On the discrepancy of some generalized Kakutani's sequences of partitions

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Abstract

In this paper we study a class of generalized Kakutani's sequences of partitions of $[0,1]$, constructed by using the technique of successive $\rho-$refinements. Our main focus is to derive bounds for the discrepancy of these sequences. The approach that we use is based on a tree representation of the sequence of partitions which is precisely the parsing tree generated by Khodak's coding algorithm. With the help of this technique we derive (partly up to a logarithmic factors) optimal upper bound in the so-called rational case. The upper bounds in the irrational case that we obtain are weaker, since they depend heavily on Diophantine approximation properties of a certain irrational number. Finally, we present an application of these results to a class of fractals.

1 Introduction

In this paper we will study uniformly distributed sequences of partitions of $[0,1]$, a concept which has been introduced in 1976 by Kakutani, [13].

Definition 1.1. *Let $\{\pi_n\}$ be a sequence of interval partitions of $[0,1]$ represented by $\pi_n = \{[t_{i-1}^{(n)}, t_i^{(n)}] : 1 \leq i \leq k(n)\}$, where $0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{k(n)}^{(n)} = 1$. The sequence $\{\pi_n\}$ is said to be uniformly distributed (u.d.) if for any continuous function f on $[0,1]$ we have*

$$\lim_{n\to\infty} \frac{1}{k(n)} \sum_{i=1}^{k(n)} f(t_i^{(n)}) = \int_0^1 f(t)\ dt.$$

Equivalently, $\{\pi_n\}$ is u.d. if the sequence of discrepancies

$$D_n = \sup_{0\leq a<b\leq 1} \left| \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_{[a,b[}(x_i^{(n)}) - (b-a) \right| \tag{1}$$

tends to 0 as $n \to \infty$ (for more details on the theory of uniform distribution see [17] or [9]; χ_M denotes the characteristic function of the set M).

Kakutani's sequence of partitions is defined in the following way. Let $\alpha \in]0,1[$ be given and start with the unit interval $I = [0,1]$. In the first step this interval is divided into the two intervals

$[0,\alpha],[\alpha,1]$ of lengths α and $1-\alpha$. In the second step the larger interval is partitioned into two subintervals of lengths proportional to α and $1-\alpha$ respectively. For example, if $\alpha=\frac{1}{3}$ then the interval $[\frac{1}{3},1]$ is split into $[\frac{1}{3},\frac{5}{9}],[\frac{5}{9},1]$. In this way one proceeds further. Note that one always considers all intervals of maximal lengths at once.

Definition 1.2. *If $\alpha\in]0,1[$ and $\pi=\{[t_{i-1},t_i]:1\le i\le k\}$ is any interval partition of $[0,1]$, then Kakutani's α-refinement of π (which will be denoted by $\alpha\pi$) is obtained by splitting all the intervals of π having maximal length in two parts, proportional to α and $1-\alpha$ respectively.*

Kakutani's sequence of partitions κ_n can be then written as $\kappa_n=\alpha^n\omega$, where $\omega=\{[0,1]\}$. His observation was that for every $\alpha\in]0,1[$, the sequence of partitions $\{\kappa_n\}$ of $[0,1]$ is u.d. ([13]).

In a recent paper [19], Kakutani's splitting procedure has been generalized by splitting the longest intervals of a partition π into a finite number of parts homothetically to a given finite interval partition ρ of $[0,1]$. The resulting interval partition $\rho\pi$ is called ρ-refinement of π. As for the α-refinement (that corresponds to $\rho=\{[0,\alpha],[\alpha,1-\alpha]\}$) the following result holds (cf. [19]):

Theorem 1.3. *The sequence $\{\rho^n\omega\}$ of successive ρ-refinements of the trivial partition $\omega=\{[0,1]\}$ is u.d.*

A natural problem which is interesting for possible applications, posed in [19], is to estimate the behaviour of the discrepancy as n tends to infinity. The only known discrepancy bounds for sequences of this kind have been obtained by Carbone [4] by a direct and elementary approach, who considered so-called LS-sequences that evolve from partitions ρ with L subintervals of $[0,1]$ of length α and S subintervals of length α^2 (where α is given by the equation $L\alpha+S\alpha^2=1$).

In this paper, we analyze this problem with a new approach that is based on a parsing tree (related to the Khodak coding algorithm [15]) that represents the successive ρ-refinements. In particular we will use refinements of the results obtained in [8] about Khodak's algorithm to give an estimate of the discrepancy for a class of sequences of partitions constructed by successive $\rho-$refinements. Suppose that ρ consists of m subintervals of lengths $p_1,\ldots,p_m$. In the so-called rational case (which means that all fractions $(\log p_i)/(\log p_j)$ are rational, see Definition 2.1) we will provide very precise bounds for the discrepancy. Note that LS-sequences are rational, therefore we generalize the results of [4]. However, we are also able to cover several irrational cases (which means that at least one of the fractions $(\log p_i)/(\log p_j)$ is irrational).

Let us give a brief outline of the structure of the paper. In Section 2 we introduce Khodak's algorithm and analyze the correspondence between subintervals of $[0,1]$ and nodes of the parsing tree. Moreover, we extend an asymptotic result from [8]. In Section 3 we present our main results in the rational case. In particular, we obtain an upper bound of the form

$$D_n=O\left((\log k(n))^d k(n)^{-\eta}\right) \tag{2}$$

where η is a positive constant ≤ 1 and $d\ge 0$ an integer (both values are explicit). Furthermore, this upper bound is best possible (despite a logarithmic factor in a special case).

In Section 4 we discuss some instances in the irrationally related case for $m = 2$. They are much more involved than in the rational case.

Finally, in Section 5 we give some examples and applications including LS-sequences and u.d. sequences of partitions on a class of fractals. Some auxiliary results that are used in Section 2 are collected in Section 6.

2 ρ-refinements and Khodak algorithm

From now on, consider a partition ρ of $[0, 1]$ consisting of m intervals of lengths $p_1, \ldots, p_m$ and the sequence of ρ-refinements of the trivial partition $\omega = \{[0, 1]\}$.

Our goal is to construct recursively an m-ary tree T. An m-ary tree is an ordered rooted tree, where each node has either m (ordered) successors (we call such a node *internal node*) or it is a leaf with no successors (which we call also *external node*). The numbers $p_1, \ldots, p_m$ induce a natural labelling on the nodes. Suppose that the unique path from the root to a node x at level l is encoded by the sequence $(j_1, j_2, \ldots, j_l)$, $1 \leq j_i \leq m$, then we set $P(x) = p_{j_1} p_{j_2} \cdots p_{j_l}$. This can be also considered as the probability of reaching the node x with a random walk that starts at the root and moves away from it according to the probabilities $p_1, \ldots, p_m$. For completeness the root r is labelled with $P(r) = 1$. If T is a finite m-ary tree then the labels of the external nodes sum up to 1 (which follows easyly by induction). Hence, the shape of an m-ary tree (together with $p_1, \ldots, p_m$) gives rise of a probability distribution. Note that if we have j internal nodes then there are $M = (m-1)j + 1$ external ones.

The start of our iteration is a tree that only consists of the root which is then an external node (with probability 1). In the first step the root is replaced by an internal node together with m (ordered) successing leaves that are given the probability distribution $p_1, \ldots, p_m$. At each further iteration we select all leaves y with largest label $P(y)$ and grow m children out of each of them. This procedure describes the construction of the parsing trees of the Tunstall code [8] (the words $(j_1, j_2, \ldots, j_l)$ that encode the paths from the root to the leaves are the phrases of the dictionary). Actually this construction corresponds precisely to the ρ-refinement procedure of the sequence $\kappa_n = \rho^n \omega$. The leaves of the tree correspond to the intervals and the labels of the leaved to the lengths of the intervals.

