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Dirichlet Problems for several nonlocal operators via variational and topological methods

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PhD Student: Silvia Frassu

Supervisor: Professor Antonio Iannizzotto

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Introduction

The main topic of the thesis is the study of elliptic differential equations with fractional order driven by nonlocal operators, which can be expressed in a compact way by (see [56])

$$\mathcal{L}^{K,\phi}u(x) = \int_{\mathbb{R}^N} \phi(u(x)-u(y))K(x-y)\,dy,$$

where $\phi:\mathbb{R}\to\mathbb{R}$ is a nondecreasing, continuous, unbounded odd function, and $K:\mathbb{R}^N\to\mathbb{R}^+$ is a measurable function satisfying

$$\begin{cases} K(z)>0, \quad K(z)=K(-z), \quad K\notin L^1(B_1), \\ \int_{\mathbb{R}^N}\min\{1,|z|^{q_0}\}K(z)\,dz<\infty, \text{ for some } q_0>0. \end{cases}$$

The power case $\phi(t)=c|t|^{p-2}t$ for some $p>1$, $K(z)=|z|^{-(N+ps)}$ for some $s\in(0,1)$, reduces to the fractional p-Laplacian $(-\Delta)_p^s$ for $c=2$ and to the fractional Laplacian $(-\Delta)^s$ for $p=2$, $c=C(N,s)$. Moreover, by taking $\phi(t)=t$ and K satisfying suitable assumptions, we obtain the general nonlocal operator L_K. Finally, for $\phi(t)=t$ and $K(z)=a\Big(\dfrac{z}{|z|}\Big)\dfrac{1}{|z|^{N+2s}}$ with a suitable function a, we get the nonlocal anisotropic L_K (see [36, 41, 72, 145, 173]).

Now, let us see the origins and the reasons of the importance of such operators. Recently, great attention has been focused on the study of fractional and nonlocal operators of elliptic type, both for pure mathematical research and in view of concrete real-world applications. This type of operators arises in a quite natural way in many different contexts, such as, among others, the thin obstacle problem, game theory, image processing, optimization, phase transition, stratified materials, anomalous diffusion, crystal dislocation, soft thin films, semipermeable membranes, flame propagation, conservation laws, ultrarelativistic limits of quantum mechanics, quasi-geostrophic flows, multiple scattering, minimal surfaces, materials science, water waves, chemical reactions of liquids, population dynamics and geophysical fluid dynamics. In all these cases, the nonlocal effect is modeled by the singularity at infinity.

The main reason is that nonlocal operators are the infinitesimal generators of Lévy-type stochastic processes. A Lévy process is a stochastic process with independent and stationary increments, it represents the random motion of a particle whose successive displacements are independent and statistically identical over different time intervals of the same length. These processes extend the concept of Brownian motion, where the infinitesimal generator is the Laplace operator, and may contain jump discontinuities.

By the Lévy-Khintchine Formula, the infinitesimal generator of any Lévy processes is an operator of the form

$$-Lu(x) = \sum_{i,j} a_{ij}\partial_{ij}u + \sum_{j} b_j \partial_j u + \int_{\mathbb{R}^N} \{u(x+y) - u(x) - y \cdot \nabla u(x)\chi_{B_1}(y)\}\, d\nu(y),$$

where ν is the Lévy measure, and satisfies $\int_{\mathbb{R}^N} \min\{1, |y|^2\}\, d\nu(y) < \infty$. When the process has no diffusion or drift part, this operator takes the form

$$-Lu(x) = \int_{\mathbb{R}^N} \{u(x+y) - u(x) - y \cdot \nabla u(x)\chi_{B_1}(y)\}\, d\nu(y).$$

Furthermore, if one assumes the process to be symmetric, and the Lévy measure to be absolutely continuous, then L can be written as

$$Lu(x) = \text{P.V.} \int_{\mathbb{R}^N} \{u(x) - u(x+y)\} d\nu(y),$$

where P.V. denotes that the integral has to be understood in the principal value sense and $d\nu(y) = K(y)\, dy$ with $K(y) = K(-y)$.

In the context of integro-differential equations, Lévy processes play the same role that Brownian motion plays in the theory of second order equations. Notice that an important difference and difficulty when studying integro-differential equations is that the "boundary data" is not given on the boundary, as in the classical case, but in the complement $\mathbb{R}^N \setminus \Omega$, where $\Omega \subset \mathbb{R}^N$ is a bounded domain. This exhibits the fact that paths of the associated processes fail to be continuous.

As we have seen, a simple example of such operator is the linear operator L_K, defined for any sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}^N$ by

$$L_K u(x) = \text{P.V.} \int_{\mathbb{R}^N} (u(x) - u(y)) K(x-y)\, dy, \tag{0.0.1}$$

where the kernel K satisfies necessary assumptions, as we will see later. When the singularity of the kernel is not integrable, these operators are also called *integro-differential* operators. This is because, due to the singularity of K, the operator (0.0.1) differentiates (in some sense) the function u.

Throughout the thesis, we will also deal with the anisotropic operator L_K, which is obtained from (0.0.1) by taking

$$K(y) = a\Big(\frac{y}{|y|}\Big) \frac{1}{|y|^{N+2s}} \quad s \in (0,1),$$

here, a is any non-negative function (or, more generally, any finite misure) defined on S^{N-1}. Such operator is the infinitesimal generator of the so-called stable Lévy processes, which satisfy self-similarity properties. Note that the structural condition on the kernel K is equivalent to saying that the Lévy measure is homogeneous. This is also equivalent to the fact that the operator L_K is scale invariant. In the particular case $a \equiv 1$ we obtain the fractional Laplacian $(-\Delta)^s$, which is the most canonical example of nonlocal elliptic integro-differential operator

$$(-\Delta)^s u(x) = C(N,s)\ \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy, \quad s \in (0,1).$$

Note that $(-\Delta)^s$ is the only (up to a multiplicative constant) infinitesimal generator of the radially symmetric and stable Lévy process of order $2s$. The Fourier symbol of this operator is $|\xi|^{2s}$ and, thus, one has that $(-\Delta)^t \circ (-\Delta)^s = (-\Delta)^{s+t}$, this is why we adopt the power-like notation $(-\Delta)^s$.

Roughly speaking, we note that the relation between the fractional p-Laplacian and the fractional Laplacian is equivalent to that of the local counterpart between the p-Laplacian and the Laplacian; we can say that $(-\Delta)^s_p : (-\Delta)^s \ = -\Delta_p : -\Delta$.

From the physical point of view, equations, driven by these operators, take into account long-range particle interactions with a power-law decay. When the decay at infinity is sufficiently weak, the long-range phenomena may prevail and the nonlocal effects persist even on large scale. There are many situations in which a nonlocal equation gives a significantly better model than a PDE, as we see now in several applications (see [172]).

In mathematical finance it is particularly important to study models involving jump processes, since the prices of assets are frequently modeled following a Lévy process. Observe that jump processes are very natural in this situation, since asset prices can have sudden changes. These models have become increasingly popular for modeling market fluctuations since the work of Merton [142] in 1976, both for risk management and option pricing purposes. For example, the obstacle problem for the fractional Laplacian can be used to model the pricing of American options [134, 166].

Integro-differential equations appear also in ecology. Indeed, optimal search theory predicts that predators should adopt search strategies based on long jumps where prey is sparse and distributed unpredictably, Brownian motion being more efficient only for locating abundant prey (see [109, 170, 192]).

Roughly speaking, it is not unreasonable that a predator may decide to use a nonlocal dispersive strategy to hunt its prey more efficiently (or, equivalently, that the natural selection may favor some kind of nonlocal diffusion): small fishes will not wait to be eaten by a big fish once they have seen it, so it may be more convenient for the big fish just to pick up a random direction, move rapidly in that direction, stop quickly and eat the small fishes there (if any) and then go on with the hunt. And this "hit and run" hunting procedure seems quite related to that described in Figure 1.

This kind of optimization problems also arises in mathematical biology. We consider the dynamics of a population modeled by the following reaction-diffusion equation

$$v_t + d(-\Delta)^s v = \rho(x)v - v^2 \quad \text{in } \Omega \times (0, \infty) \tag{0.0.2}$$

with Dirichlet boundary condition and a non-negative nontrivial function on Ω as initial data. In (0.0.2) Ω represents the environment of the population, $v(x,t)$ is the population density in position x at the time t, d is a constant describing the diffusion rate of the population and $\rho(x)$ represents the local growth rate. The local growth rate $\rho(x)$ is positive on favourable habitats and negative on unfavourable ones. When $s = 1$ the diffusion operator is the usual Laplacian operator and, from the biological point of view, it means that the dispersal of the population is modeled by a random walk of Brownian type. While, if $s \in (0,1)$, we have the fractional Laplacian operator which is associated to a random walk of Levy flight type. The Dirichlet boundary condition biologically corresponds to the assumption that the environment Ω is surrounded by an uninhabitable region. Finally, the initial data on Ω represents the population at the initial time $t = 0$. The behaviour of $v(x,t)$ as t goes to ∞ gives information about the survival of the population. It is known (see [19, 20, 44, 162]) that the model (0.0.2) predicts persistence of the population as $t \to \infty$ if $\lambda_1(\rho) < 1/d$, where $\lambda_1(\rho)$ denotes the first eigenvalue of fractional weighted eigenvalue problem (4.0.1). Therefore, finding the best location of favourable and unfavourable habitats within Ω in order to achieve the survival of the population for a given class of local growth rates ρ, is mathematically equivalent to minimize $\lambda_1(\rho)$ in that class (see Chapter 4). For more details and biological discussion of this model we refer the reader to [44, 162].

In fluid mechanics, many equations are nonlocal in nature. A clear example is the surface quasi-geostrophic equation, which is used in oceanography to model the temperature on the surface [55]. Another important example is the Benjamin-Ono equation

$$(-\Delta)^{\frac{1}{2}}u = -u + u^2,$$

which describes one-dimensional internal waves in deep water [6, 86]. Also, the half-Laplacian $(-\Delta)^{\frac{1}{2}}$ plays a very important role in the understanding of the gravity water waves equations in dimensions 2 and 3 (see [97]).

In elasticity, there are also many models that involve nonlocal equations. An important example is the Peierls-Nabarro equation, arising in crystal dislocation models [73, 138, 190]. Also, other nonlocal models are used to take into account that in many materials the stress at a point depends on the strains in a region near that point [79, 127].

In quantum physics, the fractional Schrödinger equation arises when the Brownian quantum paths are replaced by the Lévy ones in the Feynman path integral [131, 132]. Similar nonlocal dispersive equations describe the dynamics and gravitational collapse of relativistic boson stars (see [78, 104, 135]).

For more details and applications see [10, 36, 41, 173, 191] and the references therein.

Mathematically speaking, a motivation for studying integro-differential equations is trying to extend some important results which are well known for the classical case of the

Laplacian to a nonlocal setting. Indeed, some partial differential equations are a limit case (as $s \to 1$) of integro-differential equations.

In the next lines we will present two interpretations of evolutive and stationary nonlocal equations, following [36].

Nonlocal evolutive equation

In order to explain our choice, we observe that the *nonlocal evolutive equation*

$$u_t(x,t) + L_K u(x,t) = 0$$

naturally arises from a probabilistic process in which a particle moves randomly in the space subject to a probability that allows long jumps with a polynomial tail [36].

We consider a particle that moves in $\mathbb{R}^N$ according to a probabilistic process, that will be discrete both in time and space (in the end, we will formally take the limit when these time and space steps are small). We denote by τ the discrete time step, and by h the discrete space step. We will take the scaling $\tau = h^{2s}$ and we denote by $u(x,t)$ the probability of finding the particle at the point x at time t. At each time step τ, the particle selects randomly both a direction $v \in S^{N-1}$ with probability density a on S^{N-1}, and a natural number $k \in \mathbb{N}$, according to the probability law p ($p(k) = \frac{c_s}{|k|^{1+2s}}$ where c_s is a normalization constant), and it moves by a discrete space step khv. We point out that long jumps are allowed with small probability. If the particle is at time t at the point x_0 and, following the probability law, it picks up a direction $v \in S^{N-1}$ and a natural number $k \in \mathbb{N}$, then the particle at time $t + \tau$ will lie at $x_0 + khv$. Now, the probability $u(x, t + \tau)$ of finding the particle at x at time $t + \tau$ is the sum of the probabilities of finding the particle at $x + khv$ for some direction $v \in S^{N-1}$ and some natural number $k \in \mathbb{N}$, times the probability of having selected such a direction and such a natural number.

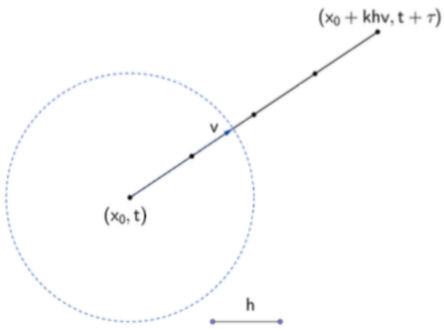

Figure 1: The random walk with jumps

This translates into

$$u(x,t+\tau) = c_s \sum_{k\in\mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv,t)}{|k|^{1+2s}}\, da(v),$$

where a is an absolutely continuous measure on S^{N-1}, namely $da(v) = a(v)dS^{N-1}$.

By subtracting $u(x,t)$, we obtain

$$u(x,t+\tau) - u(x,t) = c_s \sum_{k\in\mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv,t) - u(x,t)}{|k|^{1+2s}} a(v)\, dS^{N-1}$$

By symmetry, we can change v to $-v$ in the integral above, so we get

$$u(x,t+\tau) - u(x,t) = c_s \sum_{k\in\mathbb{N}} \int_{S^{N-1}} \frac{u(x-khv,t) - u(x,t)}{|k|^{1+2s}} a(v)\, dS^{N-1}.$$

Hence, we can sum up these two expressions (and divide by 2) and obtain that

$$u(x,t+\tau) - u(x,t) = \frac{c_s}{2} \sum_{k\in\mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv,t) + u(x-khv,t) - 2u(x,t)}{|k|^{1+2s}} a(v)\, dS^{N-1}.$$

Now we divide by $\tau = h^{2s}$, we recognize a Riemann sum, we take a formal limit and we use polar coordinates, thus we have

$$\begin{aligned} u_t(x,t) &\simeq \frac{u(x,t+\tau) - u(x,t)}{\tau} \\ &= \frac{c_s}{2} \int_{\mathbb{R}^N} \frac{u(x+y,t) + u(x-y,t) - 2u(x,t)}{|y|^{N+2s}} a\left(\frac{y}{|y|}\right)\, dy \\ &= -c_s L_K u(x,t). \end{aligned}$$

Therefore we obtain the *nonlocal evolutive equation*

$$u_t(x,t) + L_K u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,+\infty).$$

We point out that when $a \equiv 1$, $da(v) = dS^{N-1}$, the particle selects randomly a direction $v \in S^{N-1}$ according to the uniform distribution on S^{N-1}, and we obtain the *fractional heat equation*

$$u_t(x,t) + (-\Delta)^s u(x,t) = 0 \quad \text{in} \quad \mathbb{R}^N \times (0,+\infty),$$

(see [36]).

A payoff model

Another probabilistic motivation for the operator L_K arises from a *pay-off* approach [36,173]. Suppose to move in a domain Ω according to a random walk with jumps as in the previous case and assume also that, exiting the domain Ω for the first time by jumping to an outside point $y \in \mathbb{R}^N \setminus \Omega$, means earning $u_0(y)$ (see Figure 2).

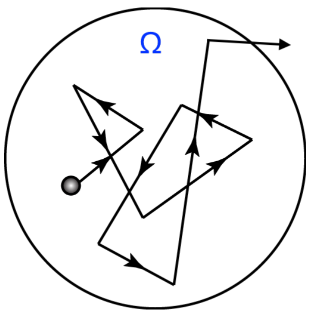

Figure 2: A walk with jumps

If we start at a given point $x \in \Omega$ and we denote by $u(x)$ the amount that we expect to gain, is there a way to obtain information on u ?

The expected payoff u is determined by the equation

$$\begin{cases} L_K u = 0 & \text{in } \Omega \\ u = u_0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{0.0.3}$$

Fix a point $x \in \Omega$, the expected value of the payoff at x is the average of all the payoffs at the points $\tilde{x}$ from which one can reach x, weighted by the probability of the jumps. By setting $\tilde{x} = x + khv$ and by keeping the same notations of the previous case, we have that the probability of jump is $\frac{c_s}{|k|^{1+2s}}$. This leads to the formula

$$u(x) = c_s \sum_{k \in \mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv)}{|k|^{1+2s}} a(v)\, dS^{N-1}.$$

By changing v to $-v$ in the expression above and by summing up, we obtain

$$2u(x) = c_s \sum_{k \in \mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv) + u(x-khv)}{|k|^{1+2s}} a(v)\, dS^{N-1}.$$

Since the total probability is 1, we can subtract $2u(x)$ to both sides and have that

$$0 = c_s \sum_{k \in \mathbb{N}} \int_{S^{N-1}} \frac{u(x+khv) + u(x-khv) - 2u(x)}{|k|^{1+2s}} a(v)\, dS^{N-1}.$$

We can now divide by h^{1+2s} and recognize a Riemann sum, which, after passing to the limit as $h \to 0$, gives $0 = L_K u(x)$, that is (0.0.3).

If we take $v = u - u_0$ in problem (0.0.3) then v solves the following homogeneous Dirichlet problem

$$\begin{cases} L_K v = -L_K u_0 & \text{in } \Omega \\ v = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

for a sufficiently smooth function u_0.

This thesis is devoted to the study of nonlinear Dirichlet problems driven by a linear nonlocal operator L_K of the type

$$\begin{cases} L_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial\Omega$, $N > 2s$ and $s \in (0,1)$. Such problems can be seen as a stationary counterpart of the nonlocal evolutive equations, involving a nonlinear reaction. So they are suitable to describe steady states or equilibria of nonlocal evolution phenomena. Also as we have seen, they arise naturally from the above described payoff model. In both frameworks the nonlinear reaction $f(x,u)$ may describe how the environment and properties of the particle affect the stochastic process. Our aim is to show existence and multiplicity results for such nonlinear elliptic Dirichlet problems by applying variational and topological methods. Such methods usually exploit the special form of the nonlinearities f entering the problem, for instance its symmetries, and offer complementary information. They are powerful tools to show the existence of multiple solutions and establish qualitative results on these solutions, for instance information regarding their location. The topological and variational approach provides not just existence of a solution, usually several solutions, but allow to achieve relevant knowledge about the behavior and properties of the solutions, which is extremely useful because generally the problems cannot be effectively solved, so the precise expression of the solutions is unknown. As a specific example of property of a solution that we look for is the sign of the solution, for example to be able to determine whether it is positive, or negative, or nodal (i.e., sign changing).

The study of problem above via variational methods started from the work of Servadei and Valdinoci [183, 185]. They have proved an existence result for equations driven by integro-differential operator L_K, with a general kernel K, satisfying "structural properties". Moreover, they have shown that such problem admits a Mountain Pass type solution, not identically zero, under the assumptions that the nonlinearity f satisfies a subcritical growth, the Ambrosetti-Rabinowitz condition and f is superlinear at 0.

Ros Oton and Valdinoci have studied the linear Dirichlet problem, proving existence of solutions, maximum principles and constructing some useful barriers, moreover they focus on the regularity properties of solutions, under weaker hypotheses on the function a in the kernel K, see [173, 176]. Regarding the works about Dirichlet problems driven by the nonlocal operator L_K we also mention [84] and the monograph [145].

Dirichlet problems driven by the fractional Laplacian have been intensively studied in the recent literature, we recall the fine regularity results of [174], the existence and multiplicity results obtained for instance in [76, 94, 113], and the study on extremal solutions in [175] (see also the contributions of [11, 17, 21, 39, 40, 43, 51, 101, 118, 153, 183, 185, 193] and the monograph [145]). Moreover, in [113] Iannizzotto, Mosconi, Squassina have proved in the case of the fractional Laplacian that, for the corresponding functional J, being a local minimizer for J with respect to a suitable weighted C^0-norm, is equivalent to being an $H^s_0(\Omega)$-local minimizer. Such result represents an extension to the fractional setting of the classical result by Brezis and Nirenberg for Laplacian operator [33] and it is one of the main tools in our results.

In this thesis, we also consider the following Dirichlet problems driven by the nonlinear fractional p-Laplacian

$$\begin{cases} (-\Delta)^s_p u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial\Omega$, $N > ps$, $p > 1$ and $s \in (0,1)$. The nonlinear case is obviously more involved: spectral properties of $(-\Delta)^s_p$ were studied in [29, 87, 105, 137], a detailed regularity theory was developed in [28, 114, 115, 128, 129] (some results about Sobolev and Hölder regularity being only proved for the degenerate case $p > 2$), maximum and comparison principles have appeared in [67, 122], while existence and multiplicity of solutions have been obtained for instance in [52, 66, 111, 195] (see also the surveys [148, 157]). Of the vast literature, we also mention the results of [15, 22, 67, 121, 122, 147, 164, 167, 180]. For the purposes of the present study, we recall in particular [116], where it was proved that the local minimizers of the energy functional corresponding to problem (2.3.14) in the topologies of $W_0^{s,p}(\Omega)$ and of the weighted Hölder space $C_s^0(\overline{\Omega})$, respectively, coincide (namely, a nonlinear fractional analogue of the classical result of [33]).

If on the one hand nonlocal equations exhibit many common features with elliptic partial differential equations, on the other there are significative differences given, for instance, by the regularity of the solutions and by the presence of integral terms, which are difficult to estimate. Regarding the first point, it is well known that the regularity up to the boundary of solutions in the local case is C^1, while this result does not hold in the nonlocal case, where the best regularity that we can obtain is only C^s ($s \in (0,1)$) on the boundary, as we will see in Chapter 2. For this reason the usual space C^1 is replaced by the weighted Hölder type space. Regarding the second, the reason is that in the nonlocal equations functions may have no gradient at all, so such term is replaced by the Gagliardo seminorm, an integral term, which is often prohibitive to compute. For instance let see some proofs of Chapters 6 - 7 - 9.

When we deal with discontinuous nonlinearities ($f(x,\cdot) \notin C(\mathbb{R})$) the corresponding energy functional may be nondifferentiable, but only locally Lipschitz continuous. Hence our goal is to extend some results seen in the smooth case to pseudo-differential inclusions, as the following

$$\begin{cases} L_K u \in \partial F(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$ and $\partial j(x,\cdot)$ denotes the Clarke generalized subdifferential of a potential $j : \Omega \times \mathbb{R} \to \mathbb{R}$.

Since Chang's pioneering work (see [50]), variational methods based on nonsmooth critical point theory are used to study nonsmooth problems driven by nonlinear operators, such as the p-Laplacian. Such variational technique allows to establish several existence and multiplicity results for problems related to locally Lipschitz potentials, which can be equivalently formulated as either differential inclusions or hemivariational inequalities, see [14, 54, 110, 112, 117, 130, 151, 159–161] and the monographs [95, 152, 155].

In the last decade hemivariational inequalities have been actively studied through employing the techniques of nonlinear analysis (including degree theory and minimax methods), see [47, 143, 155, 158, 169] and the references therein. Such variational-hemivariational inequalities are called in this way, because in them appear a maximal monotone term which is not in general everywhere defined (variational inequality), and a nonmonotone, but everywhere defined term (hemivariational inequality). Furthermore hemivariational inequalities can be naturally applied in problems of mechanics and engineering, taking into account more realistic laws which involve multivalued (nonsmooth potential) and nonmonotone (nonconvex potential) operators, see [45, 152, 155].

In [189] Teng studies hemivariational inequalities driven by nonlocal elliptic operator and he shows the existence of two nontrivial solutions, by applying critical point theory for nonsmooth functionals, while in [184] Servadei and Valdinoci prove Lewy-Stampacchia type estimates for variational inequalities driven by nonlocal operators. In [196] Xiang considers a variational inequality involving nonlocal elliptic operators, proving the existence of one solution, by exploiting variational methods combined with a penalization technique and Schauder's fixed point theorem. In [1] Aizicovici, Papageorgiou and Staicu study the degree theory for the operator $\partial J + \partial I$, where ∂J is the Clarke generalized subdifferential of a nonsmooth and locally Lipschitz functional J, and ∂I the subdifferential of I, a proper, convex and lower semicontinuous functional, in the sense of convex analysis. They show a result regarding the degree of an isolated minimizer for Euler functionals of the form

$J + I$. Such extension allow to study nonlinear variational inequalities with a nonsmooth potential function (variational-hemivariational inequalities).

Also connected with the variational-hemivariational inequalities is the study of the obstacle problem at zero driven by the fractional p-Laplacian, presented in the last chapter of this work.

The thesis is divided into nine Chapters, each Chapter, except the first two, corresponds to a paper, suitably adapted to the structure of the thesis.

- **Chapter 1** This chapter is devoted to some recalls of nonlinear functional analysis. We collect together the main results regarding variational and topological methods in the cases of smooth and nonsmooth functionals.

- **Chapter 2** In the first part of this chapter we introduce the fractional Sobolev spaces and the nonlocal operators, with which we will work.

