Dynamical Phenomena and Their Models: Truth and Empirical Correctness

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Abstract

In the epistemological tradition, there are two main interpretations of the semantic relation that an empirical theory may bear to the real world. According to realism, the theory-world relationship should be conceived as truth; according to instrumentalism, instead, it should be limited to empirical adequacy. Then, depending on how empirical theories are conceived, either syntactically as a class of sentences, or semantically as a class of models, the concepts of truth and empirical adequacy assume different and specific forms. In this paper, we review two main conceptions of truth (one sentence-based and one model-based) and two of empirical adequacy (one sentence-based and one model-based), we point out their respective difficulties, and we give a first formulation of a new general view of the theory-world relationship, which we call Methodological Constructive Realism (MCR). We then show how the content of MCR can be further specified and expressed in a definite and precise form. The bulk of the paper shows in detail how it is possible to accomplish this goal for the special case of deterministic dynamical phenomena and their correlated deterministic models. This special version of MCR is formulated as an axiomatic extension of set theory, whose specific axioms constitute a formal ontology that provides an adequate framework for analyzing the two semantic relations of truth and empirical correctness, as well as their connections.

Keywords Realism · Instrumentalism · Truth · Empirical adequacy · Syntactic view of theories · Semantic view of theories

1 Introduction

In this paper, by an *empirical theory* we mean any scientific theoretical construct, not necessarily of a linguistic type, which is expressly designed to describe or explain specific aspects of the real world. This usage of the term thus presupposes the scientific character of an empirical theory. In this acceptance, empirical theories are naturally contrasted with *mathematical ones*, which are taken to be all those scientific theoretical constructs that are

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not empirical theories. The exact nature of the semantic relations that an empirical theory may bear to the real world then depends on how the theory and the real world itself are further conceived or analyzed.

From a philosophical point of view, the relationship between an empirical theory and the real world is often understood in two opposing ways. According to *realism*, either theories are *true* representations of those aspects of reality that they are designed to describe or, even if we cannot tell whether theories are actually true or false, they may be objectively true and, in some circumstances, we may have good reasons to conjecture that they are not (or are). The second alternative is the one set forth and forcibly defended by (Popper 1969, ch. 3, sect. 6), while the first one is traditionally attributed to Galileo, as well as to many other leading figures of the Scientific Revolution (Popper 1969, ch. 3, sect. 1). According to *instrumentalism*, instead, theories are not true representations of reality, as they are just convenient tools to describe, summarize, or forecast our observations. Empirical theories do not give us any insight into reality, for they do not have an epistemic value, but only an instrumental or practical one. As Duhem put it, empirical theories just save the phenomena (Duhem 1985, p. 117). Therefore, the relevant relationship between a theory and the real world is not truth, but, at most, *empirical adequacy*.

In the philosophy of science, empirical theories have been traditionally analyzed according to two main alternative approaches, the *syntactic* or sentential view, and the *semantic* one. For the syntactic view, an empirical theory consists of a purely formal system (an axiomatized theory), together with a set of correspondence rules (an interpretative system), which is supposed to fix the references of the observational terms of the theory, but does not typically suffice to set the references of the theoretical ones (Hempel 1952, 1958, sect. 8). The theory itself is then identified with *the class of all theorems* of the formal system, that is, all those sentences that can be logically deduced from its axioms. The syntactic view used to be predominant in the philosophy of science for quite a long time, to the extent that it came to be called *the received view*. It was mainly defended and developed by the logical empiricists, but even Popper, one of their most prominent critics, would in fact subscribe to a form of the syntactic view (Popper 1959, sect. 16).

Starting from the end of the 1950s, in conjunction with the crisis of Logical Empiricism and the rise of what is sometimes called (Sindjelić 2008) the New Philosophy of Science (Hanson 1958; Feyerabend 1958, 1965, 1970; Kuhn 1962; Toulmin 1972), there was growing concern that even the syntactic view might be irremediably flawed. By the end of the 1960s, a semantic alternative (Beth 1948, 1949, 1960; Adams 1955, 1959; Suppes 1957, 1967, 1969a, b, 2002; Przełęcki 1969, 1974a, b; Suppe 1967, 1972a, b, 1977, 1989; van Fraassen 1970, 1972, 1980, 1989, 2008; Sneed 1971; Dalla Chiara and Toraldo di Francia 1973, 1976, 1979, 1981; Wójcicki 1974, 1975; Stegmüller 1976, 1979; Balzer et al. 1987; Bickle 1998) had been basically devised, even though, unlike its syntactic competitor, the semantic view has never been a substantially unitary and well delimited conception (Giunti et al. 2016, pp. 15–16). Nevertheless, one basic tenet is shared by many of its advocates: An empirical theory is *not* a class of *sentences*, but a class of *models*, which are not linguistic entities, but set-theoretical ones.

It is quite obvious that, depending on how an empirical theory is conceived, either syntactically as a class of sentences, or semantically as a class of models, the kind of relation that the theory may bear to the real world changes accordingly. For, in the first case, it is a sentences/world relation, while in the second one the relation is models/world. But the syntactic and the semantic view also differ on how they tend to conceive of the world itself. If *sentences* are related to the world, it is quite natural to think of it as a collection of *facts*, which may either be regularities (laws) or singular events. Instead, if *models* are related to

the world, it is more suitable to think of it as a collection of *phenomena*, where each phenomenon cannot be identified with any single fact, but rather with an organized complex of events or regularities (Giunti 1992, pp. 136–7). We are now going to briefly review different forms that the traditional instrumentalism/realism opposition may assume, depending on whether empirical theories are construed syntactically or semantically.

1.1 Instrumentalism Versus Realism from the Syntactic Point of View

We have mentioned above that, for the received view, the interpretative system of an empirical theory can, at most, fix the references of its observational terms, but is not sufficient to set the references of the theoretical ones. In this framework, the empirical content of the theory is defined as the class of all theorems in which only observational terms occur, and a theory is *empirically adequate* if all sentences in its empirical content turn out to be true or, at least, consistent with all actual experiments or observations. The syntactic view is thus most naturally conjoined with an instrumentalist epistemology, according to which the only significant part of a theory is its empirical content, and the relevant theory/world relation hence reduces to empirical adequacy.

Even though an instrumentalist reading of a syntactically conceived empirical theory is most straightforward, and it was in fact usually associated with the received view, a realist interpretation is nevertheless possible. According to standard Tarskian semantics, a sentence of a formal language is either true or false just in case the references of all non-logical terms occurring in it are fixed. If the sentence belongs to the language of an empirical theory, the references of its observational terms can be taken to be fixed by an appropriately chosen interpretative system, but, as mentioned, such a system is not usually thought to be sufficient to fix the references of the theoretical terms. This limitation of the interpretative system, however, is not absolute, but itself depends on how the nature of the real world is conceived. If only observable entities are thought to be real, then no interpretative system can fix the real world references of theoretical terms. But, if this empiricist assumption is forsaken, theoretical terms may very well refer to unobservable real entities, as well as observational terms refer to observable ones. Thus, an empirical theory turns out to be *true* if all its theorems are true with respect to the real world references of all its terms (either observational or theoretical), false otherwise. According to (Popper 1969, ch. 10, sect. VII), this kind of realist interpretation of a syntactically conceived empirical theory finally became available to philosophers when Tarski's semantic conception of truth (Tarski 1944, 1956) rehabilitated the classic or objective theory of truth as correspondence to the facts.

1.2 Instrumentalism Versus Realism from the Semantic Point of View

We have mentioned above that, for the semantic view, an empirical theory is a class of models, while the world is typically conceived as a class of phenomena. Thus, the relation that a theory bears to the world can in general be characterized as a models/phenomena relation. We are now going to sketch how, within a semantic framework, the models/phenomena relation can be interpreted as empirical adequacy or truth, according to, respectively, an instrumentalist or a realist epistemological stance.

A semantic account of the empirical adequacy of a theory has been proposed by (van Fraassen 1980, ch. 3). He agrees with Suppes (Suppes 1967, pp. 62–4) that empirical theories are classes of models that typically have a quite complex mathematical structure and,

for this reason, they cannot be directly related to phenomena. He also agrees with Suppes that phenomena only provide empirical models (Suppes 1967, pp. 58–9; Suppes 2002, pp. 4-5; Suppes 1957, pp. 266–71) of lower complexity, whose base entities and relational structures are entirely observable in nature; van Fraassen calls these models *appearances* (van Fraassen 1980, p. 45), and he explicitly recognizes that they are essentially identical to Suppes' empirical models (van Fraassen 1980, pp. 64–5).

According to van Fraassen, customary presentations of an empirical theory usually play two different roles. On the one hand, similar to Suppes' definitions of set theoretical predicates (Suppes 1957, sect. 12.2), they specify the models of the theory, but, on the other hand, they also specify a particular class of substructures of such models (van Fraassen 1980, p. 65). These substructures are in fact the parts of a model that can be related to appearances and, for this reason, they are called *empirical substructures* (van Fraassen 1980, p. 64). Empirical adequacy then obtains just in case empirical substructures and appearances turn out to be isomorphic: "the theory is empirically adequate if it has some model such that all appearances are isomorphic to empirical substructures of that model" (van Fraassen 1980, p. 64).

A realist interpretation of the models/phenomena relation has been put forth by Suppe. His conception of the structure of empirical theories is a special version of the semantic view that is known as the *state-space approach* (van Fraassen 1980, p. 67). The origin of this approach may be traced back to Birkhoff's and von Neumann's studies on the logic and the foundations of quantum mechanics (Birkhoff and von Neumann 1936; von Neumann 1955, 1962). The general lines of the state-space approach were then laid down by (Beth 1948, 1949, 1960), and later independently developed by (van Fraassen 1970, 1972, 1980, 2008) and (Suppe 1967, 1972a, b, 1977, 1989). In what follows, we refer to Suppe's formulation of the state-space approach, which we call the *state-space view*.

For the state-space view, an empirical theory is first of all characterized by its *intended scope*, the class of the *causally possible phenomena* that it is designed to describe or explain (Suppe 1972b, p. 130; Suppe 1989, p. 82). However, the description is only possible by selecting a number of parameters $p_1, p_2, \ldots, p_n$, which are supposed to be the only relevant ones for the study of such phenomena. As a matter of fact, the phenomena in the intended scope of the theory also depend on other factors, not included in the selected parameters. Nevertheless, the theory does not attempt to study such phenomena in their full complexity, but only in so far as they *would have been*, if the selected parameters had been the only causally relevant factors.

Thus, in the first place, the selection of the parameters somehow abstracts from the phenomena, or constructs, a class of idealized and counterfactual entities, which Suppe calls the *causally possible physical systems* (Suppe 1972b, p. 132; Suppe 1989, p. 84). But, in the second place, it also fixes their *state space*, that is to say, the class of all possible states of such systems. The theory then describes the behavior of such systems by specifying appropriate relations on the state space. This is obtained by configuring the state space by means of specific laws that, in agreement with a traditional classification, may be laws of succession, laws of coexistence, or laws of interaction, as well as deterministic or statistic (Suppe 1977, p. 226). According to Suppe, the configuration of the state space is tantamount to determining a third class of systems, the *theory-induced physical systems* (Suppe 1972b, p. 132; Suppe 1989 p. 84).

Hence, on the one hand, the state space, together with its relational structure imposed by the theoretical configuration, is in fact a model in the standard set-theoretical sense; such a model defines the class of the theory-induced physical systems, and it can in fact be identified with that class. On the other hand, each theory-induced physical system is, so to

speak, the model-theoretic version of a corresponding causally possible physical system. On this basis, Suppe finally claims that, if the theory is true, the class of the theory-induced physical systems is *identical* to the class of the causally possible physical systems and, conversely, if the two classes are distinct, the theory is false (Suppe 1972b, pp. 132, 145-8; Suppe 1989, pp. 84, 96-99). This realist interpretation of the models/phenomena relation is known as *modified realism* (Suppe 1972b, p. 150) or *quasi-realism* (Suppe 1989, p. 101).

1.3 The Theory-World Relationship. Open Problems

We have seen that the theory-world relationship is typically intended as a sentences/facts relation, or a models/phenomena one, depending on whether theories are conceived syntactically or semantically. Furthermore, depending on whether an instrumentalist or a realist stance is taken, such a relation is conceived as empirical adequacy or truth. We have then shortly reviewed two main conceptions of empirical adequacy (the sentential one proper of the received view, and the model-based one set forth by van Fraassen), and two of truth (the sentential one proposed by Popper, and the model-based one put forth by Suppe). It is well known that all four positions involve some serious difficulties, which we in turn review below.

1.3.1 Problems with the Sentential and the Model-Based Views of Empirical Adequacy

For the received view, the empirical content of a theory is the class of all its theorems in which only observational terms occur, and the theory is empirically adequate just in case all sentences in its empirical content turn out to be true or, at least, consistent with all actual experiments or observations (see Sect. 1.1, par. 1). One of the most serious problems with this kind of definition of empirical adequacy is that it is based on a purely syntactic notion of empirical content, which is not tenable. A simple argument brings about this point (van Fraassen 1980, pp. 54-55). As long as a theory asserts the existence of non-observable entities and negation belongs to the language of the theory, one of its theorems will be the sentence $\omega := \exists x (\neg O_1(x) \wedge \neg O_2(x) \wedge \ldots \wedge \neg O_n(x))$, where $O_1, O_2, \ldots, O_n$ are *all* the *observational* predicates of the theory. Note that ω belongs to the empirical content of the theory, for it is one of its theorems and only observational terms occur in it. But ω's meaning is just that a non-observable entity exists. Therefore, as ω does not make any assertion about observables, it should not belong to the empirical content of the theory.

For van Fraassen, a theory is empirically adequate if it has some model such that all appearances are isomorphic to empirical substructures of that model (see Sect. 1.2, pars. 2-3). Thus, for this notion of empirical adequacy to be applicable to a specific theory, it is crucial that the empirical substructures of all its models be clearly specified. However, van Fraassen has never set forth a general definition of empirical substructure, nor has he given explicit and sufficiently detailed indications as to how empirical substructures should be individuated or specified. Furthermore, similar to Suppes' account of empirical models, van Frassen's appearances are models with a simple mathematical structure, entirely made up of observable individual entities and observable relations or operations. Appearances are the only phenomenal referents to which a theoretical model, through its empirical substructures, is related and, as previously remarked (Sect. 1.2, par. 2), appearances are essentially identical to Suppes' empirical models. Thus, van Fraassen shares with Suppes the strong empiricist assumption that the nature of the phenomenal referents of theories is purely observational or empirical. But such an assumption does not seem to square well

with concrete scientific practice and methodology. Finally, (van Fraassen 1980) does not make clear whether the simple mathematical structure of appearances is something that belongs to phenomena in themselves, or it is rather constructed or determined in the process of connecting theoretical models to phenomena.[1]

1.3.2 Problems with the Sentential and the Model-Based Views of Truth

According to Popper's realistic interpretation of Tarskian semantics, a syntactically conceived empirical theory is true if all its theorems are true with respect to the real world references of all its non-logical terms (either observational or theoretical), false otherwise (see Sect. 1.1, par. 2). It should first of all be noted that, by this interpretation, it is the real world itself that provides a model for the theory, in the standard sense of Tarskian semantics. For the theory is either true or false in the model determined by the real world references of all the non-logical terms. But, as already remarked, the models of an empirical theory typically have a quite complex mathematical structure. Therefore, such a complex mathematical structure belongs to the world itself. Popper's realistic interpretation of Tarskian semantics thus entails a strong form of mathematical realism, according to which all the mathematical structure of an empirical theory is physically real. It is safe to say that even a scientific realist might not be willing to accept this awkward consequence of Popper's view.

There is a second objection against his view. Popper takes for granted that Tarski's definition of truth equally applies to either mathematical theories or empirical ones. In order to formulate Tarski's definition of truth for a given theory—the so called "object-theory"—, it is necessary that both the object-theory and the meta-theory—the theory in which truth is going to be defined—have a well specified structure. This condition can be easily satisfied if both the object-theory and the meta-theory can be thought as purely formal systems, that is to say, linguistic systems such that their (1) primitive terms and (2) sentences are formally specified, and (3) the conditions for a sentence to be a theorem, as well as (4) formal rules of definition, are also given. In a formal system, "theorems are the only sentences which can be asserted" (Tarski 1944, p. 346). This is all it is needed for the definition of truth to be applied to a mathematical theory, but, if the theory is empirical, the latter requirement is far too restrictive; asserting a sentence cannot reduce to its being a theorem, as sentences may also be asserted on the basis of factual or empirical considerations (Tarski 1944, p. 347, par. 2; p. 368, par. 3). But then, it is false that Tarski's definition of truth equally applies to mathematical or empirical theories. For, in the case of an empirical theory, its meta-theory cannot be a purely formal system, as the possibility of asserting a sentence on the basis of empirical or factual considerations must also be allowed. Unfortunately, however, it is not at all clear what methodological principles might grant such a kind of empirical and non-formal assertions.

A well known related difficulty of Popper's methodology concerns the corroboration of a hypothesis. A hypothesis is corroborated if there is no severe test that falsifies it or, equivalently, if any such test is consistent with the hypothesis (Popper 1959, ch. 10).

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[1] However, more recently van Fraassen seems to lean towards the second alternative: "construction of a data model is precisely the selective relevant depiction of the phenomena by the user of the theory required for the possibility of representation of the phenomenon." (van Fraassen 2008, p. 253). Note that, according to van Fraassen's new terminology (van Fraassen 2008, ch. 12, note 24), the term "data model" in the previous passage is to be intended as a synonym of "appearances".

However, one of the most serious problems of Popper's falsificationism consists in adequately defining the consistency between a test and a hypothesis. According to the Duhem-Quine thesis (Duhem 1954; Quine 1951), the requirement that all severe tests be consistent with a hypothesis can always be trivially satisfied, by imputing an inconsistency not to the hypothesis itself, but to other hypotheses that are assumed in conjunction with it. Since these auxiliary hypotheses are always present, Popper's concept of corroboration is not well defined, unless one is able to specify under what conditions auxiliary hypotheses are to be ignored (or considered).

According to Suppe's state-space view, an empirical theory has three main components. First, the intended scope, which is the class of the causally possible phenomena that the theory is designed to describe or explain. Second, the class of the causally possible physical systems, which are idealized systems obtained by selecting a number of causally relevant parameters and by considering how the causally possible phenomena would have been if the selected parameters had been the only causally relevant factors. Third, the theoretical model, which consists of the set of all possible states of the causally possible physical systems—the state space—together with the relations specified by the laws that configure the state space. The theoretical model thus defines the class of the theory-induced physical systems, and it can in fact be identified with such a class. If the theory is true, the class of the theory-induced physical systems is identical to the class of the causally possible physical systems and, conversely, if the two classes are distinct, the theory is false (see Sect. 1.2, par. 7). Therefore, if a theory is true, its theoretical model is identical to a class of idealized systems (the causally possible physical systems) that are an abstract and counterfactual counterpart of the real phenomena in its intended scope.