There is a second way to describe this tree evolution process, namely by Khodak's algorithm [15]. Fix a real number $r \in]0, p_{min}[$, where $p_{min} = \min\{p_1, \ldots, p_m\}$, and consider all nodes x among in an infinte m-ary tree with $P(x) \geq r$. Let us denote these nodes by $\mathcal{I}(r)$. Of course, if $P(x) \geq r$ then all nodes y on the path from the root to x satisfy $P(y) \geq r$, too. Hence, these nodes of $\mathcal{I}(r)$ constitute a finite subtree. These nodes will be the *internal nodes* of Khodak's construction. Finally, we append to these internal nodes all successor nodes d. By construction all these nodes satisfy $p_{min} r \leq P(d) < r$ and we denote them by $\mathcal{E}(r)$. These nodes are the *external nodes* of Khodak's construction. We denote by $M_r = |\mathcal{E}(r)|$ the number of external nodes. Obviously we have got a finite m-ary tree $\mathcal{T}(r) = \mathcal{I}(r) \cup \mathcal{E}(r)$ and it is clear that these trees grow when r decreases. For certain values r, precisely the external nodes y of largest value $P(y) = r$ turn into internal nodes and all their successors become new external nodes. Actually, the tree $\mathcal{T}(r)$ grows in correspondence to a decreasing sequence of values $\{r_j\}$. When $r \in]r_j, r_{j-1}]$ the tree remains the same, i.e $\mathcal{T}(r_{j-1}) = \mathcal{T}(r)$.

In our correspondence between Khodak's algorithm and the procedure of successive $\rho-$refinements the values r_j correspond to the partition $\rho^{j+1}\omega$. Consequently, the number of external nodes in $\mathcal{E}(r_j)$ equals the number of points defining the partition $\rho^{j+1}\omega$, i.e. $M_{r_j} = k(j+1)$. Moreover, if $r \in]r_j, r_{j-1}]$ then $M_r = M_{r_{j-1}} = k(j)$.

In the following we denote by H the entropy of the probability distribution $p_1, \ldots, p_m$, which is defined as

$$H = p_1 \log\left(\frac{1}{p_1}\right) + \cdots + p_m \log\left(\frac{1}{p_m}\right).$$

Definition 2.1. *We say that* $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are* rationally related *if there exists a positive real number* Λ *such that* $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are integer multiples of* Λ*, that is*

$$\log\left(\frac{1}{p_j}\right) = n_j\Lambda, \quad \textit{with } n_j \in \mathbb{Z} \textit{ for } j = 1, \ldots, m.$$

Without loss of generality we can assume that Λ *is as large as possible which is equivalent to assume that* $\gcd(n_1, \ldots, n_m) = 1$*. Equivalently, all fractions* $(\log p_i)/(\log p_j)$ *are rational.*

Similarly we say that $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are* irrationally related *if they are not rationally related.*

One of main result from [8] provides asymptotic information on the numbers M_r of external nodes in Khodak's construction. Actually these relations can be used to prove Theorem 1.3. However, in order to obtain bounds for the discrepancy we need more precise information on the error terms. Therefore we have extend the analysis of [8].

Theorem 2.2. *Let* M_r *be the number of the external nodes generated at the step corresponding to the parameter* r *in Khodak's construction, that is, the number of nodes in* $\mathcal{E}(r)$*.*

1. *If* $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are rationally related, let* $\Lambda > 0$ *be the largest real number for which* $\log\left(\frac{1}{p_j}\right)$ *is an integer multiple of* Λ *(for* $j = 1, \ldots, m$*). Then there exists a real number* $\eta > 0$ *and an integer* $d \geq 0$ *such that*

$$M_r = \frac{(m-1)}{rH} Q_1\left(\log\left(\frac{1}{r}\right)\right) + O\left((\log r)^d r^{-(1-\eta)}\right), \tag{3}$$

where

$$Q_1(x) = \frac{\Lambda}{1 - e^{-\Lambda}} e^{-\Lambda\left\{\frac{x}{\Lambda}\right\}}$$

and $\{y\}$ *is the fractional part of the real number* y*. Furthermore, the error term is optimal.*

2. If $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ *are irrationally related, then*

$$M_r = \frac{(m-1)}{rH} + o\left(\frac{1}{r}\right). \tag{4}$$

In particular, if $m = 2$ *and* $\gamma = (\log p_1)/(\log p_2)$ *is badly approximable then*

$$M_r = \frac{(m-1)}{rH}\left(1 + O\left(\frac{(\log\log 1/r)^{1/4}}{(\log 1/r)^{1/4}}\right)\right). \tag{5}$$

and if p_1 *and* p_2 *are algebraic then there exists an effectively computable constant* $\kappa > 0$ *with*

$$M_r = \frac{(m-1)}{rH}\left(1 + O\left(\frac{(\log\log 1/r)^{\kappa}}{(\log 1/r)^{\kappa}}\right)\right). \tag{6}$$

Proof. Set $v = \frac{1}{r}$ and denote by $A(v)$ the number of internal nodes in Khodak's construction with parameter $r = 1/v$, that is,

$$A(v) = \sum_{x: P(x) \geq \frac{1}{v}} 1.$$

Hence, the number of external nodes generated at the step corresponding to the parameter r is

$$M_r = (m-1)A(v) + 1. \tag{7}$$

The key relation is that that $A(v)$ satisfies the following recurrence, (see Lemma 2, [8]):

$$A(v) = \begin{cases} 0 & v < 1 \\ 1 + \sum_{j=1}^m A(p_j v) & v \geq 1 \end{cases} \tag{8}$$

For the asymptotic analysis of $A(v)$ (and consequently that of M_r) we distinguish between the rational and the irrational case. If the $\log(1/p_j)$ are rationally related then $A(v)$ is constant for $v \in [e^{\Lambda n}, e^{\Lambda(n+1)}[$ (for every integer n). Hence, it suffices to study the behaviour of the sequence $G(n) = A(e^{\Lambda n})$ which satisfies the recurrence

$$G(n) = 1 + \sum_{j=1}^m G(n - n_j)$$

with initial conditions $G(n) = 0$ for $n < 0$. The generating function $g(z) = \sum_{n \geq 0} G(n) z^n$ is then given by

$$g(z) = \frac{1}{(1-z)f(z)},$$

where $f(z) = 1 - z^{n_1} + \cdots - z^{n_m}$. By Definition 2.1, it follows that $e^{-\Lambda}$ is a positive real root of f and it is proved in [5] that if we denote by $\omega_1, \ldots, \omega_h$ all the other (different) roots (with multiplicities

μ_i) of f then $|\omega_i| > e^{-\Lambda}$ for $i = 1, \ldots, h$. (Here we use the assumption that $n_1, \ldots, n_m$ are coprime). Hence, it follows that

$$G(n) = \frac{\Lambda e^{\Lambda n}}{H(1-e^{-\Lambda})} + \sum_{i=1}^{h} P_i(n)\omega_i^{-n} - \frac{1}{m-1},$$

where P_i are polynomials of degree smaller than μ_i. Obviously this implies the representation (3) of Theorem 2.2 for some $\eta > 0$. Note that in view of (7) the constant term $-1/(m-1)$ disappears when we translate the asymptotics of $G(n)$ to M_r.

Next we study the error term (without the constant term $-1/(m-1)$) in more detail. W.l.o.g. we can assume that $\omega_1, \ldots, \omega_k$ (with $k \le h$) are those roots of $f(z)$ with smallest modulus $|\omega_i| = e^{-\Lambda(1-\eta)}$ (with some $\eta > 0$) such that $P_i \ne 0$, $1 \le i \le k$, and where the degrees of P_i are maximal and all equal to $d \ge 0$. This means that the difference between $G(n)$ and the asymptotic leading term is bounded by

$$\delta(n) = \left| G(n) - \frac{\Lambda e^{\Lambda n}}{H(1-e^{-\Lambda})} + \frac{1}{m-1} \right| \le C n^d e^{\Lambda(1-\eta)n}$$

for some constant $C > 0$. More precisely $\delta(n)$ can be written as

$$\delta(n) = \left| n^d \sum_{i=1}^{k} \overline{c}_i\, \omega_i^{-n} \right| + O\left(n^{d-1} e^{\Lambda(1-\eta)n} \right).$$

where $\overline{c}_i \ne 0$, $1 \le i \le k$. Since all roots of $f(z)$ are either real or appear in conjugate pairs of complex numbers we can rewrite the sum $\sum_{i=1}^{k} \overline{c}_i\, \omega_i^{-n}$ to

$$n^d e^{\Lambda(1-\eta)n} \sum_{i=1}^{k'} c_i' \cos(2\pi\theta_i n + \alpha_i)$$

with real numbers $c_i' \ne 0$. By Lemma 6.1 it follows that there exists $\delta > 0$ and infinitely many n such that $|\sum_{i=1}^{k'} c_i' \cos(2\pi\theta_i n + \alpha_i)| \ge \delta$. This shows that

$$\delta(n) \ge C' n^d e^{\Lambda(1-\eta)n}$$

for infinitely many n with some constant $C' > 0$. This means that the error term in (8) is optimal.