  In the second we give the variational formulation and we present well known results regarding Dirichlet problems driven by the nonlocal anisotropic operator L_K for the linear case, and by the fractional p-Laplacian for the nonlinear case.

- **Chapter 3** In this chapter we deal with nonlocal eigenvalue problems. In particular we consider the weighted eigenvalue problem for a general nonlocal pseudo-differential operator, depending on a bounded weight function. For such problem, we prove that strict (decreasing) monotonicity of the eigenvalues with respect to the weight function is equivalent to the unique continuation property of eigenfunctions. In addition, we discuss some unique continuation results for the special case of the fractional Laplacian (see [89]). Finally, we conclude such chapter with some spectral properties of the fractional p-Laplacian.

- **Chapter 4** In this chapter we focus on the weighted eigenvalue problem driven by the fractional Laplacian with homogeneous Dirichlet boundary conditions in a bounded domain Ω, depending on a weight function $\rho \in L^\infty(\Omega)$. We study weak* continuity, convexity and Gâteaux differentiability of the map $\rho \mapsto 1/\lambda_1(\rho)$, where $\lambda_1(\rho)$ is the first positive eigenvalue. Moreover, denoting by $\mathcal{G}(\rho_0)$ the class of rearrangements of ρ_0, we prove the existence of a minimizer of $\lambda_1(\rho)$ when ρ varies on $\mathcal{G}(\rho_0)$. Finally, we show that, if Ω is Steiner symmetric, then every minimizer shares the same symmetry (see [9]).

- **Chapter 5** In this chapter we deal with an equation driven by a nonlocal anisotropic operator with homogeneous Dirichlet boundary conditions. We find at least three nontrivial solutions: one positive, one negative and one of unknown sign, using variational methods and Morse theory. Finally, we present a Hopf's lemma, where we consider a slightly negative nonlinearity (see [88]).

- **Chapter 6** This chapter is devoted to the study of a Dirichlet type problem for an equation involving the fractional Laplacian and a reaction term subject to either subcritical or critical growth conditions, depending on a positive parameter. Applying a critical point result of Bonanno, we prove existence of one or two positive solutions as soon as the parameter lies under a (explicitly determined) value. As an application, we find two positive solutions for a fractional Ambrosetti-Brezis-Cerami problem (see [91]).

- **Chapter 7** In this chapter we study a pseudo-differential equation driven by the degenerate fractional p-Laplacian, under Dirichlet type conditions in a smooth domain. First we show that the solution set within the order interval given by a sub-supersolution pair is nonempty, directed, and compact, hence endowed with extremal elements. Then, we prove existence of a smallest positive, a biggest negative and a nodal solution, combining variational methods with truncation techniques (see [90]).

- **Chapter 8** This chapter is devoted to the study of a pseudo-differential inclusion driven by a nonlocal anisotropic operator and a Clarke generalized subdifferential of a nonsmooth potential, which satisfies nonresonance conditions both at the origin and at infinity. We prove the existence of three nontrivial solutions: one positive, one negative and one of unknown sign, using variational methods based on nonsmooth critical point theory, more precisely applying the second deformation theorem and spectral theory. Here, a nonsmooth anisotropic version of the Hölder versus Sobolev minimizers relation play an important role (see [92]).

- **Chapter 9** In this chapter we deal with the nonlocal obstacle problem at zero driven by the nonlinear fractional p-Laplacian and a nonsmooth potential (variational-hemivariational inequalities). We show that such problem admits at least two nontrivial solutions, by using a combination of degree theory, based on the degree map for specific multivalued perturbations of $(S)_+$- nonlinear operators, and variational methods (see [93]).

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Notations

Throughout the thesis, for any $A \subset \mathbb{R}^N$ we shall set $A^c = \mathbb{R}^N \setminus A$.

Unless otherwise specified, measurable means Lebesgue measurable and $|A|$ denotes the N-dimensional Lebesgue measure of a set $A \subseteq \mathbb{R}^N$. For any two measurable functions u, v, $u = v$ in A will stand for $u(x) = v(x)$ for a.e. $x \in A$ (and similar expressions). The positive (resp., negative) part of u is denoted u^+ (resp., u^-).

We will often write $t^\nu = |t|^{\nu-1}t$ for $t \in \mathbb{R}$, $\nu > 1$. For any $t \in \mathbb{R}$ we set $t^\pm = \max\{\pm t, 0\}$. By $B_r(x)$ we denote the open ball centered at $x \in \mathbb{R}^N$ of radius $r > 0$. If X is an ordered Banach space, then X_+ will denote its non-negative order cone. For all $q \in [1, \infty]$, $\|\cdot\|_q$ denotes the standard norm of $L^q(\Omega)$ (or $L^q(\mathbb{R}^N)$, which will be clear from the context). Moreover, we denote by p' the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.

Moreover, C will denote a positive constant (whose value may change line by line).

Contents

Chapter 1

Some recalls of nonlinear functional analysis

In this preliminary chapter, we collect some basic results about nonlinear functional analysis that will be used in the forthcoming chapters. As mentioned above in the Introduction, a distinct feature of this thesis is that it combines variational and topological methods as: spectral theory, regularity, maximum principles, Morse theory and degree theory. For instance, this can be seen in the study of multiple solutions, where usually every solution is obtained through a different approach and method.
In the following, for any Banach space $(X, \|\cdot\|)$ and any functional $J \in C^1(X)$ or J locally Lipschitz we will denote by K_J the set of all critical points of J, i.e., those points $u \in X$ such that $J'(u) = 0$ in X^* (dual space of X), while for all $c \in \mathbb{R}$ we set

$$K_J^c = \{u \in K_J : J(u) = c\},$$

$$J^c = \{u \in X : J(u) \leq c\} \quad (c \in \mathbb{R}),$$

beside we set

$$\overline{B}_\rho(u_0) = \{u \in X : \|u - u_0\| \leq \rho\} \quad (u_0 \in X, \rho > 0).$$

1.1 Preliminaries on smooth critical point theory and Morse theory

This section is devoted to smooth critical point theory and Morse theory, that will be applied in the next chapters in the case of nonlinear Dirichlet problems to show our existence and multiplicity results.

First of all, we recall the following *Palais-Smale compactness condition* for a functional $J \in C^1(X)$ on a Banach space X:

(PS) Any sequence (u_n) in X, such that $(J(u_n))$ is bounded in $\mathbb{R}$ and $J'(u_n) \to 0$ in X^*, admits a strongly convergent subsequence.

Most results require the following *Cerami compactness condition* (for short *(C)-condition*, a weaker version of the Palais-Smale condition):

(C) Any sequence (u_n) in X, such that $(J(u_n))$ is bounded in $\mathbb{R}$ and $(1+\|u_n\|)J'(u_n) \to 0$ in X^* admits a strongly convergent subsequence.

Now, we recall the definition and some basic properties of critical groups, referring the reader to the monograph [150] for a detailed account on the subject. Let X be a Banach space, $J \in C^1(X)$ be a functional, and let $u \in X$ be an isolated critical point of J, i.e., there exists a neighbourhood U of u such that $K_J \cap U = \{u\}$, and $J(u) = c$. For all $k \in \mathbb{N}_0$, the *k-th critical group of J at u* is defined as

$$C_k(J,u) = H_k(J^c \cap U, J^c \cap U \setminus \{u\}),$$

where $H_k(\cdot,\cdot)$ is the k-th (singular) homology group of a topological pair with coefficients in $\mathbb{R}$.
The definition above is well posed, since homology groups are invariant under excision, hence $C_k(J,u)$ does not depend on U. Moreover, critical groups are invariant under homotopies preserving isolatedness of critical points.
We recall some special cases in which the computation of critical groups is immediate ($\delta_{k,h}$ is the Kronecker symbol).

Proposition 1.1.1. *[150, Example 6.45] Let X be a Banach space, $J \in C^1(X)$ a functional and $u \in K_J$ an isolated critical point of J. The following hold:*

- *if u is a local minimizer of J, then $C_k(J,u) = \delta_{k,0}\mathbb{R}$ for all $k \in \mathbb{N}_0$,*
- *if u is a local maximizer of J, then $C_k(J,u) = \begin{cases} 0 & \text{if } \dim(X) = \infty \\ \delta_{k,m}\mathbb{R} & \text{if } \dim(X) = m \end{cases}$ for all $k \in \mathbb{N}_0$.*

Next we pass to critical points of mountain pass type.

Definition 1.1.2. *[150, Definition 6.98] Let X be a Banach space, $J \in C^1(X)$ and $x \in K_J$, u is of mountain pass type if, for any open neighbourhood U of u, the set $\{y \in U : J(y) < J(u)\}$ is nonempty and not path-connected.*

Throughout this thesis, one of the most important tool is the Mountain Pass Theorem, which can be stated in several different ways, here we mention one of this and we will specify when we will use a different version. The following result is a variant of the Mountain Pass Theorem [106, 168] and establishes the existence of critical points of mountain pass type.

Theorem 1.1.3. *[150, Theorem 6.99] If X is a Banach space, $J \in C^1(X)$ satisfies the (C)-condition, $x_0, x_1 \in X$, $\Gamma := \{\gamma \in C([0,1],X) : \gamma(0) = x_0, \gamma(1) = x_1\}$, $c := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$, and $c > \max\{J(x_0), J(x_1)\}$, then $K_J^c \neq \emptyset$ and, moreover, if K_J^c is discrete, then we can find $u \in K_J^c$ which is of mountain pass type.*

We now describe the critical groups for critical points of mountain pass type.

Proposition 1.1.4. *[150, Proposition 6.100] Let X be a reflexive Banach space, $J \in C^1(X)$, and $u \in K_J$ isolated with $c := J(u)$ isolated in $J(K_J)$. If u is of mountain pass type, then $C_1(J,u) \neq 0$.*

If the set of critical values of J is bounded below and J satisfies the (C)-condition, we define for all $k \in \mathbb{N}_0$ the *k-th critical group at infinity of J* as

$$C_k(J,\infty) = H_k(X, J^a),$$

where $a < \inf_{u \in K_J} J(u)$.
We recall the *Morse identity*.

Proposition 1.1.5. *[150, Theorem 6.62 (b)] Let X be a Banach space and let $J \in C^1(X)$ be a functional satisfying (C)-condition such that K_J is a finite set. Then, there exists a formal power series $Q(t) = \sum_{k=0}^{\infty} q_k t^k$ ($q_k \in \mathbb{N}_0$ $\forall k \in \mathbb{N}_0$) such that for all $t \in \mathbb{R}$*

$$\sum_{k=0}^{\infty} \sum_{u \in K_J} \dim C_k(J,u) t^k = \sum_{k=0}^{\infty} \dim C_k(J,\infty) t^k + (1+t) Q(t).$$

1.1.1 Abstract theorems about existence of critical points for differentiable functionals

In this subsection we deal with functionals of the form $J_\lambda = \Phi - \lambda \Psi$ ($\lambda > 0$), where $\Phi, \Psi \in C^1(X)$, defined on a Banach space X. In particular, in Chapter 6 we will apply the following abstract result, slightly rephrased from [25, Theorem 2.1].

Theorem 1.1.6. *Let X be a Banach space, $\Phi, \Psi \in C^1(X)$, $J_\lambda = \Phi - \lambda\Psi$ ($\lambda > 0$), $r \in \mathbb{R}$, $\bar{u} \in X$ satisfy*

(i) $\inf\limits_{u \in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$;

(ii) $0 < \Phi(\bar{u}) < r$;

(iii) $\sup\limits_{\Phi(u) \leqslant r} \dfrac{\Psi(u)}{r} < \dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$;

(iv) $\inf\limits_{u \in X} J_\lambda(u) = -\infty$ *for all* $\lambda \in I_r = \left(\dfrac{\Phi(\bar{u})}{\Psi(\bar{u})}, \left[\sup\limits_{\Phi(u) \leqslant r} \dfrac{\Psi(u)}{r} \right]^{-1} \right)$.

Then, for all $\lambda \in I_r$ for which J_λ satisfies (PS), there exist $u_\lambda, v_\lambda \in X$ such that

$$J'_\lambda(u_\lambda) = J'_\lambda(v_\lambda) = 0, \quad J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda).$$

As we will see, Theorem 1.1.6 can not be applied in some cases, for instance when the functional J_λ does not satisfy the Palais-Smale condition in general. For this reason we introduce the following *local Palais-Smale condition* for functionals of the type $J_\lambda = \Phi - \lambda\Psi$, with $\Phi, \Psi \in C^1(X)$, $\lambda > 0$, defined on a Banach space X, and $r > 0$:

$(PS)^r$ Every sequence (u_n) in X, such that $(J_\lambda(u_n))$ is bounded in $\mathbb{R}$, $J'(u_n) \to 0$ in X^*, and $\Phi(u_n) \leqslant r$ for all $n \in \mathbb{N}$, has a convergent subsequence.

In this case, our main tool will be the following local minimum result, slightly rephrased from [26, Theorem 3.3].

Theorem 1.1.7. *Let X be a Banach space, $\Phi, \Psi \in C^1(X)$, $J_\lambda = \Phi - \lambda\Psi$ $(\lambda > 0)$, $r \in \mathbb{R}$; $\bar{u} \in X$ satisfy*

(i) $\inf_{u\in X} \Phi(u) = \Phi(0) = \Psi(0) = 0$;

(ii) $0 < \Phi(\bar{u}) < r$;

(iii) $\sup_{\Phi(u)\leqslant r} \dfrac{\Psi(u)}{r} < \dfrac{\Psi(\bar{u})}{\Phi(\bar{u})}$.

Let

$$I_r = \Big(\frac{\Phi(\bar{u})}{\Psi(\bar{u})}, \Big[\sup_{\Phi(u)\leqslant r} \frac{\Psi(u)}{r}\Big]^{-1}\Big).$$

Then, for all $\lambda \in I_r$ for which J_λ satisfies $(PS)^r$, there exists $u_\lambda \in X$ such that

$$0 < \Phi(u_\lambda) < r, \quad J_\lambda(u_\lambda) = \min_{0<\Phi(u)<r} J_\lambda(u).$$

1.2 Nonlinear operators

This section focuses on important classes of nonlinear operators stating an abstract result that offers a powerful tool for establishing existence of solutions for nonlinear Dirichlet equations [45], as we will see in Chapter 7.
The norm convergence in X and X^* is denoted by $\to$ and the weak convergence by $\rightharpoonup$.

Definition 1.2.1. *Let $A : X \to X^*$; then A is called*

(i) continuous iff $u_n \to u$ implies $A(u_n) \to A(u)$,

(ii) demicontinuous iff $u_n \to u$ implies $A(u_n) \rightharpoonup A(u)$,

(iii) hemicontinuous iff the real function $t \to \langle A(u + tv), w\rangle$ is continuous on $[0, 1]$ for all $u, v, w \in X$,

(iv) strongly continuous iff $u_n \rightharpoonup u$ implies $A(u_n) \to A(u)$,

(v) bounded iff A maps bounded sets into bounded sets,

(vi) coercive iff $\lim_{\|u\|\to\infty} \frac{\langle A(u),u\rangle}{\|u\|} = +\infty$.

Definition 1.2.2. *(Operators of Monotone Type) Let $A : X \to X^*$; then A is called*

(i) monotone (respectively, strictly monotone) iff $\langle A(u) - A(v), u - v\rangle \geq 0$ (respectively, > 0) for all $u, v \in X$ with $u \neq v$,

(*ii*) *maximal monotone iff A is monotone and the condition*

$$(u, u^*) \in X \times X^* : \quad \langle u^* - A(v), u - v \rangle \quad \textit{for all } v \in X$$

implies $u \in X$ and $u^* = A(u)$,

(*iii*) *pseudomonotone iff $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply $\langle A(u), u - w\rangle \le \liminf_{n\to\infty}\langle A(u_n), u_n - w\rangle$ for all $w \in X$,*

(*iv*) *to satisfy $(S)_+$-condition iff $u_n \rightharpoonup u$ and $\limsup_{n\to\infty}\langle A(u_n), u_n - u\rangle \le 0$ imply $u_n \to u$.*

Remark 1.2.3. For our future purposes we give the definition of maximal monotone operator also in the case of set-valued map (or multimap, or multifunction). Let $A : X \to 2^{X^*}$ be a set-valued map, i.e., to each $u \in X$, there is assigned a subset $A(u)$ of X^*, which may be empty if $u \notin D(A)$, where $D(A)$ is the domain of A given by

$$D(A) = \{u \in X : A(u) \neq \emptyset\}.$$

The graph of A, denoted by $Gr(A)$, is given by

$$Gr(A) = \{(u, u^*) \in X \times X^* : u^* \in A(u)\}.$$

The map $A : X \to 2^{X^*}$ is called

(*i*) monotone iff

$$\langle u^* - v^*, u - v\rangle \ge 0 \quad \text{for all } (u, u^*), (v, v^*) \in Gr(A),$$

(*ii*) maximal monotone if it is monotone and for $(u, u^*) \in X \times X^*$ the inequality

$$\langle u^* - v^*, u - v\rangle \ge 0 \quad \text{for all } (v, v^*) \in Gr(A)$$

implies that $(u, u^*) \in Gr(A)$.

Finally, we conclude this section with the main theorem on pseudomonotone operators due to Brezis.

Theorem 1.2.4. *[45, Theorem 2.99] Let X be a real, reflexive Banach space, and let $A : X \to X^*$ be a pseudomonotone, bounded, and coercive operator, and $b \in X^*$. Then a solution of the equation $Au = b$ exists.*

1.3 Brief recalls of nonsmooth analysis

In this section, we collect some basic definitions and results from nonsmooth and nonlinear analysis, containing all that is necessary in this direction for the rest of the thesis. We begin with significant results of convex analysis , especially related to the convex subdifferential such as its property to be a maximal monotone operator. Then we focus on the subdifferentiability theory for locally Lipschitz functionals. For a fuller treatment we refer the reader to [1, 13, 53, 69, 70, 95, 199].

Let $(X, \|\cdot\|)$ be a reflexive Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X^* and X, and by $2^X \setminus \{\varnothing\}$ the family of all nonempty subsets of X.

By $\Gamma_0(X)$ we indicate the cone of all proper (not identically $+\infty$), convex and lower semicontinuous functions $I : X \to \mathbb{R} \cup \{+\infty\}$.

Let C be a nonempty, closed convex subset of X, the *indicator function of* C is defined by

$$i_C : X \to \mathbb{R} \cup \{+\infty\} \quad i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{if } u \notin C. \end{cases}$$

If $C \neq \varnothing$, then $i_C \in \Gamma_0(X)$.

Given $I \in \Gamma_0(X)$, the *subdifferential of* I in the sense of convex analysis is given by the multifunction $\partial I : X \to 2^{X^*}$

$$\partial I(u) = \{u^* \in X^* : \langle u^*, v - u \rangle \leq I(v) - I(u) \text{ for all } v \in X\}.$$

Regarding the properties of the subdifferential of I in the sense of convex analysis, we refer the reader to [150] and the references therein. We stress that if $I \in \Gamma_0(X)$ is Gâteaux differentiable at $u \in X$, then $\partial I(u) = \{I'(u)\}$. Moreover we note that the subdifferential in the sense of convex analysis $\partial I : X \to 2^{X^*}$ of a function $I \in \Gamma_0(X)$ is a maximal monotone operator.

If I coincides with i_C, the indicator function of $C \subseteq X$, then we obtain a closed convex cone, called the *normal cone to* C *at* u, defined by

$$\partial i_C(u) = \left\{ u^* \in X^* : \langle u^*, u \rangle = \sigma(u^*; C) = \sup_{v \in C} \langle u^*, v \rangle \right\}.$$

Now we recall some basic definitions and results of nonsmooth critical point theory (see [53, 95, 152]). A functional $J : X \to \mathbb{R}$ is said to be *locally Lipschitz continuous* if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that

$$|J(v) - J(w)| \leq L\|v - w\| \text{ for all } v, w \in U.$$

From now on, we assume J to be locally Lipschitz continuous. The *generalized directional derivative* of J at u along $v \in X$ is defined by

$$J^\circ(u; v) = \limsup_{\substack{w \to u \\ t \to 0^+}} \frac{J(w + tv) - J(w)}{t}$$

(see [95, Proposition 1.3.7]). The *Clarke generalized subdifferential* of J at u is the set

$$\partial J(u) = \{u^* \in X^* : \langle u^*, v \rangle \leq J^\circ(u; v) \text{ for all } v \in X\}.$$

(Although we use the same notation, it will be clear from the context when we will refer to the Clarke generalized subdifferential or the subdifferential in the sense of convex analysis). A point u is said to be a *critical point* of J if $0 \in \partial J(u)$. In the following lemma we recall some useful properties of ∂J (see [95, Propositions 1.3.8-1.3.12]).

Lemma 1.3.1. *If $J, J_1 : X \to \mathbb{R}$ are locally Lipschitz continuous, then*

(i) *$\partial J(u)$ is convex, closed and weakly* compact for all $u \in X$;*

(ii) *the multifunction $\partial J : X \to 2^{X^*}$ is upper semicontinuous with respect to the weak* topology on X^*;*

(iii) *if $J \in C^1(X)$, then $\partial J(u) = \{J'(u)\}$ for all $u \in X$;*

(iv) *$\partial(\lambda J)(u) = \lambda \partial J(u)$ for all $\lambda \in \mathbb{R}$, $u \in X$;*

(v) *$\partial(J + J_1)(u) \subseteq \partial J(u) + \partial J_1(u)$ for all $u \in X$;*

(vi) *if u is a local minimizer (or maximizer) of J, then $0 \in \partial J(u)$.*

We remark that in view of Lemma 1.3.1 *(i)*, for all $u \in X$

$$m_J(u) := \min_{u^* \in \partial J(u)} \|u^*\|_*$$

is well defined and $u \in X$ is a critical point of J if

$$m_J(u) = 0.$$

We say that a locally Lipschitz function $J : X \to \mathbb{R}$ satisfies the *Palais-Smale condition at level* $c \in \mathbb{R}$ if every sequence $(u_n) \subset X$ such that

$$J(u_n) \to c \text{ and } m_J(u_n) \to 0 \text{ as } n \to \infty$$

admits a strongly convergent subsequence. We say that J satisfies the *Palais-Smale condition* if it satisfies the Palais-Smale condition for every $c \in \mathbb{R}$.

Here, we recall the nonsmooth version of the mountain pass theorem (see [95, Theorem 2.1.1]).

Theorem 1.3.2. *Let X be a Banach space, $J : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the Palais-Smale condition, $u_0, \hat{u} \in X$, $r \in (0, \|\hat{u} - u_0\|)$ be such that*

$$\max\{J(u_0), J(\hat{u})\} < \eta_r = \inf_{\|u-u_0\|=r} J(u),$$

moreover, let

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = u_0, \gamma(1) = \hat{u}\}, \qquad c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).$$

Then $c \geq \eta_r$, and $K_J^c \neq \emptyset$.

We will use the following nonsmooth second deformation theorem (see [95, Theorem 2.1.3]).

Theorem 1.3.3. *Let X be a Banach space, $J : X \to \mathbb{R}$ be a locally Lipschitz function satisfying the Palais-Smale condition, let $a < b$ be real numbers such that $K_J^c = \emptyset$ for all $c \in (a, b)$ and K_J^a is a finite set. Then, there exists a continuous deformation*

$$h : [0, 1] \times (J^b \setminus K_J^b) \to (J^b \setminus K_J^b)$$

such that the following hold:

(i) $h(0, u) = u$, $h(1, u) \in J^a$ *for all* $u \in (J^b \setminus K_J^b)$,

(ii) $h(t, u) = u$ *for all* $(t, u) \in [0, 1] \times J^a$,

(iii) $t \mapsto J(h(t, u))$ *is decreasing in* $[0, 1]$ *for all* $u \in (J^b \setminus K_J^b)$.

In particular, by (i)-(ii) above we have that J^a is a *strong deformation retract* of J^b(see [150, Definition 5.33 (b)]). Moreover, we observe that, if a is the global minimum of J and is attained at a unique point $u_0 \in X$, and there are no critical levels of J in (a, b), then by Theorem 1.3.3 the set $J^b \setminus K_J^b$ is contractible (see [150, Definition 6.22]).
If J is continuous and convex, then J is locally Lipschitz and the generalized and convex subdifferentials coincide.
In the sequel we focus on the study of critical points of the functional $J + I$, for this purpose we mention the following facts (see [126, 152]).

Definition 1.3.4. *Let $J : X \to \mathbb{R}$ be a locally Lipschitz functional and $I : X \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. We say that $u \in X$ is a critical point of $J + I$ if*

$$J^0(u; v - u) + I(v) - I(u) \geq 0 \quad \forall v \in X,$$

where $J^0(u; z)$ is the generalized directional derivative of J at the point $u \in X$ in the direction $z \in X$.