This way, Suppe's quasi-realism avoids the strong form of mathematical realism implied by Popper's conception of truth. Nevertheless, there are two main objections that can be addressed to Suppe's view. First, even though the class of the causally possible physical systems is a crucial ingredient of his conception of truth, Suppe has not sufficiently explained how such abstract and counterfactual entities can be *constructed*, and how they may turn out to be *identical* to the ones defined by the theoretical model. Second, and more important, Suppe takes for granted that such an identity involves the *whole* mathematical structure of the theoretical model, even though it may be very complex and possibly redundant.

1.4 Methodological Constructive Realism (MCR)

Thus far, we have briefly reviewed four different conceptions of the theory/world relationship (Sects. 1.1, and 1.2), and we have found that each of them incurs in some serious difficulty (Sect. 1.3). In this Section, we are going to formulate, in an intuitive and informal way, the main theses of a new conception, which we call Methodological Constructive Realism (MCR). The rest of this paper will then be devoted to further specify this view and express its precise and detailed content by appropriate formal means.

MCR shares with the semantic approach the conviction that models are essential for an adequate understanding of the structure of empirical theories, and that the theory/world relationship is thus to be intended as a models/phenomena relation. MCR is a form of *realism*, because it primarily interprets such relation as *truth*, and not empirical adequacy. However, MCR is not subject to the problems of Suppe's quasi-realism, nor does it subscribe to Popper's strong mathematical realism. Rather, MCR agrees with Suppes' and van Fraassen's tenet that theoretical models cannot be directly related to phenomena, because

of their complex and rich mathematical structure, which typically exceeds the structural complexity that can be granted at the phenomenal level.

More precisely, MCR is a form of *model-based* and *constructive* realism. The representational relation between a model and the real phenomenon that it is designed to explain is taken to be *truth*—not just empirical adequacy—and the *model's* truth is based on an identity relation between the mathematical structure of the phenomenon under investigation and an appropriate substructure of the relative model. However, both the phenomenal structure and the model substructure are not independently given, but they are rather *constructed* by means of an appropriate *interpretation* of the model on the phenomenon. This interpretation, which in general is not merely empirical, presupposes, besides the model, also a low level theoretical element—a functional description—, which is constitutive of the phenomenon itself.

MCR is a form of *methodological* realism, for MCR recognizes that truth is not the only relevant semantic relation between a theoretical model and the corresponding phenomenon; rather, the *empirical correctness* of the model is important as well. Nevertheless, it should be remarked that empirical correctness is not to be thought as a new version of the instrumentalist concept of empirical adequacy. Notwithstanding their terminological similarity, the two concepts are in fact very different. According to instrumentalism, empirical adequacy is the only relevant relation between a theory and the real world, and it is thus completely independent from truth. Instead, according to MCR, empirical correctness and truth are not independent notions, and specifying their relations should be taken as one of the main goals of methodology.

In this respect, empirical correctness of a model is much more similar to Popper's corroboration, than to empirical adequacy. For Popper, a hypothesis is corroborated if no severe test falsifies it or, equivalently, if all severe tests are consistent with the hypothesis. But we have seen that giving an adequate definition of consistency between a severe test and a hypothesis is a very serious problem for Popper's falsificationism (Sect. 1.3.2, par. 3). Nevertheless, it should be noted that this problem of the falsificationist methodology depends on the fact that consistency is intended as a relation between experimental results and a *hypothesis*, that is to say, a *sentence*. If, as MCR maintains, consistency is instead a relation between an experiment and a *model*, the problem does not arise, or at least, not in such a way as to block an adequate formal development of this methodology.

The following eleven theses provide an informal and intuitive formulation of MCR.

1. *Phenomena* are not given in themselves, but are individuated by low level *theoretical descriptions* (functional descriptions). These descriptions are not sufficient to determine a definite mathematical structure for any phenomenon.
2. Each model is *interpreted* on a corresponding phenomenon.
3. The interpretation links *aspects of the model* with corresponding *aspects of the phenomenon*. These aspects may be observational, but also theoretical (this is possible because of 1).
4. The interpretation *induces* two structures: (1) a *substructure of the model* and (2) a definite mathematical *structure* of the phenomenon.
5. If the two induced structures are *identical* (or isomorphic) the model is *true*, otherwise it is *false*.
6. The interpretation also induces a definite *empirical content* of the model.
7. The empirical content of the model is a subset of all the *experiments* that are actually performed on the phenomenon under investigation.

8. If all experiments in the empirical content of the model are consistent with the model itself, the model is *empirically correct*; otherwise, it is *empirically incorrect*.
9. Truth and empirical correctness are not independent notions, for they both presuppose a common ontological ground—the phenomenon—to which both are related, even though in different ways.
10. Because of 9, if a model *S* is true of a certain phenomenon *H*, then *S* is empirically correct with respect to the same phenomenon *H* and, conversely, if *S* is empirically incorrect w.r.t. *H*, *S* is false of *H*.
11. Thus, by 10, truth of a model entails its empirical correctness, and empirical incorrectness entails its falsehood. However, in general, empirical correctness is not sufficient for truth.

As stated above, MCR is a quite general methodological theory, whose content needs to be further specified and expressed in a more precise and definite form. In the subsequent sections of this paper we are going to show in detail how it is possible to accomplish this goal by focusing on an important class of phenomena, namely, *dynamical phenomena* and their correlated models. Some paradigmatic examples of dynamical phenomena are those that Galileo explained: free fall, the motion of a sphere on an inclined plane, the motion of a pendulum, projectile motion, etc. But dynamical phenomena are not limited to the domain of mechanics or physics. Many other sciences try to explain phenomena of this kind. For example, demography is interested in the laws that govern the growth of a population under specified conditions. Chemistry may be interested in studying how the concentration of some substance varies during a certain reaction. Cognitive science attempts to explain the succession of the mental operations of a subject who performs some cognitive task. Very probably, all sciences are interested in the explanation of some dynamical phenomenon. Obviously, this is not the only type of phenomenon that an empirical science deals with. Nevertheless, if we were able to precisely analyze the semantic relations that hold between dynamical phenomena and their correlated models, we could demonstrate a deep structural identity between scientific disciplines that at a first, superficial, glance may seem worlds apart (Giunti 1992, pp. 135–36).

For Suppes, methodology is a formal discipline whose principles are not mathematical in nature, for they do not in general follow from the axioms of set theory (Suppes 1969b, p. 34). MCR agrees with this view. As far as methodology is concerned with relevant models/ phenomena relations, set theory is fundamental in order to deal with the model-theoretic side of such relations. However, in order to appropriately treat the phenomenal side, it is necessary to consider a specific ontological level for which set theory is no longer sufficient. Set theory must then be supplemented with an adequate *formal ontology* whose axioms are intended to specify and precisely describe basic features of the phenomenal level.

In the following sections we are going to develop the details of MCR for the special case of *deterministic dynamical phenomena* and their correlated *deterministic models*. This special version of MCR will be formulated as an axiomatic extension of set theory, whose specific axioms constitute a formal ontology that provides an adequate framework for analyzing the two semantic relations of *truth* and *empirical correctness*, as well as their connections.

2 Mathematical Models of Deterministic Motion

In Sect. 2, we consider the mathematical models that can be used to describe and explain deterministic motion. We treat them at a very high level of abstraction, for we are not interested in the details of any specific concept of deterministic motion, but in the most basic and essential features of this notion. In particular, in Sect. 2.1, we consider the mathematical models that express the most general concept of a *partial deterministic dynamics*, that is to say, a dynamics in which a state transition of any given duration is defined for *some* states (or even no state) of the system. Then, in Sect. 2.2, we consider the special case of a total deterministic dynamics, which obtains when a state transition of any duration is defined for *all* states. The two theories presented in Sects. 2.1 and 2.2 are purely mathematical theories, whose respective classes of models are specified by Definitions 2 and 9, which define appropriate set-theoretical predicates in Suppes' sense (Suppes 1957, sect. 12.2). The only mathematical theory that is presupposed is set theory. As usual in mathematical practice, set theory is employed in a semi-formal, intuitive, formulation; its axioms are not explicitly stated, and its language is the natural language supplemented with the usual symbols and terminological conventions of such a theory.

2.1 Partial Dynamical Systems on Monoids and Partial Deterministic Systems on monoids

A partial dynamical system DS_L on a monoid L is a kind of mathematical model which is intended to express the most general intuitive notion of a system that evolves in time with a possibly partial, or incomplete, deterministic dynamics, and whose evolutions can be obtained by composition of arbitrary sequences of time steps. The monoid L is intended to model the minimal additive structure of temporal durations. We recall below the definition of a monoid.

Definition 1 (Monoid)
Let T be an arbitrary non-empty set.
L is a monoid $:=$

1. $L = (T, +)$;
2. $+ : T \times T \to T$;
3. (a) $\exists u \in T \, \forall t \in T \, (u + t = t \wedge t + u = t)$;
   (b) $\forall w, v, t \in T \, ((w + v) + t = w + (v + t))$.

    We also recall that, for any monoid $L = (T, +)$, the element u that satisfies condition 3a is in fact unique. Such an element is called *the identity element of the + operation*, and it is indicated by 0.

Notation 1 (Notation for partial functions)
Let $h : X \to Y$ be a partial function.
$Pim(h) := \{x \in X : \exists y \in Y \, (h(x) = y)\}$. In other words, $Pim(h)$ is the set of all $x \in X$ for which h is defined; therefore, "$x \in Pim(h)$" can be read as "h is defined on x". ("$Pim(h)$" is a mnemonic abbreviation for "preimage of h".)
It should be noticed that *we do not assume* that, for an arbitrary partial function $h : X \to Y$, $Pim(h) \neq \emptyset$.

$Im(h) := \{y \in Y : \exists x \in X \ (h(x) = y)\}$. ("$Im(h)$" is a mnemonic abbreviation for "image of h".)

Definition 2 (Partial dynamical system on a monoid)

Let $L = (T, +)$ be a given monoid, and M an arbitrary non-empty set.

DS_L *is a partial dynamical system on* $L :=$

1. $DS_L = (M, (g^t)_{t \in T})$;
   - any $x \in M$ is called *a state*, and *M the state space*;
   - any $t \in T$ is called *a duration*, *T the time set*, and $L = (T, +)$ *the time model* of DS_L;
   - $+ : T \times T \to T$ is called *the operation of addition of durations* or, briefly, *duration addition*;
2. $\forall t \in T$, $g^t : M \to M$ is a partial function;
   - for any $t \in T$, g^t is called *the (state) transition of duration t* or, briefly, *the t-transition* or *the t-advance*;
3. (a) $\forall x \in M \ (x \in Pim(g^0) \to g^0(x) = x)$;
   (b) $\forall x \in M \ \forall v, t \in T((x \in Pim(g^{v+t}) \wedge x \in Pim(g^t) \wedge g^t(x) \in Pim(g^v)) \to g^{v+t}(x) = g^v(g^t(x)))$

Definition 2 expresses the intuitive notion of a *composable deterministic partial dynamics* in the following sense.

In the first place, condition 2 should be interpreted as telling us that, at instant $t + i$, the system will be in state $g^t(x)$, provided that x is the state of the system at instant i and g^t is defined on x; in other words, if at instant i the system is in state $x \in M$ and $g^t(x)$ is defined, then at instant $t + i$ the system is in state $g^t(x)$. Condition 2 thus expresses the idea of a *deterministic partial dynamics*. Note, however, that Condition 2 does not exclude the possibility that some, or even all, state transitions be total functions. Partial dynamical systems on monoids thus include as a special case dynamical systems on monoids (see Definition 9).

In the second place, condition 3a tells us that, whatever state the system is in, the transition of duration 0 does not modify that state, provided that such a transition is defined on it; and, finally, condition 3b tells us that any transition of duration $v + t$ can be decomposed in two successive transitions, the first one of duration t, and the second one of duration v, provided that all three state transitions are defined on the respective states. Conditions 3a and 3b thus express the idea of a *composable partial dynamics*.

According to the usual definition of a dynamical system (Arnold 1977; Szłenk 1984; Giunti 1997; Hirsch et al. 2004) durations are taken to be either *continuous* or *discrete* quantities. In the first case, the time set T is identified with either the set of the real numbers $\mathbb{R}$ or the non-negative real numbers $\mathbb{R}^{\geq 0}$, and the operation $+$ of addition over durations is the usual addition of two real numbers. In the second case, T is identified with either the set of the integers $\mathbb{Z}$ or the non-negative integers $\mathbb{Z}^{\geq 0}$, and the operation $+$ of addition over durations is the usual addition of two integer numbers. The following are all examples of partial dynamical systems with either a discrete or a continuous time model, as well as a discrete or a continuous state space.

Example 1 (**Partial dynamical systems with discrete or continuous time model, or state space**)

1. Discrete time model $L = (\mathbb{Z}^{\geq 0}, +)$ and discrete state space: finite state automata, Turing machines, cellular automata restricted to finite configurations.[2]
2. Discrete time model $L = (\mathbb{Z}^{\geq 0}, +)$ and continuous state space: many systems specified by difference equations, iterated mappings on $\mathbb{R}$, cellular automata not restricted to finite configurations.
3. Continuous time model $L = (\mathbb{R}, +)$ and continuous state space: systems specified by ordinary differential equations, many neural nets.

Definition 2 is a formal rendition of the most general notion of a composable deterministic partial dynamics. However, in this paper, we are going to mainly focus on those partial dynamical systems whose state space can be factorized into a finite number $n \in \mathbb{Z}^{\geq 1}$ of components. A n-component partial dynamical system is defined as follows.

Definition 3 (**n-component partial dynamical system on a monoid**)
Let $DS_L = (M, (g^t)_{t \in T})$ and, for any i $(1 \leq i \leq n)$, X_i be a non-empty set.
DS_L *is a n-component partial dynamical system on L* := DS_L is a partial dynamical system on L and $M \subseteq X_1 \times \ldots \times X_n$.
Furthermore, for any i, the set $C_i := \{x_i \in X_i$: for some n-tuple $x \in M, x_i$ is the i-th element of $x\}$ is called *the i-th component of M.*[3]

As already noted, Definition 2 is a formal rendition of the most general notion of a *composable* deterministic partial dynamics. The most general notion of a *deterministic partial dynamics* is instead expressed by Definition 4, which only retains the first two conditions of Definition 2.

Definition 4 (**Partial deterministic system on a monoid**)
Let $L = (T, +)$ be a given monoid, and M an arbitrary non-empty set.
DS_L *is a partial deterministic system on L* := DS_L satisfies the first two conditions of Definition 2.

It is important to realize that the class of the partial deterministic systems on a monoid is a kind of model for which it is appropriate to define a *specific isomorphism*. For Definition 4 defines an axiom free, or *purely structural*, set-theoretical predicate in Suppes' sense (Suppes 1957, pp. 255, 260), as it only specifies the *type* of the structural entities of the model, without imposing any additional requirement on such entities. Below is the definition of the specific isomorphism for this kind of model.

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[2] The state space of a cellular automaton is discrete (*i.e.*, finite or countably infinite) if all its states are finite configurations, that is to say, configurations where all but a finite number of cells are non-empty. If non-finite configurations are allowed, the state space has the power of the continuum.

[3] Note that, by Definition 3, $C_i = proj_i(M)$, where $proj_i : X_1 \times \ldots \times X_n \to X_i$ is the i-th projection map, that is to say, the function such that, for any $(x_1, \ldots, x_n) \in X_1 \times \ldots \times X_n$, $proj_i(x_1, \ldots, x_n) := x_i$.

Definition 5 (Isomorphism for partial deterministic systems on a given monoid)
Let $DS_{1_L} = (M, (g^t)_{t \in T})$ and $DS_{2_L} = (N, (h^t)_{t \in T})$ be partial deterministic systems on a given monoid $L = (T, +)$.
u *is an isomorphism of* DS_{2_L} *in* DS_{1_L} $:= u : N \to M$ *is a bijection; for any* $t \in T$,
$u(Pim(h^t)) = Pim(g^t)$; *for any* $t \in T$, *for any* $y \in Pim(h^t)$, $u(h^t(y)) = g^t(u(y))$.

It is not difficult to prove that being a partial dynamical system on a monoid is preserved by the specific isomorphism just defined or, in other words, that the property of having a composable dynamics is a purely structural property of the partial deterministic systems on a given monoid.

Theorem 1 (Being a partial dynamical system on a monoid is preserved by isomorphism)
Let $DS_{1_L} = (M, (g^t)_{t \in T})$ and $DS_{2_L} = (N, (h^t)_{t \in T})$ be partial deterministic systems on monoid $L = (T, +)$.
If u is an isomorphism of DS_{2_L} in DS_{1_L} and DS_{2_L} is a partial dynamical system on L, then DS_{1_L} is a partial dynamical system on L.

Proof See the Appendix. □

Given a partial deterministic system DS_L on monoid L, we now consider the set C of all states for which at least one state transition is defined. Any state in C can thus be thought as an origin of some state transition and, for this reason, we call C *the set of all original states*. Also note that C is is fact the union of the preimages of all state transitions of DS_L. The exact definition is as follows.

Definition 6 (The set C of all original states)
Let $DS_L = (M, (g^t)_{t \in T})$ be a partial deterministic system on monoid $L = (T, +)$.
$C := \{x \in M : \exists t \in T, x \in Pim(g^t)\}$
C is called *the set of all original states*.

Note that the following proposition holds.

Proposition 1
$C \neq \emptyset \leftrightarrow \exists t \in T, Pim(g^t) \neq \emptyset$.

Proof By Definition 6, if $C \neq \emptyset$, then $\exists t \in T, Pim(g^t) \neq \emptyset$, and conversely. □

For any $x \in M$, we then consider the set $q(x)$ of all durations t whose corresponding state transition g^t is defined on x. Thus, from an intuitive point of view, $q(x)$ can be thought as the temporal span during which x is active or alive. For this reason, we call $q(x)$ *the life span of x*. The exact definition is below.

Notation 2 (Notation for the power set)
For any set X, the power set of X is denoted by $\mathcal{P}(X)$.

Definition 7 (The life span $q(x)$ of state x)
Let $DS_L = (M, (g^t)_{t \in T})$ be a partial deterministic system on monoid $L = (T, +)$.
$q : M \to \mathcal{P}(T)$

$\forall x \in M, q(x) := \{t \in T : x \in Pim(g^t)\}$
$q(x)$ is called *the life span of x*.

Note that, by Definitions 6 and 7, $\forall x \in M, q(x) \neq \emptyset \leftrightarrow x \in C$, that is to say, x has a non-empty life span iff x is an original state. Furthermore, by means of C and q, we can retrieve the preimage of any state transition g^t, for it holds: $\forall t \in T, Pim(g^t) = \{x \in M : x \in C \wedge t \in q(x)\}$. The (obvious) proof of these two facts is in the Appendix.

Proposition 2
(1) $\forall x \in M, q(x) \neq \emptyset \leftrightarrow x \in C$;
(2) $\forall t \in T, Pim(g^t) = \{x \in M : x \in C \wedge t \in q(x)\}$.

Proof See the Appendix. □

We define below the concept of a *partial subsystem* of a partial deterministic system on a monoid, and we then show that this concept preserves the kind of model to which it applies, even when the model is a partial *dynamical* system.