The analysis in the irrationally related case is much more involved. Instead of using power series we use the Mellin transform

$$A^*(s) = \int_0^\infty A(v) v^{s-1}\, dv.$$

By using the fact that the Mellin transform of $A(av)$ is $a^{-s}A^*(s)$, a simple analysis of recurrence (8) reveals that the Mellin transform $A^*(s)$ of $A(v)$ is given by

$$A^*(s) = \frac{-1}{s\left(1 - p_1^{-s} - \cdots - p_m^{-s}\right)}, \qquad \Re(s) < -1.$$

In order to find asymptotics of $A(v)$ as $v \to \infty$ one can directly use the Tauberian theorem (for the Mellin transform) by Wiener-Ikehara [16, Theorem 4.1]. For this purpose we have to check that $s_0 = -1$ is the only (polar) singularity on the line $\Re(s) = -1$ and that $(s+1)A^*(s)$ can be analytically extended to a region that contains the line $\Re(s) = -1$. However, in the irrationally related case this is granted a lemma of Schachinger [18]. In particular, one finds $A(v) \sim v/H$ but this (simple) procedure does not provide any information about the error term.

In order to make our presentation as simple as possible we will restrict ourselves to the case $m = 2$ and we will also assume certain conditions on the Diophantine properties of the irrational number

$$\gamma = \frac{\log p_1}{\log p_2}.$$

We use the simplified notation $p = p_1$ and $q = p_2$.

The principle idea to obtain error terms for $A(v)$ is to use the formula for the inverse Mellin transfrom

$$A(v) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma-iT}^{\sigma+iT} A^*(s) v^{-s} ds, \qquad \sigma < -1, \tag{9}$$

and to shift the line of integration to the right. Of course, all polar singularities of $A^*(s)$ (which are given by the solutions of the equation $p^{-s} + q^{-s} = 1$ and $s = 0$) give rise to a polar singularity. Unfortunately, the order of magnitude of $A^*(s)$ is just of order $O(1/s)$. Hence the integral in (9) is not absolutely convergent. It is therefore convenient to *smooth* the problem and to study the function $A_1(v) = \int_0^v A(w)\,dw$ which is given by

$$A_1(v) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} A^*(s) \frac{v^{-s+1}}{1-s}\,ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{1}{s(s-1)(1-p^{-s}-q^{-s})} v^{-s+1}\,ds, \qquad \sigma < -1.$$

By [18] we know that all zeros of the equation $p^{-s} + q^{-s} = 1$ that are different from -1 satisfy $-1 < \Re(s) \le \sigma_0$ for some σ_0. Furthermore there is $\kappa > 0$ such that in each box of the form

$$B_k = \{s \in \mathbb{C} : -1 < \Re(s) \le \sigma_0,\, (2k-1)\tau \le \Im(s) < (2k+1)\tau\}, \quad k \in \mathbb{Z} \setminus \{0\},$$

there is precisely one zero of $p^{-s} + q^{-s} = 1$ that we denote by s_k Hence, by shifting the line of integration to the right and by collecting all residues we obtain (for some $\sigma_1 > \max\{\sigma_0 + 1, 1\}$)

$$\begin{aligned} A_1(v) &= \frac{v^2}{2H} + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)} - v - \frac{1}{1-p^{-1}-q^{-1}} \\ &\quad + \frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{1}{s(s-1)(1-p^{-s}-q^{-s})} v^{-s+1}\,ds, \end{aligned}$$

where $H(s) = p^{-s}\log(1/p) + q^{-s}\log(1/q)$. Clearly the integral can be estimated by

$$\frac{1}{2\pi i} \int_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{1}{s(s-1)(1-p^{-s}-q^{-s})} v^{-s+1}\,ds = O\left(v^{-\sigma_1+1}\right).$$

Hence we just have to deal with the sum of residues $\sum v^{s_k}/(s_k(s_k-1)H(s_k))$. First it is an easy exercise to show that there exists $\delta > 0$ such that $|H(s_k)| \ge \delta$ for all $k \in \mathbb{Z} \setminus \{0\}$. Thus, we do not have to care about this factor.

Next assume that γ is a badly approximable irrational number which means that γ has a bounded continued fraction representation. Here Lemma 6.2 shows that all zeros $s_k \ne -1$ of the equation $p^{-s} + q^{-s} = 1$ satisfy $\Re(s_k) > -1 + c/\Im(s_k)^2$ for some constant $c > 0$. Hence it follows that $\Re(s_k) > -1 + c_1/k^2$ for some constant $c_1 > 0$ and we can estimate the sum of residues by

$$\begin{aligned}\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)}\right| &\le \left|\sum_{0<|k|\le K} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)}\right| + \left|\sum_{|k|>K} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)}\right| \\ &\le C_1 v^{2-c_1/K^2} \sum_{0<|k|\le K} \frac{1}{k^2} + C_2 v^2 \sum_{|k|>K} \frac{1}{k^2} \\ &\le C_3 v^2 \left(v^{-c_1/K^2} + \frac{1}{K}\right).\end{aligned}$$

By choosing $K = \sqrt{c_1(\log v)/(\log\log v)}$ we, thus, obtain the upper bound

$$\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)} = O\left(v^2 \frac{\sqrt{\log\log v}}{\sqrt{\log v}}\right)$$

and consequently

$$A_1(v) = \frac{v^2}{2H}\left(1 + O\left(\frac{\sqrt{\log\log v}}{\sqrt{\log v}}\right)\right).$$

Finally by an application of Lemma 6.5 this implies

$$A(v) = \frac{v}{H}\left(1 + \frac{(\log\log v)^{1/4}}{(\log v)^{1/4}}\right).$$

Similarly we can deal with the case if we know that all solutions of the equation $p^{-s} + q^{-s} = 1$ (that are different from -1) satisfy $\Re(s_k) > -1 + D/\Im(s_k)^{2C}$ for some positive constants C, D (this is satisfied if p and q are algebraic, see with Lemma 6.3). Then we obtain (as above)

$$\left|\sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{v^{s_k}}{s_k(s_k-1)H(s_k)}\right| \le C_4 v^2 \left(v^{-c_2 K^{-2C}} + \frac{1}{K}\right).$$

Hence, if we choose $K = (c_2(\log v)/(\log\log v))^{1/(2C)}$ we obtain (after a second application of Lemma 6.5

$$A(v) = \frac{v}{H}\left(1 + \frac{(\log\log v)^{1/(4C)}}{(\log v)^{1/(4C)}}\right).$$

This completes the proof of Theorem 2.2. □

3 Discrepancy bounds in the rational case

In this section, we are going to consider a partition ρ of $[0,1]$ consisting of m intervals of lengths $p_1, \dots, p_m$ such that $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are rationally related.