Proposition 1.3.5. *An element $u \in X$ is a critical point of $J + I$ if and only if $0 \in \partial J(u)+\partial I(u)$, where ∂J is the Clarke generalized subdifferential and ∂I is the subdifferential in the sense of convex analysis.*

1.4 Notions about degree theory

Now, we introduce the degree theory for set-valued mappings which are the sum of a generalized subdifferential and a convex subdifferential, that we will use in Chapter 9 to study existence and multiplicity of solutions of nonlinear problems . For a deeper discussions we refer the reader to [1, 35, 107, 150] and the references therein.
Since X is a reflexive Banach space, by the Troyanski renorming theorem (see [95, Theorem A.3.9]), we can equivalently renorm X in such a way that both X and X^* are locally

uniformly convex with Fréchet differentiable norms. Therefore, in the following, we suppose that both X and X^* are reflexive and locally uniformly convex.
From [150, Theorem 2.46, Proposition 2.70], the duality map $\mathcal{F} : X \to X^*$, defined by

$$\mathcal{F}(u) = \left\{u^* \in X^* : \langle u^*, u\rangle = \|u\|^2 = \|u^*\|_*^2\right\},$$

is single-valued, strictly monotone, a homeomorphism and a $(S)_+$-operator.
A multifunction $G : X \to 2^{X^*}$ *belongs to class* (P) if it is upper semicontinuous with closed, convex nonempty values and such that

$$G(A) = \bigcup_{u\in A} G(u)$$

is relatively compact in X^* for any bounded subset U of X.
Let U be a bounded open subset in X, $S : \overline{U} \to X^*$ a bounded, demicontinuous operator of type $(S)_+$ and $A : D(A) \subseteq X \to 2^{X^*}\setminus\{\varnothing\}$ a maximal monotone operator with $0 \in A(0)$, then for every $\lambda > 0$, the operator $S + A_\lambda$ is a bounded, demicontinuous operator of type $(S)_+$. For every $u^* \notin (S + A)(\partial U)$, $\deg_0(S + A, U, u^*)$ is defined by

$$\deg_0(S + A, U, u^*) = \deg_{(S)_+}(S + A_\lambda, U, u^*)$$

for all sufficiently small $\lambda > 0$, where $A_\lambda(u) = -\frac{1}{\lambda}\mathcal{F}(v - u)$ is everywhere defined, single valued, bounded and monotone.
In addition we have a multifunction G in the class (P), then for $u^* \notin (S + A + G)(\partial U)$, $\deg(S + A + G, U, u^*)$ is defined by

$$\deg(S + A + G, U, u^*) = \deg_0(S + A + g_\varepsilon, U, u^*)$$

for $\varepsilon > 0$ small, where g_ε is a continuous $\varepsilon-$approximate selection of G (see [48, Cellina's approximate selection Theorem], [108, Theorem 4.41]).
Concerning the degree maps $\deg_{(S)_+}$ and $\deg_0$ we refer the reader to [35], while for the degree map deg we refer to [107]. The degree map preserves the usual properties: normalization, domain additivity, homotopy invariance, excision and solution property. One of such properties is the homotopy invariance with respect to a certain class of admissible homotopies. Now we introduce the admissible homotopies for the maps S, A and G (see [1]).

Definition 1.4.1. *The admissible homotopies for the maps S, A and G are defined in the following way.*

*(i) A one-parameter family $\{S_t\}_{t\in[0,1]}$ of maps from $\overline{U}$ into X^* is a **homotopy of class** $(S)_+$, if for any $(u_n) \subseteq \overline{U}$ such that $u_n \rightharpoonup u$ in X, and for any $(t_n) \subseteq [0, 1]$ with $t_n \to t$ for which*

$$\limsup_{n\to\infty} \langle S_{t_n}(u_n), u_n - u\rangle \leq 0,$$

we have that $u_n \to u$ in X and $S_{t_n}(u_n) \rightharpoonup S_t(u)$ in X^*.

(*ii*) *A family* $\{A^t\}_{t\in[0,1]}$ *of maximal monotone maps from* X *into* X^* *such that* $(0,0) \in \mathrm{Gr}(A^t)$ *for all* $t \in [0,1]$ *is a* ***pseudomonotone homotopy****, if it satisfies the following mutually equivalent conditions*

- *if* $t_n \to t$ *in* $[0,1]$, $u_n \rightharpoonup u$ *in* X, $u_n^* \rightharpoonup u^*$ *in* X^*, $u_n^* \in A^{t_n}(u_n)$ *and*
$$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0,$$
then $(u,u^*) \in \mathrm{Gr}(A^t)$ *and* $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$*;*
- $(t,u^*) \mapsto \xi(t,u^*) = (A^t + \mathcal{F})^{-1}(u^*)$ *is continuous from* $[0,1]\times X^*$ *into* X*, where both* X *and* X^* *are equipped with their respective norm topologies;*
- *for every* $u^* \in X^*$, $t \mapsto \xi(t,u^*) = (A^t + \mathcal{F})^{-1}(u^*)$ *is continuous from* $[0,1]$ *into* X *endowed with the norm topology;*
- *if* $t_n \to t$ *in* $[0,1]$ *and* $u^* \in A^t(u)$*, then there exist sequences* (u_n) *and* (u_n^*) *such that* $u_n^* \in A^{t_n}(u_n)$, $u_n \to u$ *in* X *and* $u_n^* \to u^*$ *in* X^**.*

(*iii*) *A one-parameter family* $\{G_t\}_{t\in[0,1]}$ *of multifunctions* $G_t : \overline{U} \to 2^{X^*}\setminus\{\varnothing\}$ *is a* ***homotopy of class*** (P) *if* $(t,u) \mapsto G_t(u)$ *is usc from* $[0,1]\times\overline{U}$ *into* $2^{X^*}\setminus\{\varnothing\}$*, for every* $(t,u) \in [0,1]\times\overline{U}$, $G_t(u) \subseteq X^*$ *is closed and convex and*
$$\overline{\bigcup\{G_t(u) : t\in[0,1],\ u\in\overline{U}\}}$$
is compact in X^**.*

Therefore the homotopy invariance of the degree map "deg", can be expressed in the following way.
If $\{S_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ such that each S_t is bounded, $\{A^t\}_{t\in[0,1]}$ is a pseudomonotone homotopy of maximal monotone operators with $0 \in A^t(0)$ for all $t\in[0,1]$, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $u^* : [0,1] \to X^*$ is a continuous map such that
$$u_t^* \notin (S_t + A_t + G_t)(\partial U)$$
for all $t\in[0,1]$, then $\deg(S_t + A_t + G_t, U, u_t^*)$ is independent of $t\in[0,1]$. (This is the meaning of admissible homotopy for us in Chapter 9.)
Now, we identify another class of pseudomonotone homotopies (see [1, Lemma 15]).

Lemma 1.4.2. *Let* $A : X \to X^*$ *be a bounded demicontinuous operator of type* $(S)_+$ *and* $I \in \Gamma_0(X)$*. Then*
$$(t,u) \mapsto h(t,u) = A(u) + t\partial I(u), \quad (t,u) \in [0,1]\times X$$
is a pseudomonotone homotopy.

Chapter 2

Fractional framework

2.1 Fractional Sobolev spaces

One of our purposes is to study nonlocal problems driven by the fractional p-Laplacian $(-\Delta)_p^s$, which includes for $p=2$ the fractional Laplacian $(-\Delta)^s$, or the general (anisotropic) operator L_K and with Dirichlet boundary data via variational methods. For this, the choice of the functional space where to work plays an important role. In order to correctly encode the Dirichlet boundary datum in the variational formulation, we need to work in a suitable functional analytical setting, which coincides with the fractional Sobolev spaces when the leading operator is $(-\Delta)_p^s$, while is inspired by the fractional Sobolev spaces in the case of L_K. In this section we recall some basic notions about fractional Sobolev spaces (for details we refer the reader to [72, 111, 145]).

2.1.1 The fractional Sobolev space $W_0^{s,p}(\Omega)$

For any $s \in (0,1)$ and $p \in (1,\infty)$ we introduce the Gagliardo seminorm by setting for all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$

$$[u]_{s,p}^p = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+ps}}\, dxdy.$$

Accordingly, we define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

an intermediary Banach space between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$, endowed with the natural norm

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left(\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{s,p}^p\right)^{\frac{1}{p}}.$$

Unless otherwise stated, the numbers $p > 1$ and $s \in (0,1)$ will be fixed as the order of summability and the order of differentiability.

As in the classical case with s being an integer, any function in the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ can be approximated by a sequence of smooth functions with compact support. Indeed,

$$\overline{C_0^\infty(\mathbb{R}^N)}^{\|\cdot\|_{W^{s,p}(\mathbb{R}^N)}} = W^{s,p}(\mathbb{R}^N),$$

that is, the space $C_0^\infty(\mathbb{R}^N)$ is dense in $W^{s,p}(\mathbb{R}^N)$(see [72, Theorem 2.4]).

Fractional Sobolev spaces enjoy quite a number of important functional inequalities. It is almost impossible to list here all the results and the possible applications, therefore we will only present the fractional Sobolev inequality and the embedding inequalities (see [72, 145]), that are fundamental for our goals. The fractional Sobolev inequality can be written as follows (see [72, Theorem 6.5], [145, Theorem 1.4]).
Let p_s^* be the fractional critical Sobolev exponent given by

$$p_s^* := \begin{cases} \frac{Np}{N-ps} & \text{if } N > ps \\ +\infty & \text{if } N \le ps. \end{cases}$$

Proposition 2.1.1. *Let $s \in (0,1)$ and $p \in [1,\infty)$ be such that $N > ps$. Then there exists a positive constant $C = C(N,p,s)$ such that, for any $u \in W^{s,p}(\mathbb{R}^N)$, we have*

$$\|u\|^p_{L^{p_s^*}(\mathbb{R}^N)} \le C \left(\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x)-u(y)|^p}{|x-y|^{N+sp}}\,dxdy \right).$$

Consequently, the space $W^{s,p}(\mathbb{R}^N)$ is continuously embedded in $L^q(\mathbb{R}^N)$ for any $q \in [p, p_s^*]$.

We note that, when $s = 1$ the exponent p_s^* reduces to the classical critical Sobolev exponent.

Let Ω be an open and bounded set in $\mathbb{R}^N$, we consider the subspace

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \Omega^c\},$$

this last one is a separable, uniformly convex (hence, reflexive) Banach space, endowed with the norm $\|u\|_{W_0^{s,p}(\Omega)} = [u]_{s,p}$.
Since $W_0^{s,p}(\Omega)$ is a space of functions defined in $\mathbb{R}^N$, in this section we denote by $C_0^\infty(\Omega)$ the space

$$C_0^\infty(\Omega) = \{u : \mathbb{R}^N \to \mathbb{R} : u \in C^\infty(\mathbb{R}^N), \text{Supp } u \text{ is compact and Supp } u \subseteq \Omega\}, \tag{2.1.1}$$

where Supp u is the support of the function u, given by Supp $u := \overline{\{x \in \mathbb{R}^N : u(x) \neq 0\}}$.
We note that here $W_0^{s,p}(\Omega)$ is not the closure of $C_0^\infty(\Omega)$ in $W^{s,p}(\Omega)$ (for more details see [72]).
Now, we mention the important density property of $W_0^{s,p}(\Omega)$, for a detailed proof see [85, Theorem 6] and [145, Theorem 2.6].

Proposition 2.1.2. *Let Ω be an open subset of $\mathbb{R}^N$, with continuous boundary $\partial\Omega$. Then, for any $u \in W_0^{s,p}(\Omega)$, there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $W_0^{s,p}(\Omega)$ as $n \to \infty$. In other words, $C_0^\infty(\Omega)$ is a dense subspace of $W_0^{s,p}(\Omega)$.*

Moreover, we denote by $(W^{-s,p'}(\Omega), \|\cdot\|_{W^{-s,p'}(\Omega)})$ the topological dual of $(W_0^{s,p}(\Omega), \|\cdot\|_{W_0^{s,p}(\Omega)})$ and by $\langle\cdot,\cdot\rangle$ the duality pairing between $W^{-s,p'}(\Omega)$ and $W_0^{s,p}(\Omega)$.

Remark 2.1.3. We note that since $W_0^{s,p}(\Omega)$ is a reflexive Banach space, applying the Troyanski's renorming theorem (see [95, Theorem A.3.9]), such space can be equivalently renormed so that both $W_0^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$ are locally uniformly convex (and thus also strictly convex) and with Fréchet differentiable norms.

Now we recall the embedding properties of $W_0^{s,p}(\Omega)$ when $N > ps$ (about the proof see [72, Theorem 6.7, Corollary 7.2]).

Proposition 2.1.4. *Let $s \in (0,1)$ and $p \in [1,+\infty)$ be such that $N > ps$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$ boundary. Then there exists a positive constant $C = C(N,p,s,\Omega)$ such that, for any $u \in W_0^{s,p}(\Omega)$, we have*

$$\|u\|_q \leq C\|u\|_{W_0^{s,p}(\Omega)},$$

for any $q \in [1,p_s^*]$, i.e. the space $W_0^{s,p}(\Omega)$ is continuously embedded in $L^q(\Omega)$ for any $q \in [1,p_s^*]$. In addition, such embedding is compact for all $q \in [1,p_s^*)$.

(In the sequel we denote by $c_q > 0$ the embedding constant and in particular we will use the embedding of $W_0^{s,p}(\Omega)$ in $L^p(\Omega)$.)

Furthermore, we introduce the positive order cone

$$W_0^{s,p}(\Omega)_+ = \{u \in W_0^{s,p}(\Omega) : u(x) \geq 0 \text{ for a.e. } x \in \Omega\},$$

which has an empty interior with respect to the $W_0^{s,p}(\Omega)-$ topology.
Moreover, let $\Omega \subset \mathbb{R}^N$ be bounded. We denote by $\widetilde{W}^{s,p}(\Omega)$ the space of all $u \in L^p_{\mathrm{loc}}(\mathbb{R}^N)$ such that $u \in W^{s,p}(U)$ for some open $U \subseteq \mathbb{R}^N$, $\overline{\Omega} \subset U$, and

$$\int_{\mathbb{R}^N} \frac{|u(x)|^{p-1}}{(1+|x|)^{N+ps}}\,dx < \infty.$$

This last condition holds if $u \in L^\infty(\mathbb{R}^N)$ or $[u]_{C^s(\mathbb{R}^N)} < \infty$. The space $\widetilde{W}^{s,p}(\Omega)$ can be endowed with a topological vector space structure as inductive limit, but we will not use it. Finally, we point out that $W_0^{s,p}(\Omega) \subset \widetilde{W}^{s,p}(\Omega)$.

2.1.2 The space $H_0^s(\Omega)$

In this subsection we focus on the case $p = 2$. This is quite an important case since the fractional Sobolev spaces $W^{s,2}(\mathbb{R}^N)$ and $W_0^{s,2}(\Omega)$ turn out to be Hilbert spaces. They are usually denoted by $H^s(\mathbb{R}^N)$ and $H_0^s(\Omega)$.
As well known, the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^{2_s^*}(\mathbb{R}^N)$ is continuous (see [72, Theorem 6.5]), but in this case we explicitly know the embedding constant, the so-called fractional Talenti constant, which is given by the following lemma (see [58, Theorem 1.1] and [72, Proposition 3.6]).

Lemma 2.1.5. *We have*

$$T(N,s) = \max_{u \in H^s(\mathbb{R}^N)\setminus\{0\}} \frac{\|u\|_{2_s^*}}{[u]_{s,2}} = \frac{s^{\frac{1}{2}}\Gamma(\frac{N-2s}{2})^{\frac{1}{2}}\Gamma(N)^{\frac{s}{N}}}{2^{\frac{1}{2}}\pi^{\frac{N+2s}{4}}\Gamma(1-s)^{\frac{1}{2}}\Gamma(\frac{N}{2})^{\frac{s}{N}}} > 0,$$

the maximum being attained at the functions

$$u(x) = \frac{a}{(b+|x-x_0|^2)^{\frac{N-2s}{2}}} \quad (a, b > 0,\, x_0 \in \mathbb{R}^N).$$

Let $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ be a bounded domain with $C^{1,1}$ boundary. For our purposes we work in the subspace

$$H_0^s(\Omega) = \{u \in H^s(\mathbb{R}^N) : u = 0 \text{ in } \Omega^c\},$$

a Hilbert space under the inner product

$$\langle u, v\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\, dx\, dy$$

and the corresponding norm $\|u\|_{H_0^s(\Omega)} = [u]_{s,2}$ (see [113], [185, Lemma 7]).
We denote by $H^{-s}(\Omega)$ and $\|\cdot\|_{H^{-s}(\Omega)}$ the topological dual of $H_0^s(\Omega)$ and its norm.
By Lemma 2.1.5 and Hölder's inequality, for any $q \in [1, 2_s^*]$ the embedding $H_0^s(\Omega) \hookrightarrow L^q(\Omega)$ is continuous and for all $u \in H_0^s(\Omega)$ we have

$$\|u\|_q \leqslant T(N,s)|\Omega|^{\frac{2_s^*-q}{2_s^* q}}\|u\|_{H_0^s(\Omega)}. \tag{2.1.2}$$

Further, the embedding is compact iff $q < 2_s^*$ (see [185, Lemma 8]). As in the case of the usual Sobolev spaces, the following inclusions

$$i : H_0^s(\Omega) \hookrightarrow L^2(\Omega)$$
$$j : L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$$

are compact and dense and there exists a positive constant C such that

$$\|u\|_2 \leqslant C\|u\|_{H_0^s(\Omega)} \quad \forall\, u \in H_0^s(\Omega) \tag{2.1.3}$$

$$\|u\|_{H^{-s}(\Omega)} \leqslant C\|u\|_2 \quad \forall\, u \in L^2(\Omega). \tag{2.1.4}$$

Moreover $H_0^s(\Omega)$ contains $C_0^\infty(\Omega)$ as a dense subset (see Proposition 2.1.2).

2.1.3 The space $X_K(\Omega)$

As mentioned before, in this thesis we consider also Dirichlet problems driven by the general nonlocal operator L_K and in some cases by its anisotropic version. In order to study such problems by variational methods we have to fix a suitable functional analytic framework, that takes into account the boundary condition in the weak formulation, for

this the usual fractional Sobolev space is not enough.
For all measurable $u : \mathbb{R}^N \to \mathbb{R}$ set

$$[u]_K^2 = \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))^2 K(x-y)\, dxdy,$$

where the kernel K satisfies the following hypotheses:

($\mathbf{H}_K$) $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ such that

- (i) $mK \in L^1(\mathbb{R}^N)$, where $m(x) = \min\{|x|^2, 1\}$;
- (ii) $K(x) \geq \alpha |x|^{-(N+2s)}$ in $\mathbb{R}^N \setminus \{0\}$ ($\alpha > 0, s \in (0,1)$ such that $N > 2s$);
- (iii) $K(-x) = K(x)$ in $\mathbb{R}^N \setminus \{0\}$.

The set of hypotheses ($\mathbf{H}_K$) is standard in the current literature (see [145, 173, 185]) and in the sequel we figure out the importance of these three conditions. More precisely, the symmetry condition ($\mathbf{H}_K$) (iii) is not necessary to show our results, it was assumed only for the sake of simplicity, as noted in [187, footnote 3 pag. 70] (see also [145]). Indeed, let $K(x) = \frac{\tilde{K}(x) + \tilde{K}(-x)}{2}$ satisfy ($\mathbf{H}_K$) where the kernel $\tilde{K}$ fulfills only ($\mathbf{H}_{\tilde{K}}$) (i)-(ii). By a change of variables, we deduce that $[u]_K^2 = [u]_{\tilde{K}}^2$.
Let $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ be a bounded domain with $C^{1,1}$ boundary. Now we define the Hilbert space (see [145, 183, 185])

$$X_K(\Omega) = \big\{ u \in L^2(\mathbb{R}^N) : [u]_K < \infty,\ u = 0 \text{ in } \Omega^c \big\},$$

endowed with the scalar product

$$\langle u, v \rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(v(x) - v(y)) K(x-y)\, dxdy$$

and the corresponding norm $\|u\|_{X_K(\Omega)} = [u]_K$.
Moreover, we denote by $(X_K(\Omega)^*, \|\cdot\|_*)$ the topological dual of $(X_K(\Omega), \|\cdot\|)$ and by $\langle \cdot, \cdot \rangle$ the duality pairing between $X_K(\Omega)^*$ and $X_K(\Omega)$.

By the fractional Sobolev inequality and the continuous embedding of $X_K(\Omega)$ in $H_0^s(\Omega)$, thanks to condition (ii) (see [183, Subsection 2.2], [185, Lemmas 5-8]), we have that the embedding $X_K(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, 2_s^*]$ and compact if $q \in [1, 2_s^*)$ (see [72, Theorem 6.7, Corollary 7.2]). Indeed, we have

$$\|u\|_{2_s^*}^2 \leq C \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\, dxdy \leq \frac{C}{\alpha} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\, dxdy,$$

namely

$$X_K(\Omega) \hookrightarrow H_0^s(\Omega) \hookrightarrow L^q(\Omega).$$

According to the definition of $C_0^\infty(\Omega)$ given in (2.1.1), we have the following.

Lemma 2.1.6. *Let Ω be an open subset of $\mathbb{R}^N$ and let $(\mathbf{H}_K)$ (i), (iii) hold. Then, $C_0^\infty(\Omega) \subseteq X_K(\Omega)$.*

Proof. We refer the reader to [83, Lemma 1.3.1], [145, Lemma 1.20]. □

Finally, we mention the important density property of $X_K(\Omega)$, for a detailed proof see [85, Theorem 6], [145, Theorem 2.6].

Proposition 2.1.7. *Let Ω be an open subset of $\mathbb{R}^N$, with continuous boundary and let K satisfy $(\mathbf{H}_K)$ (i), (ii). Then, for any $u \in X_K(\Omega)$ there exists a sequence $u_n \in C_0^\infty(\Omega)$ such that $u_n \to u$ in $X_K(\Omega)$ as $n \to \infty$. In other words, $C_0^\infty(\Omega)$ is a dense subspace of $X_K(\Omega)$.*

For the sake of simplicity, when we will deal with Dirichlet problem driven by the nonlocal anisotropic operator L_K we will always denote the functional space in which we work as $X_K(\Omega)$ with the tacit assumption that the kernel K has the explicit expression:

$$K(x) = a\Big(\frac{x}{|x|}\Big)\frac{1}{|x|^{N+2s}}$$

with $a \in L^1(S^{N-1})$ even with $\inf_{S^{N-1}} a > 0$, $N > 2s$ and $0 < s < 1$. Clearly, such kernel still satisfies the set of hypotheses $(\mathbf{H}_K)$, so all embedding properties continue to hold. We note that also in this case the remark about the symmetry of the kernel, previously stated, still holds for the function a, by setting $a\big(\frac{x}{|x|}\big) = \frac{1}{2}\left(\tilde{a}\big(\frac{x}{|x|}\big) + \tilde{a}\big(\frac{-x}{|x|}\big)\right)$ (see [173]).

2.2 Nonlocal operators

In this section we present the nonlocal operators we will be working with in this thesis. Let us begin by giving the definitions and the main properties of linear nonlocal operators as the fractional Laplacian $(-\Delta)^s$, the general nonlocal L_K and its anisotropic version. Then we focus our attention on the fractional p-Laplacian $(-\Delta)^s_p$, a nonlinear nonlocal operator.

As mentioned on many occasions, a typical feature of such operators is the *nonlocality*, in the sense that the value of $(-\Delta)^s u(x)$ (or $L_K u(x)$ or $(-\Delta)^s_p u(x)$) at any point $x \in \Omega$ depends not only on the values of u on a neighbourhood of x, but actually on the whole $\mathbb{R}^N$, since $u(x)$ represents the expected value of a random variable tied to a process randomly jumping arbitrarily far from the point x. In this sense, the natural Dirichlet boundary condition consists in assigning the values of u in Ω^c rather than merely on $\partial\Omega$ (a general reference on the theory can be found in [72, 145]).

Such operators are the local counterparts of the well known elliptic operators. Indeed when the fractional parameter $s \in (0,1)$ goes to 1 we obtain that the fractional Laplacian $(-\Delta)^s$ converges to the Laplacian $-\Delta$ (see [72, Proposition 4.4]), the nonlocal anisotropic L_K tends to the operator $-\sum_{ij} a_{ij}(\nabla u) D^2_{ij} u$ where the matrix $(a_{ij})(\nabla u)$ is positive definite and (D^2_{ij}) is the Hessian matrix (see [136]), and the general nonlocal L_K converges to the linear elliptic second order operator $-\sum_{ij} a_{ij}\partial_{ij}u$ (see [174]). Finally, the fractional p-Laplacian $(-\Delta)^s_p$ tends to the p-Laplacian $-\Delta_p$ (see [136]).