Definition 8 (Partial subsystem of a partial deterministic system on a monoid)
Let $DS_{1_L} = (M, (g^t)_{t \in T})$ be a partial deterministic system on monoid $L = (T, +)$.
DS_{2_L} *is a partial subsystem of* $DS_{1_L} :=$
$DS_{2_L} = (N, (h^t)_{t \in T}) \wedge \forall t \in T \, (\, h^t : N \rightarrow N$ is a partial function $\wedge$
$Pim(h^t) \subseteq Pim(g^t) \wedge \forall x \in Pim(h^t) \, h^t(x) = g^t(x) \,)$.

Proposition 3
(1) *If DS_{1_L} is a partial deterministic system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}, then DS_{2_L} is a partial deterministic system on L;*
(2) *If DS_{1_L} is a partial dynamical system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}, then DS_{2_L} is a partial dynamical system on L.*

Proof See the Appendix. □

2.2 Dynamical Systems on Monoids and Deterministic Systems on Monoids

A dynamical system DS_L on a monoid L is the kind of mathematical model which is intended to express the most general intuitive notion of a system that evolves in time with a *total*, or *complete*, deterministic dynamics, and whose evolutions can be obtained by composition of arbitrary sequences of time steps (Giunti and Mazzola 2012; Mazzola and Giunti 2012). Dynamical systems on monoids are a special case of partial dynamical systems on monoids. They are those partial dynamical systems whose transition functions are all total. Therefore, in this special case, the definition of partial dynamical system on a monoid (Definition 2) reduces to the following one.

Definition 9 (Dynamical system on a monoid)
Let $L = (T, +)$ be a given monoid, and M an arbitrary non-empty set.
DS_L *is a dynamical system on* $L :=$
1. $DS_L = (M, (g^t)_{t \in T})$;
2. $\forall t \in T, g^t : M \to M$;
3. (a) $\forall x \in M, g^0(x) = x$;
   (b) $\forall x \in M \ \forall v, t \in T, g^{v+t}(x) = g^v(g^t(x))$.

Definition 9 expresses the intuitive notion of a *composable deterministic total dynamics*. Accordingly, in this special case, the definition of a *n*-component partial dynamical system on a monoid (Definition 3) becomes the following.

Definition 10 (*n*-component dynamical system on a monoid)
Let $DS_L = (M, (g^t)_{t \in T})$ and, for any i $(1 \leq i \leq n)$, X_i be a non-empty set.
DS_L *is a n-component dynamical system on* $L := DS_L$ is a dynamical system on L and $M \subseteq X_1 \times \ldots \times X_n$.
Furthermore, for any i, the set $C_i := \{x_i \in X_i$: for some *n*-tuple $x \in M, x_i$ is the *i*-th element of $x\}$ is called *the i-th component of M*.

***Example 2* (The 4-component dynamical system DS_{L_p})**
A paradigmatic example of a 4-component dynamical system is the system DS_{L_p} (see Eq. 2 below), which is individuated by the equation of motion of a projectile:

$$\left(\frac{dx(t)}{dt} = \dot{x}(t), \ \frac{dy(t)}{dt} = \dot{y}(t), \ \frac{d\dot{x}(t)}{dt} = 0, \ \frac{d\dot{y}(t)}{dt} = -\mathbf{g} \right) \tag{1}$$

where $\mathbf{g} \in \mathbb{R}$ is a fixed positive constant. The solutions of this ordinary differential equation univocally determine the 4-component dynamical system:

$$DS_{L_p} = (X \times Y \times \dot{X} \times \dot{Y}, (g^t)_{t \in T}), \tag{2}$$

where $X = Y = \dot{X} = \dot{Y} = T = \mathbb{R}$, $L_p = (T, +)$ is the additive group of the real numbers and, for any $t \in T$, for any $(x, y, \dot{x}, \dot{y}) \in X \times Y \times \dot{X} \times \dot{Y}$,

$$g^t(x, y, \dot{x}, \dot{y}) = \left(\dot{x}t + x, \ -\frac{1}{2}\mathbf{g}t^2 + \dot{y}t + y, \ \dot{x}, \ -\mathbf{g}t + \dot{y} \right) \tag{3}$$

In the special case of a total dynamics, the definition of a partial deterministic system on a monoid (Definition 4) in turn reduces to the following.

Definition 11 (Deterministic system on a monoid)
Let $L = (T, +)$ be a given monoid, and M an arbitrary non-empty set.
DS_L *is a deterministic system on* $L := DS_L$ satisfies the first two conditions of Definition 9.

Note that $DS_L = (M, (g^t)_{t \in T})$ is a deterministic system on $L = (T, +)$ only if its set of original states $C = M$. Also note that DS_L is a deterministic system on L iff $\forall x \in M$,

$q(x) = T$, that is to say, DS_L is a deterministic system iff the life span of any state x is the whole time set T. The (obvious) proof of these facts is in the Appendix.

Proposition 4

(1)   *If DS_L is a deterministic system on L, then $C = M$;*
(2)   *DS_L is a deterministic system on L $\leftrightarrow \forall x \in M, q(x) = T$.*

Proof See the Appendix.   □

3 Real Systems and Their Motions

In Sect. 3, we set up a formal ontology that describes the phenomenal side of the models/phenomena relationship. In particular, in Sect. 3.1, we state two Axioms (1, 2) about the primitive notions of real system and temporal evolution, and one more Axiom (3) concerning the theoretical part of a dynamical phenomenon. These three axioms then allow us to explicitly define the general concept of a dynamical phenomenon. In Sect. 3.2, we state four more Axioms (4, 5, 6, 7) concerning the magnitudes of a dynamical phenomenon and their instantaneous values. On this basis, in Sect. 3.3, we are then able to define the concept of a deterministic dynamical phenomenon with respect to n ($1 \leq n$) of its magnitudes, and we prove that the kind of dynamical structure that can be found at the phenomenal level is the simplest one, namely, a partial deterministic system on a monoid (Proposition 5). In Sect. 3.4, we complete the formal description of the phenomenal level by stating the last three ontological Axioms (8, 9, 10), which concern the notions of setting (or preparation), measurement, and experiment, with respect to any given dynamical phenomenon.

3.1 Dynamical Phenomena

From an intuitive point of view, we take a *deterministic* dynamical phenomenon to be any manifestation of the real world that a *n*-component partial dynamical system on a monoid can represent. An exact definition of this kind of phenomenon will be given in Sect. 3.3 (Definition 25). In this Section, we are considering dynamical phenomena in the most general sense, irrespectively of their being deterministic or not.

As a first approximation, we think of an arbitrary dynamical phenomenon H as a pair (F, B_F) of two distinct elements, a theoretical part F and a real part B_F. The real part B_F is a non-empty set of *real dynamical systems*, while the theoretical part F is a *functional description* that allows us to define such a set.

Intuitively, a real dynamical system can be thought as a real system that has at least one temporal evolution, or a motion. Furthermore, any temporal evolution of any real dynamical system cannot be shared by any other system. In order to precisely state these conditions, we first of all assume the following two axioms. We take as primitive the binary predicate *has temporal evolution* (or *has motion*). We also take as primitive the two constants R and E, whose intended meanings are, respectively, the class of all *real systems* and the class of all *temporal evolutions*. Throughout this paper, we always take the term "motion" as a synonym of "temporal evolution".

Axiom 1 (Real systems and temporal evolutions)
$\forall r \forall e (r$ *has temporal evolution* $e \rightarrow r \in R \land e \in E)$;
$E \neq \emptyset \land \forall e \in E \, \exists r \in R \, (r$ *has temporal evolution* $e)$.

Definition 12 (The set $E(r)$ of all temporal evolutions of real system r)
Let $r \in R$.
$E(r) := \{e \in E : r$ *has temporal evolution* $e\}$.
$E(r)$ is called *the set of all temporal evolutions of r.*

Axiom 2 (Different real systems do not share any temporal evolution)
$\forall r_1, r_2 \in R \, (r_1 \neq r_2 \rightarrow E(r_1) \cap E(r_2) = \emptyset)$.

We now define the set B of all real systems that have at least one temporal evolution. By Axiom 1, B is not empty and, by Axiom 2, no two of its elements share any temporal evolution. We can thus identify B with the set of all real dynamical systems.

Definition 13 (Real dynamical systems) $B := \{b \in R : E(b) \neq \emptyset\}$. The set B is called *the set of all real dynamical systems*, and any $b \in B$ is called *a real dynamical system.*

Axiom 3 (Theoretical part $F = (AS_F, CS_F)$ of a dynamical phenomenon)
A theoretical part $F = (AS_F, CS_F)$ of a dynamical phenomenon is a functional description that consists of:
1. a sufficiently detailed specification of the internal constitution, organization and operation of any real dynamical system of a certain *functional type AS_F*.
    Formally, AS_F is a primitive unary predicate for which it holds:
    $\forall b(AS_F(b) \rightarrow b \in B)$;
2. a sufficiently detailed specification of a *causal scheme CS_F* of the external interactions that a real dynamical system of functional type AS_F is allowed to undergo during its temporal evolutions. In particular, the specification of the causal scheme CS_F must include the specification of (a) the *initial conditions* that a temporal evolution of a real system of functional type AS_F must satisfy, (b) the *running (intermediate or boundary) conditions* during the whole subsequent evolution, and possibly (c) the *final conditions* under which the evolution terminates.
    Formally, CS_F is a primitive unary predicate for which it holds:
    $\forall e(CS_F(e) \rightarrow \exists b(AS_F(b) \land e \in E(b)))$.

Once a theoretical part F of a dynamical phenomenon is given, its real part B_F is the set of all real dynamical systems which satisfy the functional description F or, more precisely, B_F is the set of all real dynamical systems b such that b satisfy the specified functional type AS_F and at least some temporal evolution $e \in E(b)$ satisfy the specified causal scheme CS_F. Thus, we define B_F as follows.

Definition 14 (The realization domain B_F of a theoretical part F of a dynamical phenomenon)
Let $F = (AS_F, CS_F)$ be a theoretical part of a dynamical phenomenon, where AS_F is its specified functional type, and CS_F its specified causal scheme.
$B_F := \{b \in B : AS_F(b) \land \exists e \in E(b)(CS_F(e))\}$.

B_F is called *the realization domain* (or *application domain*) *of F*.[4] Any real dynamical system $b \in B_F$ is called *a F-realizer*.

We can now define a dynamical phenomenon as follows.

Definition 15 (Dynamical phenomenon $H = (F, B_F)$)
H is a dynamical phenomenon $:= H = (F, B_F)$, *F is a theoretical part of a dynamical phenomenon*, B_F *is the realization domain of F, and* $B_F \neq \emptyset$.

As previously remarked (Sect. 1.4, par. 7), the phenomena of motion that Galileo explained can be considered paradigmatic examples of dynamical phenomena. Below we show in detail how Definition 15 can be applied to projectile motion. By following this paradigm, the reader will then be able to similarly apply that definition to any other example of her/his choice.[5]

Example 3 **(The phenomenon of projectile motion)**
We refer to the *phenomenon of projectile motion* by the symbol $H_{p,\phi\theta} = (F_{p,\phi\theta}, B_{F_{p,\phi\theta}})$, where p is an abbreviation for *projectile*, while ϕ and θ are two non-negative real parameters on which the functional description $F_{p,\phi\theta}$ depends, and whose meaning is explained below.
Theoretical part—Functional description $F_{p,\phi\theta} = (AS_{F_{p,\phi\theta}}, CS_{F_{p,\phi\theta}})$
1. Specification of any real dynamical system of functional type $AS_{F_{p,\phi\theta}}$:
    any medium size body in the proximity of the earth.
2. Specification of the causal scheme $CS_{F_{p,\phi\theta}}$ of the external interactions that a real dynamical system of functional type $AS_{F_{p,\phi\theta}}$ is allowed to undergo during its temporal evolutions:
    (a) initial conditions: the body is released at an arbitrary instant, with an initial velocity and position such that the body hits the earth surface at a later instant, the maximum vertical distance reached by the body with respect to the earth surface is not greater than ϕ, and the maximum horizontal distance is not greater than θ;
    (b) running conditions: during the whole motion the only force acting on the body is its weight;
    (c) final conditions: the motion terminates immediately after the impact of the body with the earth surface.

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[4] Since the functional description *F* typically contains several idealizations, it might be claimed that no real dynamical system *exactly* satisfies *F*, but it rather fits *F* up to a *certain degree*. If we take this point of view, the realization domain B_F of a dynamical phenomenon $H = (F, B_F)$ might be more faithfully described as a *fuzzy* set.

[5] For free fall and satellite motion, see (Giunti 2014, sects. 4.1 and 5.2); for pendulum motion, (Giunti 2010, sect. 5.2). In the cognitive domain, all phenomena of human computation, as defined in Giunti and Pinna (Giunti and Pinna 2016, sect. 5), are examples of dynamical phenomena. In the domain of computation theory, all computational setups, as defined in (Giunti 2017, sect. 2.1), are further examples of dynamical phenomena. Computational setups are the real objects described by computational systems. Computational systems are discrete *n*-component (partial) dynamical systems that can be effectively represented or described (Giunti and Giuntini 2007, pp. 56–7).

Real part—Realization domain $B_{F_{p,\phi\theta}}$

By the definition of realization domain (Definition 14),

$B_{F_{p,\phi\theta}} = \{b \in B : AS_{F_{p,\phi\theta}}(b) \land \exists e \in E(b)(CS_{F_{p,\phi\theta}}(e))\}$.

Thus, $B_{F_{p,\phi\theta}}$ is the set of all medium size bodies in the proximity of the earth such that some of their motions satisfy the causal interaction scheme $CS_{F_{p,\phi\theta}}$. Any body $b \in B_{F_{p,\phi\theta}}$ is called a *projectile*.

Definition 16 (The set $E_F(b)$ of all temporal evolutions of a F-realizer b)
Let $H = (F, B_F)$ be a dynamical phenomenon, and $b \in B_F$.
$E_F(b) := \{e \in E(b) : CS_F(e)\}$.
$E_F(b)$ is called *the set of all temporal evolutions of F-realizer b*.

Note that, by the definition of realization domain (Definition 14), for any $b \in B_F$, for some evolution $e \in E(b)$, $CS_F(e)$ holds, so that, by Definition 16, $E_F(b) \neq \emptyset$. However, for some $e \in E(b)$, $CS_F(e)$ may not hold, so that $E_F(b) \subseteq E(b)$, but not necessarily $E_F(b) = E(b)$.

Definition 17 (The set $\bar{E}_F$ of all temporal evolutions of a dynamical phenomenon)
Let $H = (F, B_F)$ be a dynamical phenomenon, and $b \in B_F$.
$\bar{E}_F := \bigcup_{b \in B_F} E_F(b)$.
$\bar{E}_F$ is called *the set of all temporal evolutions of H*.

3.2 Magnitudes of a Dynamical Phenomenon and Their Instantaneous Values

From an intuitive point of view, we take a *magnitude of a dynamical phenomenon* $H = (F, B_F)$ to be a property M of every F-realizer $b \in B_F$ such that, at different instants of any temporal evolution of b, it can assume different values. In order to precisely formulate this idea, we first introduce the following axiom.

Axiom 4 (Magnitudes of a dynamical phenomenon and its time magnitude)
We assume that, to any dynamical phenomenon H, *the set of all its magnitudes* is uniquely associated, and it always contains *the time magnitude of H*.
The set of all magnitudes of H is indicated by $\mathbf{M}_H$, its time magnitude is indicated by T_H, while M or M_j ($j \in \mathbb{Z}^{\geq 1}$) stands for some magnitude member of $\mathbf{M}_H$ (if not explicitly excluded, such a magnitude may be T_H).
We further assume that, to each magnitude $M \in \mathbf{M}_H$, *the set of all its possible values* is uniquely associated.[6]
The set of all possible values of an arbitrary magnitude $M \in \mathbf{M}_H$ is indicated by $V(M)$. Elements of the set of all possible values of the time magnitude, $V(T_H)$, are called *instants* or *durations*.
We finally assume that a binary operation $\hat{+}$ is uniquely associated to $V(T_H)$, and that $(V(T_H), \hat{+})$ is at least a monoid.

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[6] It should be noticed that we do not require that the magnitudes of a phenomenon be observable, or even measurable. Furthermore, the nature of the possible values of a magnitude is not specified as well. This, in particular, means that there may be magnitudes whose possible values are not real numbers.

The monoid $(V(\boldsymbol{T}_H), \hat{+})$ is indicated by L_H, and the identity element of the $\hat{+}$ operation by $\hat{0}$.

We then assume one further axiom concerning the *initial* instants of the temporal evolutions of an arbitrary F-realizer $b \in B_F$. Axiom 5 stipulates that any temporal evolution of any F-realizer has exactly one initial instant, and that the same F-realizer cannot have two different temporal evolutions with the same initial instant.

Axiom 5 (The initial instant $\iota_0^H(e)$ in phenomenon H of a temporal evolution e)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ the set of all temporal evolutions of H, $V(\boldsymbol{T}_H)$ the set of all possible values of the time magnitude of H, and $E_F(b)$ the set of all temporal evolutions of F-realizer $b \in B_F$.
$\iota_0^H : \bar{E}_F \to V(\boldsymbol{T}_H)$;
$\forall b \in B_F \, \forall e_1, e_2 \in E_F(b) \, (e_1 \neq e_2 \to \iota_0^H(e_1) \neq \iota_0^H(e_2))$.
For any $e \in \bar{E}_F$, $\iota_0^H(e)$ is called *the initial instant in H of temporal evolution e*.

We also assume (Axiom 6 below) that, to any temporal evolution e of a dynamical phenomenon H, a *set* of *instants* is uniquely associated, and that the initial instant of e belongs to this set.

Axiom 6 (The set $J_H(e)$ of all instants in phenomenon H of a temporal evolution e)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ be the set of all temporal evolutions of H, $V(\boldsymbol{T}_H)$ be the set of all possible values of the time magnitude of H, and $\iota_0^H(e)$ be the initial instant in H of temporal evolution $e \in \bar{E}_F$.
$J_H : \bar{E}_F \to \mathcal{P}(V(\boldsymbol{T}_H))$;
$\forall e \in \bar{E}_F \, (\iota_0^H(e) \in J_H(e))$.
For any $e \in \bar{E}_F$, $J_H(e)$ is called *the set of all instants in H of temporal evolution e*.

Axiom 6 allows us to define the set of all instants in a phenomenon H as the union, for all temporal evolutions e of H, of the set of all instants of e. This is expressed by the definition below.

Definition 18 (The set $\bar{J}_H$ of all instants in a phenomenon H)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ be the set of all temporal evolutions of H, and $e \in \bar{E}_F$.
$\bar{J}_H := \bigcup_{e \in \bar{E}_F} J_H(e)$.
$\bar{J}_H$ is called *the set of all instants in phenomenon H*.

Axioms 5 and 6 allow us to define the set of all *durations* of an arbitrary temporal evolution e of a dynamical phenomenon. Intuitively, any such duration is any time that it takes to go from the initial instant of the evolution e to some (other) of its instants. The exact definition is below.

Definition 19 (The set $D_H(e)$ of all durations in phenomenon H of a temporal evolution e)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ be the set of all temporal evolutions of H, $V(\boldsymbol{T}_H)$ be the set of all possible values of the time magnitude of H, $\iota_0^H(e)$ be the initial instant in H of temporal evolution $e \in \bar{E}_F$, and $J_H(e)$ be the set of all instants in H of evolution e.