By Theorem 2.2 we know that M_r is asymptotically given by

$$M_{r_n} = \frac{c'}{r_n} + O\left((\log r_n)^d r_n^{-(1-\eta)}\right), \quad r = r_n = e^{-\Lambda n}, \tag{10}$$

for some $\eta > 0$ and some integer $d \geq 0$, where $c' = (m-1)\Lambda/(H(1-e^{-\Lambda}))$ and the error term is optimal. Recall also that $k(n) = M_{r_{n-1}}$ which gives an asmyptotic expansion for $k(n)$ of the form

$$k(n) \sim \frac{(m-1)\Lambda}{H(e^{\Lambda}-1)} e^{\Lambda n}.$$

Theorem 3.1. *Suppose that the lengths of the intervals of a partition ρ are $p_1, \dots, p_m$ and suppose that $\log\left(\frac{1}{p_1}\right), \dots, \log\left(\frac{1}{p_m}\right)$ are rationally related. Furthermore let $\eta > 0$ and $d \geq 0$ be given as in Theorem 2.2.*

Then the discrepancy of the sequence of partitions $\{\rho^n \omega\}$ is bouded by

$$D_n = \begin{cases} O\left((\log k(n))^d k(n)^{-\eta}\right) & \text{if } 0 < \eta < 1, \\ O\left((\log k(n))^{d+1} k(n)^{-1}\right) & \text{if } \eta = 1, \\ O\left(k(n)^{-1}\right) & \text{if } \eta > 1. \end{cases} \tag{11}$$

Furthermore there exists $\delta > 0$ and infinitely many n such that

$$D_n \geq \begin{cases} \delta\,(\log k(n))^d k(n)^{-\eta} & \text{if } 0 < \eta < 1, \\ \delta\,(\log k(n))^d k(n)^{-1} & \text{if } \eta = 1, \\ \delta\, k(n)^{-1} & \text{if } \eta > 1. \end{cases} \tag{12}$$

Proof. For notational convenience we set

$$\Delta_n = \sup_{0 \leq y \leq 1} \left| \sum_{i=1}^{k(n-1)} \chi_{[0,y[}(x_i^{(n-1)}) - k(n-1)y \right|$$

Then we have $D_n \leq 2\Delta_{n+1}/k(n)$.

Fix a step in the algorithm corresponding to a certain parameter r of the form $r = e^{-n\Lambda}$ for some integer $n \geq 0$, and consider an interval $A = [0, y[\subset [0,1]$. We want to estimate the number of elementary intervals belonging to $\mathscr{E}(r)$ which are contained in A.

For this purpose, let us fix another parameter $\overline{r}$, of the form $\overline{r} = e^{-\overline{n}\Lambda}$ with an integer $0 \leq \overline{n} \leq n$, corresponding to a previous step in Khodak's construction. At this previous step, we have $M_{\overline{r}}$ intervals I_j generated by the construction. If we denote by $l(I_j)$ the lengths of the intervals I_j, then we have that

$$p_{min}\overline{r} \leq l(I_j) < \overline{r}, \quad \text{for} \quad j = 1, \dots, M_{\overline{r}}, \tag{13}$$

(since the lengths of $I \in \mathscr{E}(\overline{r})$ correspond to the values $P(d)$ of the external nodes in $\mathcal{E}(\overline{r})$).

Suppose that precisely the first h of these intervals I_j are contained in A, so $U = I_1 \cup \ldots \cup I_h \subset A$.

Now, we want to estimate the number of elementary intervals in $\mathscr{E}(r)$ contained in I_j. Khodak's construction shows that this equals precisely the number of external nodes in the subtree of the node x that is related to the interval I_j. An important feature of Khodak's construction is that subtrees of $\mathcal{T}(r)$ rooted at an internal node $x \in \mathcal{I}(r)$ are parts of a self-similar infinite tree and therefore they are constructed in the same way as the whole tree. So, one just has to replace r by $\frac{r}{P(x)}$. Hence, by using this remark in (10), the number N_{I_j} of subintervals of I_j (corresponding to the value r) equals

$$N_{I_j} = M_{\frac{r}{l(I_j)}} = \frac{c'}{r} l(I_j) + \mathcal{O}\left((\log r)^d \frac{l(I_j)^{1-\eta}}{r^{1-\eta}} \right).$$

Therefore, we have that the number N_U of elementary intervals in $\mathscr{E}(r)$ contained in U is

$$N_U = N_{I_1} + \ldots + N_{I_h} = \frac{c'}{r}(l(I_1) + \ldots + l(I_h)) + \mathcal{O}\left(\frac{|\log r|^d}{r^{1-\eta}} \sum_{j=1}^{h} l(I_j)^{1-\eta} \right).$$

By using (13) and the fact that $h \le M_{\overline{r}} = O(1/\overline{r})$ we obtain

$$N_U = \frac{c'}{r}(l(I_1) + \ldots + l(I_h)) + \mathcal{O}\left((\log r)^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right)$$

Since the total number of intervals equals $M_r = c'/r + O(|\log r)|^d r^{-1+\eta})$ it follows that

$$N_U - M_r l(U) = \mathcal{O}\left((\log r)^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right) + O(|\log r)|^d r^{-1+\eta}) = \mathcal{O}\left(|\log r|^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right).$$

Since $N_A - M_r l(A) = (N_U - M_r l(U)) + (N_{A\setminus U} - M_r l(A \setminus U))$ it remains to study the difference

$$\begin{aligned} N_{A\setminus U} - M_r l(A \setminus U) &= N_{A\setminus U} - M_{r/l(I_{h+1})} \frac{l(A \setminus U)}{l(I_{h+1})} \\ &\quad + M_{r/l(I_{h+1})} \frac{l(A \setminus U)}{l(I_{h+1})} - M_r l(A \setminus U). \end{aligned}$$

The second term can be directly estimated by

$$\left| M_{r/l(I_{h+1})} \frac{l(A \setminus U)}{l(I_{h+1})} - M_r l(A \setminus U) \right| = O\left(|\log r|^d \frac{\overline{r}^{(1-\eta)}}{r^{(1-\eta)}} \right),$$

whereas the first term is bounded by

$$\left| N_{A\setminus U} - M_{r/l(I_{h+1})} \frac{l(A \setminus U)}{l(I_{h+1})} \right| \le \Delta_{n-\overline{n}}$$

Summing up and taking the supremum over all sets $A = [0, y[$ we obtain the recurrence relation

$$\Delta_n \le \Delta_{n-\overline{n}} + \mathcal{O}\left(|\log r|^d \frac{\overline{r}^{(-\eta)}}{r^{(1-\eta)}} \right). \tag{14}$$

We now set $\overline{n} = 1$ and recall that $r = e^{-\Lambda n}$ (and also $\overline{r} = e^{-\Lambda \overline{n}} = e^{-\Lambda}$). Thus, we get

$$\Delta_n \le \Delta_{n-1} + \mathcal{O}\left(n^d e^{\Lambda n(1-\eta)}\right). \tag{15}$$

We distinguish between three cases.

1. $0 < \eta < 1$. In this case we get

$$\Delta_n = O\left(\sum_{k\le n} k^d e^{\Lambda k(1-\eta)}\right) = O\left(n^d e^{\Lambda n(1-\eta)}\right)$$

   which implies $D_n = O\left((\log k(n))^d k(n)^{-\eta}\right)$.

2. $\eta = 1$. In this case we get $\Delta_n = O(n^{d+1})$ and consequently $D_n = O\left((\log k(n))^{d+1} k(n)^{-1}\right)$.

3. $\eta > 1$. Here we have

$$\Delta_n = O\left(\sum_{k\le n} k^d e^{-\Lambda k(\eta-1)}\right) = O\left(1\right)$$

   which rewrites to $D_n = O\left(k(n)^{-1}\right)$.

