is necessary for I-realizability on $\mathbb{R}_+$ and hence also on $\mathbb{N}_0$ (see Lemma A.1(b)), so that we have necessary conditions and sufficient conditions of a very similar structure.

For $n = 1$ the necessary condition of Lemma A.1(b), the sufficient condition of Theorem 7.1, and the exact condition of Theorem 4.4 all coincide: each is $m_1 > 0$. For $n = 2$, when the exact condition is from Theorem 5.1, they are respectively $m_2 - m_1^2 > 0$, $m_2 - m_1^2 > 1/4$, and $m_2 - m_1^2 > \theta_1(1-\theta_1)$, where θ_1 is the fractional part of m_1; note that the necessary condition is also sufficient when m_1 is an integer and that the sufficient condition is also necessary when it is a half integer. For $n \geq 2$ the condition of Theorem 7.1 is not necessary for I-realizability; for example, the vector $m^{(2)} = (1, 1/5, 1/4)$ is realized by the measure $(33\delta_0 + 6\delta_1 + \delta_2)/40$ but does not satisfy $D_2 > 0$ (or even $D_2 \geq 0$).

As noted in the introduction, Theorem 7.1 may be useful in establishing realizability for conditions of the form $m_j \geq f_j(m_1, \ldots, m_i)$, $i < j$.

8 A more general realization problem

The truncated moment problem on an infinite discrete semi-bounded subset of $\mathbb{R}$ can be solved in the same way, if we adapt our arguments. Specifically, instead of $\mathbb{N}_0$ we consider a set $\mathbb{M} \subset \mathbb{R}$ which is discrete and bounded below; without loss of generality we may assume that $0 \in \mathbb{M} \subset \mathbb{R}_+$. All the arguments presented previously apply if one uses, instead of $\lfloor y \rfloor$ and $\lfloor y \rfloor + 1$, the largest element of $\mathbb{M}$ not greater than y, which we denote $l(y)$, and the smallest element of $\mathbb{M}$ larger than y, which we denote $u(y)$.

In the case $n = 2$ one thus must replace the polynomial in (5.3) by

$$P_2^{(m)}(x) = (x - l(m_1))(x - u(m_1)), \tag{8.1}$$

and the corresponding condition (5.4) becomes

$$m_2 - m_1^2 \geq (u(m_1) - m_1)(m_1 - l(m_1)). \tag{8.2}$$

In the case $n = 3$ one must replace similarly the polynomial in (5.5) by

$$P_3^{(m)}(x) = x(x - l(m_2/m_1))(x - u(m_2/m_1)), \tag{8.3}$$

and the condition in (5.6) becomes

$$\frac{m_3}{m_1} - \left(\frac{m_2}{m_1}\right)^2 \geq (u(m_2/m_1) - m_2/m_1)(m_2/m_1 - l(m_2/m_1)). \tag{8.4}$$

When $n = 4$, (6.10) and (6.11) must be modified to

$$t_1 = \frac{m_3 - (l(y_2) + u(y_2))m_2 + l(y_2)u(y_2)m_1}{m_2 - (l(y_2) + u(y_2))m_1 + l(y_2)u(y_2)m_0}, \tag{8.5}$$

$$t_2 = \frac{m_3 - (l(y_1) + u(y_1))m_2 + l(y_1)u(y_1)m_1}{m_2 - (l(y_1) + u(y_1))m_1 + l(y_1)u(y_1)m_0}, \tag{8.6}$$

and then in the analogue of Theorem 6.8 one obtains that

$$\mathrm{supp}(\mu) \subset \{l(t_1), u(t_1), l(t_2), u(t_2)\}$$

and that the analogue of the minimizing polynomial in (6.12) is

$$P_4^{(m)}(x) = (x - l(t_1))(x - u(t_1))(x - l(t_1))(x - u(t_2)). \tag{8.7}$$

The generalization for $n = 5$ of Theorem 6.10 is analogous. The iterative procedure in Theorem 4.4 and Subsection 6.1 can be adapted easily.