$\forall e \in \bar{E}_F, D_H(e) := \{t \in V(\boldsymbol{T}_H) : t \hat{+} \iota_0^H(e) \in J_H(e)\}$.

For any $e \in \bar{E}_F$, $D_H(e)$ is called *the set of all durations in H of temporal evolution e*.

Analogously to Definition 18, we then define the set of all durations in a phenomenon H as the union, for all temporal evolutions e of H, of the set of all durations of e. This is expressed by the definition below.

Definition 20 (The set $\bar{D}_H$ of all durations in a phenomenon H)

Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ be the set of all temporal evolutions of H, and $e \in \bar{E}_F$.

$\bar{D}_H := \bigcup_{e \in E_F} D_H(e)$.

$\bar{D}_H$ is called *the set of all durations in phenomenon H*.

We are now ready to specify under what conditions a magnitude $\boldsymbol{M}$ of a dynamical phenomenon H has a value. In the first place, the magnitude $\boldsymbol{M}$ may have different values depending on the F-realizer b, the evolution e and the instant i that are considered. Thus, it makes sense to think of a 4-place function, val_H, which applies to an arbitrary magnitude $\boldsymbol{M}$, F-realizer b, time-evolution e, and instant i of the phenomenon H, and returns the value of $\boldsymbol{M}$. The range of this function will thus be the union, for all magnitudes $\boldsymbol{M}$, of the set of values of $\boldsymbol{M}$. This function, however, must be partial, because, whenever $e \notin E_F(b)$ or $i \notin J_H(e)$, $val_H(\boldsymbol{M}, b, e, i)$ should not be defined. In addition, if $val_H(\boldsymbol{M}, b, e, i)$ is defined, it must belong to the set of values of $\boldsymbol{M}$. Finally, for the special case of the time magnitude $\boldsymbol{T}_H$, we assume that, for any $b \in B_F$, $e \in E_F(b)$, and $i \in J_H(e)$, $val_H(\boldsymbol{T}_H, b, e, i)$ is defined, and that $val_H(\boldsymbol{T}_H, b, e, i) = i$. All this is precisely stated by Axiom 7 below.

Axiom 7 (The H-value of magnitude M of realizer b in evolution e at instant i)

Let $H = (F, B_F)$ be a dynamical phenomenon, $\mathbf{M}_H$ the set of all magnitudes of H, $\bar{E}_F$ the set of all temporal evolutions of H, $\bar{J}_H$ the set of all instants in H and, for any magnitude $\boldsymbol{M} \in \mathbf{M}_H$, $V(\boldsymbol{M})$ be the set of all its possible values.

$val_H : \mathbf{M}_H \times B_F \times \bar{E}_F \times \bar{J}_H \to \bigcup_{M \in \mathbf{M_H}} V(\boldsymbol{M})$ is a partial function;

$\forall \boldsymbol{M} \in \mathbf{M}_H, \forall b \in B_F, \forall e \in \bar{E}_F, \forall i \in \bar{J}_H$,

if $val_H(\boldsymbol{M}, b, e, i)$ is defined, then $e \in E_F(b)$, $i \in J_H(e)$, and $val_H(\boldsymbol{M}, b, e, i) \in V(\boldsymbol{M})$;

$\forall b \in B_F, \forall e \in E_F(b), \forall i \in J_H(e)$,

$val_H(\boldsymbol{T}_H, b, e, i)$ is defined and $val_H(\boldsymbol{T}_H, b, e, i) = i$.

$val_H(\boldsymbol{M}, b, e, i)$ is called *the H-value of magnitude M of realizer b in evolution e at instant i*.

3.3 The H-State Space w.r.t. Magnitudes $M_1, \ldots, M_n$, and H's being a deterministic dynamical phenomenon w.r.t. magnitudes $M_1, \ldots, M_n$

Given a dynamical phenomenon H and $n \geq 1$ of its magnitudes, we define the state space of H *with respect to those magnitudes* as the cartesian product of their sets of values. The exact definition is below.

Definition 21 (The H-state space w.r.t. magnitudes $M_1, \ldots, M_n$)

Let $H = (F, B_F)$ be a dynamical phenomenon, and $M_1, \ldots, M_n$ be $n \in \mathbb{Z}^{\geq 1}$ different magnitudes of H.

$M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] := (V(\boldsymbol{M}_1) \times \ldots \times V(\boldsymbol{M}_n))$.

$M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is called *the H-state space w.r.t. magnitudes* $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$; any $x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is called *a possible H-state w.r.t. magnitudes* $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.

Given $n \geq 1$ magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ of a dynamical phenomenon H, Axiom 7 and Definition 21 allow us to define the instantaneous state, *with respect to those magnitudes*, of a realizer b, in evolution e, at instant i. The instantaneous state can be thought as the output of a partial function, $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, which applies to an arbitrary F-realizer b, time-evolution e, and instant i of the phenomenon H, and returns the values of the n magnitudes, whenever all such values are defined. Hence, $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, i)$ is defined just in case, for each magnitude $\boldsymbol{M}_k$ $(1 \leq k \leq n)$, $val_H(\boldsymbol{M}_k, b, e, i)$ is defined. This is precisely stated by the following definition.

Definition 22 (**The H-state, w.r.t. magnitudes** $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, **of realizer** b **in evolution** e **at instant** i)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\bar{E}_F$ the set of all temporal evolutions of H, $\bar{J}_H$ the set of all instants in H, and $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ be $n \in \mathbb{Z}^{\geq 1}$ different magnitudes of H.
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] : B_F \times \bar{E}_F \times \bar{J}_H \to M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is a partial function;
$\forall b \in B_F, \forall e \in \bar{E}_F, \forall i \in \bar{J}_H,$
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, i)$ is defined iff $val_H(\boldsymbol{M}_1, b, e, i), \ldots, val_H(\boldsymbol{M}_n, b, e, i)$ are all defined;
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, i) := (val_H(\boldsymbol{M}_1, b, e, i), \ldots, val_H(\boldsymbol{M}_n, b, e, i)).$
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, i)$ is called *the (instantaneous) H-state, w.r.t. magnitudes* $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, *of realizer* b *in evolution* e *at instant* i.

We now define, with respect to $n \geq 1$ given magnitudes, the set $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ of all those possible states $x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ that actually are initial states of some evolution of some F-realizer.

Definition 23 (**The set** $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ **of the H-initial states w.r.t. magnitudes** $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$)
Let $H = (F, B_F)$ be a dynamical phenomenon, and $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ be $n \in \mathbb{Z}^{\geq 1}$ different magnitudes of H.
$C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] := \{x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] :$
$\exists b \in B_F \, \exists e \in E_F(b)(\, st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, \iota_0^H(e))$ is defined and
$x = st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, \iota_0^H(e))\,)\}.$
$C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is called *the set of the H-initial states w.r.t. magnitudes* $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.

Note that, depending on the choice of the n magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, the set $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ may be empty. For, by Definitions 23 and 22, $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] = \emptyset$ if, for any $b \in B_F$ and any evolution $e \in E_F(b)$, some magnitude $\boldsymbol{M}_i$ does not have a value at the initial instant of e.

In Example 4 below, we consider again the phenomenon of projectile motion $H_{p,\phi\theta}$ (Example 3), and we show how to apply Definition 23 in order to specify the set of the initial states of such a phenomenon, with respect to the horizontal and vertical components of the position and velocity of an arbitrary projectile.

Example 4 (**The set** $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}]$ **of the** $H_{p,\phi\theta}$**-initial states w.r.t. magnitudes** $X, Y, \dot{X}, \dot{Y}$)

Let $H_{p,\phi\theta} = (F_{p,\phi\theta}, B_{F_{p,\phi\theta}})$ be the phenomenon of projectile motion, $b \in B_{F_{p,\phi\theta}}$ an arbitrary projectile, and r_b the point where the projectile b is initially released. Let us then consider the plane that contains the initial velocity vector of b and the earth center. On this plane, we fix the axes X and Y of a Cartesian coordinate system with origin in the earth center, and whose Y axis passes through r_b. Accordingly, we call the Y axis *vertical* and the X axis *horizontal*. We take the positive direction of the Y axis to be the one from the earth center to the point r_b.

Let us then consider the following four magnitudes of $H_{p,\phi\theta}$:

X = the horizontal component of the position of b,
Y = the vertical component of the position of b,
$\dot{X}$ = the horizontal component of the velocity of b,
$\dot{Y}$ = the vertical component of the velocity of b.

Let $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}]$ be the set of the $H_{p,\phi\theta}$-initial states w.r.t. magnitudes $X, Y, \dot{X}, \dot{Y}$. By Definition 23, such a set turns out to be:

1. $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}] = \{(x_0, y_0, \dot{x}_0, \dot{y}_0) \in M_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}] :$
   $\exists b \in B_{F_{p,\phi\theta}} \exists e \in E_{F_{p,\phi\theta}}(b) \, (\, st_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](b, e, \iota_0^{H_{p,\phi\theta}}(e))$ is defined and
   $(x_0, y_0, \dot{x}_0, \dot{y}_0) = st_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](b, e, \iota_0^{H_{p,\phi\theta}}(e)) \,)\}$.

2. Let s_E be the distance of the surface of the earth from the earth center. By the equation above, because the origin of the coordinate system is in the earth center and the Y axis passes through the point r_b where the projectile b is initially released, we get:

3. for any $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}]$, $x_0 = 0$ and $y_0 \geq s_E$.

By Definition 16, for an arbitrary projectile $b \in B_{F_{p,\phi\theta}}$, any of its motions $e \in E_{F_{p,\phi\theta}}(b)$ satisfy the causal scheme $CS_{F_{p,\phi\theta}}$ (see Example 3, 2). Thus, in particular, by the specification of the initial conditions (Example 3, 2a), the motion e of b has both an initial position and an initial velocity. Therefore, the initial values $0, y_0 \geq s_E, \dot{x}_0, \dot{y}_0$ of the four magnitudes above are defined and, consequently, $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}] \neq \emptyset$.

Also note that the three initial values $y_0 \geq s_E, \dot{x}_0, \dot{y}_0$ are not completely arbitrary because, according to the causal scheme $CS_{F_{p,\phi\theta}}$, they depend on the two parameters ϕ and θ.

Given $n \geq 1$ magnitudes $M_1, \dots, M_n$ of a dynamical phenomenon H, it is important to consider the set of all durations that take the initial state x of a F-realizer b, in evolution e, to some (other) state. This set can be thought as the output of a partial function $\bar{q}_H[M_1, \dots, M_n]$ that applies to an arbitrary state $x \in M_H[M_1, \dots, M_n]$, F-realizer $b \in B_F$, and evolution $e \in \bar{E}_F$, and is defined for all and only those triples (x, b, e) such that x is the initial state of b in e. This is precisely expressed by Definition 24.

Definition 24 (**The set** $\bar{q}_H[M_1, \dots, M_n](x, b, e)$ **of all durations that transform the initial** H**-state** x **of realizer** b **in evolution** e **into some (other) state**)

Let $H = (F, B_F)$ be a dynamical phenomenon, and $M_1, \dots, M_n$ be $n \in \mathbb{Z}^{\geq 1}$ different magnitudes of H.

$\bar{q}_H[M_1, \dots, M_n] : M_H[M_1, \dots, M_n] \times B_F \times \bar{E}_F \to \mathcal{P}(V(\boldsymbol{T}_H))$ is a partial function;
$\forall x \in M_H[M_1, \dots, M_n], \forall b \in B_F, \forall e \in \bar{E}_F$,
$\bar{q}_H[M_1, \dots, M_n](x, b, e)$ is defined iff $x \in C_H[M_1, \dots, M_n]$,
$st_H[M_1, \dots, M_n](b, e, \iota_0^H(e))$ is defined, and $x = st_H[M_1, \dots, M_n](b, e, \iota_0^H(e))$;
$\bar{q}_H[M_1, \dots, M_n](x, b, e) := \{t \in V(\boldsymbol{T}_H) : st_H[M_1, \dots, M_n](b, e, t \hat{+} \iota_0^H(e))$ is defined$\}$.

$\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x, b, e)$ is called *the set of all durations that transform the initial H-state x of realizer b in evolution e into some (other) state.*

We are now in a position to define what it means for a dynamical phenomenon $H = (F, B_F)$ to be deterministic. In the first place, such a notion is not absolute, but it must always be relativized to $n \geq 1$ given magnitudes. In the second place, if the phenomenon H is deterministic w.r.t. n of its magnitudes, the existence and identity of the instantaneous state, at any stage $(t \hat{+} \iota_0^H(e))$ of an evolution $e \in E_F(b)$ of a realizer $b \in B_F$, should not depend on either the identity of b, or the particular evolution e considered, but only on the initial state x. Thus we define:[7]

Definition 25 (Deterministic dynamical phenomenon w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$)
Let $H = (F, B_F)$ be a dynamical phenomenon, $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ be $n \in \mathbb{Z}^{\geq 1}$ different magnitudes of H, and $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] \neq \emptyset$.
H is a deterministic dynamical phenomenon w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$:=
for any state $x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, for any $t \in V(\boldsymbol{T}_H)$, for any two realizers $b_1, b_2 \in B_F$, for any two evolutions $e_1 \in E_F(b_1)$, $e_2 \in E_F(b_2)$, if $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_1, e_1, \iota_0^H(e_1))$ is defined, $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_2, e_2, \iota_0^H(e_2))$ is defined, $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_1, e_1, \iota_0^H(e_1)) = x = st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_2, e_2, \iota_0^H(e_2))$, and $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_1, e_1, t \hat{+} \iota_0^H(e_1))$ is defined, then $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_2, e_2, t \hat{+} \iota_0^H(e_2))$ is defined and $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_1, e_1, t \hat{+} \iota_0^H(e_1) = st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b_2, e_2, t \hat{+} \iota_0^H(e_2))$.

By Definition 24, the output of the partial function $\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ in general depends on the initial state x, the realizer b, and the evolution e to which it applies. However, by Definition 25, if H is deterministic with respect to $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, $\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ only depends on x. Thus, for the special case of deterministic phenomena, we can define (see below) a one-place total function $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ whose only argument is the state $x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, and whose output coincides with the output of $\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, whenever the latter is defined. When it is not, $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ outputs the empty set.

Definition 26 (The set $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$ of all durations that transform the initial H-state x into some (other) state)
Let $H = (F, B_F)$ be a deterministic dynamical phenomenon w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.
$q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] : M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] \to \mathcal{P}(V(\boldsymbol{T}_H))$;
$\forall x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$,
if $x \notin C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x) := \emptyset$,
if $x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x) := \bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x, b, e)$, where
$b \in B_F$ and $e \in \bar{E}_F$ satisfy: $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, \iota_0^H(e))$ is defined and
$x = st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, \iota_0^H(e))$.
$q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$ is called *the set of all durations that transform the initial H-state x into some (other) state.*

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[7] Montague (Montague 1974, p. 322) defines determinism for a formal *theory*, while Earman (Earman 1986, p. 13) similarly defines determinism for a physically *possible world*; however, neither of the two definitions is intended to directly apply to *dynamical phenomena*. Wójcicki's definition of determinism (Wójcicki 1975, p. 221) applies to a set theoretic representation Π of an *empirical phenomenon P*, but Wójcicki's Π is quite different from a dynamical phenomenon as defined by Definition 15.

Note that the definition above is well given, because, if $b \in B_F$ and $e \in \bar{E}_F$ satisfy the previous condition, then, by Definition 24, (i) $\bar{q}_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x, b, e)$ is defined and (ii) since H is a deterministic dynamical phenomenon w.r.t. $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$, $\bar{q}_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x, b, e)$ does not depend on either b or e, but only on x.

In Example 5 below, we examine again the phenomenon of projectile motion $H_{p,\phi\theta}$ (Example 3), and we show how to apply Definition 26 in order to specify the set of all durations that transform the initial state of an arbitrary projectile into a subsequent state of such a phenomenon, when the magnitudes considered are the horizontal and vertical components of the position and velocity of the projectile.

Example 5 (**The set** $q_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]((x_0, y_0, \dot{x}_0, \dot{y}_0))$ **of all durations that transform the initial** $H_{p,\phi\theta}$**-state** $(x_0, y_0, \dot{x}_0, \dot{y}_0)$ **of the phenomenon of projectile motion into some (other) state)**

Let $H_{p,\phi\theta} = (F_{p,\phi\theta}, B_{F_{p,\phi\theta}})$ be the phenomenon of projectile motion (see Example 3). We recall that $C_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]$ is the set of the $H_{p,\phi\theta}$-initial states w.r.t. magnitudes $\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}$, and that $C_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}] \neq \emptyset$ (see Example 4).

We also notice that the phenomenon of projectile motion $H_{p,\phi\theta}$ is usually assumed to be deterministic w.r.t. magnitudes $\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}$, for the existence and identity of the instantaneous state $(x, y, \dot{x}, \dot{y})$, at any stage $(t \hat{+} \iota_0^H(e))$ of a motion $e \in E_{F_{p,\phi\theta}}(b)$ of a projectile $b \in B_{F_{p,\phi\theta}}$, is not supposed to depend on either the identity of b, or the particular motion e considered, but only on the initial state $(x_0, y_0, \dot{x}_0, \dot{y}_0)$. Therefore, by Definition 26, for any $(x_0, y_0, \dot{x}_0, \dot{y}_0) \in C_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]$,

1. $q_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]((x_0, y_0, \dot{x}_0, \dot{y}_0)) = \bar{q}_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]((x_0, y_0, \dot{x}_0, \dot{y}_0), b, e)$,
   where $b \in B_{F_{p,\phi\theta}}$ and $e \in \bar{E}_{F_{p,\phi\theta}}$ satisfy: $st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, \iota_0^H(e))$ is defined, and $(x_0, y_0, \dot{x}_0, \dot{y}_0) = st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, \iota_0^H(e))$.
   By 1 and Definition 24,

2. $q_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]((x_0, y_0, \dot{x}_0, \dot{y}_0)) = \{t \in V(\boldsymbol{T}_{H_{p,\phi\theta}}) : st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, t \hat{+} \iota_0^{H_{p,\phi\theta}}(e))$ is defined$\}$,
   where $b \in B_{F_{p,\phi\theta}}$ and $e \in \bar{E}_{F_{p,\phi\theta}}$ satisfy: $st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, \iota_0^H(e))$ is defined, and $(x_0, y_0, \dot{x}_0, \dot{y}_0) = st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, \iota_0^H(e))$.

As it is customary, we assume that the set $V(\boldsymbol{T}_{H_{p,\phi\theta}})$ of all possible values of the time magnitude of the phenomenon of projectile motion is the set $\mathbb{R}$ of the real numbers and, accordingly, the associated binary operation $\hat{+}$ is the usual operation $+$ of addition over the real numbers.

Let $\mathrm{x}_0 = (x_0, y_0, \dot{x}_0, \dot{y}_0) \in C_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}]$, and $\bar{u}(\mathrm{x}_0, b, e)$ be the final instant of the motion e of b that starts at instant $\iota_0^{H_{p,\phi\theta}}(e)$ in state x_0. That is to say, $\bar{u}(\mathrm{x}_0, b, e)$ is the instant at which the projectile b hits the earth surface.