In order to give a lower bound of the discrepancy it is sufficient to handle the case $0 < \eta \le 1$. If $\eta > 1$ we just use the trivial lower bound $D_n \ge 1/k(n)$ which meets the upper bound. For the remaining case $0 < \eta \le 1$ we consider the interval $A = [0, p_1[$. We also recall that we can write M_r (for $r = r_n = e^{-\Lambda n}$) as

$$M_r = c'\, e^{\Lambda n} + \delta_n,$$

where δ_n has an representation of the form

$$\delta_n = n^d e^{\Lambda n(1-\eta)} \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i) + O\left(n^{d-1} e^{Ln(1-\eta)}\right).$$

Similarly to the above we obtain

$$\begin{aligned} N_A - M_r l(A) &= M_{r/p_1} - M_r p_1 \\ &= \delta_{n-n_1} - p_1 \delta_n \\ &= n^d e^{\Lambda n(1-\eta)} \left(\sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i - 2\pi\theta_i n_1) - p_1 \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i)\right) \\ &+ O\left(n^{d-1} e^{\Lambda n(1-\eta)}\right). \end{aligned}$$

By applying Lemma 6.1 it follows that there exists $\delta > 0$ and infinitely many n with

$$|N_A - M_r l(A)| \ge \delta n^d e^{\Lambda n(1-\eta)}.$$

Consequently

$$D_n \ge \frac{1}{M_r} |N_A - M_r l(A)| \ge \delta' n^d e^{-\Lambda n \eta}$$

for some $\delta' > 0$. This proves completes the proof of the lower bound (12). □

4 The irrational case

As mentioned above, the case where $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are rationally related is much more difficult to handle, since the error term in the asymptotic expansion for M_r is not that explicit in general, see Theorem 2.2. Nevertheless, we can provide upper bounds in some cases of interest.

Suppose that $m = 2$, set $p = p_1$ and $q = p_2$ and $\gamma = (\log p)/(\log q)$. It is an easy exercise to show that the number of intervals $k(n)$ is given asymptotically by

$$k(n) \sim \frac{m-1}{H} \exp\left(\sqrt{2n \log\frac{1}{p}\,\log\frac{1}{q}}\right).$$

This follows from the fact that the equation $k \log p + \ell \log q = x$ has at most one solution in integer pairs (k, ℓ). Hence, if we fix r in Khodak's construction then the corresponding number n of steps equals the number of non-negative integral lattice points (k, ℓ) with $k \log p + \ell \log q \geq \log r$ which is given by

$$n = \frac{\left(\log\frac{1}{r}\right)^2}{2\log\frac{1}{p}\,\log\frac{1}{q}} + O\left(\log\frac{1}{r}\right).$$

We have considered the case when γ is badly approximable, that is, the continued fractional expansion of γ is bounded, and the case where p and q are algebraic.

Theorem 4.1. *If $\gamma \notin \mathbb{Q}$ and it is badly approximable, then the discrepancy is estimated by*

$$D_n^* = \mathcal{O}\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\frac{1}{8}}\right) \quad \text{as } n \to \infty.$$

Furthermore, if p and q are algebraic and $\gamma \notin \mathbb{Q}$ then

$$D_n^* = \mathcal{O}\left(\left(\frac{\log\log(k(n))}{\log(k(n))}\right)^{\kappa}\right) \quad \text{as } n \to \infty,$$

where $\kappa > 0$ is an effectively computable constant.

Note that the upper bounds for the discrepancy we obtained are worse than $k(n)^{-\beta}$ for any $\beta > 0$. Actually it seems that we cannot do really better in the irrationally related cases. This is due to the fact that $\liminf_{k\neq 0} \Re(s_k) = -1$ where s_k, $k \neq 0$, runs through all the zeros of the equation $p^{-s} + q^{-s} = 1$ different from $s_0 = -1$. Actually it seems that the continued fraction expansion of $\gamma = (\log p)/(\log q)$ could be used to obtain more explicit upper bounds. However, since they are all rather poor it is probably not worth working them out in detail. The case $m > 2$ is even more involved, compare with the discussion of [10].

Proof. We use a procedure similar to that of the proof of Theorem 3.1. However, we have to use the asymptotic expansion

$$M_r = \frac{c''}{r} + O\left(\frac{1}{r}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\kappa}\right)$$

with $c'' = (m-1)/H$ and with a suitable $\kappa > 0$.

First it follows that

$$\begin{aligned} N_U - M_r l(U) &= \frac{c''}{r}(l(I_1) + \ldots + l(I_h)) + \mathcal{O}\left(\frac{1}{r\,\overline{r}}\left(\frac{\log\log\frac{\overline{r}}{r}}{\log\frac{\overline{r}}{r}}\right)^{\kappa}\right) \\ &- \frac{c''}{r}(l(I_1) + \ldots + l(I_h)) + O\left(\frac{1}{r}\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\kappa}\right) \\ &= \mathcal{O}\left(\frac{1}{r\,\overline{r}}\left(\frac{\log\log\frac{\overline{r}}{r}}{\log\frac{\overline{r}}{r}}\right)^{\kappa}\right). \end{aligned}$$

For the remaining interval $A \setminus U$ we use the (trivial) bounds $N_{A\setminus U} \le M_{r/l(I_{h+1})} = O(\overline{r}/r)$ and $l(I_{h+1}) = O(\overline{r})$ to end up with the upper bound

$$D_n = \mathcal{O}\left(\frac{1}{\overline{r}}\left(\frac{\log\log\frac{\overline{r}}{r}}{\log\frac{\overline{r}}{r}}\right)^{\kappa}\right) + O\left(\overline{r}\right).$$

Hence, by choosing

$$\overline{r} = \left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\kappa/2}$$

we finally obtain

$$D_n = O\left(\left(\frac{\log\log\frac{1}{r}}{\log\frac{1}{r}}\right)^{\kappa/2}\right).$$

This completes the proof of Theorem 4.1. □

5 Applications

5.1 LS-sequences

We recall that LS-sequences of partitions are iterative ρ-refinements of $\omega = [0,1]$, where ρ consists of L subintervals of $[0,1]$ of length α and S subintervals of length α^2 and α is given by the equation $L\alpha + S\alpha^2 = 1$.

For instance, if $L = S = 1$ then $\alpha = \frac{\sqrt{5}-1}{2}$ and we obtain the so-called Kakutani-Fibonacci sequence. Here we have $p_1 = \alpha$ and $p_2 = 1 - \alpha = \alpha^2$ and consequently

$$\log\left(\frac{1}{\alpha}\right) = n_1\Lambda \quad \text{and} \quad \log\left(\frac{1}{\alpha^2}\right) = n_2\Lambda.$$

for $\Lambda = -\log\alpha$, $n_1 = 1$ and $n_2 = 2$. Since the roots of the equation $1 - z - z^2 = 0$ are given by $z_1 = \frac{\sqrt{5}-1}{2} = \alpha = e^{-\Lambda}$ and $z_2 = \frac{-\sqrt{5}-1}{2}$ it follows that $d = 0$ and

$$\eta = 1 + \frac{\log|z_2|}{\Lambda} = 1 + \frac{\log\left|\frac{-\sqrt{5}-1}{2}\right|}{-\log\left(\frac{\sqrt{5}-1}{2}\right)} = 2$$

This shows that the discrepancy is of the order of $1/k(n)$ (and therefore it is optimal).

In the general case set $L+S=m$. Of course we are in the rational case since $p_j=\alpha$ or $p_i=\alpha^2$. More precisely we have $\Lambda=\log(1/\alpha)$ and $n_i\in\{1,2\}$ corresponding to $p_i=\alpha^{n_i}$. The zeros of the equation

$$1-Lz-Sz^2=0$$

are given by $z_1=\frac{-L+\sqrt{L^2+4S}}{2S}=\alpha$ and $z_2=\frac{-L-\sqrt{L^2+4S}}{2S}$. Hence,

$$\eta = 1+\frac{\log\left|\frac{-L-\sqrt{L^2+4S}}{2S}\right|}{L} = 1+\frac{\log\left(\frac{L+\sqrt{L^2+4S}}{2S}\right)}{L}. \tag{16}$$

Consequently we have $\eta<1$ if and only if $\frac{L+\sqrt{L^2+4S}}{2S}<1$ or if $S>L+1$. Similarly we have $\eta=1$ if and only if $S=L+1$ and $\eta>1$ if and only if $S<L+1$. This is in perfect accordance with the results of Carbone [4]. The discrepancy bounds are (or course) also of the same kind.