2.2.1 The fractional Laplacian

Nonlocal equations have attracted much attention in recent decades. The basic operator involved in this kind of problems is the so-called fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$. Such operator can be defined in different ways: as a singular integral operator, as a pseudo-differential operator via the Fourier transform and as a generator of a stable Lévy process, just to name a few (for a general introduction we refer to [36, 39, 40, 72]).
The fractional Laplacian operator is defined for any sufficiently smooth function [1] $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}^N$ by

$$(2.2.1) \qquad \begin{aligned}(-\Delta)^s\, u(x) &= C(N,s)\,\mathrm{P.V.}\int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy \\ &= C(N,s)\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{N+2s}}\,dy,\end{aligned}$$

or, equivalently (thanks to the factor $\frac{C(N,s)}{2}$) by

$$(2.2.2) \qquad (-\Delta)^s\, u(x) = \frac{C(N,s)}{2}\int_{\mathbb{R}^N} \frac{2u(x)-u(x+y)-u(x-y)}{|y|^{N+2s}}\,dy$$

where $C(N,s)$ is a positive normalization constant given by

$$C(N,s) = \frac{2^{2s}s\Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}}\Gamma(1-s)},$$

or, equivalently by [36, Lemma 3.1.3]

$$C(N,s) = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\omega_1)}{|\omega|^{N+2s}}\,d\omega\right)^{-1}$$

with $\omega = (\omega_1, \omega')$, $\omega' \in \mathbb{R}^{N-1}$ (for a precise evaluation of $C(N,s)$ see [36, 39, 72]). Here P.V. is a commonly used abbreviation for "in the principal value sense" (as defined by the latter equation). From (2.2.2) we see that $(-\Delta)^s$ is an operator of order $2s$, namely, it arises from a differential quotient of order $2s$ weighted in the whole space. Furthermore, for $u \in \mathcal{S}(\mathbb{R}^N)$ the fractional Laplacian operator can be expressed in Fourier frequency variables multiplied by $(2\pi|\xi|)^{2s}$, namely

$$(-\Delta)^s\, u(x) = \mathcal{F}^{-1}((2\pi|\xi|)^{2s}(\mathcal{F}u))$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform (see [36, Lemma 3.1.1]).
Different definitions of the fractional Laplacian consider different normalizing constants. The constant $C(N,s)$ chosen here is the one that ensures the equivalence of the integral

[1] To define the operator $(-\Delta)^s : \mathcal{S}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ it is sufficient, for simplicity, to take here u in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ of smooth and rapidly decaying functions, or in $C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ in (2.2.1) or (2.2.2).

definition of $(-\Delta)^s$ with that given by the Fourier transform (see [39, Remark 2.11], [72, 145]). Moreover, $C(N,s)$ has the following additional properties:

$$\lim_{s\to 1^-} (-\Delta)^s u = -\Delta u \qquad \text{and} \qquad \lim_{s\to 0^+} (-\Delta)^s u = u,$$

here $-\Delta$ denotes the classical Laplacian operator.
For the sake of completeness, the fractional Laplacian can be interpreted as a (generalized) Dirichlet-to-Neumann map (see [43]), but we will not employ this notion in the sequel. In addition, we recall that in the literature the fractional Laplacian is often defined by

$$\langle (-\Delta)^s u, v\rangle = \frac{C(N,s)}{2} \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,dxdy, \qquad \forall u,v\in H_0^s(\Omega).$$

We point out that, in the current literature, there are several notions of fractional Laplacian, all of which agree when the problems are set on the whole $\mathbb{R}^N$, but some of them disagree in a bounded domain. We refer the reader to [186] for a discussion on the comparison between the integral fractional Laplacian and the regional (or spectral) notion obtained by taking the s-powers of the Laplacian operator $-\Delta$ with zero Dirichlet boundary conditions (see also [154]).

2.2.2 The nonlocal anisotropic operator L_K

Throughout this thesis, we also consider a linear operator, which is a generalization of the fractional Laplacian, the main difference is that such operator is anisotropic, as we will see below.

Definition 2.2.1. *The linear operator L_K is defined for any u in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$ as*

$$\begin{aligned} L_K u(x) &= P.V.\int_{\mathbb{R}^N} (u(x)-u(y))K(x-y)\,dy \\ &= \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N\setminus B_\epsilon(x)} (u(x)-u(y))K(x-y)\,dy, \end{aligned} \tag{2.2.3}$$

where the singular kernel $K:\mathbb{R}^N\setminus\{0\}\to(0,+\infty)$ is given by

$$K(y) = a\Big(\frac{y}{|y|}\Big)\frac{1}{|y|^{N+2s}}, \qquad a\in L^1(S^{N-1}),\ \inf_{S^{N-1}} a>0,\ \textit{even}.$$

This operator is said *anisotropic*, because the role of the function a in the kernel is to weight differently the different spacial directions. As stated in Subsection 2.1.3, the symmetry of the kernel K, namely of the function a, can be easily removed (see [173]). We stress that the operators L_K and $L_{\tilde{K}}$ (where K is even and $\tilde{K}$ is not even) does not coincide, they are different operators ($L_K u \neq L_{\tilde{K}} u$), hence the classical solutions of Dirichlet problems driven by such operators (that we will study in the sequel) are different. Although, by taking into account the remark in Subsection 2.1.3, we will find out that the weak solutions

of equations, driven by L_K or $L_{\tilde{K}}$, are the same, because the weak formulation of such problems does not require the symmetry of the kernel as stated in [145, 187](see Section 2.3).
In general, the u's we will be dealing with, do not belong in $S(\mathbb{R}^N)$, as the optimal regularity for solutions of nonlocal problems is only $C^s(\mathbb{R}^N)$. We give a weaker definition of $L_K : X_K(\Omega) \to X_K(\Omega)^*$ as

$$\langle L_K(u), v\rangle = \iint_{\mathbb{R}^N\times\mathbb{R}^N} (u(x) - u(y))(v(x) - v(y))K(x - y)\, dx\, dy$$

for all $u, v \in X_K(\Omega)$.

We notice that the kernel of the operator L_K satisfies the set $(\mathbf{H}_K)$, which is useful for our goals. The typical example is $K(y) = |y|^{-(N+2s)}$, which corresponds to $a \equiv 1$, namely $L_K = (-\Delta)^s$, the fractional Laplacian. We remark that we do not assume any regularity on the kernel $K(y)$. As we will see in Section 2.3, there is an interesting relation between the regularity properties of solutions and the regularity of kernel $K(y)$.

We recall some special properties of the case $a \in L^\infty(S^{N-1})$.

Remark 2.2.2. Due to the singularity at 0 of the kernel, the right-hand side of (2.2.3) is not well defined in general. In the case $s \in (0, \frac{1}{2})$ the integral in (2.2.3) is not really singular near x. Indeed, for any $u \in S(\mathbb{R}^N)$, $a \in L^\infty(S^{N-1})$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|u(x) - u(y)|}{|x - y|^{N+2s}}\, a\Big(\frac{x - y}{|x - y|}\Big)\, dy \\
&\le C\|a\|_\infty \int_{B_R} \frac{|x - y|}{|x - y|^{N+2s}}\, dy + C\|a\|_\infty\|u\|_\infty \int_{B_R^c} \frac{1}{|x - y|^{N+2s}}\, dy \\
&= C\left(\int_{B_R} \frac{1}{|x - y|^{N+2s-1}}\, dy + \int_{B_R^c} \frac{1}{|x - y|^{N+2s}}\, dy\right) < \infty,
\end{aligned}$$

where C is a positive constant depending only on the dimension and on the L^∞ norms of u and a, see [72, Remark 3.1] in the case of the fractional Laplacian.

The singular integral given in Definition 2.2.1 can be written as a weighted second-order differential quotient as follows (see [72, Lemma 3.2] for the fractional Laplacian):

Lemma 2.2.3. *For all $u \in S(\mathbb{R}^N)$ L_K can be defined as*

$$L_K u(x) = \frac{1}{2}\int_{\mathbb{R}^N} (2u(x) - u(x + z) - u(x - z))K(z)\, dz, \quad x \in \mathbb{R}^N. \tag{2.2.4}$$

Remark 2.2.4. We notice that the expression in (2.2.4) does not require the P.V. formulation since, for instance, taking $u \in L^\infty(\mathbb{R}^N)$ and locally C^2, $a \in L^\infty(S^{N-1})$, using a

Taylor expansion of u in B_1, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{|2u(x)-u(x+z)-u(x-z)|}{|z|^{N+2s}}\, a\Big(\frac{z}{|z|}\Big)\, dz \\
&\le c\|a\|_\infty \|u\|_\infty \int_{B_1^c} \frac{1}{|z|^{N+2s}}\, dz + \|a\|_\infty \|D^2 u\|_{L^\infty(B_1)} \int_{B_1} \frac{1}{|z|^{N+2s-2}}\, dz \\
&< \infty.
\end{aligned}$$

We show that the two definitions are equivalent, hence we have

$$\begin{aligned}
L_K u(x) =& \frac{1}{2} \int_{\mathbb{R}^N} (2u(x)-u(x+z)-u(x-z)) K(z)\, dz \\
=& \frac{1}{2} \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (2u(x)-u(x+z)-u(x-z)) K(z)\, dz, \\
=& \frac{1}{2} \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (u(x)-u(x+z)) K(z)\, dz \\
&+ \frac{1}{2} \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (u(x)-u(x-z)) K(z)\, dz,
\end{aligned}$$

we make a change of variables $\tilde{z} = -z$ in the second integral and we set $\tilde{z} = z$

$$= \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c} (u(x)-u(x+z)) K(z)\, dz,$$

we make another change of variables $z = y - x$ and we obtain the first definition

$$= \lim_{\epsilon \to 0^+} \int_{B_\epsilon^c(x)} (u(x)-u(y)) K(x-y)\, dy.$$

It is important stressing that this holds only if the kernel is even, more precisely if the function a is even.

There exists a third definition of L_K that uses a Fourier transform, we can define it as

$$L_K u(x) = \mathcal{F}^{-1}(S(\xi)(\mathcal{F}u))$$

where $\mathcal{F}$ is a Fourier transform and $S : \mathbb{R}^N \to \mathbb{R}$ is a multiplier, $S(\xi) = \int_{\mathbb{R}^N} (1 - \cos(\xi \cdot z)) a\big(\frac{z}{|z|}\big)\, dz$. We consider (2.2.4) and we apply the Fourier transform to obtain

$$\begin{aligned}
\mathcal{F}(L_K u) &= \mathcal{F}\left(\frac{1}{2} \int_{\mathbb{R}^N} (2u(x)-u(x+z)-u(x-z)) a\Big(\frac{z}{|z|}\Big)\, dz \right) \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (\mathcal{F}(2u(x)-u(x+z)-u(x-z)) a\Big(\frac{z}{|z|}\Big)\, dz \\
&= \frac{1}{2} \int_{\mathbb{R}^N} (2 - e^{i\xi\cdot z} - e^{-i\xi\cdot z})(\mathcal{F}u)(\xi) a\Big(\frac{z}{|z|}\Big)\, dz \\
&= \frac{1}{2} (\mathcal{F}u)(\xi) \int_{\mathbb{R}^N} (2 - e^{i\xi\cdot z} - e^{-i\xi\cdot z}) a\Big(\frac{z}{|z|}\Big)\, dz \\
&= (\mathcal{F}u)(\xi) \int_{\mathbb{R}^N} (1 - \cos(\xi \cdot z)) a\Big(\frac{z}{|z|}\Big)\, dz.
\end{aligned}$$

We recall that in the case $a \equiv 1$, namely for the fractional Laplacian (see [72, Proposition 3.3]), $S(\xi) = |\xi|^{2s}$, as we have seen above.

2.2.3 The general nonlocal operator L_K

The third linear operator that we consider in this thesis is the general nonlocal operator L_K, from which we can obtain as particular cases the anisotropic version and the fractional Laplacian, by choosing explicitly the kernel. The general nonlocal operator L_K is defined by

$$L_K u(x) = \text{P.V.} \int_{\mathbb{R}^N} (u(x) - u(y))K(x - y)\, dy,$$

whose kernel K satisfies the set of hypotheses $(\mathbf{H}_K)$. Condition $(\mathbf{H}_K)$ *(i)* is required to prove an integration by parts formula. Condition $(\mathbf{H}_K)$ *(ii)* ensures the compact embedding of the Sobolev-type space $X_K(\Omega)$ into $L^2(\Omega)$ (as we have seen in Section 2.1). Finally, the symmetry condition $(\mathbf{H}_K)$ *(iii)* allows to rephrase equivalently the operator L_K as

$$L_K u(x) = \frac{1}{2} \int_{\mathbb{R}^N} (2u(x) - u(x + z) - u(x - z))K(z)\, dz,$$

for any conveniently regular u. All these play a role in the weak formulation of Dirichlet problems driven by such operator, in this regard the remark about $(\mathbf{H}_K)$ *(iii)*, stated in Subsection 2.2.2, still holds. A typical example of a kernel satisfying $(\mathbf{H}_K)$ is $K(x) = |x|^{-N-2s}$, in this case we have $L_K = (-\Delta)^s$ (the fractional Laplacian).

2.2.4 The fractional p-Laplacian

One of the nonlinear nonlocal operators is the fractional p-Laplacian, which is the gradient of the functional $\frac{[u]_{s,p}^p}{p}$ defined on $W_0^{s,p}(\Omega)$ (see [114]). Under suitable smoothness conditions on any function $u : \mathbb{R}^N \to \mathbb{R}$ and for all $x \in \mathbb{R}^N$ such operator can be written as

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \to 0^+} \int_{B_\varepsilon^c(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\, dy \tag{2.2.5}$$

where $s \in (0, 1)$, $p \in (1, \infty)$. Up to some normalization constant depending on N, p and s, this definition is consistent with the one of the fractional Laplacian $(-\Delta)^s$ in the case $p = 2$ (see [39, 40, 72]). Furthermore, such nonlinear operator is degenerate when $p > 2$ and singular when $1 < p < 2$.
By [114, Lemma 2.3], for any $u \in \widetilde{W}^{s,p}(\Omega)$ we can define $(-\Delta)_p^s\, u \in W^{-s,p'}(\Omega)$ by setting for all $v \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u, v\rangle = \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dxdy. \tag{2.2.6}$$

The definition above agrees with (2.2.5) when u lies in the Schwartz space $\mathcal{S}(\mathbb{R}^N)$. In the next lemma we recall some useful properties of $(-\Delta)_p^s$ in $W_0^{s,p}(\Omega)$ (see also Definitions 1.2.1 - 1.2.2).

Lemma 2.2.5. *$(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a monotone, continuous, $(S)_+$-operator.*

Proof. By [116, Lemma 2.3] (with $q = 1$) we have for all $u, v \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u - (-\Delta)_p^s\, v, u - v\rangle \geqslant 0,$$

hence $(-\Delta)_p^s$ is monotone. Plus, $(-\Delta)_p^s$ is continuous as the Gâteaux derivative of the C^1-functional $u \mapsto \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p}$. Finally, if $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$ and

$$\limsup_n \langle (-\Delta)_p^s\, u_n, u_n - u\rangle \leqslant 0,$$

then for all $n \in \mathbb{N}$ we have

$$\begin{aligned}
&(\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)} - \|u\|^{p-1}_{W_0^{s,p}(\Omega)})(\|u_n\|_{W_0^{s,p}(\Omega)} - \|u\|_{W_0^{s,p}(\Omega)}) \\
&= \|u_n\|^p_{W_0^{s,p}(\Omega)} - \|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}\|u\|_{W_0^{s,p}(\Omega)} - \|u_n\|_{W_0^{s,p}(\Omega)}\|u\|^{p-1}_{W_0^{s,p}(\Omega)} + \|u\|^p_{W_0^{s,p}(\Omega)} \\
&\leq \langle (-\Delta)_p^s\, u_n, u_n\rangle - \langle (-\Delta)_p^s\, u_n, u\rangle - \langle (-\Delta)_p^s\, u, u_n\rangle + \langle (-\Delta)_p^s\, u, u\rangle \\
&= \langle (-\Delta)_p^s\, u_n, u_n - u\rangle + \langle (-\Delta)_p^s\, u, u - u_n\rangle \leq o(1),
\end{aligned}$$

hence $\|u_n\|_{W_0^{s,p}(\Omega)} \to \|u\|_{W_0^{s,p}(\Omega)}$. By uniform convexity of $W_0^{s,p}(\Omega)$, $u_n \to u$ in $W_0^{s,p}(\Omega)$. Therefore, $(-\Delta)_p^s$ is an $(S)_+$-operator. □

Remark 2.2.6. In this remark we point out other properties of the fractional p-Laplacian operator, useful for the results of Chapters 7 and 9.

(i) We note that the operator $(-\Delta)_p^s$ is odd, $(p-1)-$homogeneous, and satisfies for all $u \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u, u\rangle = \|u\|^p_{W_0^{s,p}(\Omega)}, \quad \|(-\Delta)_p^s\, u\|_{W^{-s,p'}(\Omega)} \leq \|u\|^{p-1}_{W_0^{s,p}(\Omega)}.$$

Hence, $(-\Delta)_p^s$ is bounded (see [111]) and by [165, Lemma 3.3] it is strictly monotone.

(ii) Since $(-\Delta)_p^s$ is a continuous and monotone operator, then, by [150, Corollary 2.42], it is maximal monotone from $W_0^{s,p}(\Omega)$ into $W^{-s,p'}(\Omega)$.

(iii) For the sake of completeness, the continuity of the operator $(-\Delta)_p^s$ can be shown in a different way. In order to do this, we define a support mapping $f(u) = \frac{(-\Delta)_p^s\, u}{\|u\|^{p-2}_{W_0^{s,p}(\Omega)}}$ for every $u \in \partial B_1(0) \subset W_0^{s,p}(\Omega)$ (for definition and properties we refer the reader to [71]). Recalling Remark 2.1.3, we obtain that the norm of $W^{-s,p'}(\Omega)$ is Fréchet differentiable and, applying [71, Theorem 1], we obtain that $f : \partial B_1(0) \subset W_0^{s,p}(\Omega) \to \partial B_1(0) \subset W^{-s,p'}(\Omega)$ is continuous. Hence, by definition of f, $(-\Delta)_p^s$ is continuous in $W_0^{s,p}(\Omega) \setminus \{0\}$. Indeed, we suppose that $v_n = \frac{u_n}{\|u_n\|_{W_0^{s,p}(\Omega)}}$ strongly converges to $v = \frac{u}{\|u\|_{W_0^{s,p}(\Omega)}}$ in $W_0^{s,p}(\Omega)$. Hence,

$$\begin{aligned}
(-\Delta)_p^s\, u_n &= (-\Delta)_p^s\, (\|u_n\|_{W_0^{s,p}(\Omega)} v_n) = \|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}(-\Delta)_p^s\, v_n \\
&= \|u_n\|^{p-1}_{W_0^{s,p}(\Omega)} f(v_n) \to \|u\|^{p-1}_{W_0^{s,p}(\Omega)} f(v) \\
&= \|u\|^{p-1}_{W_0^{s,p}(\Omega)} f\left(\frac{u}{\|u\|_{W_0^{s,p}(\Omega)}}\right) = \|u\|^{p-2}_{W_0^{s,p}(\Omega)} f(u) = (-\Delta)_p^s\, u
\end{aligned}$$

as n goes to infinity. The continuity in the origin is trivial, then $(-\Delta)_p^s$ is continuous in the whole space $W_0^{s,p}(\Omega)$.

2.3 Variational formulation of the Dirichlet problems driven by nonlocal operators

As we have stated more than once, we consider Dirichlet problem driven by the different nonlocal operators, introduced in the previous section. Here, for reasons of simplicity, we give the variational formulation in the cases of the linear operator L_K and of the nonlinear operator $(-\Delta)_p^s$. It is clear that, by changing the operator also will change the corresponding functional space and the energy functional associated to the Dirichlet problem.

2.3.1 Dirichlet problem driven by the anisotropic L_K

Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial\Omega$, $N > 2s$ and $s \in (0,1)$. We consider the following Dirichlet problem driven by the nonlocal anisotropic operator L_K (defined in (2.2.3))

$$\begin{cases} L_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.3.1}$$

We remark that the Dirichlet datum is given in Ω^c and not simply on $\partial\Omega$, consistently with the nonlocal character of the operator L_K.
The nonlinearity $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which satisfies the growth condition

$$|f(x,t)| \le C(1 + |t|^{q-1}) \text{ a.e. in } \Omega, \forall t \in \mathbb{R} \ (C > 0, q \in [1, 2_s^*]) \tag{2.3.2}$$

(here $2_s^* := 2N/(N-2s)$ is the fractional critical exponent). Condition (2.3.2) is referred to as a subrictical or critical growth if $q < 2_s^*$ or $q = 2_s^*$, respectively.

In analogy with the classical cases, if $N < 2s$ then $X_K(\Omega)$ is embedded in $C^\alpha(\overline{\Omega})$ with $\alpha = \frac{2s-N}{2}$ [72, Theorem 8.2], while in the limit case $N = 2s$ it is embedded in $L^q(\Omega)$ for all $q \ge 1$. Therefore, due to Corollary 4.53 and Theorem 4.54 in [68], we can state that the results of this thesis hold true even when $N \le 2s$, but we only focus on the case $N > 2s$, with subcritical or critical nonlinearities, to avoid trivialities (for instance, the L^∞ bounds are obvious for $N < 2s$). Note that $N \le 2s$ requires $N = 1$, hence this case falls into the framework of ordinary nonlocal equations. In the limit case $N = 1$, $s = \frac{1}{2}$ the critical growth for the nonlinearity is of exponential type, according to the fractional Trudinger-Moser inequality. Such case is open for general nonlocal operators, though some results are known for the operator $(-\Delta)^{\frac{1}{2}}$, see [120].

We set for all $u \in X_K(\Omega)$

$$J(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F(x, u(x))\, dx,$$

where the function F is the primitive of f with respect to the second variable, that is

$$F(x,t) = \int_0^t f(x,\tau)\, d\tau, \quad x \in \Omega, t \in \mathbb{R}.$$

Then, $J \in C^1(X_K(\Omega))$ and all its critical points are weak solutions of (2.3.1), namely they satisfy

$$\langle u, v\rangle = \int_\Omega f(x, u(x))v(x)\, dx \tag{2.3.3}$$

for all $v \in X_K(\Omega)$. Here, it is convenient to assume $(\mathbf{H}_K)$ *(iii)* for writing (2.3.3), but, as already stated, this is not necessary. Indeed, as we have seen in Subsection 2.2.2, the weak formulation (2.3.3) does not depend on the symmetry of the kernel. (The same argument continues to be true if we replace the nonlocal anisotropic operator L_K by the general nonlocal L_K, see [145, 187].)
As first result, we prove a priori bounds on the weak solution of problem (2.3.1) (in the subcritical case such bound is uniform) [88, Theorem 4.1].

Theorem 2.3.1. *If f satisfies the growth condition (2.3.2), then for any weak solution $u \in X_K(\Omega)$ of (2.3.1) we have $u \in L^\infty(\Omega)$. Moreover, if $q < 2^*_s$ in (2.3.2), then there exists a function $M \in C(\mathbb{R}_+)$, only depending on the constants C, n, s and Ω, such that*

$$\|u\|_\infty \le M(\|u\|_{2^*_s}).$$

Proof. Let $u \in X_K(\Omega)$ be a weak solution of (2.3.1) and set $\gamma = (2^*_s/2)^{1/2}$ and $t_k = \operatorname{sgn}(t)\min\{|t|, k\}$ for all $t \in \mathbb{R}$ and $k > 0$. We define $v = u|u|^{r-2}_k$, for all $r \ge 2$, $k > 0$, $v \in X_K(\Omega)$. By $(\mathbf{H}_K)$ *(ii)* and applying the fractional Sobolev inequality we have that

$$\|u|u|_k^{\frac{r}{2}-1}\|^2_{2^*_s} \le C\|u|u|_k^{\frac{r}{2}-1}\|^2_{H^s_0(\Omega)} \le \frac{C}{\beta}\|u|u|_k^{\frac{r}{2}-1}\|^2_{X_K(\Omega)}.$$

By [113, Lemma 3.1] and assuming v as test function in (2.3.3), we obtain

$$\begin{aligned}\|u|u|_k^{\frac{r}{2}-1}\|^2_{2^*_s} &\le C\|u|u|_k^{\frac{r}{2}-1}\|^2_{X_K(\Omega)} \le \frac{Cr^2}{r-1}\langle u, v\rangle \\ &\le Cr\int_\Omega |f(x,u)||v|\, dx \le Cr\int_\Omega (|u||u|_k^{r-2} + |u|^q|u|_k^{r-2})\, dx,\end{aligned} \tag{2.3.4}$$

for some $C > 0$ independent of $r \ge 2$ and $k > 0$, where in the last inequality we have exploited (2.3.2). Applying the Fatou Lemma as $k \to \infty$ yields

$$\|u\|_{\gamma^2 r} \le Cr^{1/r}\left(\int_\Omega (|u|^{r-1} + |u|^{r+q-2})\, dx\right)^{1/r} \tag{2.3.5}$$

(where the right hand side may be ∞). We want to develop from (2.3.5) a suitable bootstrap argument to show that $u \in L^p(\Omega)$ for all $p \geq 1$. We define recursively a sequence (r_n) by choosing $\mu > 0$ and setting

$$r_0 = \mu, \quad r_{n+1} = \gamma^2 r_n + 2 - q.$$

The only fixed point of $t \to \gamma^2 t + 2 - q$ is

$$\mu_0 = \frac{q-2}{\gamma^2 - 1},$$

hence we get $r_n \to +\infty$ iff $\mu > \mu_0$. We separately consider the subcritical and critical cases.