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A The truncated Stieltjes moment problem

In this appendix we consider the solution of the truncated Stieltjes moment problem given in [5], with the goal of re-expressing it in a form parallel to that given in the current paper for the truncated moment problem on $\mathbb{N}_0$. Thus in this appendix realizability always refers to realizability by a measure supported on $\mathbb{R}_+$. We say that a realizable moment vector $m^{(n)}$ is *I-realizable* if it lies in the interior of the set of realizable moment vectors, and *B-realizable* if it lies on the boundary of this set.

Now we suppose that $m^{(n)}$ is a given moment vector and let C_i, $0 \leq i \leq n$, be the corresponding Hankel matrices (1.1).

Lemma A.1 *(a) If $m^{(n)}$ is realizable then for all $j \leq n$, $m^{(j)}$ is realizable and $C_j \geq 0$.*

(b) The following are equivalent:

(i) $m^{(n)}$ is I-realizable;

(ii) $m^{(j)}$ is I-realizable for all $j \leq n$;

(iii) $C_j > 0$ for all $j \leq n$.

Proof: (a) Realizability of $m^{(n)}$ trivially implies that for $j \leq n$, $m^{(j)}$ is realizable, and by Theorems 5.1 and 5.3 of [5] (or see Section 1, equations (1.2) and (1.3), for a direct proof), the latter implies that $C_j \geq 0$.

(b) First, (i) implies (ii), since I-realizability of $m^{(n)}$ implies that for some $\epsilon > 0$ all $\bar{m}^{(n)}$ with $|\bar{m}_i - m_i| < \epsilon$ for $1 \leq i \leq n$ are realizable, and this implies I-realizability of $m^{(j)}$, $j \leq n$. Next, (ii) implies (iii), since if (ii) holds then we may assume inductively that $C_j > 0$ for $j < n$ and Theorem 5.1 or 5.3 of [5] implies that $C_n \geq 0$, so that we need only show that C_n is nonsingular. But if $\det C_n = 0$ then, since $\det C_{n-2} > 0$ and

$$\det C_n = m_n \det C_{n-2} + (\text{terms independent of } m_n), \tag{A.1}$$

any small perturbation $m_n \to \bar{m}_n < m_n$ would render the corresponding Hankel matrix $\bar{C}_n$ non-positive and hence, from (a), $\bar{m}^{(n)}$ non-realizable, contradicting the I-realizability of $m^{(n)}$. Finally, (iii) implies (i), for if (iii) holds then also for some sufficiently small perturbation $\bar{m}^{(n)}$ of $m^{(n)}$ the corresponding Hankel matrices $\bar{C}_j$ also satisfy $\bar{C}_j > 0$ for $j \leq n$, and then Theorems 5.1 and 5.3 of [5] imply that $\bar{m}^{(n)}$ is realizable. ■

Proposition A.2 *(a) If the moment vector $m^{(n)}$ is realizable then either (i) $C_i > 0$ for $0 \leq i \leq n$ or (ii) there exist an index j, $1 \leq j \leq n$, and constants $\varphi_0, \ldots, \varphi_{r-1}$, where $r = \lfloor (j+1)/2 \rfloor$, such that $C_i > 0$ for $i < j$, $C_i \geq 0$ with C_i singular for $i \geq j$, and*

$$m_{r+k} = \sum_{i=0}^{r-1} \varphi_i m_{k+i} \qquad \text{for } k = 0, \ldots, n-r. \tag{A.2}$$

(b) Conversely, if either (i) or (ii) holds then $m^{(n)}$ is realizable.

(c) In case (ii) of (a) the realizing measure is uniquely determined by $m^{(n)}$ and its support consists of r points; the support includes 0 if and only if j is odd.

Note that by Lemma A.1 the cases (i) and (ii) of (a) correspond respectively to the I-realizability and B-realizability of $m^{(n)}$.

Proof: (a,c) Suppose that $m^{(n)}$ is realizable, with realizing measure ν, but that (i) does not hold. By Lemma A.1(a), $C_i \geq 0$ for $0 \leq i \leq n$, and since $C_0 = [1] > 0$ there is an index j, $1 \leq j \leq n$, with $C_i > 0$ for $0 \leq i < j$ and $C_j \geq 0$ with C_j singular.