We can safely assume that $st_{H_{p,\phi\theta}}[\boldsymbol{X}, \boldsymbol{Y}, \dot{\boldsymbol{X}}, \dot{\boldsymbol{Y}}](b, e, i)$ is defined for any instant i such that $\iota_0^{H_{p,\phi\theta}}(e) \leq i \leq \bar{u}(\mathrm{x}_0, b, e)$, while it is not at any later instant $i > \bar{u}(\mathrm{x}_0, b, e)$, because, by the causal scheme of projectile motion (Example 3, 2c), the motion e terminates immediately after $\bar{u}(\mathrm{x}_0, b, e)$.

Let $\bar{d}(\mathrm{x}_0, b, e)$ be the duration of the motion e of b that starts at instant $\iota_0^{H_{p,\phi\theta}}(e)$ in state x_0, that is to say, $\bar{d}(\mathrm{x}_0, b, e) := \bar{u}(\mathrm{x}_0, b, e) - \iota_0^{H_{p,\phi\theta}}(e)$. It is not difficult to show that, under the

assumption that $H_{p,\phi\theta}$ be deterministic w.r.t. magnitudes $X, Y, \dot{X}, \dot{Y}$, such a duration does not depend on either the projectile b or its motion e, but only on the initial state x_0.[8]

Let us then define $d(\mathrm{x}_0) := \bar{d}(\mathrm{x}_0, b, e)$. $d(\mathrm{x}_0)$ is thus the duration of any motion of any projectile whose initial state is x_0. Therefore, the second equation in the list above reduces to Equation 4 below (Giunti 2014, par. 5.3):

$$q_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](\mathrm{x}_0) = \{t \in \mathbb{R} : 0 \le t \le d(\mathrm{x}_0)\}. \tag{4}$$

For an arbitrary state $x \in M_H[M_1, \dots, M_n]$, it is convenient to consider the set of all realizers that have at least one evolution whose initial state is x. This set can be thought as the output of a total function, which is defined below.

Definition 27 (**The set $B_F[M_1, \dots, M_n](x)$ of all realizers with an evolution whose initial H-state w.r.t. magnitudes $M_1, \dots, M_n$ is x**)

Let $H = (F, B_F)$ be a dynamical phenomenon, and $M_1, \dots, M_n$ be $n \in \mathbb{Z}^{\ge 1}$ different magnitudes of H.

$B_F[M_1, \dots, M_n] : M_H[M_1, \dots, M_n] \to \mathcal{P}(B_F)$;

$\forall x \in M_H[M_1, \dots, M_n]$,

if $x \notin C_H[M_1, \dots, M_n]$, $B_F[M_1, \dots, M_n](x) := \emptyset$,

if $x \in C_H[M_1, \dots, M_n]$,

$B_F[M_1, \dots, M_n](x) := \{b \in B_F : \exists e \in E_F(b)(st_H[M_1, \dots, M_n](b, e, \iota_0^H(e))$ is defined and $x = st_H[M_1, \dots, M_n](b, e, \iota_0^H(e)))\}$.

$B_F[M_1, \dots, M_n](x)$ is called *the set of all realizers with an evolution whose initial H-state w.r.t. magnitudes $M_1, \dots, M_n$ is x*.

Note that, if $C_H[M_1, \dots, M_n] \ne \emptyset$, by Definitions 27 and 23, for any $x \in C_H[M_1, \dots, M_n]$, $B_F[M_1, \dots, M_n](x) \ne \emptyset$ as well.

For an arbitrary realizer $b \in B_F$, and any state $x \in M_H[M_1, \dots, M_n]$, it is also convenient to consider the set of all evolutions of b whose initial state is x. This set can be thought as the output of a total function, which is defined below.

Definition 28 (**The set $E_F[M_1, \dots, M_n](b, x)$ of all evolutions of realizer b whose initial H-state w.r.t. magnitudes $M_1, \dots, M_n$ is x**)

Let $H = (F, B_F)$ be a dynamical phenomenon, and $M_1, \dots, M_n$ be $n \in \mathbb{Z}^{\ge 1}$ different magnitudes of H.

$E_F[M_1, \dots, M_n] : B_F \times M_H[M_1, \dots, M_n] \to \mathcal{P}(E_F(b))$;

$\forall b \in B_F, \forall x \in M_H[M_1, \dots, M_n]$,

if $x \notin C_H[M_1, \dots, M_n]$, $E_F[M_1, \dots, M_n](b, x) := \emptyset$,

if $x \in C_H[M_1, \dots, M_n]$, $E_F[M_1, \dots, M_n](b, x) :=$

$\{e \in E_F(b) : st_H[M_1, \dots, M_n](b, e, \iota_0^H(e))$ is defined and $x = st_H[M_1, \dots, M_n](b, e, \iota_0^H(e))\}$.

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[8] Let us assume for *reductio* that, for some projectile b^* and evolution e^*, $\bar{d}(\mathrm{x}_0, b^*, e^*) \ne \bar{d}(\mathrm{x}_0, b, e)$. Suppose $\bar{d}(\mathrm{x}_0, b^*, e^*) < \bar{d}(\mathrm{x}_0, b, e)$. Since $st_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](b, e, \bar{u}(\mathrm{x}_0, b, e))$ is the state of b in e at $\bar{u}(\mathrm{x}_0, b, e) = \bar{d}(\mathrm{x}_0, b, e) + \iota_0^{H_{p,\phi\theta}}(e)$ and $H_{p,\phi\theta}$ is deterministic w.r.t. magnitudes $X, Y, \dot{X}, \dot{Y}$, by Definition 25, $st_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](b, e, \bar{d}(\mathrm{x}_0, b, e) + \iota_0^{H_{p,\phi\theta}}(e)) = st_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](b^*, e^*, \bar{d}(\mathrm{x}_0, b, e) + \iota_0^{H_{p,\phi\theta}}(e^*))$. It follows that $\bar{u}(\mathrm{x}_0, b^*, e^*) = \bar{d}(\mathrm{x}_0, b^*, e^*) + \iota_0^{H_{p,\phi\theta}}(e^*)$ is not the final instant of the motion e^* of b^* that starts at instant $\iota_0^{H_{p,\phi\theta}}(e^*)$ in state x_0, because the projectile b^*, in motion e^*, has still a state at the later instant $\bar{d}(\mathrm{x}_0, b, e) + \iota_0^{H_{p,\phi\theta}}(e^*)$. Analogously for the case $\bar{d}(\mathrm{x}_0, b, e) < \bar{d}(\mathrm{x}_0, b^*, e^*)$.

$E_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, x)$ is called *the set of all evolutions of realizer b whose initial H-state w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ is x.*

If $C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] \neq \emptyset$, for some $x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ and $b \in B_F$, $E_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, x)$ may be empty. However note that, by Definitions 28 and 27, for any $x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$, for any $b \in B_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$, $E_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, x) \neq \emptyset$.

We are now ready to show that, whenever H is a deterministic dynamical phenomenon w.r.t. to magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, for the state space $M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ we can define a structure $(g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t)_{t \in V(\boldsymbol{T}_H)}$, in such a way that $(M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n], (g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t)_{t \in V(\boldsymbol{T}_H)})$ turns out to be a partial deterministic system on the monoid $L_H = (V(\boldsymbol{T}_H), \hat{+})$. This is shown in detail by Definition 29 and Proposition 5 below.

Definition 29 (The partial deterministic system of H w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$)
Let $H = (F, B_F)$ be a deterministic dynamical phenomenon w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.
$DS_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] := (M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n], (g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t)_{t \in V(\boldsymbol{T}_H)})$;
$g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t : M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] \to M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is a partial function;
$\forall t \in V(\boldsymbol{T}_H), Pim(g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t) :=$
$\{x \in M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] : x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] \wedge t \in q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)\}$;
$\forall t \in V(\boldsymbol{T}_H), \forall x \in Pim(g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t)$,
$g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t(x) := st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, t \hat{+} \iota_0^H(e))$, where
$b \in B_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$ and $e \in E_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, x)$.
$DS_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ is called *the partial deterministic system of H w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.*

Note that the previous definition is well given; in the first place, $\forall x \in Pim(g_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]^t)$, $x \in C_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ and $t \in q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$; in the second place, if $b \in B_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$ and $e \in E_F[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, x)$, then, by the definitions of $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x)$ (Definition 26) and $\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x, b, e)$ (Definition 24), (i) $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, t \hat{+} \iota_0^H(e))$ is defined and (ii) $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, e, t \hat{+} \iota_0^H(e))$ does not depend on either b or e, as H is a deterministic dynamical phenomenon w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.

Proposition 5
$DS_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ *is a partial deterministic system on the monoid $L_H = (V(\boldsymbol{T}_H), \hat{+})$.*

Proof The thesis is an immediate consequence of Definitions 29 and 4. □

3.4 Settings, Measurements, and Experiments of a Dynamical Phenomenon

In this Section, we complete the formal description of the phenomenal level by stating the last three ontological Axioms (8, 9, 10) which introduce, respectively, the notions of setting (or preparation) of a F-realizer b with respect to $n \geq 1$ magnitudes, measurement on b in evolution e of $m \geq 1$ magnitudes, and experiment of a dynamical phenomenon $H = (F, B_F)$.

In what follows, we are going to use a compact and uniform notation for intervals of integers, as specified below.

Notation 3 (Notation for intervals of integers)
Let $\mathbb{Z}$ be the set of the integers, and $m, n \in \mathbb{Z}$.
With the notation $\mathbb{Z}^{[m,n]}$ we intend the set $\{x \in \mathbb{Z} : m \leq x \leq n\}$.

With the notation $\mathbb{Z}^{\geq n}$, we intend the set $\{x \in \mathbb{Z} : n \geq x\}$;
analogously for the notations $\mathbb{Z}^{>n}$, $\mathbb{Z}^{\leq n}$, and $\mathbb{Z}^{<n}$.

Given a dynamical phenomenon $H = (F, B_F)$ and $n \geq 1$ of its magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, let us consider an arbitrary F-realizer $b \in B_F$. As a first approximation, we think of a setting, or a preparation, of b relative to the n given magnitudes as a procedure that (i) starts an evolution e of b and, with respect to this evolution e of b, (ii) simultaneously sets the initial values of each of the n magnitudes.

In general, the initial setting of a magnitude should not be thought to exactly fix the initial value of the magnitude at issue. A better picture is to think of an initial setting as producing an approximate result. As we are not presupposing any special property of the set of possible values of a magnitude, we think of an approximate result as generally as possible, namely, as a set of values that contains the initial value of the magnitude as one of its members.

According to this picture, we can then think of each setting procedure as a partial function $\sigma_k[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$ that applies to an arbitrary realizer $b \in B_F$ and, if it is defined on b, it returns a pair of outputs. The first output is an evolution $e \in E_F(b)$, and the second one is a subset of the state space $M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n]$. For any $r \in \mathbb{Z}^{[1,n]}$, the r-projection of this subset is to be thought as the approximate result of the initial setting of magnitude $\boldsymbol{M}_r$ of realizer b in evolution e. Therefore, with respect to the n magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, the initial state of b in e should be defined, and it should belong to the second output. All this is precisely stated by Axiom 8 below.

Axiom 8 (**The k-th setting w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ of realizer b**)
Let $H = (F, B_F)$ be a dynamical phenomenon and, $\forall r \in \mathbb{Z}^{[1,n]}$, $\boldsymbol{M}_r \in \mathbf{M}_H$.
$\forall k \in \mathbb{Z}^{\geq 1}$, $\sigma_k[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n] : B_F \to \bar{E}_F \times \mathcal{P}(M_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n])$ is a partial function.
$\forall k \in \mathbb{Z}^{\geq 1}$, $\forall b \in B_F$, if $b \in Pim(\sigma_k[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n])$, then
$\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b) \in E_F(b)$,
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))$ is defined, and
$st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))) \in \sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)$,
where, for $s \in \mathbb{Z}^{[1,2]}$, $\sigma_k^s[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)$ is the s-th element of $\sigma_k[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)$.
$\sigma_k[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)$ is called *the k-th setting w.r.t. magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$ of b.*

Given a dynamical phenomenon $H = (F, B_F)$ and $n \geq 1$ of its magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, let us consider an arbitrary F-realizer $b \in B_F$ and an arbitrary evolution $e \in E_F(b)$. We think of a measurement of $m \leq n$ of the n magnitudes, on realizer b in evolution e, as a procedure that (i) measures a duration of the evolution e of b and (ii) simultaneously measures the values of each of the $m \leq n$ magnitudes at the instant of e that corresponds to such a duration.

Analogously to an initial setting, a measurement of a magnitude does not detect the exact instantaneous value of the magnitude, but it rather produces an approximate result (Dalla Chiara and Toraldo di Francia 1973, 1981). We think of an approximate result as a set of values that contains the instantaneous value of the magnitude as one of its members.

Given $n \geq 1$ magnitudes $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$, we can then think of each measurement procedure of $m \leq n$ of the given magnitudes $\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}$, as a partial function $\mu_l[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}]$ that applies to an arbitrary realizer $b \in B_F$ and evolution $e \in \bar{E}_F$ and, whenever it is defined on b and e, $e \in E_F(b)$ and it returns a pair of outputs. The first output is a subset of the set of possible values $V(\boldsymbol{T}_H)$ of the time magnitude, and the second one is a subset of the state space $M_H[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}]$. The first output is to be thought as the approximate result of

a measurement of a duration t of e such that the state, w.r.t. $M_1, \ldots, M_n$, of b in e at the corresponding instant $t \hat{+} \iota_0^H(e)$ is defined. And, for any $p \in \mathbb{Z}^{[1,m]}$, the p-projection of the second output is to be thought as the approximate result of the measurement of magnitude $M_{i(p)}$ on realizer b in evolution e, at the instant $t \hat{+} \iota_0^H(e)$. Therefore, (i) the duration t should belong to the first output, (ii) with respect to the n magnitudes $M_1, \ldots, M_n$, the state of b in e at $t \hat{+} \iota_0^H(e)$ should be defined, and (iii) with respect to the m magnitudes $M_{i(1)}, \ldots, M_{i(m)}$, the state of b in e at $t \hat{+} \iota_0^H(e)$ should belong to the second output. All this is precisely stated by Axiom 9 below.

Axiom 9 (The l-th measurement w.r.t. magnitudes $M_{i(1)}, \ldots, M_{i(m)}$ on realizer b in evolution e)

Let $H = (F, B_F)$ be a dynamical phenomenon, $m, n \in \mathbb{Z}^{\geq 1}$, $m \leq n$, $i : \mathbb{Z}^{[1,m]} \to \mathbb{Z}^{[1,n]}$, $\forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))$, and $\forall r \in \mathbb{Z}^{[1,n]}, M_r \in \mathbf{M}_H$.
$\forall l \in \mathbb{Z}^{\geq 1}, \mu_l[M_{i(1)}, \ldots, M_{i(m)}] : B_F \times \bar{E}_F \to \mathcal{P}(V(\mathbf{T}_H)) \times \mathcal{P}(M_H[M_{i(1)}, \ldots, M_{i(m)}])$ is a partial function.
$\forall l \in \mathbb{Z}^{\geq 1}, \forall b \in B_F, \forall e \in \bar{E}_F$, if $(b, e) \in Pim(\mu_l[M_{i(1)}, \ldots, M_{i(m)}])$, then
$e \in E_F(b)$ and $\exists t$ such that
$t \in \mu_l^1[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ and
$st_H[M_1, \ldots, M_n](b, e, t \hat{+} \iota_0^H(e))$ is defined, and
$st_H[M_{i(1)}, \ldots, M_{i(m)}](b, e, t \hat{+} \iota_0^H(e)) \in \mu_l^2[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ where, for $s \in \mathbb{Z}^{[1,2]}$, $\mu_l^s[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ is the s-th element of $\mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, e)$.
$\mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ is called *the l-th measurement w.r.t. magnitudes $M_{i(1)}, \ldots, M_{i(m)}$ on realizer b in evolution e*.

On the basis of Axioms 8 and 9, we can now think of a *possible* experiment of a dynamical phenomenon $H = (F, B_F)$ as a setting, w.r.t. $n \geq 1$ magnitudes $M_1, \ldots, M_n$, of a realizer $b \in B_F$, together with a measurement, on b in evolution e, w.r.t. $m \leq n$ of the n magnitudes $M_{i(1)}, \ldots, M_{i(m)}$. The setting and the measurement must be conveniently correlated, in the sense that the setting should be defined on the realizer b, while the measurement should be defined on the same realizer b and the evolution e which is the first output of the setting. This is precisely expressed by Definition 30.

Definition 30 (Possible experiment of a dynamical phenomenon H)

Let $H = (F, B_F)$ be a dynamical phenomenon, $b \in B_F$, $e \in \bar{E}_F$, $k, l, m, n \in \mathbb{Z}^{\geq 1}$, $m \leq n$, $i : \mathbb{Z}^{[1,m]} \to \mathbb{Z}^{[1,n]}, \forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))$, and $\forall r \in \mathbb{Z}^{[1,n]}, M_r \in \mathbf{M}_H$.
exp is a possible experiment of $H :=$
$exp = (\sigma_k[M_1, \ldots, M_n](b), \mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, e))$,
$\sigma_k[M_1, \ldots, M_n](b)$ is the k-th setting w.r.t. $M_1, \ldots, M_n$ of b,
$\mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ is the l-th measurement w.r.t. $M_{i(1)}, \ldots, M_{i(m)}$ on b in e,
$b \in Pim(\sigma_k[M_1, \ldots, M_n])$,
$(b, e) \in Pim(\mu_l[M_{i(1)}, \ldots, M_{i(m)}])$, and
$e = \sigma_k^1[M_1, \ldots, M_n](b)$.

Finally, according to Axiom 10 below, we think of the experiments that are *actually* performed with respect to a dynamical phenomenon H as a special subclass Exp_H of its possible experiments.

Axiom 10 (The set Exp_H of all experiments of a dynamical phenomenon H)
We assume that, to any dynamical phenomenon H, *the set of all (actual) experiments of H is uniquely associated.* Such a set is indicated by Exp_H. We further assume that, for any $exp \in Exp_H$, exp is a possible experiment of H. Any $exp \in Exp_H$ is called *an (actual) experiment of H.*

4 Real World Semantics for Partial Dynamical Systems with Finitely Many Components

In Sect. 4,[9] we develop a *real world* semantics for *models* (in contrast with the usual *possible worlds* semantics for *sentences*), in the case of a widely used class of dynamical models, namely, the *n*-component partial dynamical systems. In particular, we define an interpretation $I_{DS_L,H}$ of a *n*-component partial dynamical system DS_L on a dynamical phenomenon H. The interpretation $I_{DS_L,H}$ then allows us to define what it means, for the interpreted system $(DS_L, I_{DS_L,H})$, to be a *true/false model of H* (Definition 38). We also show how such interpretation induces, on the one hand, a substructure of DS_L (Definition 34 and Proposition 6) and, on the other one, a mathematical structure of H (Definition 35). Finally, we prove that $(DS_L, I_{DS_L,H})$ is a true model of H if, and only if, the substructure of DS_L induced by $I_{DS_L,H}$ is identical to the structure of H induced by $I_{DS_L,H}$ (Theorem 2 and Corollary 1).