- **Case** $q < 2_s^*$. We fix

$$\mu = 2_s^* + 2 - q > \max\{2, \mu_0\}, \tag{2.3.6}$$

and bootstrap on the basis of (2.3.5). Since $r_0 + q - 2 = 2_s^*$, we obtain that $u \in L^{r_0+q-2}(\Omega)$ (in particular $u \in L^{r_0-1}(\Omega)$). Therefore, by taking $r = r_0$ in (2.3.5), we have a finite right hand side, hence $u \in L^{\gamma^2 r_0}(\Omega) = L^{r_1+q-2}(\Omega)$, and so on. By iterating this argument and noting that $r \mapsto r^{\frac{1}{r}}$ is bounded in $[2, \infty)$, for all $n \in \mathbb{N}$ we deduce that $u \in L^{\gamma^2 r_n}(\Omega)$ and

$$\|u\|_{\gamma^2 r_n} \leq H(n, \|u\|_{2_s^*})$$

(henceforth, H will denote a continuous function of one or several real variables, whose definition may change case by case). By (2.3.6) we get that $\gamma^2 r_n \to \infty$ as $n \to \infty$, hence for all $p \geq 1$ there exists $n \in \mathbb{N}$ such that $\gamma^2 r_n \geq p$. The Hölder inequality implies for all $p \geq 1$ that $u \in L^p(\Omega)$ and

$$\|u\|_p \leq H(p, \|u\|_{2_s^*}). \tag{2.3.7}$$

Such L^p-estimate is not sufficient to prove our claim, as the right hand side may not be bounded as $p \to \infty$. Therefore, we need to improve (2.3.7) to a uniform L^p-bound. Set $\gamma' = \gamma/(\gamma - 1)$. By (2.3.7) and by using Hölder inequality, we obtain that

$$\|1 + |u|^{q-1}\|_{\gamma'} \leq H(\|u\|_{2_s^*}).$$

Hence, for any $r \geq 2$ we get

$$\begin{aligned}\int_\Omega |u|^{r-1}(1 + |u|^{q-1})\,dx &\leq \|1 + |u|^{q-1}\|_{\gamma'} \||u|^{r-1}\|_\gamma \\ &\leq H(\|u\|_{2_s^*})\|u\|_{\gamma(r-1)}^{r-1} \leq H(\|u\|_{2_s^*})|\Omega|^{\frac{1}{\gamma r}}\|u\|_{\gamma r}^{r-1}.\end{aligned}$$

Since $r \mapsto |\Omega|^{\frac{1}{\gamma r}}$ is bounded in $[2, \infty)$, we obtain that

$$\int_\Omega |u|^{r-1}(1 + |u|^{q-1})\,dx \leq H(\|u\|_{2_s^*})\|u\|_{\gamma r}^{r-1}.$$

By collecting the inequality above with (2.3.5), we deduce the following estimate:

$$\|u\|_{\gamma^2 r}^r \leq H(\|u\|_{2_s^*})\|u\|_{\gamma r}^{r-1}.$$

Set $v = u/H(\|u\|_{2_s^*})$ and $r = \gamma^{n-1}$ ($\gamma^{n-1} \geq 2$ for $n \in \mathbb{N}$ big enough), we obtain the following nonlinear recursive relation:

$$\|v\|_{\gamma^{n+1}} \leq \|v\|_{\gamma^n}^{1-\gamma^{1-n}}$$

which, iterated, yields

$$\|v\|_{\gamma^n} \leq \|v\|_{\gamma}^{\Pi_{i=1}^{n-2}(1-\gamma^{-i})} \quad n \in \mathbb{N}.$$

We note that the sequence $(\Pi_{i=1}^{n-2}(1-\gamma^{-i}))$ is bounded in $\mathbb{R}$, so for all $n \in \mathbb{N}$ we get

$$\|v\|_{\gamma^n} \leq H(\|u\|_{2_s^*}).$$

By going back to u, and recalling that $\gamma^n \to \infty$ as $n \to \infty$, we can find $M \in C(\mathbb{R}_+)$ such that for all $p \geq 1$

$$\|u\|_p \leq M(\|u\|_{2_s^*}),$$

i.e., from classical results in functional analysis, $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq M(\|u\|_{2_s^*}).$$

- **Case** $q = 2_s^*$. We start from (2.3.4) with $r = q+1 > 2$ and fix $\sigma > 0$ such that $Cr\sigma < 1/2$. Then we can find $K_0 > 0$ (depending on u) such that

$$\left(\int_{\{|u|>K_0\}} |u|^q \, dx\right)^{1-\frac{2}{q}} \leq \sigma. \tag{2.3.8}$$

The Hölder inequality and (2.3.8) imply

$$\begin{aligned}
\int_\Omega |u|^q |u|_k^{r-2} \, dx &\leq K_0^{q+r-2}|\{|u| \leq K_0\}| + \int_{\{|u|>K_0\}} |u|^q |u|_k^{r-2} \, dx \\
&\leq K_0^{q+r-2}|\Omega| + \left(\int_\Omega (|u|^2|u|_k^{r-2})^{\frac{q}{2}} \, dx\right)^{\frac{2}{q}} \left(\int_{\{|u|>K_0\}} |u|^q \, dx\right)^{1-\frac{2}{q}} \\
&\leq K_0^{q+r-2}|\Omega| + \sigma \|u|u|_k^{\frac{r}{2}-1}\|_q^2.
\end{aligned}$$

By recalling that $Cr\sigma < 1/2$ and (2.3.4) holds, we have

$$\frac{1}{2}\|u|u|_k^{\frac{q-1}{2}}\|_q^2 \leq C(q+1)\left(\int_\Omega |u||u|_k^{q-1} \, dx + K_0^{2q-1}|\Omega|\right).$$

By passing to the limit for $k \to \infty$, we obtain

$$\|u\|_{\frac{q(q+1)}{2}} \leq \left(C(q+1)(\|u\|_q^q + K_0^{2q-1}|\Omega|)\right)^{\frac{1}{q+1}} \leq \tilde{H}(K_0, \|u\|_q)$$

(where, as above, $\tilde{H}$ is a continuous function). Now the bootstrap argument can be applied through (2.3.5), starting with

$$r_0 = \mu = \frac{q(q+1)}{2} + 2 - q > \mu_0 = 2,$$

since $u \in L^{r_0+q-2}(\Omega)$. The rest of the proof follows as in the previous case, providing in the end $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq \tilde{M}(K_0, \|u\|_{2_s^*})$$

for a suitable function $\tilde{M} \in C(\mathbb{R}^2)$.

The main difference is that such bound is uniform only in the subcritical case and not in the critical case. □

Theorem 2.3.1 allows to set $g(x) := f(x, u(x)) \in L^\infty(\Omega)$ and now we rephrase the problem as a linear Dirichlet problem

$$\begin{cases} L_K u = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.3.9}$$

Now, we recall some preliminary results, including the weak and strong maximum principles, and a Hopf lemma.

Proposition 2.3.2. *[173, Proposition 4.1, Weak maximum principle] Let u be any weak solution to (2.3.9), with $g \geq 0$ in Ω. Then, $u \geq 0$ in Ω.*

We observe that the weak maximum principle also holds when the Dirichlet datum is given by $u = h$, with $h \geq 0$ in Ω^c.
For problem (2.3.9), the interior regularity of solutions depends on the regularity of g, but it also depends on the regularity of $K(y)$ in the y-variable. Furthermore, if the kernel K is not regular, then the interior regularity of u will in addition depend on the boundary regularity of u.

Theorem 2.3.3. *[173, Theorem 6.1, Interior regularity] Let $\alpha > 0$ be such that $\alpha + 2s$ is not an integer, and $u \in L^\infty(\mathbb{R}^N)$ be any weak solution to $L_K u = g$ in B_1. Then,*

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C(\|g\|_{C^\alpha(B_1)} + \|u\|_{C^\alpha(\mathbb{R}^N)}).$$

It is important to remark that the previous estimate is valid also in case $\alpha = 0$ (in which the C^α norm has to be replaced by the L^∞). With no further regularity assumption on the kernel K, this estimate is sharp, in the sense that the norm $\|u\|_{C^\alpha(\mathbb{R}^N)}$ can not be replaced by a weaker one. Under the extra assumption that the kernel $K(y)$ is C^α outside the origin, the following estimate holds

$$\|u\|_{C^{2s+\alpha}(B_{1/2})} \leq C(\|g\|_{C^\alpha(B_1)} + \|u\|_{L^\infty(\mathbb{R}^N)}).$$

We focus now on the boundary regularity of solutions to (2.3.9).

Proposition 2.3.4. *[173, Proposition 7.2, Optimal Hölder regularity] Let $g \in L^\infty(\Omega)$, and u be the weak solution of (2.3.9). Then,*

$$\|u\|_{C^s(\overline{\Omega})} \leq C\|g\|_{L^\infty(\Omega)},$$

for some positive constant c.

Finally, we conclude that the solutions to (2.3.9) are C^{3s} inside Ω whenever $g \in C^s$, but only C^s on the boundary, and this is the best regularity that we can obtain. For instance, we consider the following torsion problem

$$\begin{cases} L_K u = c & \text{in } B_1 \\ u = 0 & \text{in } B_1^c \end{cases}$$

for some positive constant $c > 0$. The solution $u_0 := (1-|x|^2)_+^s$ belongs to $C^s(\overline{B_1})$, but $u_0 \notin C^{s+\epsilon}(\overline{B_1})$ for any $\epsilon > 0$, as a consequence we can not expect solutions to be better than $C^s(\overline{\Omega})$.
The regularity theory for fractional Dirichlet problems was essentially developed in [174] (see also [17, 113]). While solutions of fractional equations exhibit good interior regularity properties, they may have a singular behaviour on the boundary. Therefore, instead of the usual space $C^1(\overline{\Omega})$, they are better embedded in the following weighted Hölder-type spaces $C_s^0(\overline{\Omega})$ and $C_s^\alpha(\overline{\Omega})$ as defined here below.
We set $\mathrm{d}_\Omega(x) = \mathrm{dist}(x, \Omega^c)$ with $x \in \overline{\Omega}$ and we define

$$C_s^0(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \text{ admits a continuous extension to } \overline{\Omega} \right\},$$

$$C_s^\alpha(\overline{\Omega}) = \left\{ u \in C^0(\overline{\Omega}) : \frac{u}{\mathrm{d}_\Omega^s} \text{ admits a } \alpha\text{- Hölder continuous extension to } \overline{\Omega} \right\} \quad (\alpha \in (0,1)),$$

endowed with the norms

$$\|u\|_{0,s} = \left\| \frac{u}{\mathrm{d}_\Omega^s} \right\|_\infty, \quad \|u\|_{\alpha,s} = \|u\|_{0,s} + \sup_{x \neq y} \frac{|u(x)/\mathrm{d}_\Omega^s(x) - u(y)/\mathrm{d}_\Omega^s(y)|}{|x-y|^\alpha},$$

respectively. Clearly, any function $u \in C_s^0(\overline{\Omega})$ vanishes on $\partial\Omega$, so it can be naturally extended by 0 on $\mathbb{R}^N \setminus \overline{\Omega}$. In this way, we will always consider elements of $C_s^0(\overline{\Omega})$ as defined on the whole $\mathbb{R}^N$. Moreover, for all $0 \leq \alpha < \beta < 1$ the embedding $C_s^\beta(\overline{\Omega}) \hookrightarrow C_s^\alpha(\overline{\Omega})$ is continuous and compact. In this case, unlike in $W_0^{s,p}(\Omega)$, the positive order cone $C_s^0(\overline{\Omega})_+$ has a nonempty interior given by

$$\mathrm{int}(C_s^0(\overline{\Omega})_+) = \left\{ u \in C_s^0(\overline{\Omega}) : \frac{u(x)}{\mathrm{d}_\Omega^s(x)} > 0 \text{ in } \overline{\Omega} \right\} \tag{2.3.10}$$

(see [111, Lemma 5.1]). The function $\frac{u}{\mathrm{d}_\Omega^s}$ on $\partial\Omega$ plays sometimes the role that the normal derivative $\frac{\partial u}{\partial \nu}$ plays in second order equations. Furthermore, we recall that another fractional normal derivative can be considered, namely the one in formula (1.2) of [74].

Lemma 2.3.5. *[173, Lemma 7.3, Hopf's lemma] Let u be any weak solution to (2.3.9), with $g \geq 0$. Then, either*

$$u \geq c\mathrm{d}_\Omega^s \qquad \text{in } \overline{\Omega} \text{ for some } \ c > 0 \quad \text{or} \quad u \equiv 0 \text{ in } \overline{\Omega}.$$

Furthermore, the quotient $\frac{u}{\mathrm{d}_\Omega^s}$ is not only bounded, but it is also Hölder continuous up to the boundary. Using the explicit solution u_0 and similar barriers, it is possible to show that solutions u satisfy $|u| \leq C\mathrm{d}_\Omega^s$ in Ω.

Theorem 2.3.6. *[173, Theorem 7.4] Let $s \in (0,1)$, and u be any weak solution to (2.3.9), with $g \in L^\infty(\Omega)$. Then,*

$$\left\| \frac{u}{\mathrm{d}_\Omega^s} \right\|_{C^\alpha(\overline{\Omega})} \leq C\|g\|_{L^\infty(\overline{\Omega})}, \quad \alpha \in (0,s).$$

Remark 2.3.7. The results, in [173], hold even if $a \geq 0$ in the kernel K.

We observe that Hopf's lemma involves strong maximum principle and we will see another general version of Hopf's lemma, where the nonlinearity is slightly negative, but this requires an higher regularity for f (see Chapter 5). Moreover, we recall [76, Proposition 2.5] for the fractional Laplacian analogy.

Now, we present a useful topological result, relating the minimizers of the energy functional J in the $X_K(\Omega)$-topology and in $C_s^0(\overline{\Omega})$-topology, respectively [88, Theorem 4.5]. This is an anisotropic version of the result of [113, Theorem 1.1], independently proved in [17, Proposition 2.5], which in turn is inspired from [33]. In the proof of Theorem 2.3.8 the critical case, i.e. $q = 2_s^*$ in (2.3.2), presents a twofold difficulty: a loss of compactness which prevents minimization of J, and the lack of uniform a priori estimate for the weak solutions of (2.3.1).

Theorem 2.3.8. *Let (2.3.2) hold, J be defined as above, and $u_0 \in X_K(\Omega)$. Then, the following conditions are equivalent:*

(i) there exists $\rho > 0$ such that $J(u_0 + v) \geq J(u_0)$ for all $v \in X_K(\Omega) \cap \mathrm{C}_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leq \rho$;

(ii) there exists $\epsilon > 0$ such that $J(u_0 + v) \geq J(u_0)$ for all $v \in X_K(\Omega)$, $\|v\|_{X_K(\Omega)} \leq \epsilon$.

We remark that, contrary to the result of [33] in the local case $s = 1$, there is no relationship between the topologies of $X_K(\Omega)$ and $C_s^0(\overline{\Omega})$.

Proof. We define $J \in C^1(X_K(\Omega))$ as in the Subsection 2.3.1.
We argue as in [113, Theorem 1.1].
(i) ⇒ (ii)
We suppose $u_0 = 0$. By noting that $J(u_0) = 0$, hence we can rewrite the hypothesis as

$$\inf_{u \in X_K(\Omega) \cap \overline{B}_\rho^s} J(u) = 0, \tag{2.3.11}$$

where $\overline{B}_\rho^s$ denotes the closed ball in $C_s^0(\overline{\Omega})$ centered at 0 with radius ρ.
We consider two cases.

- Let $q < 2_s^*$ in (2.3.2). By contradiction, we assume (i) and that there exists a sequence $(\epsilon_n) \in (0,\infty)$ such that $\epsilon_n \to 0$ and for all $n \in \mathbb{N}$

$$\inf_{u \in \overline{B}_{\epsilon_n}^X} J(u) = m_n < 0.$$

The subcritical growth condition (2.3.2) and the compact embedding $X_K(\Omega) \hookrightarrow L^q(\Omega)$ imply that J is sequentially weakly lower semicontinuous in $X_K(\Omega)$, hence m_n is attained at some $u_n \in \overline{B}_{\epsilon_n}^X$ for all $n \in \mathbb{N}$.
Claim: for all $n \in \mathbb{N}$ there exists $\mu_n \leq 0$ such that for all $v \in X_K(\Omega)$

$$\langle J'(u_n), v\rangle = \mu_n \langle u_n, v\rangle. \tag{2.3.12}$$

Indeed, if $u_n \in B_{\epsilon_n}^X$, then u_n is a local minimizer of J in $X_K(\Omega)$, so a critical point, hence (2.3.12) holds with $\mu_n = 0$. If $u_n \in \partial B_{\epsilon_n}^X$, then u_n minimizes J restricted to the C^1- Banach manifold

$$\left\{ u \in X_K(\Omega) : \frac{\|u\|^2_{X_K(\Omega)}}{2} = \frac{\epsilon_n^2}{2} \right\},$$

hence we can find a Lagrange multiplier $\mu_n \in \mathbb{R}$ such that (2.3.12) holds. Moreover, testing (2.3.12) with $-u_n$ and recalling that $J(u) \geq J(u_n)$ for all $u_n \in B_{\epsilon_n}^X$, we obtain

$$0 \leq \langle J'(u_n), -u_n\rangle = -\mu_n \|u_n\|^2_{X_K(\Omega)},$$

therefore $\mu_n \leq 0$.
The relation (2.3.12) is equivalent to $u_n \in X_K(\Omega)$ for all $n \in \mathbb{N}$ being a weak solution of

$$\begin{cases} L_K u = C_n f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

with $C_n = (1-\mu_n)^{-1} \in (0,1]$ and the nonlinearity satisfies (2.3.2) uniformly with respect to $n \in \mathbb{N}$. By Theorem 2.3.1 (and recalling that (u_n) is bounded in $L^{2_s^*}(\Omega)$), there exists $M > 0$ such that for all $n \in \mathbb{N}$ we get $u_n \in L^\infty(\Omega)$ with $\|u_n\|_\infty \leq M$. From which we deduce that for all $n \in \mathbb{N}$

$$\|C_n f(\cdot, u_n(\cdot))\|_\infty \leq C(1 + M^{q-1}).$$

Hence, by Proposition 2.3.4 and by Theorem 2.3.6, we have that $u_n \in C_s^\alpha(\overline{\Omega})$ and $\|u_n\|_{\alpha,s} \leq C(1+M^{q-1})$. By the compact embedding $C_s^\alpha(\overline{\Omega}) \hookrightarrow C_s^0(\overline{\Omega})$, passing to a subsequence, (u_n) converges in $C_s^0(\overline{\Omega})$ and uniformly in $\overline{\Omega}$. Since $u_n \to 0$ in $X_K(\Omega)$, up to a subsequence, $u_n(x) \to 0$ a.e. in Ω, consequently this implies that $u_n \to 0$ in $C_s^0(\overline{\Omega})$. Therefore, for $n \in \mathbb{N}$ big enough we obtain that $\|u_n\|_{0,s} \leq \rho$ together with $J(u_n) = m_n < 0$, a contradiction to (2.3.11).

- Let $q = 2^*_s$ in (2.3.2). We suppose by contradiction that there exist sequences (ϵ_n) in $(0,\infty)$ and (w_n) in $X_K(\Omega)$ such that for all $n \in \mathbb{N}$ we have $w_n \in \overline{B}^X_{\epsilon_n}$ and $J(w_n) < 0$. For all $k > 0$ we define the truncated functional $J_k \in C^1(X_K(\Omega))$

$$J_k(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F_k(x, u(x))\, dx \quad \forall u \in X_K(\Omega),$$

with $f_k, F_k : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, given by $f_k(x,t) = f(x, \mathrm{sgn(t)} \min\{|t|, k\})$, $F_k(x,t) = \int_0^t f_k(x,\tau)\, d\tau$ for all $(x,t) \in \overline{\Omega} \times \mathbb{R}$.
The dominated convergence Theorem implies that for all $u \in X_K(\Omega)$ $J_k(u) \to J(u)$ as $k \to \infty$. Hence, for all $n \in \mathbb{N}$ we can find $k_n \geq 1$ such that $J_{k_n}(w_n) < 0$. Since f_k has subcritical growth, for all $n \in \mathbb{N}$ there exists $u_n \in \overline{B}^X_{\epsilon_n}$ such that

$$J_{k_n}(u_n) = \inf_{u \in \overline{B}^X_{\epsilon_n}} J_{k_n}(u) \leq J_{k_n}(w_n) < 0.$$

By reasoning as in the previous case, we find a sequence $(C_n) \in (0,1]$ such that u_n is a weak solution of

$$\begin{cases} L_K u = C_n f_{k_n}(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

and the nonlinearities $C_n f_{k_n}$ satisfy (2.3.2) uniformly with respect to $n \in \mathbb{N}$. We know that $u_n \to 0$ in $X_K(\Omega)$, so in $L^{2^*_s}(\Omega)$; therefore (2.3.8) holds with $K_0 = 0$ and $n \in \mathbb{N}$ big enough. By Theorem 2.3.1, we have that $u_n \in L^\infty(\Omega)$ and $\|u_n\|_\infty \leq M$ for some $M > 0$ independent of $n \in \mathbb{N}$. Now, by arguing as in the subcritical case, we prove that (up to a subsequence) $u_n \to 0$ in $C^0_s(\overline{\Omega})$ and uniformly in $\overline{\Omega}$. Consequently, for $n \in \mathbb{N}$ big enough we obtain that $\|u_n\|_{0,s} \leq \rho$ and $\|u_n\|_\infty \leq 1$, hence

$$J(u_n) = J_{k_n}(u_n) < 0,$$

a contradiction to (2.3.11).

Now suppose $u_0 \neq 0$. Since $C^\infty_0(\Omega)$ is a dense subspace of $X_K(\Omega)$ (see Proposition 2.1.7) and $J'(u_0) \in X_K(\Omega)^*$,

$$\langle J'(u_0), v \rangle = 0 \tag{2.3.13}$$

holds, not only for all $v \in C^\infty_0(\Omega)$ (in particular $v \in X_K(\Omega) \cap C^0_s(\overline{\Omega})$), but for all $v \in X_K(\Omega)$, i.e., u_0 is a weak solution of (2.3.1). By L^∞- bounds, we have $u_0 \in L^\infty(\Omega)$, hence $f(., u_0(.)) \in L^\infty(\Omega)$. Now Proposition 2.3.4 and Theorem 2.3.6 imply that $u_0 \in C^0_s(\overline{\Omega})$. We set for all $v \in X_K(\Omega)$

$$\tilde{J}(v) = \frac{\|v\|^2_{X_K(\Omega)}}{2} - \int_\Omega \tilde{F}(x, v(x))\, dx,$$

with for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{F}(x,t) = F(x, u_0(x) + t) - F(x, u_0(x)) - f(x, u_0(x))t.$$

We note that $\tilde{J} \in C^1(X_K(\Omega))$ and the mapping $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ defined by $\tilde{f}(x,t) = \partial_t \tilde{F}(x,t)$ satisfies a growth condition of the type (2.3.2). Besides, by (2.3.13), we have for all $v \in X_K(\Omega)$

$$\begin{aligned}\tilde{J}(v) &= \frac{1}{2}(\|u_0 + v\|^2_{X_K(\Omega)} - \|u_0\|^2_{X_K(\Omega)}) - \int_\Omega (F(x, u_0 + v) - F(x, u_0))\, dx \\ &= J(u_0 + v) - J(u_0),\end{aligned}$$

in particular $\tilde{J}(0) = 0$. The hypothesis (i) thus rephrases as

$$\inf_{v \in X_K(\Omega) \cap \overline{B}^s_\rho} \tilde{J}(v) = 0$$

and by the previous cases, we obtain the thesis.

(ii) ⇒ (i)

By contradiction: we assume (ii) and we suppose there exists a sequence (u_n) in $X_K(\Omega) \cap C^0_s(\overline{\Omega})$ such that $u_n \to u_0$ in $C^0_s(\overline{\Omega})$ and $J(u_n) < J(u_0)$. Since

$$\int_\Omega F(x, u_n)\, dx \to \int_\Omega F(x, u_0)\, dx,$$

and $J(u_n) < J(u_0)$, we deduce that

$$\limsup_n \|u_n\|^2_{X_K(\Omega)} \le \|u_0\|^2_{X_K(\Omega)}.$$

In particular (u_n) is bounded in $X_K(\Omega)$, so (up to a subsequence) $u_n \rightharpoonup u_0$ in $X_K(\Omega)$, hence, by [30, Proposition 3.32], $u_n \to u_0$ in $X_K(\Omega)$. For $n \in \mathbb{N}$ big enough we have $\|u_n - u_0\|_{X_K(\Omega)} \le \epsilon$, a contradiction. □

Finally, notice that when u is regular enough then for all $\varphi \in X_K(\Omega)$ we have

$$\begin{aligned}\langle u, \varphi \rangle &= \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dxdy \\ &= \text{P.V.} \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(x) - u(y))\varphi(x) K(x-y)\, dxdy \\ &+ \text{P.V.} \iint_{\mathbb{R}^N \times \mathbb{R}^N} (u(y) - u(x))\varphi(y) K(x-y)\, dxdy \\ &= \int_{\mathbb{R}^N} L_K u(x)\varphi(x)\, dx + \int_{\mathbb{R}^N} L_K u(y)\varphi(y)\, dy \\ &= 2\int_\Omega L_K u(x)\varphi(x)\, dx,\end{aligned}$$

where we used that $K(y) = K(-y)$ and that $\varphi \equiv 0$ in Ω^c. This means that, when u is regular enough, the weak formulation (2.3.3) reads as

$$2\int_\Omega L_K u(x)\varphi(x)\, dx = \int_\Omega f(x,u)\varphi(x)\, dx \quad \text{for all } \varphi \in X_K(\Omega)$$

and thus is equivalent to $2L_K u = f(x,u)$ in Ω. As we will see, when the right hand side f is Hölder continuous then all solutions are classical solutions (in the sense that the operator L_K can be evaluated pointwisely).