Case 1: $j = 2r$ **even**. In this case $C_j = A(r)$ is singular but $C_{j-2} = A(r-1)$ is not, so that C_j has a null vector of the form $Q = (-\varphi_0, \ldots, -\varphi_{r-1}, 1)^T$. Let $g(x) = x^r - \sum_{i=0}^{r-1} \varphi_i x^i$; then a computation as in (1.2) shows that $Q^T C_j Q = E_\nu[g(X)^2] = 0$, so that the support of ν must be contained in the zero set of $g(x)$. Indeed, the support must be precisely this zero set and must consist of r points, since otherwise there would be a polynomial $\widetilde{g}(x)$ with $\deg \widetilde{g} < r$ vanishing on the support of ν, and from $E_\nu[\widetilde{g}(X)^2] = 0$ we could conclude, again as in (1.2), that $C_{2(\deg \widetilde{g})}$ was singular, a contradiction. Moreover, then $0 \in \operatorname{supp} \nu$ if and only if $\varphi_0 = 0$, and then with $\widetilde{Q} = (-\varphi_1, \ldots, -\varphi_{r-1}, 1)^T$ and $h(x) = x^{r-1} - \sum_{i=1}^{r-1} \varphi_i x^{i-1}$ we would have $\widetilde{Q}^T B(r-1) \widetilde{Q} = E_\nu[X h(X)^2] = 0$, so that $B(r-1) = C_{j-1}$ would be singular, again a contradiction. This establishes (c). Next,

$$E_\nu[X^k g(X)] = m_{r+k} - \sum_{i=0}^{r-1} \varphi_i m_{k+i} = 0 \qquad \text{for } 0 \leq k \leq n-r, \tag{A.3}$$

which verifies (A.2). Finally, (A.2) implies that if for some $i \geq j$, $\mathbf{v}_0, \ldots, \mathbf{v}_{\lfloor i/2 \rfloor}$ are the columns of C_i, then $\mathbf{v}_l = \sum_{q=0}^{r-1} \varphi_q \mathbf{v}_q$ for $l \geq r$, so that C_i is singular.

Case 2: $j = 2r-1$ **odd**. The proof is similar. Now $C_j = B(r-1)$ has a null vector $Q = (-\varphi_1, \ldots, -\varphi_{r-1}, 1)^T$, and if $h(x) = x^{r-1} - \sum_{i=1}^{r-1} \varphi_i x^{i-1}$ then $Q^T C_j Q = E_\nu[x h(X)^2] = 0$, so that the support of ν must be contained in the zero set of $g(x) = x h(x) = x^r - \sum_{i=0}^{r-1} \varphi_i x^i$. As above, the support must be precisely this zero set and must consist of r points, so that (c) holds. Now (A.2) follows from $E_\nu[X^k g(X)] = 0$, and the argument that C_i is singular for $i \geq j$ is the same.

(b) If (i) holds then $m^{(n)}$ is realizable by Lemma A.1. If (ii) holds, then according to Theorems 5.1 and 5.3 of [5], realizability of $m^{(n)}$ requires positive semidefiniteness of C_n and C_{n-1} and that a certain vector $\mathbf{v}$ lie in the range of C_{n-1}, where $\mathbf{v} = (m_{l+1}, \ldots, m_{2l})^T$ if $n = 2l$ and $\mathbf{v} = (m_{l+1}, \ldots, m_{2l+1})^T$ if $n = 2l+1$; the latter condition follows immediately from (A.2) for $k = l+1-r, \ldots, n-r$, which expresses $\mathbf{v}$ as a linear combination of the last r columns of C_{n-1}. ■