4.1 Correct Interpretation of a *n*-Component Partial Dynamical System on a Dynamical Phenomenon

Let us now see how a *n*-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ can be interpreted on a dynamical phenomenon $H = (F, B_F)$. The key point of the interpretation consists in establishing a correspondence between each component C_i $(1 \leq i \leq n)$ of the state space M of DS_L and a different magnitude M_i of the phenomenon H, as well as between the time model $L = (T, +)$ of the dynamical system and the monoid $L_H = (V(T_H), \hat{+})$ of the possible values of the time magnitude T_H of the phenomenon (see Axiom 4).

More specifically, an interpretation is obtained by stating that (i) each component C_i of the state space M of DS_L is included in, or is equal to, the set $V(M_i)$ of the possible values of a magnitude M_i of the phenomenon H and (ii) the time model L of the partial dynamical system is identical to the monoid L_H of the time values of the phenomenon. Thus, an interpretation $I_{DS_L,H}$ of a dynamical system DS_L on a dynamical phenomenon H can always be identified with a set of $n + 1$ special statements. The exact definition is below.

Definition 31 (Interpretation of a *n*-component partial dynamical system on a dynamical phenomenon)
Let DS_L be a *n*-component partial dynamical system on a monoid L, $H = (F, B_F)$ be a dynamical phenomenon, and L_H be the monoid of the possible values of the time magnitude of H.

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[9] The main ideas and results of Sect. 4 were first presented in Giunti (2016).

$I_{DS_L,H}$ is an interpretation of DS_L on H := $I_{DS_L,H} = \{C_1 \subseteq V(\boldsymbol{M}_1), \dots, C_n \subseteq V(\boldsymbol{M}_n), L = L_H\}$, where C_i is the i-th component of the state space of DS_L, $\boldsymbol{M}_i$ is a magnitude of H and, for any i, j $(1 \leq i, j \leq n)$, if $i \neq j$, then $\boldsymbol{M}_i \neq \boldsymbol{M}_j$.
$\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$ are called *the n magnitudes of H specified by $I_{DS_L,H}$*.

Going back to Example 2, we notice that the 4-component dynamical system DS_{L_p} is not usually thought as a pure mathematical system. Instead, it is conceived *together with* a largely implicit *intended interpretation*, which makes it a model of the phenomenon $H_{p,\phi\theta}$ of projectile motion (Example 3). This interpretation is made explicit in the following example.

Example 6 (**The intended interpretation $I_{DS_{L_p},H_{p,\phi\theta}}$ of the dynamical system DS_{L_p} on the phenomenon $H_{p,\phi\theta}$ of projectile motion**)

We use the symbol $I_{DS_{L_p},H_{p,\phi\theta}}$ to indicate the intended interpretation of the dynamical system $DS_{L_p} = (X \times Y \times \dot{X} \times \dot{Y}, (g^t)_{t \in T})$ on the phenomenon $H_{p,\phi\theta} = (F_{p,\phi\theta}, B_{F_{p,\phi\theta}})$ of projectile motion. Let $X, Y, \dot{X}, \dot{Y}$ be the four magnitudes of $H_{p,\phi\theta}$ specified in Example 4.
We now let the four components $X, Y, \dot{X}, \dot{Y}$ of the state space of DS_{L_p} and its time model $L_p = (T, +)$ correspond, respectively, to these four magnitudes of $H_{p,\phi\theta}$, and to the monoid $L_{H_{p,\phi\theta}} = (V(T_{H_{p,\phi\theta}}), \hat{+})$ of the possible values of the time magnitude $T_{H_{p,\phi\theta}}$ of the phenomenon. The intended interpretation of the dynamical system DS_{L_p} on the phenomenon $H_{p,\phi\theta}$ of projectile motion is thus the following set of five statements:

$$I_{DS_{L_p},H_{p,\phi\theta}} = \{X = V(X), Y = V(Y), \dot{X} = V(\dot{X}), \dot{Y} = V(\dot{Y})), L_p = L_{H_{p,\phi\theta}}\}. \tag{5}$$

From an intuitive point of view, as soon as an interpretation $I_{DS_L,H}$ is fixed, we expect that the n-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ would provide a representation of some, or even all, temporal evolutions $e \in \bar{E}_F$ of the phenomenon H. However, for such a representation to obtain, four obvious conditions must be satisfied.

First, if $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$ are the n magnitudes of H specified by $I_{DS_L,H}$, then the set $C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$ of the H-initial states w.r.t. those magnitudes must not be empty. For, if $C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n] = \emptyset$, there is no evolution $e \in \bar{E}_F$ whose initial state, w.r.t. $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$, is defined. Second, the phenomenon H must be deterministic w.r.t. $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$, for DS_L is a partial dynamical system, and it is thus intended to represent a deterministic dynamics. Third, the set $C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$ of the H-initial states w.r.t. $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$ must be included in the set C of the original states of DS_L. For, if some $x \in C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$ and $x \notin C$, no evolution $e \in \bar{E}_F$ whose initial state is x can be represented by any state transition g^t of DS_L, as no state transition g^t is defined on x. Fourth, for any $x \in C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$, the set $q_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x)$ of all durations that transform x into some (other) state must be a subset of the life span $q(x)$. For, if $t \in q_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x)$ and $t \notin q(x)$, the state transition g^t is not defined on x, so that it cannot represent how the duration t transforms state x into some (other) state.

It is thus quite clear that, whenever the interpretation $I_{DS_L,H}$ satisfies the four conditions just stated, the n-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ can be thought to provide a representation of any temporal evolution $e \in \bar{E}_F$ whose initial state is member of $C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$. We call any interpretation $I_{DS_L,H}$ that satisfies those four conditions *admissible*. The formal definition is below.

Definition 32 (Admissible interpretation of a n-component partial dynamical system on a dynamical phenomenon)

Let $DS_L = (M, (g^t)_{t \in T})$ be a n-component partial dynamical system on a monoid L, C be the set of all original states, $q(x)$ be the life span of state $x \in M$, H be a dynamical phenomenon, $I_{DS_L,H}$ be an interpretation of DS_L on H, and $M_1, \ldots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

$I_{DS_L,H}$ *is an admissible interpretation of* DS_L *on* $H := C_H[M_1, \ldots, M_n] \neq \emptyset$, H *is a deterministic dynamical phenomenon w.r.t.* $M_1, \ldots, M_n$, $C_H[M_1, \ldots, M_n] \subseteq C$ *and,* $\forall x \in C_H[M_1, \ldots, M_n], q_H[M_1, \ldots, M_n](x) \subseteq q(x)$.

We now show that the intended interpretation (Example 6) of the dynamical system DS_{L_p} (Example 2) on the phenomenon of projectile motion (Example 3) is admissible.

***Example 7* (Admissibility of the intended interpretation** $I_{DS_{L_p}, H_{p,\phi\theta}}$ **of the dynamical system** DS_{L_p} **on the phenomenon** $H_{p,\phi\theta}$ **of projectile motion)**

Let us consider again (see Example 6) the intended interpretation $I_{DS_{L_p}, H_{p,\phi\theta}}$ of the dynamical system DS_{L_p} on the phenomenon $H_{p,\phi\theta}$ of projectile motion.

We already know that $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}] \neq \emptyset$ (Example 4), and that $H_{p,\phi\theta}$ is usually assumed to be a deterministic dynamical phenomenon w.r.t. $X, Y, \dot{X}, \dot{Y}$ (Example 5).

Recall that $DS_{L_p} = (X \times Y \times \dot{X} \times \dot{Y}, (g^t)_{t \in T})$ is a dynamical system on $L_p = (T, +)$. Let C_p be its set of all original states (see Definition 6) and, $\forall x \in X \times Y \times \dot{X} \times \dot{Y}$, $q_p(x)$ be the life span of state x (see Definition 7). Thus, by Proposition 4, $C_p = X \times Y \times \dot{X} \times \dot{Y}$ and, $\forall x \in X \times Y \times \dot{X} \times \dot{Y}, q_p(x) = T$. We then notice that the intended interpretation $I_{DS_{L_p}, H_{p,\phi\theta}}$ entails $C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}] \subseteq X \times Y \times \dot{X} \times \dot{Y} = C_p$, for all its component sentences are identities (see Equation 5 above). Also, $\forall x \in C_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}], q_{H_{p,\phi\theta}}[X, Y, \dot{X}, \dot{Y}](x) \subseteq T = q_p(x)$. It thus follows that $I_{DS_{L_p}, H_{p,\phi\theta}}$ is an admissible interpretation of DS_{L_p} on $H_{p,\phi\theta}$.

We remarked above that, given an admissible interpretation $I_{DS_L,H}$, the n-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ can be thought to provide a representation of any temporal evolution $e \in \bar{E}_F$ whose initial state is member of $C_H[M_1, \ldots, M_n]$. In more detail, for any initial state $x \in C_H[M_1, \ldots, M_n]$, for any F-realizer $b \in B_F[M_1, \ldots, M_n](x)$, for any evolution $e \in E_F[M_1, \ldots, M_n](b, x)$, such a representation is provided by all state transitions g^t such that t belongs to the set $q_H[M_1, \ldots, M_n](x)$ of all durations that transform x into some (other) state. The representation will then be *correct* if, for any such t, $g^t(x)$ is identical to the state $st_H[M_1, \ldots, M_n](b, e, t + \iota_0^H(e))$ into which the initial state x is transformed by duration t. Accordingly, we call any admissible interpretation $I_{DS_L,H}$ that satisfies this condition a *correct interpretation*. Below is the exact definition.

Definition 33 (Correct interpretation of a n-component partial dynamical system on a dynamical phenomenon)

Let $DS_L = (M, (g^t)_{t \in T})$ be a n-component partial dynamical system on a monoid L, H be a dynamical phenomenon, $I_{DS_L,H}$ be an interpretation of DS_L on H, and $M_1, \ldots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

$I_{DS_L,H}$ *is a correct interpretation of* DS_L *on* $H :=$
$I_{DS_L,H}$ *is an admissible interpretation of* DS_L *on* H, *and*
$\forall x \in C_H[M_1, \ldots, M_n], \forall t \in q_H[M_1, \ldots, M_n](x),$

$\forall b \in B_F[M_1, \dots, M_n](x), \forall e \in E_F[M_1, \dots, M_n](b, x),$
$g^t(x) = st_H[M_1, \dots, M_n](b, e, t + t_0^H(e)).$

We are now going to show that, whenever $I_{DS_L,H}$ is an admissible interpretation of $DS_L = (M, (g^t)_{t \in T})$ that specifies magnitudes $M_1, \dots, M_n$ of H, for the state space $M_H[M_1, \dots, M_n]$, we can define a structure $(g[M_1, \dots, M_n]^t)_{t \in T}$ such that $(M_H[M_1, \dots, M_n], (g[M_1, \dots, M_n]^t)_{t \in T})$ turns out to be a partial subsystem of DS_L. The pair $(M_H[M_1, \dots, M_n], (g[M_1, \dots, M_n]^t)_{t \in T})$ is called the partial subsystem of DS_L induced by the admissible interpretation $I_{DS_L,H}$. This is shown in detail by Definition 34 and Proposition 6 below.

Definition 34 (The partial subsystem $S_{DS_L}[I_{DS_L,H}]$ of n-component partial dynamical system DS_L, induced by admissible interpretation $I_{DS_L,H}$)

Let $DS_L = (M, (g^t)_{t \in T})$ be a n-component partial dynamical system on a monoid L, H be a dynamical phenomenon, $I_{DS_L,H}$ be an admissible interpretation of DS_L on H, and $M_1, \dots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.
$S_{DS_L}[I_{DS_L,H}] := (M_H[M_1, \dots, M_n], (g[M_1, \dots, M_n]^t)_{t \in T});$
$g[M_1, \dots, M_n]^t : M_H[M_1, \dots, M_n] \to M_H[M_1, \dots, M_n]$ is a partial function;
$\forall t \in T, Pim(g[M_1, \dots, M_n]^t) :=$
$\{x \in M_H[M_1, \dots, M_n] : x \in C_H[M_1, \dots, M_n] \wedge t \in q_H[M_1, \dots, M_n](x)\};$
$\forall t \in T, \forall x \in Pim(g[M_1, \dots, M_n]^t), g[M_1, \dots, M_n]^t(x) := g^t(x).$
$S_{DS_L}[I_{DS_L,H}]$ is called *the partial subsystem of DS_L induced by admissible interpretation $I_{DS_L,H}$*.

Proposition 6

$S_{DS_L}[I_{DS_L,H}]$ *is a partial subsystem of DS_L.*

Proof See the Appendix. □

If $I_{DS_L,H}$ is an admissible interpretation of DS_L on H that specifies magnitudes $M_1, \dots, M_n$, by Definition 32, H is a deterministic dynamical phenomenon with respect to $M_1, \dots, M_n$. Thus, by Definition 29 and Proposition 5, the partial deterministic system of H w.r.t. magnitudes $M_1, \dots, M_n$, $DS_H[M_1, \dots, M_n]$, is well defined. We then call $DS_H[M_1, \dots, M_n]$ the partial deterministic system of H induced by $I_{DS_L,H}$, and we indicate it by $DS_H[I_{DS_L,H}]$. The exact definition is below.

Definition 35 (The partial deterministic system of H induced by admissible interpretation $I_{DS_L,H}$)

Let DS_L be a n-component partial dynamical system on a monoid L, H be a dynamical phenomenon, $I_{DS_L,H}$ be an admissible interpretation of DS_L on H, and $M_1, \dots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.
$DS_H[I_{DS_L,H}] := DS_H[M_1, \dots, M_n].$
$DS_H[I_{DS_L,H}]$ is called *the partial deterministic system of H induced by admissible interpretation $I_{DS_L,H}$*.

We have thus seen that any admissible interpretation $I_{DS_L,H}$ induces, on the one hand, a partial subsystem of DS_L and, on the other one, a partial deterministic system of H. We prove below that the interpretation $I_{DS_L,H}$ turns out to be correct if, and only if, the two induced systems are identical.

Theorem 2 (Correct interpretation as interpretation induced structure identity)
Let $DS_L = (M, (g^t)_{t \in T})$ be a n-component partial dynamical system on a monoid L, H be a dynamical phenomenon, $I_{DS_L,H}$ be an admissible interpretation of DS_L on H, and $M_1, \dots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.
Let $S_{DS_L}[I_{DS_L,H}] = (M_H[M_1, \dots, M_n], (g[M_1, \dots, M_n]^t)_{t \in T})$ be the partial sub-system of DS_L induced by $I_{DS_L,H}$, and $DS_H[I_{DS_L,H}] = DS_H[M_1, \dots, M_n] = (M_H[M_1, \dots, M_n], (g_H[M_1, \dots, M_n]^t)_{t \in V(T_H)})$ be the partial deterministic system of H induced by $I_{DS_L,H}$.
$I_{DS_L,H}$ is a correct interpretation of DS_L on H iff $S_{DS_L}[I_{DS_L,H}] = DS_H[I_{DS_L,H}]$.

Proof See the Appendix. $\square$

4.2 True Models of Dynamical Phenomena

We have seen in Sect. 4.1 that, whenever an interpretation $I_{DS_L,H}$ is fixed, the partial dynamical system DS_L *may* provide a representation of any temporal evolution $e \in E_F[M_1, \dots, M_n](b, x)$, for any F-realizer $b \in B_F[M_1, \dots, M_n](x)$ and any initial state $x \in C_H[M_1, \dots, M_n]$, where $M_1, \dots, M_n$ are the n magnitudes specified by the interpretation. Hence, the system DS_L together with the interpretation $I_{DS_L,H}$ can be thought as a model of the phenomenon H. This idea is precisely expressed by the definition below.

Definition 36 (Model of a dynamical phenomenon)
DS_H *is a model of H* $:= DS_H = (DS_L, I_{DS_L,H})$, DS_L is a n-component partial dynamical system on a monoid L, H is a dynamical phenomenon, and $I_{DS_L,H}$ is an interpretation of DS_L on H.

Example 8 **(The projectile model $DS_{H_{p,\phi\theta}} = (DS_{L_p}, I_{DS_{L_p},H_{p,\phi\theta}})$)**
Let DS_{L_p} be the 4-component dynamical system individuated by the equation of motion of a projectile (see Example 2), and $I_{DS_{L_p},H_{p,\phi\theta}}$ be the intended interpretation of DS_{L_p} on the phenomenon of projectile motion $H_{p,\phi\theta}$ (see Example 6). Let $DS_{H_{p,\phi\theta}} = (DS_{L_p}, I_{DS_{L_p},H_{p,\phi\theta}})$.
By Definition 36, $DS_{H_{p,\phi\theta}}$ is a model of $H_{p,\phi\theta}$. $DS_{H_{p,\phi\theta}}$ is called *the projectile model* (Giunti 2014, sect. 4.2.1).

An *admissible* model of a dynamical phenomenon is a model whose interpretation is admissible. From an intuitive point of view, an admissible model *does* provide a representation of any temporal evolution $e \in E_F[M_1, \dots, M_n](b, x)$, for any F-realizer $b \in B_F[M_1, \dots, M_n](x)$ and any initial state $x \in C_H[M_1, \dots, M_n]$, where $M_1, \dots, M_n$ are the n magnitudes specified by the interpretation.

Definition 37 (Admissible model of a dynamical phenomenon)
Let $DS_H = (DS_L, I_{DS_L,H})$ be a model of H.
DS_H *is an admissible model of H* $:= I_{DS_L,H}$ is an admissible interpretation of DS_L on H.

Example 9 (**Admissibility of the projectile model $DS_{H_{p,\phi\theta}}$**)

Let $DS_{H_{p,\phi\theta}} = (DS_{L_p}, I_{DS_{L_p},H_{p,\phi\theta}})$ be the projectile model (see Example 8).

We know that $I_{DS_{L_p},H_{p,\phi\theta}}$ is an admissible interpretation of DS_{L_p} on $H_{p,\phi\theta}$ (see Example 7). Therefore, by the definition of admissible model (Definition 37), $DS_{H_{p,\phi\theta}}$ is an admissible model of $H_{p,\phi\theta}$.

A *true* model of a dynamical phenomenon is a model whose interpretation is correct. From an intuitive standpoint, a true model provides a *correct* representation of any temporal evolution $e \in E_F[M_1, \dots, M_n](b, x)$, for any F-realizer $b \in B_F[M_1, \dots, M_n](x)$ and any initial state $x \in C_H[M_1, \dots, M_n]$, where $M_1, \dots, M_n$ are the n magnitudes specified by the interpretation. A false model is a model whose interpretation is not correct.

Definition 38 (**True/False model of a dynamical phenomenon**)

Let $DS_H = (DS_L, I_{DS_L,H})$ be a model of H.

DS_H *is a true model of* $H := I_{DS_L,H}$ is a correct interpretation of DS_L on H;

DS_H *is a false model of* $H := I_{DS_L,H}$ is not a correct interpretation of DS_L on H.

We can finally reformulate Theorem 2 as stating a necessary and sufficient condition for the *truth* of an admissible model of a phenomenon.