2.3.2 Dirichlet problem driven by the fractional p-Laplacian

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $C^{1,1}$ boundary, $p \geqslant 2$, $s \in (0,1)$ and $N > ps$. We focus on the following Dirichlet problem driven by the nonlinear nonlocal operator $(-\Delta)_p^s$ (defined in (2.2.5))

$$
\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.3.14}
$$

The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$
|f(x,t)| \leq c_0(1 + |t|^{q-1}) \qquad (c_0 > 0, q \in (p, p_s^*)) \tag{2.3.15}
$$

Definition 2.3.9. *$u \in W_0^{s,p}(\Omega)$ is a (weak) solution of (2.3.14) if for all $v \in W_0^{s,p}(\Omega)$*

$$
\langle (-\Delta)_p^s u, v \rangle = \int_\Omega f(x,u)v\,dx.
$$

Now we recall well known results about a priori bound and regularity of solutions of (2.3.14).

Lemma 2.3.10. *[116, Lemma 2.1] Let f satisfy (2.3.15), $u \in W_0^{s,p}(\Omega)$ be a solution of (2.3.14). Then, $u \in L^\infty(\Omega)$ with $\|u\|_\infty \leqslant C$, for some $C = C(\|u\|_{W_0^{s,p}(\Omega)}) > 0$.*

Consider the following Dirichlet problem, with right-hand side $g \in L^\infty(\Omega)$:

$$
\begin{cases} (-\Delta)_p^s u = g(x) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{2.3.16}
$$

We have the following regularity result.

Lemma 2.3.11. *[115, Theorem 1.1] Let $g \in L^\infty(\Omega)$, $u \in W_0^{s,p}(\Omega)$ be a solution of (2.3.16). Then, $u \in C_s^\alpha(\overline{\Omega})$ with $\|u\|_{\alpha,s} \leqslant C\|g\|_\infty^{\frac{1}{p-1}}$, for some $\alpha \in (0,s]$, $C = C(\Omega) > 0$.*

Combining Lemmas 2.3.10, 2.3.11 we see that any solution of (2.3.14), under the assumption (2.3.15), lies in $C_s^\alpha(\overline{\Omega})$, with a uniform estimate on the $C_s^\alpha(\overline{\Omega})$-norm. Now we define an energy functional for problem (2.3.14) by setting for all $(x,t) \in \Omega \times \mathbb{R}$

$$
F(x,t) = \int_0^t f(x,\tau)\,d\tau,
$$

and for all $u \in W_0^{s,p}(\Omega)$

$$
J(u) = \frac{\|u\|_{W_0^{s,p}(\Omega)}^p}{p} - \int_\Omega F(x,u)\,dx.
$$

By hypothesis (2.3.15), it is easily seen that $J \in C^1(W_0^{s,p}(\Omega))$ and the solutions of (2.3.14) coincide with the critical points of J. We will need the following equivalence result for local minimizers of J in $W_0^{s,p}(\Omega)$ and in $C_s^0(\overline{\Omega})$.

Lemma 2.3.12. *[116, Theorem 1.1] Let f satisfy (2.3.15), $u \in W_0^{s,p}(\Omega)$. Then, the following are equivalent:*

(i) there exists $\sigma > 0$ such that $J(u+v) \geqslant J(u)$ for all $v \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leqslant \sigma$;

(ii) there exists $\rho > 0$ such that $J(u+v) \geqslant J(u)$ for all $v \in W_0^{s,p}(\Omega)$, $\|v\|_{W_0^{s,p}(\Omega)} \leqslant \rho$.

Since we are mainly interested in constant sign solutions, we will need a strong maximum principle and Hopf's lemma. Consider the problem

$$\begin{cases} (-\Delta)_p^s u = -c(x)|u|^{p-2}u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{2.3.17}$$

with $c \in C^0(\overline{\Omega})_+$. Then we have the following:

Lemma 2.3.13. *[67, Theorem 1.5] Let $c \in C^0(\overline{\Omega})_+$, $u \in \widetilde{W}^{s,p}(\Omega)_+ \setminus \{0\}$ be a supersolution of (2.3.17). Then, $u > 0$ in Ω and for any $x_0 \in \partial\Omega$*

$$\liminf_{\Omega \ni x \to x_0} \frac{u(x)}{\mathrm{d}_\Omega^s(x)} > 0.$$

Chapter 3

Nonlocal eigenvalue problems

Weighted eigenvalue problems can be studied for any type of linear elliptic (ordinary or partial) differential operator and even for integro-differential operators, exhibiting some kind of uniform ellipticity, and under various boundary conditions. In most cases, the resulting problem can be written as

$$\begin{cases} Lu = \lambda\rho(x)u & \text{in } \Omega \\ u \in X_0(\Omega), \end{cases}$$

where L is the chosen operator, Ω is a bounded domain, $\rho \in L^\infty(\Omega)$ is the weight function, and $X_0(\Omega)$ is some function space defined on Ω (which includes the boundary conditions). The problem above with $\rho \equiv 1$ has been investigated by Servadei and Valdinoci in [185] for a general nonlocal operator, which includes as a special case the fractional Laplacian. Molica Bisci et al., in [145], studied the same problem with a positive and Lipschitz continuous weight ρ and Iannizzotto and Papageorgiou, in [118], considered the case of a general positive function $\rho \in L^\infty(\Omega)$.

Such problem admits a sequence of variational eigenvalues, generally unbounded both from above and below (the sequence is bounded from below if ρ is non-negative, and from above if ρ is non-positive), denoted by

$$\ldots \leqslant \lambda_{-k}(\rho) \leqslant \ldots \leqslant \lambda_{-1}(\rho) < 0 < \lambda_1(\rho) \leqslant \ldots \leqslant \lambda_k(\rho) \leqslant \ldots$$

(we refer to [64]). Even nonlinear operators, as we will see in the final section of this chapter, under some homogeneity and monotonicity properties, admit an analogous sequence of variational eigenvalues, though it is not known whether they cover the whole spectrum or not (see [163]).

Clearly, every eigenvalue depends on the weight function, and it is an easy consequence of the variational characterization of eigenvalues that the mapping $\rho \mapsto \lambda_k(\rho)$ is monotone non-increasing for all integer $k \neq 0$, with respect to the pointwise order in $L^\infty(\Omega)$. A more delicate question is whether such dependence is *strictly decreasing*.

De Figueiredo and Gossez [65] have proved that, if L is a second order elliptic operator with bounded coefficients and Dirichlet boundary conditions, strict monotonicity of the eigenvalues with respect to the weight is equivalent to the *unique continuation property* (for short, u.c.p.) of eigenfunctions, i.e., the fact that eigenfunctions vanish at most in a negligible set. The result strongly relies on min-max characterizations of the eigenvalues of both signs.

Such equivalence is extremely important in the study of *nonlinear* boundary value problems of the type

$$\begin{cases} Lu = f(x,u) & \text{in } \Omega \\ u \in X_0(\Omega), \end{cases}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping, asymptotically linear in the second variable either at zero or at infinity. Many existence/multiplicity results for nonlinear boundary value problems are obtained by locating the limits

$$\lim_{t \to 0,\infty} \frac{f(x,t)}{t}$$

in known spectral intervals of the type $[\lambda_k(\rho), \lambda_{k+1}(\rho)]$, possibly involving several weight functions, and then by using strict monotonicity to avoid resonance phenomena. Thus, it is possible to compute the critical groups of the corresponding energy functional both at zero and at infinity, and so deduce the existence of nontrivial solutions (one typical application of this approach for the fractional Laplacian can be found in [118]).

Motivated by the considerations above, we devote this chapter to proving an analog of the results of [65] for general linear nonlocal operators. We study the following eigenvalue problem:

$$\begin{cases} L_K u = \lambda \rho(x) u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{3.0.1}$$

Here $\Omega \subset \mathbb{R}^N$ is a bounded domain with a Lipschitz continuous boundary, $s \in (0,1)$, $N > 2s$, the leading operator is the general nonlocal L_K, defined in Section 2.2.

Problem (3.0.1) depends on a weight function $\rho \in L^\infty(\Omega)$, and it admits a sequence of eigenvalues $(\lambda_k(\rho))_{k \in \mathbb{Z}_0}$ ($k \in \pm\mathbb{N}_0$ if ρ has constant sign). Here we prove equivalence between the strict monotonicity of the mapping $\rho \mapsto \lambda_k(\rho)$ ($k \in \mathbb{Z}_0$), and u.c.p. of eigenfunctions. Our proof is based on the functional setting, and the observation that the norm induced by the operator L_K is strictly stronger than the L^2 norm (here, we use $(\mathbf{H}_K)$ *(ii)*).

We note that, in general, u.c.p. for solutions of nonlocal problems is a challenging open problem, though some partial results have been established, mostly regarding the case of the fractional Laplacian.

Finally, for our future purposes we conclude this chapter with a section dedicated to some spectral properties of the eigenvalues of weighted problem driven by the nonlinear operator fractional p-Laplacian.

This chapter is organized as follows: in Section 3.1 we recall the general structure of the spectrum of problem (3.0.1); in Section 3.2 we prove our equivalence result; in Section 3.3 we survey some known results about u.c.p. for nonlocal operators; and in Section 3.4 we focus on the weighted eigenvalue problem driven by the fractional p-Laplacian.

3.1 General properties of the eigenvalues

We say that $u \in X_K(\Omega)$ is a (weak) solution of (3.0.1), if for all $v \in X_K(\Omega)$

$$\langle u, v\rangle = \lambda \int_\Omega \rho(x) uv \, dx.$$

If, for a given $\lambda \in \mathbb{R}$, problem (3.0.1) has a nontrivial solution $u \in X_K(\Omega) \setminus \{0\}$, then λ is an *eigenvalue* with associated *eigenfunction* u. The *spectrum* of (3.0.1) is the set of all eigenvalues, denoted $\sigma(\rho)$.

Following the general scheme of [64], we provide a characterization of $\sigma(\rho)$. In particular, we provide four min-max formulas for eigenvalues of both signs, that will be a precious tool in the proofs of our main results.

Proposition 3.1.1. *Let $\rho \in L^\infty(\Omega)$, $\rho \not\equiv 0$. Set for all integer $k > 0$*

$$\mathcal{F}_k = \big\{F \subset X_K(\Omega) : F \text{ linear subspace}, \dim(F) = k\big\},$$

and

$$\lambda_k^{-1}(\rho) = \sup_{F\in\mathcal{F}_k} \inf_{u\in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x)u^2 \, dx = \inf_{F\in\mathcal{F}_{k-1}} \sup_{u\in F^\perp,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x)u^2 \, dx, \tag{3.1.1}$$

$$\lambda_{-k}^{-1}(\rho) = \inf_{F\in\mathcal{F}_k} \sup_{u\in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x)u^2 \, dx = \sup_{F\in\mathcal{F}_{k-1}} \inf_{u\in F^\perp,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x)u^2 \, dx, \tag{3.1.2}$$

Then,

(i) if $\rho^+ \not\equiv 0$, then

$$0 < \lambda_1(\rho) < \lambda_2(\rho) \leqslant \ldots \leqslant \lambda_k(\rho) \leqslant \ldots \to \infty,$$

and for all $k \in \mathbb{N}_0$ $\lambda_k(\rho)$ is an eigenvalue of (3.0.1) with associated eigenfunction $e_{k,\rho} \in X_K(\Omega)$;

(ii) if $\rho^- \not\equiv 0$, then

$$0 > \lambda_{-1}(\rho) > \lambda_{-2}(\rho) \geqslant \ldots \geqslant \lambda_{-k}(\rho) \geqslant \ldots \to -\infty,$$

and for all $k \in \mathbb{N}_0$ $\lambda_{-k}(\rho)$ is an eigenvalue of (3.0.1) with associated eigenfunction $e_{-k,\rho} \in X_K(\Omega)$.

Moreover, all sup's and inf's in (3.1.1), (3.1.2) are attained (at $\lambda_{\pm k}(\rho)$-eigenfunctions). If $\rho \geqslant 0$ (resp. $\rho \leqslant 0$), then (3.0.1) admits only positive (resp. negative) eigenvalues. Finally, for all $k, h \in \mathbb{Z}_0$

$$\langle e_{k,\rho}, e_{h,\rho} \rangle = \delta_{kh}.$$

Proof. Given $u \in X_K(\Omega)$, the linear functional

$$v \mapsto \int_\Omega \rho(x) uv \, dx$$

is bounded in $X_K(\Omega)$. By Riesz' representation theorem there exists a unique $T(u) \in X_K(\Omega)$ such that

$$\langle T(u), v \rangle = \int_\Omega \rho(x) uv \, dx.$$

So we define a bounded linear operator $T \in \mathcal{L}(X_K(\Omega))$, indeed for all $u \in X_K(\Omega)$

$$\begin{aligned} ||T(u)||_{\mathcal{L}(X_K(\Omega))} &= \sup_{v \in X_K(\Omega),\, \|v\|_{X_K(\Omega)}=1} \Big| \int_\Omega \rho(x) uv \, dx \Big| \\ &\leq \|\rho\|_\infty \|u\|_2 \sup_{v \in X_K(\Omega),\, \|v\|_{X_K(\Omega)}=1} \|v\|_2 \leq C\|u\|_{X_K(\Omega)}. \end{aligned}$$

Clearly T is symmetric. Moreover, T is compact. Indeed, let (u_n) be a bounded sequence in $X_K(\Omega)$, then (passing to a subsequence) $u_n \rightharpoonup u$ in $X_K(\Omega)$, $u_n \to u$ in $L^2(\Omega)$. So we have for all $v \in X_K(\Omega)$, $\|v\|_{X_K(\Omega)} \leq 1$

$$\begin{aligned} \big| \langle T(u_n) - T(u), v \rangle \big| &\leq \int_\Omega |\rho(x)(u_n - u)v| \, dx \\ &\leq \|\rho\|_\infty \|u_n - u\|_2 \|v\|_2 \leq C\|u_n - u\|_2, \end{aligned}$$

and the latter tends to 0 as $n \to \infty$. So $T(u_n) \to T(u)$ in $X_K(\Omega)$. First assume $\rho^+ \not\equiv 0$, then

$$\mu_1 = \sup_{u \in X_K(\Omega), \|u\|_{X_K(\Omega)}=1} \langle T(u), u \rangle > 0.$$

By [64, Lemma 1.1], there exists $e_{1,\rho} \in X_K(\Omega)$ such that $T(e_{1,\rho}) = \mu_1 e_{1,\rho}$, $\|e_{1,\rho}\|_{X_K(\Omega)} = 1$. Further, set for all $k > 0$

$$\mu_k = \sup_{F \in \mathcal{F}_k} \inf_{u \in F, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x) u^2 \, dx > 0.$$

Then, by [64, Propositions 1.3, 1.8], there exists $e_{k,\rho} \in X_K(\Omega)$ such that $T(e_{k,\rho}) = \mu_k e_{k,\rho}$. Applying [64, Lemma 1.4], we see that (μ_k) is a sequence of eigenvalues of T, such that $\mu_k \geqslant \mu_{k+1}$ and $\mu_k \to 0^+$. Besides, for all $k > 0$, the eigenspace associated to μ_k has finite dimension (hence it admits an orthonormal basis). So, by relabeling $(e_{k,\rho})$ if necessary, we have for all $k, h \in \mathbb{Z}_0$

$$\langle e_{k,\rho}, e_{h,\rho} \rangle = \delta_{kh},$$

which in turn implies for all $k \neq h$

$$\int_\Omega \rho(x) e_{k,\rho}, e_{h,\rho}\, dx = 0.$$

Now set $\lambda_k(\rho) = \mu_k^{-1}$. Then, (3.1.1) follows from the definition of μ_k and [64, Propositions 1.7, 1.8]. Besides, we have for all $v \in X_K(\Omega)$

$$\langle e_{k,\rho}, v\rangle = \lambda_k(\rho) \int_\Omega \rho(x) e_{k,\rho} v\, dx,$$

so $\lambda_k(\rho) \in \sigma(\rho)$ with associated eigenfunction $e_{k,\rho}$. Moreover, $\lambda_k(\rho) \to \infty$ as $k \to \infty$, all eigenspaces are finite-dimensional, and eigenfunctions associated to different eigenvalues are orthogonal. Also, all sup's and inf's in (3.1.1) are attained at (subspaces generated by) eigenfunctions, as pointed out in [64, Remark, p. 40].

Finally, reasoning as in [118, Proposition 2.8] it is easily seen that $\lambda_1(\rho) < \lambda_2(\rho)$ and that there are no positive eigenvalues other than $\lambda_k(\rho)$, $k > 0$.

Similarly, if $\rho^- \not\equiv 0$, then (3.1.2) defines a sequence $(\lambda_{-k}(\rho))$ of negative eigenvalues of (3.0.1) such that $\lambda_{-k}(\rho) \to -\infty$, with an orthonormal sequence $(e_{-k,\rho})$ of associated eigenfunctions.

By [64, Proposition 1.11], if $\rho \geq 0$ there are no negative eigenvalues, similarly if $\rho \leq 0$ there are no positive eigenvalues (see Proposition 4.1.3 in the case of fractional Laplacian). $\square$

In particular, when $\rho \in L^\infty(\Omega)_+ \setminus \{0\}$ in (3.0.1), we have the following alternative variational characterization.

Proposition 3.1.2. *The set of the eigenvalues of problem* (3.0.1) *consists of a sequence* $(\lambda_k(\rho))$ *with*

$$0 < \lambda_1(\rho) < \lambda_2(\rho) \leq \cdots \leq \lambda_k(\rho) \leq \lambda_{k+1}(\rho) \leq \cdots \quad \text{and} \quad \lambda_k(\rho) \to +\infty \quad \text{as} \quad k \to +\infty,$$

with associated eigenfunctions $e_{1,\rho}, e_{2,\rho}, \cdots, e_{k,\rho}, e_{k+1,\rho}, \cdots$ *such that*

(i) the eigenvalues can be characterized as follows:

$$\lambda_1(\rho) = \min_{u \in X_K(\Omega),\quad \|u\|_{2,\rho}=1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\, dxdy, \tag{3.1.3}$$

$$\lambda_{k+1}(\rho) = \min_{u \in \mathbb{P}_{k+1,\rho},\quad \|u\|_{2,\rho}=1} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |u(x) - u(y)|^2 K(x-y)\, dxdy \quad \forall k \in \mathbb{N}, \tag{3.1.4}$$

where $\mathbb{P}_{k+1,\rho} := \{u \in X_K(\Omega) \text{ such that } \langle u, e_{j,\rho}\rangle_{X_K(\Omega)} = 0\ \forall j = 1, \cdots, k\}$ *and* $\|u\|_{2,\rho}^2 = \int_\Omega \rho(x) u^2\, dx$;

(ii) there exists a non-negative function $e_{1,\rho} \in X_K(\Omega)$*, which is an eigenfunction corresponding to* $\lambda_1(\rho)$*, attaining the minimum in* (3.1.3)*, that is* $\|e_{1,\rho}\|_{2,\rho} = 1$*; moreover, for any* $k \in \mathbb{N}$ *there exists a nodal function* $e_{k+1,\rho} \in \mathbb{P}_{k+1,\rho}$*, which is an eigenfunction corresponding to* $\lambda_{k+1}(\rho)$*, attaining the minimum in* (3.1.4)*, that is* $\|e_{k+1,\rho}\|_{2,\rho} = 1$*;*

(iii) $\lambda_1(\rho)$ *is simple and isolated (as an element of the spectrum), namely the eigenfunctions* $u \in X_K(\Omega)$ *corresponding to* $\lambda_1(\rho)$ *are* $u = \zeta e_{1,\rho}$*, with* $\zeta \in \mathbb{R}$*;*

(iv) the sequence $(e_{k,\rho})$ *of eigenfunctions corresponding to* $\lambda_k(\rho)$ *is an orthonormal basis of* $L^2_\rho(\Omega)$ *(the space* $L^2(\Omega)$ *endowed with the equivalent weighted norm* $\|u\|_{2,\rho}$*) and an orthogonal basis of* $X_K(\Omega)$*;*

(v) each eigenvalue $\lambda_k(\rho)$ *has finite multiplicity, more precisely, if* $\lambda_k(\rho)$ *is such that*

$$\lambda_{k-1}(\rho) < \lambda_k(\rho) = \cdots = \lambda_{k+h,}(\rho) < \lambda_{k+h+1}(\rho)$$

for some $h \in \mathbb{N}_0$*, then the set of all the eigenfunctions corresponding to* $\lambda_k(\rho)$ *agrees with*

$$\mathrm{span}\{e_{k,\rho}, \dots, e_{k+h,\rho}\}.$$

When $\rho \equiv 1$ we set $\lambda_1(\rho) = \lambda_1$ and $e_{1,\rho} = e_1$. Moreover, the second eigenvalue λ_2 admits the following variational characterization

$$\lambda_2 = \inf_{\gamma \in \Gamma_1} \sup_{t \in [0,1]} \|\gamma(t)\|^2_{X_K(\Omega)}, \tag{3.1.5}$$

where Γ_1 is the family of paths $\gamma \in C([0,1], X_K(\Omega))$ such that $\gamma(0) = e_1$, $\gamma(1) = -e_1$, and $\|\gamma(t)\|_2 = 1$ for all $t \in [0,1]$ (see [100]).

Remark 3.1.3. The proof of this result can be found in [185]. Due to the kind of kernel considered, we point out the following differences. For L_K with a general kernel K, satisfying $(\mathbf{H}_K)$, the first eigenfunction e_1 is non-negative and every eigenfunction is bounded, there aren't any better regularity results [185]. While for the particular kernel $K(y) = a(\frac{y}{|y|})\frac{1}{|y|^{N+2s}}$ we note that the first eigenfunction is positive and all eigenfunctions belong to $C^s(\overline{\Omega})$, like in the case of fractional Laplacian. More precisely, $e_1 \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, by applying Lemma 2.3.5 and Theorem 2.3.6.

Now we prove continuous dependence of the eigenvalues on ρ, with respect to the norm topology of $L^\infty(\Omega)$ (in the forthcoming results, we say that $k \in \mathbb{Z}_0$ is *admissible* if the corresponding eigenvalue does exist):

Proposition 3.1.4. *Let* (ρ_n) *be a sequence in* $L^\infty(\Omega)$ *such that* $\rho_n \to \rho$ *in* $L^\infty(\Omega)$*. Then, for all admissible* $k \in \mathbb{Z}_0$ *we have* $\lambda_k(\rho_n) \to \lambda_k(\rho)$*.*

Proof. For simplicity, assume $\rho_n^+ \not\equiv 0$ for all $n \in \mathbb{N}$, $\rho^+ \not\equiv 0$, and $k > 0$ (other cases are studied similarly). Set for all $u, v \in X_K(\Omega)$

$$\langle T_n(u), v\rangle = \int_\Omega \rho_n(x) uv \, dx,$$

then $T_n \in \mathcal{L}(X_K(\Omega))$ is a bounded, symmetric, compact operator. Similarly we define $T \in \mathcal{L}(X_K(\Omega))$ using ρ. We claim that

$$T_n \to T \ \text{ in } \mathcal{L}(X_K(\Omega)). \tag{3.1.6}$$

Indeed, for any $n \in \mathbb{N}$ and $u \in X_K(\Omega)$, $\|u\|_{X_K(\Omega)} = 1$, we have by the Cauchy-Schwarz inequality

$$\begin{aligned}\|T_n(u) - T(u)\|_{\mathcal{L}(X_K(\Omega))} &= \sup_{v \in X_K(\Omega),\, \|v\|_{X_K(\Omega)}=1} \Big| \int_\Omega \rho_n(x) uv \, dx - \int_\Omega \rho(x) uv \, dx \Big| \\ &\leq \sup_{v \in X_K(\Omega),\, \|v\|_{X_K(\Omega)}=1} \|\rho_n - \rho\|_\infty \|u\|_2 \|v\|_2 \leq C \|\rho_n - \rho\|_\infty,\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. Now fix $k > 0$: we have for all $n \in \mathbb{N}$

$$|\lambda_k^{-1}(\rho_n) - \lambda_k^{-1}(\rho)| \leq \|T_n - T\|_{\mathcal{L}(X_K(\Omega))}. \tag{3.1.7}$$

We argue as in [103, Theorem 2.3.1]. Recalling (3.1.1), there exists $F \in \mathcal{F}_k$ such that

$$\lambda_k^{-1}(\rho) = \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x) u^2 \, dx.$$

By compactness, there exists $\hat{u} \in F$, $\|\hat{u}\|_{X_K(\Omega)} = 1$ such that

$$\int_\Omega \rho_n(x) \hat{u}^2 \, dx = \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho_n(x) u^2 \, dx.$$

So we have for all $n \in \mathbb{N}$

$$\begin{aligned}\lambda_k^{-1}(\rho) - \lambda_k^{-1}(\rho_n) &\leq \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x) u^2 \, dx - \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho_n(x) u^2 \, dx \\ &\leq \int_\Omega \rho(x) \hat{u}^2 \, dx - \int_\Omega \rho_n(x) \hat{u}^2 \, dx = \langle T(\hat{u}) - T_n(\hat{u}), \hat{u} \rangle \leq \|T - T_n\|_{\mathcal{L}(X_K(\Omega))}.\end{aligned}$$

An analogous argument leads to

$$\lambda_k^{-1}(\rho) - \lambda_k^{-1}(\rho_n) \geq -\|T - T_n\|_{\mathcal{L}(X_K(\Omega))},$$

proving (3.1.7). Now (3.1.6), (3.1.7) imply $\lambda_k(\rho_n) \to \lambda_k(\rho)$ as $n \to \infty$. □

Remark 3.1.5. In fact, continuous dependence can be proved even with respect to weaker types of convergence, such as weak* convergence of the weights (see Chapter 4). Anyway, continuity in the norm topology is enough for our purposes in this chapter.