5.2 Sequences Related to Pisot Numbers

A Pisot number β is an algebraic integer (larger than 1) with the property that all its conjugates have modulus smaller than 1. A prominent example of Pisot numbers are the real roots of a polynomial of the form

$$z^k-a_1z^{k-1}-a_2z^{k-2}-\cdots-a_k=0, \tag{17}$$

where a_j are positive integers with $a_1\ge a_2\ge\cdots\ge a_k$, see [3]. In this case the polynomial in (17) is also irreducible over the rationals.

Suppose now that ρ is a partition of $m=a_1+a_2+\cdots+a_k$ intervals, where a_j intervals have length α^j, $1\le j\le k$, where $\alpha=1/\beta$ and β is the Pisot number related to the polynomial (17). Note that we have

$$a_1\alpha+a_2\alpha^2+\cdots+a_k\alpha^k=1.$$

Since all conjugates of α have now modulus largen than 1 it follows that $\eta>1$. This means that the order of magnitude of the discrepancy is optimal, namely $1/k(n)$. LS-sequences are a special instance for $k=1$, $a_1=L$ and $a_2=S$ with $L\ge S$.

5.3 Multiple Zeros

In the Pisot case all zeros of the polynomial are simple, since the polynomial is irreducible. However, this is not necessarily true in less restrictive cases. For example, let $\alpha=1/5$ and consider one interval of length $\alpha=1/5$, 16 intervals of lengths $\alpha^2=1/25$ and 20 intervals of lengths $\alpha^3=1/125$. Since $\alpha+16\alpha^2+20\alpha^3=1$ we have a proper partition ρ. Here the roots of the polynomial $z+16z^2+20z^3=1$ are $z_1=\alpha=1/5$ and $z_2=z_3=-1/2$ (which is a double root). Hence, we obtain $\eta=1-(\log 2)/(\log 5)=0.56932\ldots<1$ and $d=1$. Consequently the discrepancy is bouded by

$$D_n=O((\log k(n))\,k(n)^{-\eta}),$$

and this upper bound is optimal.

5.4 The rational case on fractals

The same procedure of ρ-refinements could be used also to construct u.d. sequences of partitions on fractals generated by an iterated function system (IFS) satisfying the Open Set Condition (OSC). This class of fractals has been already considered in [12], where the authors introduced a general algorithm to produce u.d. sequences of partitions and of points on fractals generated by an IFS consisting of similarities which have the same ratio and which satisfy the OSC.

Now we can extend these results eliminating the restriction that the similarities have the same ratio. In fact, we can get the same results as obtained on $[0,1]$ in Section 3 by introducing a new correspondence between nodes and subsets of the fractal.

Let $\varphi = \{\varphi_1, \ldots, \varphi_m\}$ be a system of m similarities on $\mathbb{R}^d$ having ratios $c_1, \ldots, c_m \in]0,1[$ respectively and satisfying the Open Set Condition (OSC). Let F be the attractor of φ and let S be its Hausdorff dimension. Moreover, we will consider the normalized S-dimensional Hausdorff measure P on the fractal F, that is,

$$P(A) = \frac{\mathcal{H}^S(A)}{\mathcal{H}^S(F)} \text{ for any Borel set } A \subset F;$$

recall that P is a regular probability measure.

Start with a tree having a root node of probability 1, which corresponds to the fractal F, and m leaves corresponding to the m imagines of F through the m similarities, i.e. $\varphi_1(F), \ldots, \varphi_m(F)$. The probability of each node is given by the probability P of the corresponding subset, that is, $p_i = P(\varphi_i(F)) = c_i^S$. At each iteration we select the leaves having the highest probability and grow m children out of each of them. On the fractal, this corresponds to apply successively the m similarities only to those subsets having the highest probability at this certain step. By iterating this procedure, we obtain a parsing tree associated to the sequence of partitions on the fractal F.

Let us denote by $\{\pi_n\}$ the sequence of partitions of F generated in this way, such that

$$\pi_n = \left\{\psi_{j_{k(n)}} \psi_{j_{(k(n)-1)}} \cdots \ \psi_{j_1}(F) : j_1, \ldots, j_{k(n)} \in \{1, \ldots, m\}\right\}.$$

where $k(n)$ is the number of sets constructed at the step n.

Let us denote by $\mathscr{E}_n$ the collection of the $k(n)$ sets E_i^n belonging to the partition π_n and by $\mathscr{E}$ the union of the families $\mathscr{E}_n$, by varying n. The sets of the class $\mathscr{E}$ are called *elementary sets.*

In [12], it is proved that the class $\mathscr{E}$ is determining and consists of P-continuity sets. Now, if we choose a point $t_i^{(n)}$ in each $E_i^n \in \mathscr{E}_n$, we can consider the elementary discrepancy of this set of points on the fractal, i.e.

$$D_n^{\mathscr{E}} = \sup_{E \in \mathscr{E}} \left| \frac{1}{k(n)} \sum_{i=1}^{k(n)} \chi_E(t_i^{(n)}) - P(E) \right|.$$

By using a procedure similar to the one used in the proof of the Theorem 3.1 we get the following estimates for the elementary discrepancy if $\log\left(\frac{1}{p_1}\right), \ldots, \log\left(\frac{1}{p_m}\right)$ are rationally related:

$$D_n^{\mathscr{E}} = \begin{cases} O\left((\log k(n))^d k(n)^{-\eta}\right) & \text{if } 0 < \eta \leq 1, \\ O\left(k(n)^{-1}\right) & \text{if } \eta > 1. \end{cases} \tag{18}$$

Furthermore, both upper bounds are best possible. We just have to observe that the number $N_E^{(n)}$ of elementary sets in $\mathscr{E}_n$ that are contained in an elementary set E is given by $M_{r/P(E)}$ which implies

$$N_E^{(n)} = \frac{c'}{r} P(E) + O\left(|\log r|^d\, r^{-1+\eta} P(E)^{1-\eta}\right). \tag{19}$$

This proves (18) directly for $\eta \leq 1$ and also shows that this bound is optimal. If $\eta > 1$ then we argue recursively. The elementary interval E is either contained in $\mathscr{E}_1 = \{\varphi_1(F), \ldots, \varphi_m(F)\}$, which means that we can use (19) for $P(E) \in \{p_1, \ldots, p_n\}$, or it is part of $\varphi_j(F)$ for some j. In the latter case we rewrite $N_E - k(n)P(E)$ to

$$N_E - k(n)P(E) = \left(N_E - k(n-1)\frac{P(E)}{P(E_j)}\right) + \left(\frac{P(E)}{P(E_j)} - k(n)P(E)\right)$$

which leads to a recurrence of the form

$$\Delta_n^{\mathscr{E}} = \sup_{E \in \mathscr{E}} \left| N_E^{(n)} - k(n)P(E) \right| \leq \Delta_{n-1}^{\mathscr{E}} + O\left(n^d e^{\lambda n(1-\eta)}\right).$$

Hence $\Delta_n^{\mathscr{E}} = O(1)$ and consequently $D_n^{\mathscr{E}} = O(1/k(n))$ (which is also optimal).

In particular it follows that the sequence of partitions $\{\pi_n\}$ is u.d. with respect to P. Actually, this remains true in the irrationally related case, too. However, we can only derive effective upper bounds for the discrepancy in very specific cases.

There are few papers devoted to uniformly distributed sequences on fractals and to estimates of the discrepancy, see [7, 6, 11]. The various types of discrepancy considered depend very much on the geometric features of the fractal. Moreover, the only kind of discrepancy which makes sense for all the fractals generated by IFS and satisfying the OSC is the so-called elementary discrepancy. A unifying approach has been proposed by Albrecher, Matoušek and Tichy in [1], but it concerns the average discrepancy.

6 Auxiliary Results

In this section we collect some auxiliary results that are used in the proof of Theorem 2.2 (see Section 2).

6.1 Trigonometric Sums

Lemma 6.1. *Let* $f(n) = \sum_{i=1}^{k} c_i \cos(2\pi\theta_i n + \alpha_i)$*,* $c_i, \alpha_i, \theta_i \in \mathbb{R}$ *be defined for non-negative integers* n *and suppose that* f *is not identically zero. Then there exists* $\delta > 0$ *such that* $|f(n)| \geq \delta$ *for infinitely many non-negative integers* n*.*

Proof. We have to distinguish two cases:

Case 1 $\theta_1, \dots, \theta_k$ are rationally related.