Corollary 1 (**Truth as interpretation induced structure identity**)

Let $DS_H = (DS_L, I_{DS_L,H})$ be an admissible model of H and $M_1, \dots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

Let $S_{DS_L}[I_{DS_L,H}] = (M_H[M_1, \dots, M_n], (g[M_1, \dots, M_n]^t)_{t\in T})$ be the partial subsystem of DS_L induced by $I_{DS_L,H}$.

Let $DS_H[I_{DS_L,H}] = DS_H[M_1, \dots, M_n] =$
$(M_H[M_1, \dots, M_n], (g_H[M_1, \dots, M_n]^t)_{t\in V(T_H)})$ be the partial deterministic system of H induced by $I_{DS_L,H}$.

DS_H is a true model of H iff $S_{DS_L}[I_{DS_L,H}] = DS_H[I_{DS_L,H}]$.

Proof The thesis follows from Definitions 36, 37, 38, and Theorem 2. ◻

5 Empirical Semantics for Partial Dynamical Systems with Finitely Many Components

In Sect. 5, we work out an empirical semantics for partial dynamical systems with a finite number of state-space components. In the first place, we define under what conditions an interpretation $I_{DS_L,H}$ of a n-component partial dynamical system DS_L on a dynamical phenomenon H is *empirical* (Definition 40). The empirical interpretation $I_{DS_L,H}$ then allows us to define what it means, for the interpreted system $(DS_L, I_{DS_L,H})$, to be an *empirically correct/incorrect model of H* (Definition 45). And, finally, we are able to prove: For any empirical interpretation $I_{DS_L,H}$, if $(DS_L, I_{DS_L,H})$ is a true model of H, then $(DS_L, I_{DS_L,H})$ is an empirically correct model of H and, conversely, if $(DS_L, I_{DS_L,H})$ is an empirically incorrect model of H, then $(DS_L, I_{DS_L,H})$ is a false model of H (Theorem 3 and Corollary 3).

5.1 Empirical Interpretation of a *n*-Component Partial Dynamical System on a Dynamical Phenomenon, and Its Consistency with an Experiment

We have seen in the previous Section that any interpretation $I_{DS_L,H}$ induces, on the one hand, a partial subsystem of DS_L and, on the other one, a partial deterministic system of H. It is important to realize that the interpretation $I_{DS_L,H}$ also induces a definite subset of the set Exp_H of all experiments of H. Its elements are all those experiments whose magnitudes $M_1, \ldots, M_n$ are exactly the n magnitudes of H specified by $I_{DS_L,H}$. We call this subset of Exp_H *the empirical content of* $I_{DS_L,H}$, and we indicate it by $Exp_H(I_{DS_L,H})$. The exact definition is below.

Definition 39 (The empirical content $Exp_H(I_{DS_L,H})$ of interpretation $I_{DS_L,H}$)
Let $I_{DS_L,H}$ be an interpretation of n-component partial dynamical system DS_L on dynamical phenomenon $H = (F, B_F)$, and $M_1, \ldots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.
$Exp_H(I_{DS_L,H}) := \{exp \in Exp_H : \exists k, l \in \mathbb{Z}^{\geq 1}, \exists m \in \mathbb{Z}^{[1,n]},$
$\exists i(i : Z^{[1,m]} \to Z^{[1,n]} \wedge \forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))), \exists b \in B_F,$
$exp = (\sigma_k[M_1, \ldots, M_n](b), \mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, \sigma_k^1[M_1, \ldots, M_n](b)))\}.$
$Exp_H(I_{DS_L,H})$ is called *the empirical content of* $I_{DS_L,H}$.

Example 10 (The empirical content $Exp_H(I_{DS_{L_p},H_{p,\phi\theta}})$ of the intended interpretation of DS_{L_p} on the phenomenon of projectile motion $H_{p,\phi\theta}$)
Let $I_{DS_{L_p},H_{p,\phi\theta}}$ be the intended interpretation of the n-component dynamical system DS_{L_p} on the phenomenon $H_{p,\phi\theta} = (F_{p,\phi\theta}, B_{F_{p,\phi\theta}})$ of projectile motion (see Example 6), and $M_1 = X, M_2 = Y, M_3 = \dot{X}, M_4 = \dot{Y}$ be the four magnitudes of $H_{p,\phi\theta}$ specified by $I_{DS_{L_p},H_{p,\phi\theta}}$.
By Definition 39, the empirical content of $I_{DS_{L_p},H_{p,\phi\theta}}$ is as follows:
$Exp_{H_{p,\phi\theta}}(I_{DS_{L_p},H_{p,\phi\theta}}) = \{exp \in Exp_{H_{p,\phi\theta}} : \exists k, l \in \mathbb{Z}^{\geq 1}, \exists m \in \mathbb{Z}^{[1,4]},$
$\exists i(i : \mathbb{Z}^{[1,m]} \to \mathbb{Z}^{[1,4]} \wedge \forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))), \exists b \in B_F,$
$exp = (\sigma_k[M_1, M_2, M_3, M_4](b), \mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, \sigma_k^1[M_1, M_2, M_3, M_4](b)))\}.$
We notice that $Exp_{H_{p,\phi\theta}}(I_{DS_{L_p},H_{p,\phi\theta}}) \neq \emptyset$, because there are actual experiments that consist in (i) starting a motion of a projectile while setting its initial velocity and position, and (ii) measuring a duration of the projectile motion, as well as its corresponding instantaneous position and/or velocity.

According to the following definition, an interpretation is called *empirical* whenever its empirical content is not empty.

Definition 40 (Empirical interpretation of DS_L on H)
Let $I_{DS_L,H}$ be an interpretation of n-component partial dynamical system DS_L on dynamical phenomenon H.
$I_{DS_L,H}$ *is an empirical interpretation of* DS_L *on* $H := Exp_H(I_{DS_L,H}) \neq \emptyset.$

Example 11 ($I_{DS_{L_p},H_{p,\phi\theta}}$ is an empirical interpretation of DS_{L_p} on the phenomenon of projectile motion $H_{p,\phi\theta}$)
We argued that the empirical content $Exp_{H_{p,\phi\theta}}(I_{DS_{L_p},H_{p,\phi\theta}})$ is not empty (see Example 10). Therefore, by the definition of empirical interpretation (Definition 40), $I_{DS_{L_p},H_{p,\phi\theta}}$ is an empirical interpretation of DS_{L_p} on the phenomenon of projectile motion $H_{p,\phi\theta}$.

Given $n \geq 1$ arbitrary sets $X_1, \ldots, X_n$, a choice of $m \leq n$ of these sets $X_{i(1)}, \ldots, X_{i(m)}$, and a set of n-tuples $X \subseteq X_1 \times \ldots \times X_n$, it will be convenient to consider the subset of $X_{i(1)} \times \ldots \times X_{i(m)}$ whose elements y have, in each place $j \in \mathbb{Z}^{[1,m]}$, the $i(j)$-th element $x_{i(j)}$ of some $x \in X$. According to the following notational convention, we call this subset *the* $[i(1), \ldots, i(m)]$-*projection of X*.

Notation 4 (The $[i(1), \ldots, i(m)]$-projection of a set of n-tuples)

Let $m, n \in \mathbb{Z}^{\geq 1}$, $m \leq n$, $i : \mathbb{Z}^{[1,m]} \to \mathbb{Z}^{[1,n]}$, and $\forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))$. For any $r \in \mathbb{Z}^{[1,n]}$, let X_r be an arbitrary set, and $X \subseteq X_1 \times \ldots \times X_n$.
$proj^{[i(1),\ldots,i(m)]}(X) := \{y \in X_{i(1)} \times \ldots \times X_{i(m)} : \text{for some } (x_1, \ldots, x_n) \in X, y = (x_{i(1)}, \ldots, x_{i(m)})\}$.
$proj^{[i(1),\ldots,i(m)]}(X)$ is called *the* $[i(1), \ldots, i(m)]$-*projection of X*.

Given an empirical interpretation $I_{DS_L,H}$ and an experiment $exp \in Exp_H(I_{DS_L,H})$, we ask under what conditions the experiment is consistent with the results that are expected according to the theoretical model DS_L. The experiment $exp = (\sigma_k[M_1, \ldots, M_n](b), \mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, e))$ consists in (i) starting a motion $e \in E_F(b)$ of a realizer $b \in B_F$ while (ii) approximately setting the initial values of the n magnitudes $M_1, \ldots, M_n$ of H specified by $I_{DS_L,H}$, and then approximately measuring, w.r.t. a choice of $m \leq n$ of the n magnitudes $M_{i(1)}, \ldots, M_{i(m)}$, (iii) a duration of the motion e of b and, (iv) the value of each magnitude $M_{i(j)}$ at the instant of e that corresponds to such a duration. Thus, in accordance with the theoretical model $DS_L = (M, (g^t)_{t \in T})$, we expect that, for some t that belongs to the approximate result $\mu_l^1[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ of the duration measurement, if we apply g^t to the approximate result $\sigma_k^2[M_1, \ldots, M_n](b)$ of the setting procedure, then the $[i(1), \ldots, i(m)]$-projection of the resulting set should have a non-empty intersection with the approximate result $\mu_l^2[M_{i(1)}, \ldots, M_{i(m)}](b, e)$ of the measurements of the m chosen magnitudes. Whenever an experiment $exp \in Exp_H(I_{DS_L,H})$ satisfies the previous condition we say that the experiment is consistent with the interpretation $I_{DS_L,H}$. The exact definition is stated below.

Definition 41 (Consistency of an experiment $exp \in Exp_H(I_{DS_L,H})$ with empirical interpretation $I_{DS_L,H}$)

Let $I_{DS_L,H}$ be an empirical interpretation of n-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ on dynamical phenomenon $H = (F, B_F)$, and $M_1, \ldots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$. Let $exp \in Exp_H(I_{DS_L,H})$ and $exp = (\sigma_k[M_1, \ldots, M_n](b), \mu_l[M_{i(1)}, \ldots, M_{i(m)}](b, \sigma_k^1[M_1, \ldots, M_n](b)))$, where $b \in B_F$, $k, l \in \mathbb{Z}^{\geq 1}$, $m \in \mathbb{Z}^{[1,n]}$, $i : \mathbb{Z}^{[1,m]} \to \mathbb{Z}^{[1,n]}$, and $\forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))$.
exp is consistent with $I_{DS_L,H} := \exists t \in \mu_l^1[M_{i(1)}, \ldots, M_{i(m)}](b, \sigma_k^1[M_1, \ldots, M_n](b))$,
$(proj^{[i(1),\ldots,i(m)]}(g^t(\sigma_k^2[M_1, \ldots, M_n](b))) \bigcap \mu_l^2[M_{i(1)}, \ldots, M_{i(m)}](b, \sigma_k^1[M_1, \ldots, M_n](b))) \neq \emptyset$.

We are now able to prove an important result that, with respect to an arbitrary empirical interpretation, connects the notion of correctness of Sect. 4.1 (Definition 33) with the one of consistency just defined (Definition 41). More precisely, we prove below that, whenever an empirical interpretation is correct, it turns out to be consistent with any experiment in its empirical content.

Theorem 3 (Correctness of an empirical interpretation entails its consistency with all experiments in its empirical content)

Let $I_{DS_L,H}$ be an empirical interpretation of n-component partial dynamical system $DS_L = (M, (g^t)_{t \in T})$ on dynamical phenomenon $H = (F, B_F)$, and $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

For any $exp \in Exp_H(I_{DS_L,H})$, if $I_{DS_L,H}$ is a correct interpretation of DS_L on H, then exp is consistent with $I_{DS_L,H}$.

Proof See the Appendix. □

5.2 Galilean Models. Truth and Empirical Correctness for Empirical Models

Given a model $\boldsymbol{DS}_H = (DS_L, I_{DS_L,H})$ of a dynamical phenomenon H, we define the empirical content of $\boldsymbol{DS}_H$ as the empirical content of its interpretation $I_{DS_L,H}$.

Definition 42 (The empirical content $Exp_H(\boldsymbol{DS}_H)$ of model $\boldsymbol{DS}_H$)
Let $\boldsymbol{DS}_H = (DS_L, I_{DS_L,H})$ be a model of H.
$Exp_H(\boldsymbol{DS}_H) := Exp_H(I_{DS_L,H})$.
$Exp_H(\boldsymbol{DS}_H)$ is called *the empirical content of $\boldsymbol{DS}_H$.*

We can now define an empirical model of a dynamical phenomenon H as a model whose empirical content is not empty or, equivalently, whose interpretation is empirical. The definition of an empirical model of H can be thought as a formal explication of the concept of a Galilean model (Giunti 1995, sect. 18.4; Giunti 2009, sect. 2; Giunti 2014, sect. 4.1).

Definition 43 (Empirical, or Galilean, model of a dynamical phenomenon)
Let $\boldsymbol{DS}_H = (DS_L, I_{DS_L,H})$ be a model of H.
$\boldsymbol{DS}_H$ is an empirical model of $H := Exp_H(\boldsymbol{DS}_H) \neq \emptyset$.
Any empirical model of H is also called a *Galilean model of* H.

Example 12 (The projectile model $DS_{H_{p,\phi\theta}}$ is an empirical, or Galilean, model of the phenomenon of projectile motion $H_{p,\phi\theta}$)
Let us consider the projectile model $\boldsymbol{DS}_{H_{p,\phi\theta}} = (DS_{L_p}, I_{DS_{L_p},H_{p,\phi\theta}})$ (see Example 8). By Example 10, $Exp_{H_{p,\phi\theta}}(I_{DS_{L_p},H_{p,\phi\theta}}) \neq \emptyset$. Thus, by Definition 42 and Definition 43, $\boldsymbol{DS}_{H_{p,\phi\theta}}$ is an empirical, or Galilean, model of $H_{p,\phi\theta}$.

Given an empirical model $\boldsymbol{DS}_H = (DS_L, I_{DS_L,H})$, an experiment $exp \in Exp_H(\boldsymbol{DS}_H)$ is consistent with $\boldsymbol{DS}_H$ just in case it is consistent with the interpretation $I_{DS_L,H}$.

Definition 44 (Consistency of an experiment $exp \in Exp_H(DS_H)$ **with an empirical model** DS_H**)**
Let $DS_H = (DS_L, I_{DS_L,H})$ be an empirical model of H and $exp \in Exp_H(DS_H)$.
exp is consistent with DS_H := *exp* is consistent with $I_{DS_L,H}$.

The previous definitions allow us to reformulate Theorem 3 as stating a connection between the notion of truth of an empirical model and the one of consistency with the experiments in its empirical content. More precisely, it is shown below that, whenever an empirical model is true, it turns out to be consistent with any experiment in its empirical content.

Corollary 2 (Truth of an empirical model entails its consistency with all experiments in its empirical content)
Let $DS_H = (DS_L, I_{DS_L,H})$ *be an empirical model of H.*
For any $exp \in Exp_H(DS_H)$, *if* DS_H *is a true model of H, then exp is consistent with* DS_H.

Proof The thesis follows from Definitions 38, 42, 43, 44, and Theorem 3. □

We finally define an empirically correct model of a dynamical phenomenon as any empirical model that is consistent with all experiments of its empirical content. If it is not consistent with some of these experiments, the model is empirically incorrect.

Definition 45 (Empirically correct/incorrect model of a dynamical phenomenon)
Let $DS_H = (DS_L, I_{DS_L,H})$ be an empirical model of H.
DS_H *is an empirically correct model of H* :=
for any $exp \in Exp_H(DS_H)$, *exp* is consistent with DS_H.
DS_H *is an empirically incorrect model of H* :=
for some $exp \in Exp_H(DS_H)$, *exp* is not consistent with DS_H.

Example 13 (For large ϕ **and** θ, **the projectile model** $DS_{H_{p,\phi\theta}}$ **is an empirically incorrect model of the phenomenon of projectile motion** $H_{p,\phi\theta}$**)**
We know that the projectile model $DS_{H_{p,\phi\theta}} = (DS_{L_p}, I_{DS_{L_p},H_{p,\phi\theta}})$ is an empirical model of $H_{p,\phi\theta}$ (see Example 12). However, if the two parameters ϕ and θ are sufficiently large, we also know that there are experiments in its empirical content that are not consistent with it. These are the experiments in which the instantaneous position and/or velocity of a projectile is measured with a precision sufficient to detect the error due to the fact that the acceleration of gravity is not constant during the whole motion of the projectile, but it rather depends on its position. Therefore, for sufficiently large ϕ and θ, the projectile model is an empirically incorrect model of the phenomenon of projectile motion $H_{p,\phi\theta}$.

On the basis of Definition 45 we are finally able to rephrase Corollary 2 as stating a relation between the truth/falsehood of an empirical model and its empirical correctness/incorrectness.

Corollary 3 (Truth of an empirical model entails its empirical correctness; conversely, empirical incorrectness of an empirical model entails its falsehood)
Let $DS_H = (DS_L, I_{DS_L,H})$ be an empirical model of H.
If DS_H is a true model of H, then DS_H is an empirically correct model of H.
If DS_H is an empirically incorrect model of H, then DS_H is a false model of H.

Proof The first thesis follows from Definition 45 and Corollary 2. The second one follows from Definition 45, Corollary 2, and Definition 38. □

Example 14 (For large ϕ and θ, the projectile model $DS_{H_{p,\phi\theta}}$ is false)
Let $DS_{H_{p,\phi\theta}}$ be the projectile model, and $H_{p,\phi\theta}$ the phenomenon of projectile motion. By Example 13, for sufficiently large ϕ and θ, $DS_{H_{p,\phi\theta}}$ is an empirically incorrect model of $H_{p,\phi\theta}$. Therefore, by the second thesis of Corollary 3, for sufficiently large ϕ and θ, $DS_{H_{p,\phi\theta}}$ is a false model of $H_{p,\phi\theta}$.

6 Concluding Remarks: Further Developing MCR

This paper has focused on the semantic relations that an empirical theory may bear to the real world, and it has proposed Methodological Constructive Realism (MCR) as a new epistemological framework for dealing with this kind of problem. In Sect. 1.4, we have given a *general* and *informal* formulation of MCR, and we have then elaborated an *axiomatic* version of MCR (Sects. 2–5) for the *special* case of deterministic dynamical phenomena and their correlated deterministic dynamical models.

In agreement with the semantic view, MCR interprets the relevant aspects of the theory/world relationship as involving two semantic relations—truth and empirical correctness—which primarily apply to *models* on the one hand, and phenomena on the other one. However, both the general and the special versions of MCR worked out in this paper do not provide a definite view of the structure of empirical *theories*, and how the application domain of the two semantic relations of truth and empirical correctness can be extended to include full blown theories, and not just the models of which they are made.

These problems are the subjects of ongoing research, and they will be the topics of future dedicated papers. For the moment, we can just anticipate that MCR conceives of an empirical theory as a framework that (i) first of all specifies a particular class of phenomena—the *intended domain* of the theory—and then (ii) for each phenomenon in this class, specifies a class of models of that phenomenon—its *theoretical models*. The theoretical models of a phenomenon have the main function of constituting the theoretical search space out of which that particular model which is most likely to be empirically correct should be selected. The selected model should then undergo appropriate empirical testing, in order to actually assess its empirical correctness (Giunti and Pinna 2016, p. 567). According to this view, then, both empirical correctness and truth can be naturally extended to empirical theories. An empirical theory is empirically correct just in case, for any intended phenomenon, at least one of its theoretical models turns out to be empirically correct. From this definition, by substituting "true" for "empirically correct", we then obtain an analogous definition for truth.