3.2 Strict monotonicity and unique continuation property

This section is devoted to proving our main result, i.e., the equivalence between strict monotonicity of the map $\rho \mapsto \lambda_k(\rho)$ ($k \in \mathbb{Z}_0$) and u.c.p. of the eigenfunctions. Our definition of u.c.p. is the following:

Definition 3.2.1. *We say that $\rho \in L^\infty(\Omega) \setminus \{0\}$ satisfies u.c.p., if for any eigenfunction $u \in X_K(\Omega)$ of (3.0.1) (with any $\lambda \in \sigma(\rho)$)*

$$\big|\{u = 0\}\big| = 0.$$

We follow the approach of [65]. First we note that, by (3.1.1) and (3.1.2), given $\rho, \tilde{\rho} \in L^\infty(\Omega) \setminus \{0\}$,

$$\rho \leq \tilde{\rho} \ \Rightarrow \ \lambda_k(\rho) \geq \lambda_k(\tilde{\rho}) \text{ for all admissible } k \in \mathbb{Z}_0. \tag{3.2.1}$$

First we prove that u.c.p. implies strict monotonicity.

Theorem 3.2.2. *Let $\rho, \tilde{\rho} \in L^\infty(\Omega) \setminus \{0\}$ be such that $\rho \leq \tilde{\rho}$, $\rho \not\equiv \tilde{\rho}$, and either ρ or $\tilde{\rho}$ satisfies u.c.p. Then, $\lambda_k(\rho) > \lambda_k(\tilde{\rho})$ for all admissible $k \in \mathbb{Z}_0$.*

Proof. Assume ρ has u.c.p., $\rho^+, \tilde{\rho}^+ \not\equiv 0$, $k > 0$. By (3.1.1), there exists $F \in \mathcal{F}_k$ such that

$$\lambda_k^{-1}(\rho) = \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x) u^2 \, dx. \tag{3.2.2}$$

Fix $u \in F$, $\|u\|_{X_K(\Omega)} = 1$. Two cases may occur:

(a) if u is a minimizer in (3.2.2), then u is a $\lambda_k(\rho)$-eigenfunction, hence $|\{u = 0\}| = 0$. So we have $\rho u^2 \leq \tilde{\rho} u^2$, with strict inequality on a subset of Ω with positive measure, hence

$$\lambda_k^{-1}(\rho) = \int_\Omega \rho(x) u^2 \, dx < \int_\Omega \tilde{\rho}(x) u^2 \, dx;$$

(b) if u is not a minimizer in (3.2.2), then

$$\lambda_k^{-1}(\rho) < \int_\Omega \rho(x) u^2 \, dx \leq \int_\Omega \tilde{\rho}(x) u^2 \, dx.$$

In both cases, we have

$$\lambda_k^{-1}(\rho) < \int_\Omega \tilde{\rho}(x) u^2 \, dx.$$

Since F has finite dimension, the set of u's above is compact. Recalling also (3.1.1) with weight $\tilde{\rho}$, we have

$$\lambda_k^{-1}(\rho) < \inf_{u \in F,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \tilde{\rho}(x) u^2 \, dx \leq \lambda_k^{-1}(\tilde{\rho}).$$

Now we assume $\rho^-, \tilde{\rho}^- \not\equiv 0$ and consider negative eigenvalues, i.e., $k < 0$. Set $j = -k$ for simplicity. By (3.1.2), there exists $F \in \mathcal{F}_{j-1}$ such that

$$\lambda_{-j}^{-1}(\rho) = \inf_{u \in F^\perp,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \rho(x) u^2 \, dx.$$

Arguing as above, we see that for all $u \in F^\perp$, $\|u\|_{X_K(\Omega)} = 1$

$$\lambda_{-j}^{-1}(\rho) < \int_\Omega \tilde{\rho}(x) u^2 \, dx.$$

By (3.1.2), we have

$$\lambda_{-j}^{-1}(\tilde{\rho}) \geqslant \inf_{u \in F^\perp,\, \|u\|_{X_K(\Omega)}=1} \int_\Omega \tilde{\rho}(x) u^2 \, dx.$$

From Proposition 3.1.1 we know that there exists $\tilde{u} \in F^{\perp}$, $||\tilde{u}|| = 1$ such that

$$\lambda_{-j}^{-1}(\tilde{\rho}) = \int_{\Omega} \tilde{\rho}(x)\tilde{u}^2\, dx > \lambda_{-j}^{-1}(\rho).$$

Hence, $\lambda_{-j}(\rho) > \lambda_{-j}(\tilde{\rho})$. The case when ρ does not satisfy u.c.p. is treated similarly. $\square$

The next result establishes the reverse implication.

Theorem 3.2.3. *Assume that $\rho \in L^{\infty}(\Omega) \setminus \{0\}$ does not satisfy u.c.p. Then, there exist $\tilde{\rho} \in L^{\infty}(\Omega) \setminus \{0\}$ such that either $\rho \leq \tilde{\rho}$ or $\rho \geq \tilde{\rho}$, $\rho \not\equiv \tilde{\rho}$, and $k \in \mathbb{Z}_0$ such that $\lambda_k(\rho) = \lambda_k(\tilde{\rho})$.*

Proof. By Definition 3.2.1, we can find $k \in \mathbb{Z}_0$ and a $\lambda_k(\rho)$-eigenfunction $u \in X_K(\Omega)$ such that $|A| > 0$, where $A := \{u = 0\}$. First assume $\rho^+ \not\equiv 0$, $k > 0$, and without loss of generality $\lambda_k(\rho) < \lambda_{k+1}(\rho)$. For all $\varepsilon \in \mathbb{R}$ set

$$\rho_{\varepsilon}(x) = \begin{cases} \rho(x) & \text{if } x \in \Omega \setminus A \\ \rho(x) + \varepsilon & \text{if } x \in A, \end{cases}$$

so $\rho_{\varepsilon} \in L^{\infty}(\Omega)$ and $\rho_{\varepsilon} \to \rho$ in $L^{\infty}(\Omega)$ as $\varepsilon \to 0$. By Proposition 3.1.4

$$\lim_{\varepsilon \to 0} \lambda_{k+1}(\rho_{\varepsilon}) = \lambda_{k+1}(\rho) > \lambda_k(\rho),$$

so we can find $\varepsilon \in (0, 1)$ such that $\lambda_{k+1}(\rho_{\varepsilon}) > \lambda_k(\rho)$. Set $\tilde{\rho} = \rho_{\varepsilon} \in L^{\infty}(\Omega) \setminus \{0\}$, so $\rho \leqslant \tilde{\rho}$, $\rho \not\equiv \tilde{\rho}$. For all $v \in X_K(\Omega)$ we have

$$\langle u, v \rangle = \lambda_k(\rho) \int_{\Omega} \rho(x) uv\, dx = \lambda_k(\rho) \int_{\Omega} \tilde{\rho}(x) uv\, dx,$$

so $\lambda_k(\rho) \in \sigma(\tilde{\rho})$ with associated eigenfunction u. We can find $h \in \mathbb{N}_0$ such that

$$\lambda_k(\rho) = \lambda_h(\tilde{\rho}) < \lambda_{h+1}(\tilde{\rho}),$$

in particular $\lambda_h(\tilde{\rho}) < \lambda_{k+1}(\tilde{\rho})$, which implies $h \leq k$. Besides, by (3.2.1) we have

$$\lambda_k(\tilde{\rho}) \leq \lambda_k(\rho) = \lambda_h(\tilde{\rho}),$$

hence $k \leq h$. Summarizing, $h = k$, thus $\lambda_k(\rho) = \lambda_k(\tilde{\rho})$.

Now assume $\rho^- \not\equiv 0$ and $k < 0$. Set $j = -k$, for simplicity of notation, and without loss of generality $\lambda_{-j-1}(\rho) < \lambda_{-j}(\rho)$. Arguing as above (with $\varepsilon < 0$), we find $\tilde{\rho} \in L^{\infty}(\Omega) \setminus \{0\}$ such that $\tilde{\rho} \leq \rho$, $\tilde{\rho} \not\equiv \rho$, and $\lambda_{-j-1}(\tilde{\rho}) < \lambda_{-j}(\rho)$. For all $v \in X_K(\Omega)$ we have

$$\langle u, v \rangle = \lambda_{-j}(\rho) \int_{\Omega} \rho(x) uv\, dx = \lambda_{-j}(\rho) \int_{\Omega} \tilde{\rho}(x) uv\, dx,$$

so there exists $i \in \mathbb{N}_0$ such that $\lambda_{-j}(\rho) = \lambda_{-i}(\tilde{\rho})$, with u as an associated eigenfunction. By $\lambda_{-i}(\tilde{\rho}) > \lambda_{-j-1}(\tilde{\rho})$ we have $-i \geq -j$, while by (3.2.1) we have

$$\lambda_{-j}(\tilde{\rho}) \geq \lambda_{-j}(\rho) = \lambda_{-i}(\tilde{\rho}),$$

hence $-j \geq -i$. Thus $-i = -j$ and $\lambda_{-j}(\rho) = \lambda_{-j}(\tilde{\rho})$. Clearly, if ρ has constant sign only one of the previous argument applies. $\square$

Remark 3.2.4. A partial result for Theorem 3.2.2 was given in [118, Proposition 2.10] for the fractional Laplacian, with two positive weights one of which is in $C^1(\Omega)$.

3.3 Unique continuation for nonlocal operators

This section is devoted to a brief survey on recent results on u.c.p. for nonlocal operators. Note that, even in the local case, there are counterexamples of solutions to elliptic equations vanishing in non-negligible sets, see [139, 194], not to mention that the question of u.c.p. of eigenfunctions is still open for the p-Laplacian. Browsing the literature, many results of this type are encountered, dealing in most cases with the fractional Laplacian $(-\Delta)^s$ (which, as seen before, corresponds to our L_K with the kernel $K(x) = |x|^{-N-2s}$). First we recall the main notions of u.c.p. considered in the literature:

Definition 3.3.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a domain and $\mathcal{S}$ a family of measurable functions on Ω:*

(i) $\mathcal{S}$ satisfies the strong unique continuation property (s.u.c.p.), if no function $u \in \mathcal{S}\setminus\{0\}$ has a zero of infinite order in Ω;

(ii) $\mathcal{S}$ satisfies the unique continuation property (u.c.p.), if no function $u \in \mathcal{S} \setminus \{0\}$ vanishes on a subset of Ω with positive measure;

(iii) $\mathcal{S}$ satisfies the weak unique continuation property (w.u.c.p.), if no function $u \in \mathcal{S}\setminus\{0\}$ vanishes on an open subset of Ω.

Definition 3.2.1 corresponds to the case *(ii)*. We recall that a function $u \in L^2(\Omega)$ has a zero of infinite order at $x_0 \in \Omega$ if for all $n \in \mathbb{N}$

$$\int_{B_r(x_0)} u^2\,dx = O(r^n) \text{ as } r \to 0^+.$$

The relations between the properties depicted in Definition 3.3.1 are the following:

$$\text{s.u.c.p. or u.c.p. } \Rightarrow \text{w.u.c.p.}$$

We recall now some recent results on nonlocal unique continuation.

In [80], Fall and Felli consider fractional Laplacian equations involving regular, lower order perturbations of a Hardy-type potential, of the following type:

$$(-\Delta)^s u - \frac{\lambda}{|x|^{2s}} u = h(x)u + f(x,u) \text{ in } \Omega,$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain such that $0 \in \Omega$, $0 < s < \min\{1, N/2\}$, $\lambda < 2^{2s}\Gamma^2(\frac{N+2s}{4})/\Gamma^2(\frac{N-2s}{4})$, and $h \in C^1(\Omega \setminus \{0\})$, $f \in C^1(\Omega \times \mathbb{R})$ satisfy the estimates

$$|h(x)| + |x \cdot \nabla h(x)| \lesssim |x|^{-2s+\varepsilon} \ (\varepsilon > 0),$$
$$|f(x,t)t| + |\partial_t f(x,t)t^2| + |\nabla_x F(x,t) \cdot x| \lesssim |t|^p \ \Big(2 < p < \frac{2N}{N-2s}\Big),$$

where $F(x,\cdot)$ is the primitive of $f(x,\cdot)$. The main results asserts that, if u is a solution of the equation above and u vanishes of infinite order at 0, then $u \equiv 0$ (s.u.c.p.). The proof relies on the Caffarelli-Silvestre extension operator, exploited in order to define an adapted notion of frequency function, admitting a limit as $r \to 0^+$.

Another result of Fall and Felli [81] deals with a relativistic Schrödinger equation involving a fractional perturbation of $(-\Delta)^s$ and an anisotropic potential:

$$(-\Delta + m^2)^s u - a\Big(\frac{x}{|x|}\Big)\frac{u}{|x|^{2s}} - h(x)u = 0 \text{ in } \mathbb{R}^N,$$

where Ω, s are as above, $m \geqslant 0$, $a \in C^1(S^{N-1})$ and $h \in C^1(\Omega)$ satisfies a similar estimate as above. The authors give a precise description of the asymptotic behavior of solutions near the origin, and deduce again s.u.c.p. These results do not apply in our framework, even restricting ourselves to the fractional Laplacian, since they involve *smooth* weight functions, differentiability being required in order to derive Pohozaev-type identities.

Instead, Seo [181] considers possibly *nonsmooth* weights in the fractional inequality

$$|(-\Delta)^s u| \leqslant |V(x)u| \text{ in } \mathbb{R}^N,$$

where $N \geqslant 2$, $N-1 \leqslant 2s < N$, and the measurable weight function V satisfies

$$\lim_{r\to 0^+} \sup_{x\in\mathbb{R}^N} \int_{B_r(x)} \frac{|V(y)|}{|x-y|^{N-2s}}\, dy = 0.$$

By means of strong Carleman estimates, the author proves w.u.c.p. for solutions of the above inequality with $u, (-\Delta)^s u \in L^1(\mathbb{R}^N)$. Moreover, Seo [182] obtained a special u.c.p. result for potentials V in Morrey spaces.

The problem of nonsmooth weights is also the focus of the work of Rüland [178], dealing with the fractional Schrödinger-type equation

$$(-\Delta)^s u = V(x)u \text{ in } \mathbb{R}^N,$$

with a measurable function $V = V_1 + V_2$ satisfying

$$V_1(x) = |x|^{-2s} h\Big(\frac{x}{|x|}\Big)\ (h \in L^\infty(S^{N-1})),\ |V_2(x)| \leqslant c|x|^{-2s+\varepsilon}\ (c, \varepsilon > 0).$$

For $s < 1/2$, the following additional conditions are assumed: either $V_2 \in C^1(\mathbb{R}^N \setminus \{0\})$ satisfies $|x \cdot \nabla V_2(x)| \lesssim |x|^{-2s+\varepsilon}$, or $s \geqslant 1/4$ and $V_1 \equiv 0$. Under such assumptions, any solution $u \in H^s(\mathbb{R}^N)$ vanishing of infinite order at 0 is in fact $u \equiv 0$ (s.u.c.p.). Rüland's approach, based on Carleman estimates, allows for non-smooth weights and generalization to anisotropic operators. A w.u.c.p. result for $(-\Delta)^s$ ($s \in (0,1)$), as well as s.u.c.p. for the square root of the Laplacian $(-\Delta)^{1/2}$, with a weight in $L^{N+\varepsilon}(\mathbb{R}^N)$, appear in Rüland [177].

The result of Ghosh, Rüland, Salo, and Uhlmann [98, Theorem 3] is the closest to our framework. For any $V \in L^\infty(\Omega)$ and any $s \in [1/4, 1)$, if $u \in H^s(\mathbb{R}^N)$ solves

$$(-\Delta)^s u = V(x)u \text{ in } \Omega$$

and vanishes on a subset of Ω with positive measure, then $u \equiv 0$ (u.c.p.). Here the approach is based on Carleman estimates again, along with a boundary u.c.p. for solutions of the (local) degenerate elliptic equation

$$\nabla \cdot \left(x_{N+1}^{1-2s} \nabla u \right) = 0 \text{ in } \mathbb{R}^{N+1}_+,$$

with homogeneous Robin conditions. By combining the results of [98] with our Theorem 3.2.2, then, we have:

Corollary 3.3.2. *Let L_K be defined by $s \in [1/4, 1)$ and $K(x) = |x|^{-N-2s}$, $\rho, \tilde{\rho} \in L^\infty(\Omega)$ be such that $\rho \leqslant \tilde{\rho}$, $\rho \not\equiv \tilde{\rho}$. Then, $\lambda_k(\rho) > \lambda_k(\tilde{\rho})$ for all admissible $k \in \mathbb{Z}_0$.*

For completeness we also mention the work of Yu [198], where s.u.c.p. is proved for fractional powers of linear elliptic operators with Lipschitz continuous coefficients (the power being meant in the spectral sense). We note that the Dirichlet fractional Laplacian $(-\Delta)^s$ in a bounded domain $\Omega \subset \mathbb{R}^N$ does not fall in this class, as observed in [186].

3.4 A weighted eigenvalue problem driven by a nonlinear operator

In this final section we focus on some spectral properties of $(-\Delta)^s_p$ (see [66, 105] and [93, Proposition 3.4]). Let $\rho \in L^\infty(\Omega)_+ \setminus \{0\}$ and consider the following weighted eigenvalue problem:

$$\begin{cases} (-\Delta)^s_p\, u = \lambda \rho(x) |u|^{p-2} u & \text{in } \Omega \\ u = 0 & \text{on } \Omega^c, \end{cases} \tag{3.4.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,1}$ boundary $\partial\Omega$, $N > ps$, $p \geq 2$ and $s \in (0,1)$. Finally, λ is a real parameter. We say that λ is an *eigenvalue* of $(-\Delta)^s_p$ related to the weight ρ if (3.4.1) has a nontrivial solution $u \in W^{s,p}_0(\Omega) \setminus \{0\}$ and such solution u is called an *eigenfunction* corresponding to the eigenvalue λ.

Lemma 3.4.1. *Let $\rho \in L^\infty(\Omega)_+ \setminus \{0\}$. Then, (3.4.1) has an unbounded sequence of variational eigenvalues*

$$0 < \lambda_1(\rho) < \lambda_2(\rho) \leqslant \ldots \leqslant \lambda_k(\rho) \leqslant \ldots$$

The first eigenvalue admits the following variational characterization:

$$\lambda_1(\rho) = \inf_{u \in W^{s,p}_0(\Omega) \setminus \{0\}} \frac{\|u\|^p_{W^{s,p}_0(\Omega)}}{\int_\Omega \rho(x) |u|^p \, dx},$$

and

(i) $\lambda_1(\rho) > 0$ is simple, isolated and attained at an unique positive eigenfunction $\hat{u}_1(\rho) \in W^{s,p}_0(\Omega) \cap \operatorname{int}(C^0_s(\overline{\Omega})_+)$ such that $\int_\Omega \rho(x) |\hat{u}_1|^p \, dx = 1$;

(ii) if $u \in W_0^{s,p}(\Omega) \setminus \{0\}$ is an eigenfunction of (3.4.1) associated to any eigenvalue $\lambda > \lambda_1(\rho)$, then u is nodal;

(iii) if $\tilde{\rho} \in L^\infty(\Omega)_+ \setminus \{0\}$ is such that $\tilde{\rho} \leqslant \rho$, $\tilde{\rho} \not\equiv \rho$, then $\lambda_1(\rho) < \lambda_1(\tilde{\rho})$.

Proof. We refer to [66, 105] for the proof of (i) and (ii).
We show the third point. Let $\rho, \tilde{\rho} \in L^\infty(\Omega)_+$ be such that $\rho, \tilde{\rho} \neq 0$, $\rho(x) \leq \tilde{\rho}(x)$ for a.e. $x \in \Omega$, $\rho \not\equiv \tilde{\rho}$. Let u_1 and u_2 the first eigenfunctions corresponding to the weights ρ and $\tilde{\rho}$, respectively. By (i) such eigenfunctions are positive a.e., hence u_1 and u_2 clearly satisfy the u.c.p. From the definition of λ_1, we obtain

$$\lambda_1(\rho) = \frac{\|u_1\|^p_{W_0^{s,p}(\Omega)}}{\int_\Omega \rho(x) u_1^p \, dx} > \frac{\|u_1\|^p_{W_0^{s,p}(\Omega)}}{\int_\Omega \tilde{\rho}(x) u_1^p \, dx} \geq \lambda_1(\tilde{\rho}),$$

so $\lambda_1(\rho) > \lambda_1(\tilde{\rho})$. □

When $\rho \equiv 1$ we set $\lambda_1(\rho) = \lambda_1$ and $\hat{u}_1(\rho) = \hat{u}_1$. Moreover, the second (non-weighted) eigenvalue admits the following variational characterization:

$$\lambda_2 = \inf_{\gamma \in \Gamma_1} \max_{t \in [0,1]} \|\gamma(t)\|^p_{W_0^{s,p}(\Omega)}, \tag{3.4.2}$$

where

$$\Gamma_1 = \left\{\gamma \in C([0,1], W_0^{s,p}(\Omega)) : \gamma(0) = \hat{u}_1, \gamma(1) = -\hat{u}_1, \|\gamma(t)\|_p = 1 \text{ for all } t \in [0,1]\right\},$$

see [29, Theorem 5.3].

We stress that in the case $p \neq 2$ the spectrum is not yet completely understood. In the previous lemma we have seen that the nonlinear operator $(-\Delta)_p^s$ admits a sequence of variational eigenvalues, though it is not known whether they cover the whole spectrum or not (see [121]).

Now we show the relation between the eigenvalues of problem (3.4.1) and the degree of a suitable operator related to the fractional weighted problem above. In order to do this, we define the operators $\tilde{T}_\lambda, K_\rho : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ by

$$\langle K_\rho(u), v\rangle = \int_\Omega \rho(x) |u(x)|^{p-2} u(x) v(x) \, dx, \text{ with } \rho \in L^\infty(\Omega)_+, \rho \neq 0, \tag{3.4.3}$$

$$\langle \tilde{T}_\lambda(u), v\rangle = \langle (-\Delta)_p^s u - \lambda K_\rho(u), v\rangle, \text{ with } \lambda \in \mathbb{R}, \tag{3.4.4}$$

for any $v \in W_0^{s,p}(\Omega)$. In the sequel we will change the function ρ in (3.4.3) with a suitable function, but the definition of the operator K_ρ remains the same.
In an equivalent way, we can say that $u \in W_0^{s,p}(\Omega)$ is a (weak) solution of (3.4.1) if for all $v \in W_0^{s,p}(\Omega)$, we have

$$\langle (-\Delta)_p^s u, v\rangle = \lambda \langle K_\rho(u), v\rangle \quad \text{or} \quad \tilde{T}_\lambda(u) = 0 \text{ in } W^{-s,p'}(\Omega).$$

In the following lemma some important features of such operators are stated. We recall that the properties of $(-\Delta)_p^s$ can be found in Chapter 2, in particular for the aims of this section we use that $(-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a maximal monotone, bounded and continuous operator of type $(S)_+$.