Remark A.3 Suppose that we are in the situation of Proposition A.2(a.ii). If $j = 2r$ is even, and we define $\Phi = (\varphi_0, \ldots, \varphi_{r-1})$, $M = (m_r, \ldots, m_{2r-1})$, then $\Phi = MA(r-1)^{-1}$. Similarly if $j = 2r-1$ is odd then $\widetilde{\Phi} = \widetilde{M}B(r-2)^{-1}$, where $\widetilde{\Phi} = (\varphi_1, \ldots, \varphi_{r-1})$, $\widetilde{M} = (m_r, \ldots, m_{2r-2})$. These formulas permit the computation of the φ_i, and hence of the polynomial $g(x)$ whose zeros form the support of ν, in terms of minors of C_j. For example, when $j = 4$ and $j = 5$ the support of ν consists respectively of the roots of

$$\begin{vmatrix} m_0 & m_1 \\ m_1 & m_2 \end{vmatrix} x^2 - \begin{vmatrix} m_0 & m_1 \\ m_2 & m_3 \end{vmatrix} x + \begin{vmatrix} m_1 & m_2 \\ m_2 & m_3 \end{vmatrix} = 0 \tag{A.4}$$

and

$$\begin{vmatrix} m_1 & m_2 \\ m_2 & m_3 \end{vmatrix} x^3 - \begin{vmatrix} m_1 & m_2 \\ m_3 & m_4 \end{vmatrix} x^2 + \begin{vmatrix} m_2 & m_3 \\ m_3 & m_4 \end{vmatrix} x = 0. \tag{A.5}$$

The specific consequence of Proposition A.2 needed in Section 1 is:

Corollary A.4 *(a) If $m^{(n-1)}$ is B-realizable, then $m^{(n)}$ is realizable if and only m_n satisfies*

$$m_n = \sum_{i=0}^{r-1} \varphi_i m_{n-r+i}, \tag{A.6}$$

that is, satisfies (A.2) *with $r + k = n$, and then is B-realizable.*
(b) If $m^{(n-1)}$ is I-realizable, then $m^{(n)}$ is realizable if and only if $C_n \geq 0$: I-realizable if $C_n > 0$, B-realizable if C_n is singular.

Proof: (a) If $m^{(n-1)}$ is B-realizable then we are in case (ii) of Proposition A.2 with $j \leq n-1$; thus the φ_i are defined and Proposition A.2 implies that realizability of $m^{(n)}$ is equivalent to (A.6) together with positive semidefiniteness and singularity of C_n. But singularity of C_n follows from (A.2) (for $k+r < n$) and (A.6), since these show that the last column of C_n is a linear combination of the previous r columns, and positive semidefiniteness of C_n follows from (A.6) and Theorem 2.4 of [5] applied to $\gamma = (m_0, \ldots, m_{2l-2})$ if $n = 2l$ and to $\gamma = (m_1, \ldots, m_{2l-1})$ if $n = 2l + 1$ and $m_1 \neq 0$. Note that the theorem assumes $\gamma_0 \neq 0$ and so does not apply if $m_1 = 0$, but in that case $r = 1$, $\varphi_0 = 0$, and (A.6) becomes $m_n = 0$, easily seen to be necessary and sufficient for realizability.
(b) If $m^{(n-1)}$ is I-realizable then $C_i > 0$ for $i = 1, \ldots, n-1$ by Lemma A.1, and by the same lemma, I-realizability of $m^{(n)}$ is then equivalent to $C_n >$

0. B-realizability of $m^{(n)}$ certainly implies that C_n be positive semidefinite and singular, by Proposition A.2; conversely, the latter conditions imply realizability of $m^{(n)}$, by Theorems 5.1 and 5.3 of [5], and this must be B-realizability, by Proposition A.2. ■

B Algebraic techniques

As noted in Remark 4.5(g), some of the results of Section 4 may be obtained directly from the properties of the polynomials in $\mathcal{P}_n$, without reference to a realizing measure.

Lemma B.1 *Let $m = m^{(n)}$ be a moment vector and suppose that $L_P(m) \geq 0$ for all $P \in \mathcal{P}_n$. Then (i) $L_Q(m) \geq 0$ for all $Q \in \mathcal{P}_{n-2}$, and (ii) if $Q \in \mathcal{P}_{n-2}$ satisfies $L_Q(m) = 0$, then $L_{xQ}(m) \leq 0$.*