There exist $\Lambda \in \mathbb{R} \setminus \{0\}$ and $k_i \in \mathbb{Z}$ such that $\theta_i = \Lambda k_i$. In this case, we can rewrite the function as follows

$$f(n) = \sum_{i=1}^{k} c_i \cos(2\pi\Lambda n k_i + \alpha_i) = \sum_{i=1}^{k} c_i \cos(2\pi\{\Lambda n\} k_i + \alpha_i),$$

where $\{x\}$ denotes is the fractional part of x.

Hence, $f(n) = g(\{\Lambda n\})$ where $g(x) = \sum_{i=1}^{k} c_i \cos(2\pi k_i x + \alpha_i)$ is a periodic non-zero function of period 1.

Case 1.1 If $\Lambda \in \mathbb{Q}$, then $\Lambda = \frac{p}{q}$ for some coprime integers $p, q \in \mathbb{Z}$ and the sequence $f(n)$ attains periodically the set of values

$$g\left(\left\{\frac{pn}{q}\right\}\right), \quad n = 0, \dots, q-1.$$

Since they are not all equal to zero there exists $\delta > 0$ such that $|f(n)| = |g(\{\Lambda n\})| \geq \delta$ for infinitely many n. In particular we can use a linear subsequence $qn + r$ for which $|f(qn + r)| \geq \delta$.

Case 1.2 If $\Lambda \notin \mathbb{Q}$, then the sequence $\{\Lambda n\}$ is u.d. modulo 1 and consequently dense in $[0, 1]$. Hence, there (again) exists $\delta > 0$ such that $|f(n)| = |g(\{\Lambda n\})| \geq \delta$ for infinitely many n.

Case 2 $\theta_1, \dots, \theta_k$ are irrationally related.

Here we divide the θ_i in groups which are rationally related. Assume that we have s groups $\{\theta_i : i \in I_j\}$, $j = 1, \dots, s$, and in each group we write

$$\theta_i = \Lambda_j k_i, \quad i \in I_j$$

with $k_i \in \mathbb{Z}$ and some $\Lambda_j \in \mathbb{R} \setminus \{0\}$.

In this case, we distinguish between three different sub-cases:

Case 2.1 $1, \Lambda_1, \dots, \Lambda_s$ are linearly independent over $\mathbb{Q}$ (and consequently $\Lambda_1, \dots, \Lambda_s \notin \mathbb{Q}$). We set $f_j(x) = \sum_{i \in I_j} c_i \cos(2\pi x k_i + \alpha_i)$ (where we assume w.l.o.g. that f_j is non-zero) and $g(x_1, \dots, x_n) = \sum_{j=1}^{s} f_j(x_j)$ Then

$$f(n) = \sum_{j=1}^{s} f_j(\{n\Lambda_j\}) = g\left(\{n\Lambda_1\}, \dots, \{n\Lambda_s\}\right).$$

By Kronecher's Theorem, the sequence $(\{n\Lambda_1\},\ldots,\{n\Lambda_s\})$ is dense in the cube $[0,1]^s$. Thus, it follows (as above) that there exists $\delta > 0$ such that $|f(n)| \geq \delta$ for infinitely many n.

Note that by same reasoning it follows that for every $\varepsilon > 0$ we have $|f(n)| \leq \varepsilon$ for infinitely many n. (Here we also use that fact that f has zero mean.) This observation will be used in Case 2.3.

Case 2.2 $1, \Lambda_1, \ldots, \Lambda_s$ are linearly dependent over $\mathbb{Q}$ and $\Lambda_1, \ldots, \Lambda_s \notin \mathbb{Q}$.

In this case there exist $q, p_1, \ldots, p_s \in \mathbb{Z}$ such that $q = p_1\Lambda_1 + \ldots + p_s\Lambda_s$. Suppose (w.l.o.g.) that $p_1 > 0$ and consider the subsequence of integers $(p_1 n)$:

$$\begin{aligned} f(p_1 n) &= \sum_{j=1}^{s} f_j(n\Lambda_j p_1) \\ &= f_1(n(q - \Lambda_2 p_2 - \cdots - \Lambda_s p_s)) + \sum_{j=2}^{s} f_j(n\Lambda_j p_1). \end{aligned}$$

By using the addition theorem for cos and rewriting the sum accordingly we obtain a representation of the form

$$f(p_1 n) = \sum_{j=2}^{s} \sum_{i \in I_j} \tilde{f}_j(n\Lambda_j p_j),$$

where $\tilde{f}_j$ are certain trigonometric polynomials. This means that we have eliminated Λ_1. In this way we can proceed further. If $1, p_2\Lambda_2, \ldots, p_s\Lambda_s$ are linearly independent over $\mathbb{Q}$ then we argue as in Case 2.1. However, if $1, p_2\Lambda_2, \ldots, p_s\Lambda_s$ are linearly dependent over $\mathbb{Q}$ then we repeat the elimination procedure etc. Note that this elimination procedure terminates, since we assume that $\Lambda_1, \ldots, \Lambda_s \notin \mathbb{Q}$. Hence, we always end up in Case 2.1.

Case 2.3 $\Lambda_1, \ldots, \Lambda_s$ are not all irrationals.

Here we represent $f(n) = h_1(n) + h_2(n)$, where

$$h_1(n) = \sum_{j \in \{j : \Lambda_j \in \mathbb{Q}\}} f_j(n) \quad \text{and} \quad h_2(n) = \sum_{j \in \{j : \Lambda_j \notin \mathbb{Q}\}} f_j(n).$$

If h_1 is non-zero then we can argue as in Case 1.1. All appearing θ_i are rational and consequently there exits a linear subsequence $qn + r$ such that $|h_1(qn + r)| \geq \delta$ for some $\delta > 0$. Next we reduce the sum $h_2(qn + r)$ to a sum of the form that is discussed in Case 2.1. (possibly we have to eliminate several terms as discussed in Case 2.2.). Consequently it follows that there exists infinitely many n such that $|h_2(qn + r)| \leq \delta/2$. Hence we have $|f(n)| \geq \delta/2$ for infinitely many n.

If h_1 is zero for all non-negative integers we just have to consider h_2. But this case is precisely that of Case 2.2.

□

6.2 Zerofree Regions

The purpose of the next two lemmas is to discuss zero-free regions of the equation $1-p^{-s}-q^{-s}=0$ (where p,q are positive numbers with $p+q=1$). It is clear that $s=-1$ is a solution and that all solutions have to satisfy $\Re(s)\ge -1$. (Otherwise, we would have $|p^{-s}|+|q^{-s}|<1$.) Furthermore, it is easy to verify that there are no solutions (other than $s=-1$) of the line $\Re(s)=-1$ if and only if the ratio $\gamma=(\log p)/(\log q)$ is irrational, compare also with [18]. Furthermore it is known that there exist σ_0 and $\kappa>0$ such that in each box of the form

$$B_k=\{s\in\mathbb{C}:-1\le\Re(s)\le\sigma_0,\,(2k-1)\tau\le\Im(s)<(2k+1)\tau\},\quad k\in\mathbb{Z}\setminus\{0\},$$

there is precisely one zero of $p^{-s}+q^{-s}=1$, and there are no other zeros.

However, the positions of the zeros in B_k is by no means clear. Nevertheless, with the help of the continued fraction expansion of γ it is possible to construct (infinitely many) zeros s of the above equation with $\Re(s)<-1+\varepsilon$ (for every $\varepsilon>0$). Therefore it is natural to ask for zero-free regions of this equation. Actually one has to assume some Diophantine condition on γ to get precise information.

Lemma 6.2. *If γ is badly approximable then for every solution $s\ne -1$ of the equation*

$$1-p^{-s}-q^{-s}=0$$

we have that

$$\Re(s)>\frac{c}{(\Im(s))^2}-1 \tag{20}$$

for some positive constant c.