To sum up, the line of development described above will thus consist of two main steps. First, the general version of MCR will be supplemented with the above sketched general

conception of the structure of an empirical theory and the relative notions of truth and empirical correctness. And, second, the special axiomatic version of MCR for deterministic dynamical models will be accordingly extended to the special case of *deterministic dynamical theories*.

Once these steps are accomplished, however, a second line of development should also be considered. So far, a formal version of MCR has been given exclusively for one special type of phenomenon, namely deterministic dynamical phenomena. But, quite obviously, this is not the only type of phenomenon that scientific research is involved with. To elaborate an adequate classification of the different types of phenomena is not a trivial matter, and we believe that this is the most basic goal of methodology, on which all its subsequent developments depend. We also believe that this goal can only be achieved by employing a piece-meal strategy (Giunti 1992 p. 139). That is to say, we need to individuate other specific types of phenomena, and then produce an axiomatic version of MCR for each specific type, analogous to the one we have formulated here for deterministic dynamical phenomena. To conclude, we can just advance a guess on some plausible candidates as further phenomenal types. The first, quite natural, candidate is the class of all dynamical phenomena that are not deterministic. We then obtain two more candidates if, in accordance with the traditional classification of scientific laws (van Fraassen 1970, p. 330; Suppe 1977, p. 226), we consider those non-dynamical phenomena that are described by *coexistence* laws, either of the deterministic or the indeterministic variety.

Compliance with ethical standards

Conflict of interest The corresponding author states that there is no conflict of interest.

Appendix: Proofs of Theorems and Selected Propositions

Theorem 1 (Being a partial dynamical system on a monoid is preserved by isomorphism)

Let $DS_{1_L} = (M, (g^t)_{t \in T})$ and $DS_{2_L} = (N, (h^t)_{t \in T})$ be partial deterministic systems on monoid $L = (T, +)$.

If u is an isomorphism of DS_{2_L} in DS_{1_L} and DS_{2_L} is a partial dynamical system on L, then DS_{1_L} is a partial dynamical system on L.

Proof

It is sufficient to prove that conditions 3a and 3b of Definition 2 hold. All other conditions are satisfied by DS_{1_L}. For, by the assumed hypotheses, DS_{1_L} is a partial deterministic system on L.

- **Thesis 3a:** $\forall x \in M \ (x \in Pim(g^0) \rightarrow g^0(x) = x)$.

  Let $x \in Pim(g^0)$.

  As u is an isomorphism of DS_{2_L} in DS_{1_L}, by Definition 5,

  $u(h^0(u^{-1}(x))) = g^0(u(u^{-1}(x))) = g^0(x)$.

  On the other hand, as DS_{2_L} is a partial dynamical system on L,

  $h^0(u^{-1}(x)) = u^{-1}(x)$.

  Therefore,

  $g^0(x) = u(h^0(u^{-1}(x))) = u(u^{-1}(x)) = x$.

- **Thesis 3b:** $\forall x \in M \ \forall t_2, t_1 \in T \ ((x \in Pim(g^{t_2+t_1}) \wedge x \in Pim(g^{t_1}) \wedge g^{t_1}(x) \in Pim(g^{t_2})) \rightarrow g^{t_2+t_1}(x) = g^{t_2}(g^{t_1}(x)))$.

  Let $x \in Pim(g^{t_2+t_1}) \wedge x \in Pim(g^{t_1}) \wedge g^{t_1}(x) \in Pim(g^{t_2})$.

  As u is an isomorphism of DS_{2_L} in DS_{1_L}, by Definition 5,

  $u(h^{t_1}(u^{-1}(x))) = g^{t_1}(u(u^{-1}(x))) = g^{t_1}(x)$ and $u(h^{t_2}(u^{-1}(g^{t_1}(x)))) = g^{t_2}(u(u^{-1}(g^{t_1}(x)))) = g^{t_2}(g^{t_1}(x))$.

  From this, by substituting $u(h^{t_1}(u^{-1}(x)))$ for $g^{t_1}(x)$, and because DS_{2_L} is a partial dynamical system on a monoid,

  $g^{t_2}(g^{t_1}(x)) = u(h^{t_2}(u^{-1}(g^{t_1}(x)))) = u(h^{t_2}(u^{-1}(u(h^{t_1}(u^{-1}(x)))))) = u(h^{t_2}(h^{t_1}(u^{-1}(x)))) = u(h^{t_2+t_1}(u^{-1}(x)))$.

  As u is an isomorphism of $DS_{2_{L_2}}$ in $DS_{1_{L_1}}$, by Definition 5,

  $u(h^{t_2+t_1}(u^{-1}(x))) = g^{t_2+t_1}(u(u^{-1}(x))) = g^{t_2+t_1}(x)$.

  Therefore,

  $g^{t_2+t_1}(x) = g^{t_2}(g^{t_1}(x))$.

  $\square$

Proposition 2

(1) $\forall x \in M, q(x) \neq \emptyset \leftrightarrow x \in C$;

(2) $\forall t \in T, Pim(g^t) = \{x \in M : x \in C \wedge t \in q(x)\}$.

Proof

- **Thesis (1)**

  Let $x \in M$. If $q(x) \neq \emptyset$, then, by the definition of $q(x)$ (Definition 7), $\exists t \in T$ such that $x \in Pim(g^t)$. Therefore, by the definition of C (Definition 6), $x \in C$. Conversely, if $x \in C$, by Definition 6, $\exists t \in T$ such that $x \in Pim(g^t)$. Hence, by Definition 7, $t \in q(x)$, so that $q(x) \neq \emptyset$.

- **Thesis (2)**

  Let $x \in M$. If $x \in Pim(g^t)$, then, by the definition of C (Definition 6), $x \in C$ and, by the definition of $q(x)$ (Definition 7), $t \in q(x)$. Thus, $Pim(g^t) \subseteq \{x \in M : x \in C \wedge t \in q(x)\}$. Conversely, if $t \in q(x)$, by the definition of $q(x)$, $x \in Pim(g^t)$. Thus, $\{x \in M : x \in C \wedge t \in q(x)\} \subseteq Pim(g^t)$. Therefore, $Pim(g^t) = \{x \in M : x \in C \wedge t \in q(x)\}$.

  $\square$

Proposition 3

(1) *If DS_{1_L} is a partial deterministic system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}, then DS_{2_L} is a partial deterministic system on L;*

(2) *If DS_{1_L} is a partial dynamical system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}, then DS_{2_L} is a partial dynamical system on L.*

Proof

- **Thesis (1)**

  Assume that DS_{1_L} is a partial deterministic system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}. Then, by the definition of partial subsystem (Definition 8) and the definition of partial deterministic system on a monoid (Definition 4), DS_{2_L} is a partial deterministic system on L.

- **Thesis (2)**

  Assume that DS_{1_L} is a partial dynamical system on monoid L and DS_{2_L} is a partial subsystem of DS_{1_L}. Then, by the definition of partial subsystem (Definition 8) and the definition of partial dynamical system on a monoid (Definition 2), DS_{2_L} is a partial dynamical system on L.

  $\square$

Proposition 4

(1) *If DS_L is a deterministic system on L, then $C = M$;*

(2) *DS_L is a deterministic system on $L \leftrightarrow \forall x \in M, q(x) = T$.*

Proof

- **Thesis (1)**

  If DS_L is a deterministic system on L, then $\forall t \in T \ \forall x \in M$, g^t is defined on x. Therefore, by Definition 6, $C = M$.

- **Thesis (2)**

  If DS_L is a deterministic system on L, by the definition of deterministic system (Definition 11), $\forall t \in T$, $\forall x \in M$, $x \in Pim(g^t)$. Therefore, by the definition of $q(x)$ (Definition 7), $\forall x \in M$, $q(x) = T$. Conversely, if $\forall x \in M$, $q(x) = T$, then, by Definition 7, $\forall t \in T$, $\forall x \in M$, $x \in Pim(g^t)$. Therefore, by Definition 11, DS_L is a deterministic system on L.

  $\square$

Proposition 6

$S_{DS_L}[I_{DS_L,H}]$ *is a partial subsystem of DS_L.*

Proof

By the definition of $S_{DS_L}[I_{DS_L,H}]$ (Definition 34),
$I_{DS_L,H}$ is an admissible interpretation of DS_L on H;
thus, by the definition of admissible interpretation (Definition 32),
$C_H[M_1, \ldots, M_n] \subseteq C$ and, $\forall x \in C_H[M_1, \ldots, M_n], q_H[M_1, \ldots, M_n](x) \subseteq q(x)$;
hence, by Definition 34 and Thesis (ii) of Proposition 2, $\forall t \in T$,
$Pim(g[M_1, \ldots, M_n]^t) \subseteq Pim(g^t)$ and $\forall x \in Pim(g[M_1, \ldots, M_n]^t), g[M_1, \ldots, M_n]^t(x) = g^t(x)$.
Therefore, by the definition of partial subsystem (Definition 8),
$S_{DS_L}[I_{DS_L,H}]$ is a partial subsystem of DS_L. $\square$

Theorem 2 (Correct interpretation as interpretation induced structure identity)

Let $DS_L = (M, (g^t)_{t \in T})$ be a n-component partial dynamical system on a monoid L, H be a dynamical phenomenon, $I_{DS_L,H}$ be an admissible interpretation of DS_L on H, and $M_1, \ldots, M_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

Let $S_{DS_L}[I_{DS_L,H}] = (M_H[M_1, \ldots, M_n], (g[M_1, \ldots, M_n]^t)_{t \in T})$ be the partial sub-system of DS_L induced by $I_{DS_L,H}$, and $DS_H[I_{DS_L,H}] = DS_H[M_1, \ldots, M_n] = (M_H[M_1, \ldots, M_n], (g_H[M_1, \ldots, M_n]^t)_{t \in V(T_H)})$ be the partial deterministic system of H induced by $I_{DS_L,H}$.

$I_{DS_L,H}$ is a correct interpretation of DS_L on H iff $S_{DS_L}[I_{DS_L,H}] = DS_H[I_{DS_L,H}]$.

Proof

We prove first (i) the left-right implication, and then (ii) the right-left one.

- **Thesis (1):** If $I_{DS_L,H}$ is a correct interpretation of DS_L on H, then $S_{DS_L}[I_{DS_L,H}] = DS_H[M_1, \ldots, M_n]$.

  Assume that $I_{DS_L,H}$ is a correct interpretation of DS_L on H.

  Then, in the first place, as $I_{DS_L,H}$ is an interpretation of DS_L on H, by Definition 31,

  $L = L_H$.

  In the second place, by the definition of $S_{DS_L}[I_{DS_L,H}]$ (Definition 34), and by the definition of $DS_H[M_1, \ldots, M_n]$ (Definition 29),

  $M_H[M_1, \ldots, M_n]$ is the state space of both $S_{DS_L}[I_{DS_L,H}]$ and $DS_H[M_1, \ldots, M_n]$.

  Furthermore, by the same definitions, $\forall t \in T$,

  $g[M_1, \ldots, M_n]^t : M_H[M_1, \ldots, M_n] \to M_H[M_1, \ldots, M_n]$ is a partial function,

  $g_H[M_1, \ldots, M_n]^t : M_H[M_1, \ldots, M_n] \to M_H[M_1, \ldots, M_n]$ is a partial function,

  and $Pim(g[M_1, \ldots, M_n]^t) = Pim(g_H[M_1, \ldots, M_n]^t) = $

  $\{x \in M_H[M_1, \ldots, M_n] : x \in C_H[M_1, \ldots, M_n] \wedge t \in q_H[M_1, \ldots, M_n](x)\}$.

  Let $t \in T$, $x \in Pim(g[M_1, \ldots, M_n]^t)$, $b \in B_F[M_1, \ldots, M_n](x)$, and

  $e \in E_F[M_1, \ldots, M_n](b, x)$.

  Then, in the first place, since $x \in Pim(g[M_1, \ldots, M_n]^t)$,

  $t \in q_H[M_1, \ldots, M_n](x)$.

  Thus, by Definition 34, by the definition of correct interpretation (Definition 33), and by Definition 29,

  $g[M_1, \ldots, M_n]^t(x) = g^t(x) = st_H[M_1, \ldots, M_n](b, e, t + \iota_0^H(e)) = g_H[M_1, \ldots, M_n]^t(x)$.

  Therefore,

  $S_{DS_L}[I_{DS_L,H}] = DS_H[M_1, \ldots, M_n]$.

- **Thesis (2):** If $S_{DS_L}[I_{DS_L,H}] = DS_H[M_1, \ldots, M_n]$, then $I_{DS_L,H}$ is a correct interpretation of DS_L on H.

  Assume that $S_{DS_L}[I_{DS_L,H}] = DS_H[M_1, \ldots, M_n]$.

  By the hypotheses of the theorem,

  $I_{DS_L,H}$ is an admissible interpretation of DS_L on H.

  Let $x \in C_H[M_1, \ldots, M_n]$, $t \in q_H[M_1, \ldots, M_n](x)$, $b \in B_F[M_1, \ldots, M_n](x)$, and $e \in E_F[M_1, \ldots, M_n](b, x)$.

  Thus, by the definition of $S_{DS_L}[I_{DS_L,H}]$ (Definition 34), the assumption above, and the definition of $DS_H[M_1, \ldots, M_n]$ (Definition 29),

  $g^t(x) = g[M_1, \ldots, M_n]^t(x) = g_H[M_1, \ldots, M_n]^t(x) = st_H[M_1, \ldots, M_n](b, e, t + \iota_0^H(e))$.

  Therefore, by the definition of correct interpretation (Definition 33),

  $I_{DS_L,H}$ is a correct interpretation of DS_L on H.

$\square$

Theorem 3 (Correctness of an empirical interpretation entails its consistency with all experiments in its empirical content)

Let $I_{DS_L,H}$ be an empirical interpretation of n-component partial dynamical system $DS_L = (M, (g^t)_{t\in T})$ on dynamical phenomenon $H = (F, B_F)$, and $\boldsymbol{M}_1, \dots, \boldsymbol{M}_n$ be the n magnitudes of H specified by $I_{DS_L,H}$.

For any $exp \in Exp_H(I_{DS_L,H})$, if $I_{DS_L,H}$ is a correct interpretation of DS_L on H, then exp is consistent with $I_{DS_L,H}$.

Proof

Since $I_{DS_L,H}$ is an empirical interpretation of DS_L on H, by Definition 40, $Exp_H(I_{DS_L,H}) \neq \emptyset$. Thus, suppose

1. $exp \in Exp_H(I_{DS_L,H})$.
   Suppose
2. $I_{DS_L,H}$ is a correct interpretation of DS_L on H.
   As $I_{DS_L,H}$ is an interpretation of DS_L on H, by Definition 31,
3. $L = L_H$.
   By assumption 2 and the definition of correct interpretation (Definition 33),
4. $I_{DS_L,H}$ is an admissible interpretation of DS_L on H and
   $\forall x \in C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n], \forall t \in q_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x),$
   $\forall b \in B_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x), \forall e \in E_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, x),$
   $g^t(x) = st_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, e, t + \iota_0^H(e)).$
   By 1 and the definition of empirical content (Definition 39),
5. $exp \in Exp_H$ and
   $\exists k, l \in \mathbb{Z}^{\geq 1}, \exists m \in \mathbb{Z}^{[1,n]}, \exists i (i : Z^{[1,m]} \to Z^{[1,n]} \wedge \forall j_1, j_2 \in \mathbb{Z}^{[1,m]}(j_1 < j_2 \to i(j_1) < i(j_2))), \exists b \in B_F,$
   $exp = (\sigma_k[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b), \mu_l[\boldsymbol{M}_{i(1)}, \dots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b)))\}.$
   By 5, the Exp_H axiom (Axiom 10), and the definition of possible experiment (Definition 30),
6. $b \in Pim(\sigma_k[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b)).$
   By 6 and the k-th setting axiom (Axiom 8),
7. $\sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b) \in E_F(b),$
   $st_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b), \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b)))$ is defined, and
   $st_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b), \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b))) \in \sigma_k^2[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b).$
   Let
8. $x_0 = st_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b), \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b))).$
   By 8, the first and second conjunct of 7, 5, and the definition of $C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n]$ (Definition 23),
9. $x_0 \in C_H[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n].$
   By 9, 8, the first and second conjunct of 7, 5, and the definition of $B_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x_0)$ (Definition 27),
10. $b \in B_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](x_0).$
    By 9, 8, the first and second conjunct of 7, 5, and the definition of $E_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, x_0)$ (Definition 28),
11. $\sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b) \in E_F[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b, x_0).$
    By 5, the Exp_H axiom (Axiom 10), and the definition of possible experiment (Definition 30),
12. $(b, \sigma_k^1[\boldsymbol{M}_1, \dots, \boldsymbol{M}_n](b)) \in Pim(\mu_l[\boldsymbol{M}_{i(1)}, \dots, \boldsymbol{M}_{i(m)}]).$
    By 3, 12, and the l-th measurement axiom (Axiom 9),

13. $\exists t \in \mu_l^1[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$ such that
    $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))$ is defined and
    $st_H[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))) \in$
    $\mu_l^2[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$.
    By the first conjunct of 13, 5, the first and second conjunct of 7, 8, 9, and the definition
    of $\bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x_0, b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$ (Definition 24),

14. $t \in \bar{q}_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x_0, b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$.
    By the first conjunct of 4 and the definition of admissible interpretation (Definition 32),

15. H is a deterministic dynamical phenomenon w.r.t. $\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n$.
    By 15, 14, and the definition of $q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x_0)$ (Definition 26),

16. $t \in q_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](x_0)$.
    By the second conjunct of 4, 9, 16, 10, and 11,

17. $g^t(x_0) = st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))))$.
    By 8 and the third conjunct of 7,

18. $g^t(x_0) \in g^t(\sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$.
    By 18 and 17,

19. $st_H[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))) \in$
    $g^t(\sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$.
    By 19 and the notational convention for $proj^{[i(1),\ldots,i(m)]}(g^t(\sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))$ (Notation 4),

20. $st_H[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))) \in$
    $proj^{[i(1),\ldots,i(m)]}(g^t(\sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))$.
    By the third conjunct of 13,

21. $st_H[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b), t + \iota_0^H(\sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))) \in$
    $\mu_l^2[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))$.
    By 20 and 21,

22. $(proj^{[i(1),\ldots,i(m)]}(g^t(\sigma_k^2[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b)))) \bigcap$
    $\mu_l^2[\boldsymbol{M}_{i(1)}, \ldots, \boldsymbol{M}_{i(m)}](b, \sigma_k^1[\boldsymbol{M}_1, \ldots, \boldsymbol{M}_n](b))) \neq \emptyset$.
    By 22 and the definition of consistency of an experiment with an empirical interpretation (Definition 41),

23. exp is consistent with $I_{DS_L,H}$. $\qquad\qquad\square$

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