Proof: Suppose that $Q \in \mathcal{P}_{n-2}$. If k is such that neither k nor $k-1$ is a zero of Q then $(x-k)(x-k-1)Q(x) \in \mathcal{P}_n$, so that

$$L_{(x-k)(x-k-1)Q}(m) = L_{x^2Q}(m) - (2k+1)L_{xQ}(m) + k(k+1)L_Q(m) \geq 0. \quad \text{(B.1)}$$

Then (i) and (ii) follow by letting k become very large in (B.1). ■

Proposition B.2 *Suppose that $L_P(m) \geq 0$ for all $P \in \mathcal{P}_n \cup \mathcal{P}_{n-1}$. Then $L_P(m) \geq 0$ for all $P \in \bigcup_{k=1}^n \mathcal{P}_k$, and if $L_Q(m) = 0$ for $Q \in \mathcal{P}_k$ with $k \leq n-2$ then $L_{x^iQ}(m) = 0$ for all i with $1 \leq i \leq n-1-k$.*

Proof: The first statement follows from repeated application of Lemma B.1. Suppose then that $Q \in \mathcal{P}_k$, with $k \leq n-2$, satisfies $L_Q(m) = 0$. If α is the smallest nonnegative integer which is not a zero of Q then $(x-\alpha)Q \in \mathcal{P}_{k+1}$, and hence $L_{(x-\alpha)Q}(m) \geq 0$. On the other hand, from Lemma B.1,

$$L_{(x-\alpha)Q}(m) = L_{xQ}(m) - \alpha L_Q(m) = L_{xQ}(m) \leq 0. \quad \text{(B.2)}$$

We conclude that $L_{(x-\alpha)Q}(m) = 0$.

Thus we have constructed a linear polynomial T_1 with $T_1Q \in \mathcal{P}_{k+1}$ and $L_{T_1Q}(m) = 0$. By repeating the argument we can generate a sequence of polynomials $T_1, \ldots, T_{n-k-1}$, with $\deg T_j = j$, such that $T_jQ \in \mathcal{P}_{k+j}$ and

$L_{T_jQ}(m) = 0$. Because $\deg T_j = j$ it follows that $L_{PQ}(m) = 0$ for all polynomials P of degree less or equal to $n-k-1$. The conclusion follows by taking $P(x) = x^i$ with $1 \le i \le n-1-k$. ■

We see that Lemma 4.3 is an immediate consequence of Proposition B.2. Moreover, we can now relax the hypotheses of Theorem 3.2(a), requiring only that $L_P(m) > 0$ for $P \in \mathcal{P}_{n-2}$ and $P \in \mathcal{P}_{n-3}$, rather than for all $P \in \mathcal{P}_k$ with $k \le n-2$.

Furthermore, thanks to Proposition B.2, we can prove that (3.6) implies (3.1) and (3.4). For convenience let us restate here explicitly what we aim to show.

Proposition B.3 *Let $m = m^{(n)} \in \mathbb{R}^n$. If*

$L_P(m) \ge 0$ for all $P \in \mathcal{P}_{n-1} \cup \mathcal{P}_n$, and there exists a real number λ such that if $m' = (m_1, m_2, \ldots, m_n, \lambda)$ then $L_Q(m') \ge 0$ for all $Q \in \mathcal{P}_{n+1}$. (B.3)

then the following conditions both hold

$$L_P(m) \ge 0, \quad \forall\, P \in \mathcal{P}_n \tag{B.4}$$

$$L_P(m) \ge 0,\ \forall P \in \mathcal{P}_{n-1} \text{ and if } L_P(m) = 0 \text{ then } L_{xP}(m) = 0. \tag{B.5}$$

Proof: Suppose that (B.3) holds. Then clearly we have that (B.4) and the first part of (B.5) hold. It remains to show that if $P \in \mathcal{P}_{n-1}$ is such that $L_P(m) = 0$ then also $L_{xP}(m) = 0$. But (B.3) ensures that the vector m' fulfills the assumption of Proposition B.2 with n replaced by $n+1$ and so we get that: if $L_Q(m') = 0$ for $Q \in \mathcal{P}_k$ with $k \le n-1$ then $L_{x^iQ}(m') = 0$ for all i with $1 \le i \le n-k$. In particular, for $k = n-1$ we obtain the desired conclusion.

■

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