Proof. We recall that an irrational number γ is badly approximable if its continued fractional expansion $\gamma=[a_0;a_1;\ldots]$ is bounded, that is, there exist a positive constant D such that $\max\limits_{j\ge1}(a_j)\le D$. Equivalently we have the property that there exists a constant $d>0$ such that

$$\left|\gamma-\frac{k}{l}\right|\ge\frac{d}{l^2} \tag{21}$$

for all non-zero integers k,l, see [14].

In order to make the presentation of the proof more transparent we make a shift by 1 and consider the equation

$$p^{1-s}+q^{1-s}=1 \tag{22}$$

and show that all non-zero solutions satisfy $\Re(s)>c/\Im(s)^2$ for some positive constant c that depends on γ.

Suppose that $s=\sigma+i\tau$ is a zero of (22) with $\sigma>0$. Furthermore, we assume that $\sigma\le\varepsilon$, where ε is a sufficiently small constant (that will be fixed in a moment). Since $p+q=1$ and

$|p^{1-s}| = p^{1-\sigma} = p(1+O(\varepsilon)) > p$ and $|q^{1-s}| = q^{1-\sigma} = q(1+O(\varepsilon)) > q$ we can only have a solution if the arguments of p^{1-s} and q^{1-s} are small. (Actually they have to be of order $O(\sqrt{\varepsilon})$ if ε is chosen sufficiently small). W.l.o.g. we write

$$\arg(p^{1-s}) = \tau \log(1/p) = 2\pi k + \eta_1 \quad \text{and} \quad \arg(q^{1-s}) = \tau \log(1/q) = 2\pi l - \eta_2$$

for some integers k, l and certain positive numbers η_1, η_2 (which are of order $O(\sqrt{\varepsilon})$). More precisely, by doing a local expansion in (22) we obtain

$$\eta_2 = \frac{q}{p}\eta_1 + O(\eta_1^2) \quad \text{and} \quad \sigma = \frac{p}{2qH}\eta_1^2 + O(\eta_1^4).$$

Furthermore we have

$$\begin{aligned}\gamma &= \frac{\tau \log \frac{1}{p}}{\tau \log \frac{1}{q}} \\ &= \frac{2\pi k + \eta_1}{2\pi l - \eta_2} \\ &= \frac{k}{l} + \frac{1}{2\pi}\left(\frac{1}{l} + \frac{kp}{l^2 q}\right)\eta_1(1 + O(\eta_1/l)).\end{aligned}$$

This means that k/l is close to γ and by applying (21) it follows that

$$\eta_1 \geq \frac{d'}{|l|}$$

for some constant $d' > 0$. Consequently we obtain $\sigma \geq d''/l^2$ (for some constant $d'' > 0$) which translates directly to $\sigma > c/\tau^2$ for some positive constant c. □

Next we consider the case of algebraic number p and q with the property that $\log(p)/\log(q)$ is irrational.

Lemma 6.3. *If $p, q \in]0,1[$ are positive algebraic numbers with $p + q = 1$ and the property that $\log(p)/\log(q)$ is irrational. Then for every solution $s \neq -1$ of the equation*

$$1 - p^{-s} - q^{-s} = 0$$

we have

$$\Re(s) > \frac{D}{(\Im(s))^{2C}} - 1 \tag{23}$$

with effectively computable positive constants C, D.

The classical theorem of Gelfond-Schneider says that $\gamma = \log(p)/\log(q)$ is irrational for algebraic numbers p and q then γ is transcendental. Baker's Theorem ([2]) gives also effective bounds for Diophantine approximation of γ that will be used in the subsequent proof of Lemma 6.3. (Recall that the height of an algebraic number is the maximum of the absolute values of the relatively prime integer coefficients in its minimal defining polynomial, while its degree is the degree of this polynomial.)

Theorem 6.4 (Baker's Theorem [2]). *Let $\gamma_1, \ldots, \gamma_n$ be non-zero algebraic numbers with degrees at most d and heights at most A. Further, $\beta_0, \beta_1, \ldots, \beta_n$ are algebraic numbers with degree at most d and heights at most B (≥ 2). Then for*

$$\Lambda = \beta_0 + \beta_1 \log \gamma_1 + \ldots + \beta_n \log \gamma_n$$

we have either $\Lambda = 0$ or $|\Lambda| \geq B^{-C}$, where C is an effectively computable number depending only on n, d, and A.

Proof. (Lemma 6.3) We apply Theorem 6.4 for the algebraic number $\gamma_1 = p$ and $\gamma_1 = q$ and the integers $\beta_0 = 0$, $\beta_1 = l$, and $\beta_2 = -k$. Then $B = \max\{|k|, |l|\}$. W.l.o.g. we may assume that $p > q$ which assures that we only have to consider cases with $|k| \leq |l|$. Thus

$$|l \log p - k \log q| > B^{-C}$$

and consequently

$$\left| \frac{\log p}{\log q} - \frac{k}{l} \right| > \left(\frac{1}{\log q} \right) \frac{B^{-C}}{l} > \left(\frac{1}{\log q} \right) \frac{1}{l^{1+C}}, \tag{24}$$

where C is effectively computable.

By using (24) instead of (21) in the proof of Lemma 6.2 we complete the proof of Lemma 6.3 easily. □

6.3 Differentiating Asymptotic Expansions

Lemma 6.5. *Suppose that $f(v)$ is a non-negative increasing function for $v \geq 0$. Assume that*

$$F(v) = \int_0^v f(w)dw$$

has the asymptotic expansion

$$F(v) = \frac{v^{\lambda+1}}{(\lambda+1)} \left(1 + \mathcal{O}\left(g(v)\right)\right) \quad \text{as} \quad v \to \infty,$$

where $\lambda > -1$ and $g(v)$ is a decreasing function that tends to zero as $v \to \infty$. Then

$$f(v) = v^{\lambda} \left(1 + \mathcal{O}\left(g(v)^{\frac{1}{2}}\right)\right) \quad \text{as} \quad v \to \infty.$$

Proof.
By the assumption we have that there exist $v_0, c > 0$ such that for all $v \geq v_0$ we have

$$\left| F(v) - \frac{v^{\lambda+1}}{(\lambda+1)} \right| \leq c|g(v)| \frac{v^{\lambda+1}}{(\lambda+1)}.$$

Now, set $h = |g(v)|^{\frac{1}{2}} v$. By monotonicity, for $v \geq v_0$ we get

$$\frac{F(v+h) - F(v)}{h} = \frac{1}{h} \int_v^{v+h} f(w)dw \geq \frac{1}{h} \int_v^{v+h} f(v)dw = f(v)$$

and so

$$\begin{aligned}
f(v) &\leq \frac{F(v+h)-F(v)}{h} \\
&\leq \frac{1}{h}\left(\frac{(v+h)^{\lambda+1}}{\lambda+1}-\frac{v^{\lambda+1}}{\lambda+1}\right)+\frac{1}{h}\left(c|g(v+h)|\frac{(v+h)^{\lambda+1}}{(\lambda+1)}+c|g(v)|\frac{v^{\lambda+1}}{(\lambda+1)}\right) \\
&\leq \frac{1}{h(\lambda+1)}\left(v^{\lambda+1}+(\lambda+1)v^{\lambda}h+\mathcal{O}(v^{\lambda-1}h^2)-v^{\lambda+1}\right)+\mathcal{O}\left(|g(v)|\frac{v^{\lambda+1}}{h}\right) \\
&= v^{\lambda}+\mathcal{O}\left(v^{\lambda-1}h\right)+\mathcal{O}\left(|g(v)|\frac{v^{\lambda+1}}{h}\right) \\
&= v^{\lambda}+\mathcal{O}\left(v^{\lambda-1}|g(v)|^{\frac{1}{2}}v\right)+\mathcal{O}\left(|g(v)|\frac{v^{\lambda+1}}{|g(v)|^{\frac{1}{2}}v}\right) \\
&= v^{\lambda}+\mathcal{O}\left(v^{\lambda}|g(v)|^{\frac{1}{2}}\right).
\end{aligned}$$

Similarly we obtain a corresponding lower bound. □

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