Lemma 3.4.2. *The operators $K_\rho, \tilde{T}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$, defined above, satisfy the following properties:*

- *(i) $K_\rho : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a bounded, continuous and compact operator,*
- *(ii) $\tilde{T}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ is a bounded, continuous operator that satisfies the condition $(S)_+$.*

Proof. Now we show the first point. By Schwarz and Hölder inequalities, we get $|\langle K_\rho(u), v\rangle| \leq \|\rho\|_\infty \|u\|^{p-1}_{W_0^{s,p}(\Omega)} \|v\|_{W_0^{s,p}(\Omega)}$, hence $\|K_\rho(u)\|_{W^{-s,p'}(\Omega)} \leq c\|u\|^{p-1}_{W_0^{s,p}(\Omega)}$. Therefore, K_ρ is bounded. Let $(u_n) \subset W_0^{s,p}(\Omega)$ be bounded, we may assume, passing to a subsequence, $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, hence, by [30, Theorem 4.9], up to a subsequence, $u_n(x) \to u(x)$ a.e. on Ω and $|u_n(x)| \leq h(x)$ a.e. on Ω, with $h \in L^p(\Omega)$. Now, applying the dominated convergence Theorem, we obtain that

$$\langle K_\rho(u_n), v\rangle \to \langle K_\rho(u), v\rangle \quad \text{as } n \to \infty.$$

Hence, K_ρ is compact. Similarly, we see that K_ρ is also continuous.
Using the previous fact, we get the second assertion. By exploiting the properties of $(-\Delta)_p^s$ (cited above) and (i), we obtain that $\tilde{T}_\lambda$ is a bounded, continuous operator. Moreover, by [75, Lemma 1.2] and using again the properties of $(-\Delta)_p^s$ and K_ρ we get that $\tilde{T}_\lambda$ is an operator of type $(S)_+$. □

The following result [93, Proposition 3.5] about the degree of the operator $\tilde{T}_\lambda$ is fundamental to prove the results in Chapter 9, whose proof closely follows that of [75, Theorem 3.7]. Moreover we point out that $\tilde{T}_\lambda$ is a monotone map, so we can apply the properties of the degree for generalized monotone maps (see [75]).

Proposition 3.4.3. *Let $(-\Delta)_p^s, K_\rho, \tilde{T}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ be defined by (2.2.6), (3.4.3), (3.4.4) and $\delta > 0$ small. Then*

$$\deg(\tilde{T}_\lambda, B_r(0), 0) = 1 \textit{ for } \lambda \in (0, \lambda_1(\rho)),$$

and

$$\deg(\tilde{T}_\lambda, B_r(0), 0) = -1 \textit{ for } \lambda \in (\lambda_1(\rho), \lambda_1(\rho) + \delta).$$

Proof. From the variational characterization of $\lambda_1(\rho)$ we have that

$$\langle \tilde{T}_\lambda(u), u\rangle > 0,$$

for $\lambda \in (0, \lambda_1(\rho))$ and any $u \in W_0^{s,p}(\Omega)$ with $\|u\|_{W_0^{s,p}(\Omega)} \neq 0$. Hence, by [75, Theorem 1.5], the degree $\deg(\tilde{T}_\lambda, B_r(0), 0)$ is well defined for any $\lambda \in (0, \lambda_1(\rho))$ and any ball $B_r(0) \subset W_0^{s,p}(\Omega)$, moreover, applying [75, Theorem 1.6], we obtain

$$\deg(\tilde{T}_\lambda, B_r(0), 0) = 1 \text{ for } \lambda \in (0, \lambda_1(\rho)).$$

Now we show that $\deg(\tilde{T}_\lambda, B_r(0), 0) = -1$ for $\lambda \in (\lambda_1(\rho), \lambda_1(\rho)+\delta)$. On account of Lemma 3.4.1 there exists $\delta > 0$ such that the interval $(\lambda_1(\rho), \lambda_1(\rho)+\delta)$ does not include any eigenvalue for the problem (3.4.1). Therefore the degree $\deg(\tilde{T}_\lambda, B_r(0), 0)$ is well defined also for $\lambda \in (\lambda_1(\rho), \lambda_1(\rho)+\delta)$. Let us compute $\mathrm{Ind}(\tilde{T}_\lambda, 0)$ for $\lambda \in (\lambda_1(\rho), \lambda_1(\rho)+\delta)$. We introduce a function $\phi : \mathbb{R} \to \mathbb{R}$ by setting

$$\phi(t) = \begin{cases} 0 & \text{if } t \leq k \\ \frac{2\delta}{\lambda_1(\rho)}(t-2k) & \text{if } t \geq 3k, \end{cases}$$

for a fixed number $k > 0$. We note that $\phi(t)$ is continuously differentiable, positive and strictly convex in $(k, 3k)$.
Now we can introduce the functional

$$\Phi_\lambda(u) = \frac{1}{p}\langle(-\Delta)_p^s u, u\rangle - \frac{\lambda}{p}\langle K_\rho(u), u\rangle + \phi\left(\frac{1}{p}\langle(-\Delta)_p^s u, u\rangle\right),$$

that is Fréchet differentiable and its critical point $u_0 \in W_0^{s,p}(\Omega)$ coincides to a solution of the equation

$$(-\Delta)_p^s u_0 - \frac{\lambda}{1+\phi'\left(\frac{1}{p}\langle(-\Delta)_p^s u_0, u_0\rangle\right)} K_\rho(u_0) = 0.$$

Nevertheless, since $\lambda \in (\lambda_1(\rho), \lambda_1(\rho)+\delta)$, the only nontrivial critical points of Φ_λ turn up if

$$\phi'\left(\frac{1}{p}\langle(-\Delta)_p^s u_0, u_0\rangle\right) = \frac{\lambda}{\lambda_1(\rho)} - 1. \tag{3.4.5}$$

Owing to the definition of ϕ it follows that $\frac{1}{p}\langle(-\Delta)_p^s u_0, u_0\rangle \in (k, 3k)$ and because of (3.4.5) and the simplicity of $\lambda_1(\rho)$, it deduces that either $u_0 = -\hat{u}_1(\rho)$ or $u_0 = \hat{u}_1(\rho)$, where $\hat{u}_1(\rho) > 0$ is the first eigenfunction (which is not necessarily normed by 1). Therefore, we may conclude that for $\lambda \in (\lambda_1(\rho), \lambda_1(\rho)+\delta)$ the derivative Φ'_λ has precisely three isolated critical points $\{-\hat{u}_1(\rho), 0, \hat{u}_1(\rho)\}$ (in the sense of [75, Definition 1.2]).
We now show that Φ_λ is weakly lower semicontinuous. Indeed, suppose that $u_n \rightharpoonup \tilde{u}_0$ in $W_0^{s,p}(\Omega)$. Owing to the compactness of K_ρ, we get

$$\langle K_\rho(u_n), u_n\rangle \to \langle K_\rho(\tilde{u}_0), \tilde{u}_0\rangle, \tag{3.4.6}$$

and recalling that $\liminf_{n\to\infty} \|u_n\|_{W_0^{s,p}(\Omega)} \geq \|\tilde{u}_0\|_{W_0^{s,p}(\Omega)}$, (3.4.6) holds, and ϕ is nondecreasing, we obtain

$$\liminf_{n\to\infty}\left[\frac{1}{p}\langle(-\Delta)_p^s u_n, u_n\rangle - \frac{\lambda}{p}\langle K_\rho(u_n), u_n\rangle + \phi\left(\frac{1}{p}\langle(-\Delta)_p^s u_n, u_n\rangle\right)\right] \geq \Phi_\lambda(\tilde{u}_0).$$

Furthermore, Φ_λ is coercive, i.e. $\lim_{\|u\|_{W_0^{s,p}(\Omega)}\to\infty}\Phi_\lambda(u)=\infty$. Indeed, we get

$$\Phi_\lambda(u)=\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle-\frac{\lambda_1(\rho)}{p}\langle K_\rho(u),u\rangle+\frac{\lambda_1(\rho)-\lambda}{p}\langle K_\rho(u),u\rangle+\phi\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)$$

and, by the variational characterization of $\lambda_1(\rho)$,

$$\langle(-\Delta)_p^s u,u\rangle-\lambda_1(\rho)\langle K_\rho(u),u\rangle\geq 0 \tag{3.4.7}$$

for any $u\in W_0^{s,p}(\Omega)$. From (3.4.7) we have that

$$\begin{aligned}\frac{\lambda_1(\rho)-\lambda}{p}\langle K_\rho(u),u\rangle+\phi\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)&\geq\frac{\lambda_1(\rho)-\lambda}{p\lambda_1(\rho)}\langle(-\Delta)_p^s u,u\rangle+\phi\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)\\&\geq-\frac{\delta}{p\lambda_1(\rho)}\langle(-\Delta)_p^s u,u\rangle+\frac{2\delta}{\lambda_1(\rho)}\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle-2k\right)\to\infty\end{aligned}$$

for $\|u\|_{W_0^{s,p}(\Omega)}\to\infty$ because of the definition of ϕ. Therefore we obtain the coercivity. We observe that Φ_λ is even, the minimum of Φ_λ is achieved exactly in two points $-\hat{u}_1(\rho),\hat{u}_1(\rho)$, while the origin is an isolated critical point, but it is not a minimum. Indeed, by definition of Φ_λ and ϕ, we get that

$$\begin{aligned}\Phi_\lambda(t\hat{u}_1(\rho))&=\left(\frac{1}{p}\langle(-\Delta)_p^s\hat{u}_1(\rho),\hat{u}_1(\rho)\rangle-\frac{\lambda}{p}\langle K_\rho(\hat{u}_1(\rho)),\hat{u}_1(\rho)\rangle\right)t^p+\phi\left(\frac{t^p}{p}\langle(-\Delta)_p^s\hat{u}_1(\rho),\hat{u}_1(\rho)\rangle\right)\\&=\frac{t^p}{p}(\lambda_1(\rho)-\lambda)\langle K_\rho(\hat{u}_1(\rho)),\hat{u}_1(\rho)\rangle<0\quad\forall t\in(0,t_0).\end{aligned}$$

In accordance with [75, Theorem 1.8] we get

$$\operatorname{Ind}(\Phi'_\lambda,-\hat{u}_1(\rho))=\operatorname{Ind}(\Phi'_\lambda,\hat{u}_1(\rho))=1.$$

At the same time, we have $\langle\Phi'_\lambda(u),u\rangle>0$ for any $u\in W_0^{s,p}(\Omega)$, $\|u\|_{W_0^{s,p}(\Omega)}=\kappa$, with $\kappa>0$ large enough. Indeed

$$\begin{aligned}\langle\Phi'_\lambda(u),u\rangle&=\langle(-\Delta)_p^s u,u\rangle-\lambda\langle K_\rho(u),u\rangle+\phi'\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)\langle(-\Delta)_p^s u,u\rangle\\&=\langle(-\Delta)_p^s u,u\rangle-\lambda_1(\rho)\langle K_\rho(u),u\rangle+\phi'\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)\\&\qquad\left(\langle(-\Delta)_p^s u,u\rangle-\frac{\lambda-\lambda_1(\rho)}{\phi'\left(\frac{1}{p}\langle(-\Delta)_p^s u,u\rangle\right)}\langle K_\rho(u),u\rangle\right)\\&\geq\frac{2\delta}{\lambda_1(\rho)}\left(\langle(-\Delta)_p^s u,u\rangle-\frac{\lambda_1(\rho)}{p}\langle K_\rho(u),u\rangle\right)\to\infty\text{ as }\|u\|_{W_0^{s,p}(\Omega)}\to\infty.\end{aligned}$$

We again used the variational characterization of $\lambda_1(\rho)$ and the definition of ϕ. Then, [75, Theorem 1.6] and $\langle\Phi'_\lambda(u),u\rangle>0$ imply

$$\deg(\Phi'_\lambda,B_\kappa(0),0)=1.$$

We pick $\kappa > 0$ so large that $\pm\hat{u}_1(\rho) \in B_\kappa(0)$. By [75, Theorem 1.7] and $\mathrm{Ind}(\Phi'_\lambda, -\hat{u}_1(\rho)) = \mathrm{Ind}(\Phi'_\lambda, \hat{u}_1(\rho)) = 1$, and $\deg(\Phi'_\lambda, B_\kappa(0), 0) = 1$, we have

$$\mathrm{Ind}(\Phi'_\lambda, 0) = -1. \tag{3.4.8}$$

Furthermore, by the definition of ϕ, we have

$$\deg(\tilde{T}_\lambda, B_r(0), 0) = \mathrm{Ind}(\Phi'_\lambda, 0) \tag{3.4.9}$$

for $r > 0$ small enough. Then we deduce from (3.4.8), (3.4.9), that

$$\mathrm{Ind}(\tilde{J}_\lambda, 0) = -1 \text{ for } \lambda \in (\lambda_1(\rho), \lambda_1(\rho) + \delta).$$

It follows from the previous relations that

$$\deg(\tilde{T}_\lambda, B_r(0), 0) = -1.$$

$\square$

Chapter 4

Steiner symmetry in the minimization of the first eigenvalue of a fractional eigenvalue problem with indefinite weight

In this chapter we consider the weighted fractional eigenvalue problem

$$\begin{cases} (-\Delta)^s u = \lambda \rho u & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{4.0.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with $C^{1,1}$ boundary $\partial\Omega$, $(-\Delta)^s$, $s \in (0,1)$, denotes the fractional Laplacian operator with normalization constant $C(N,s) = 1$ and finally, $\rho \in L^\infty(\Omega)$, $\lambda \in \mathbb{R}$. We denote by $\lambda_k(\rho)$, $k \in \mathbb{Z} \setminus \{0\}$, the k-th eigenvalue of problem (4.0.1) corresponding to the weight ρ.

In this chapter we study the dependence of $\lambda_k(\rho)$ on ρ, in particular we investigate continuity and, for $k = 1$, convexity and differentiability properties. Then, we examine the minimization of $\lambda_1(\rho)$ in the class of rearrangements $\mathcal{G}(\rho_0)$ of a fixed function $\rho_0 \in L^\infty(\Omega)$. We prove the existence of minimizers and a characterization of them in terms of the eigenfunctions relative to $\lambda_1(\rho)$.

Moreover, when Ω is a Steiner symmetric domain we get that any minimizer inherits the same symmetry. In particular, if Ω is a ball, there exists a unique radially symmetric minimizer.

The analogous minimization problem with the Laplacian operator in place of the fractional Laplacian has been studied by some authors. Cox and McLaughlin in [59, 60] considered the case of the weight ρ_0 equal to a positive step function. Cosner et al. in [57] investigated the optimization of the first eigenvalue with an indefinite weight $\rho_0 \in L^\infty(\Omega)$, they proved existence of optimizers and a characterization formula of them. Related problems are studied in [7, 8]. In the first paper the eigenvalue problem is driven by the p-Laplacian operator. In the second an example of symmetry breaking of the minimizer is exhibited. For a complete survey on the optimization of eigenvalues related to elliptic problems we

refer the reader to [103].
We remark that the argument used in this chapter to prove the existence of minimizers is inspired by the approach of [103] and it is different from those used in [57, 59, 60], nevertheless it can be applied also for the corresponding problem driven by the Laplacian operator.
This kind of optimization problems arises in mathematical biology, as we have seen in the Introduction. In this sense, Theorem 4.3.6 and Theorem 4.4.6 provide information about the existence, qualitative features and symmetry of the best local growth rate ρ when ρ belongs to a class of rearrangements $\mathcal{G}(\rho_0)$.

This chapter is organized in this way: in Section 4.1 we study the eigenvalues of problem (4.0.1); in Section 4.2 we collect some known results about rearrangements of measurable functions; in Section 4.3 we prove the existence results; finally, in Section 4.4 we focus on the symmetry of the minimizers.

4.1 Fractional weighted eigenvalue problem

Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded domain with $C^{1,1}$ boundary. Now we introduce the notion of weak solution of the boundary value problem

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{4.1.1}$$

where $f \in H^{-s}(\Omega)$. A function $u \in H_0^s(\Omega)$ is called *weak solution* of problem (4.1.1) if

$$\langle u, \varphi \rangle_{H_0^s(\Omega)} = \langle f, \varphi \rangle \quad \forall \, \varphi \in H_0^s(\Omega)$$

holds, where $\langle f, g \rangle$ means the duality between $f \in H^{-s}(\Omega)$ and $g \in H_0^s(\Omega)$. By the Riesz-Fréchet representation Theorem, for every $f \in H^{-s}(\Omega)$ there exists a unique solution $u \in H_0^s(\Omega)$ of (4.1.1) and moreover

$$\|u\|_{H_0^s(\Omega)} = \|f\|_{H^{-s}(\Omega)}. \tag{4.1.2}$$

We call G,

$$G : H^{-s}(\Omega) \to H_0^s(\Omega), \tag{4.1.3}$$

the linear operator defined by $G(f) = u$. Identity (4.1.2) implies

$$\|G\|_{\mathcal{L}(H^{-s}(\Omega), H_0^s(\Omega))} = 1. \tag{4.1.4}$$

For any ρ in $L^\infty(\Omega)$, let $M_\rho : L^2(\Omega) \to L^2(\Omega)$ be the linear operator defined by $M_\rho(f) = \rho f$. Of course

$$\|\rho f\|_2 \leq \|\rho\|_\infty \|f\|_2. \tag{4.1.5}$$

Next, we introduce the linear operator

$$G_\rho : H_0^s(\Omega) \to H_0^s(\Omega) \tag{4.1.6}$$

defined by $G_\rho = G \circ j \circ M_\rho \circ i$ or, briefly, $G_\rho(f) = G(\rho f)$. Equivalently, $u = G_\rho(f)$ is the unique weak solution of the problem

$$\begin{cases} (-\Delta)^s u = \rho f & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

i.e.

$$\langle u, \varphi \rangle_{H_0^s(\Omega)} = \langle \rho f, \varphi \rangle_{L^2(\Omega)} \quad \forall \varphi \in H_0^s(\Omega). \tag{4.1.7}$$

From (2.1.3), (2.1.4), (4.1.4) and (4.1.5) it follows straightforwardly that

$$\|G_\rho\|_{\mathcal{L}(H_0^s(\Omega), H_0^s(\Omega))} \le C^2 \|\rho\|_\infty.$$

In the sequel we will use the formula

$$G_{a\rho + b\eta} = a\, G_\rho + b\, G_\eta \quad \forall \rho, \eta \in L^\infty(\Omega),\ \forall a, b \in \mathbb{R}. \tag{4.1.8}$$

In particular, (4.1.8) implies $G_{-\rho} = -G_\rho$ for all $\rho \in L^\infty(\Omega)$.

Lemma 4.1.1. *Let G_ρ be the operator (4.1.6). Then G_ρ is a self-adjoint compact operator.*

Proof. For all $f, g \in H_0^s(\Omega)$, by (4.1.7), we have

$$\langle G_\rho(f), g \rangle_{H_0^s(\Omega)} = \langle G(\rho f), g \rangle_{H_0^s(\Omega)} = \langle \rho f, g \rangle_{L^2(\Omega)} = \langle \rho g, f \rangle_{L^2(\Omega)} = \langle G_\rho(g), f \rangle_{H_0^s(\Omega)},$$

then G_ρ is self-adjoint.
The compactness of the operator G_ρ is an immediate consequence of the representation $G_\rho = G \circ j \circ M_\rho \circ i$ and the compactness of i and j. □

As we have seen in Chapter 3, by general theory of self-adjoint compact operators (see [30,64,133]) it follows that all nonzero eigenvalues of G_ρ have a finite dimensional eigenspace and they can be obtained by Fischer's Principle

$$\mu_k(\rho) = \sup_{F_k} \inf_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} = \sup_{F_k} \inf_{\substack{f \in F_k \\ f \neq 0}} \frac{\int_\Omega \rho f^2\, dx}{\|f\|^2_{H_0^s(\Omega)}}, \quad k = 1, 2, 3, \dots \tag{4.1.9}$$

and

$$\mu_{-k}(\rho) = \inf_{F_k} \sup_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} = \inf_{F_k} \sup_{\substack{f \in F_k \\ f \neq 0}} \frac{\int_\Omega \rho f^2\, dx}{\|f\|^2_{H_0^s(\Omega)}}, \quad k = 1, 2, 3, \dots,$$

where the first extrema are taken over all the subspaces F_k of $H_0^s(\Omega)$ of dimension k. As observed in [64], all the inf's and sup's in the above characterizations of the eigenvalues are actually assumed. Hence, they could be replaced by min's and max's. The sequence $(\mu_k(\rho))$ contains all the real positive eigenvalues (repeated with their multiplicity), is decreasing and converging to zero, whereas $(\mu_{-k}(\rho))$ is formed by all the real negative eigenvalues (repeated with their multiplicity), is increasing and converging to zero.

Remark 4.1.2. By Fischer's Principle it follows easily that $\mu_{-k}(\rho) = -\mu_k(-\rho)$ for all $\rho \in L^\infty(\Omega)$ and $k = 1, 2, 3, \dots$
For this reason, in the rest of the chapter, we will consider mainly positive eigenvalues.

We will write $\{\rho > 0\}$ as short form of $\{x \in \Omega : \rho(x) > 0\}$ and similarly $\{\rho < 0\}$ for $\{x \in \Omega : \rho(x) < 0\}$. The following proposition is analogous to [64, Proposition 1.11].

Proposition 4.1.3. *Let $\rho \in L^\infty(\Omega)$, G_ρ be the operator defined in (4.1.6) and $\mu_k(\rho)$, $\mu_{-k}(\rho)$ its eigenvalues. The following statements hold:*
(i) if $|\{\rho > 0\}| = 0$, then there are no positive eigenvalues;
(ii) if $|\{\rho > 0\}| > 0$, then there is a sequence of positive eigenvalues $\mu_k(\rho)$;
(iii) if $|\{\rho < 0\}| = 0$, then there are no negative eigenvalues;
(iv) if $|\{\rho < 0\}| > 0$, then there is a sequence of negative eigenvalues $\mu_{-k}(\rho)$.

Proof. (i) Let μ be an eigenvalue and u a corresponding eigenfunction. Then

$$\mu = \frac{\langle G_\rho u, u\rangle_{H_0^s(\Omega)}}{\|u\|^2_{H_0^s(\Omega)}} = \frac{\int_\Omega \rho u^2\, dx}{\|u\|^2_{H_0^s(\Omega)}} \leq 0.$$

(ii) By measure theory covering theorems, for each positive integer k there exist k disjoint closed balls $B_1, \dots, B_k$ in Ω such that $|B_i \cap \{\rho > 0\}| > 0$ for $i = 1, \dots, k$. Let $f_i \in C_0^\infty(B_i)$ such that $\int_\Omega \rho f_i^2\, dx = 1$ for every $i = 1, \dots, k$. Note that the functions f_i are linearly independent and let $F_k = \operatorname{span}\{f_1, \dots, f_k\}$. F_k is a subspace of $H_0^s(\Omega)$ and for every $f \in F_k \setminus \{0\}$, $f = \sum_{i=1}^k a_i f_i$, $a_i \in \mathbb{R}$, we have

$$\begin{aligned}
\frac{\langle G_\rho f, f\rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} &= \frac{\int_\Omega \rho f^2\, dx}{\|f\|^2_{H_0^s(\Omega)}} = \frac{\sum_{i,j=1}^k \int_\Omega \rho f_i f_j a_i a_j\, dx}{\sum_{i,j=1}^k \langle f_i, f_j\rangle_{H_0^s(\Omega)} a_i a_j} \\
&= \frac{\sum_{i=1}^k a_i^2}{\sum_{i,j=1}^k \langle f_i, f_j\rangle_{H_0^s(\Omega)} a_i a_j} = \frac{\|a\|^2_{\mathbb{R}^k}}{\langle E_k a, a\rangle_{\mathbb{R}^k}} \geq \frac{1}{\|E_k\|} > 0,
\end{aligned}$$

where $\|a\|_{\mathbb{R}^k}$, $\|E_k\|$ and $\langle E_k a, a\rangle_{\mathbb{R}^k}$ denote, respectively, the euclidean norm of the vector $a = (a_1, \dots, a_k)$, the norm of the non null matrix $E_k = \left(\langle f_i, f_j\rangle_{H_0^s(\Omega)}\right)_{i,j=1}^k$ and the inner product in $\mathbb{R}^k$. From Fischer's Principle (4.1.9) we conclude that $\mu_k(\rho) \geq \dfrac{1}{\|E_k\|} > 0$ for every k.
The cases (iii) and (iv) are similarly proved. □

Finally, we introduce the weak formulation of problem (4.0.1). A function $u \in H_0^s(\Omega) \setminus \{0\}$ is said an *eigenfunction* of (4.0.1) associated to the *eigenvalue* λ if

$$\langle u, \varphi\rangle_{H_0^s(\Omega)} = \lambda \langle \rho u, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H_0^s(\Omega). \tag{4.1.10}$$

It is easy to check that zero is not an eigenvalue of problem (4.0.1). The eigenvalues of problem (4.0.1) are exactly the reciprocal of the nonzero eigenvalues of the operator G_ρ

and the correspondent eigenspaces coincide. Indeed, if $\lambda \neq 0$ is an eigenvalue of problem (4.0.1) and u is an associated eigenfunction, by (4.1.10) we have

$$\left\langle \frac{u}{\lambda}, \varphi \right\rangle_{H^s_0(\Omega)} = \langle \rho u, \varphi \rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H^s_0(\Omega)$$

and then, by definition of G_ρ, $G_\rho(u) = \dfrac{u}{\lambda}$. Consequently, in general, the eigenvalues of problem (4.0.1) form two monotone sequences

$$0 < \lambda_1(\rho) \leq \lambda_2(\rho) \leq \ldots \leq \lambda_k(\rho) \leq \ldots$$

and

$$\ldots \leq \lambda_{-k}(\rho) \leq \ldots \leq \lambda_{-2}(\rho) \leq \lambda_{-1}(\rho) < 0,$$

where every eigenvalue appears as many times as its multiplicity, the latter being finite owing to the compactness of G_ρ.
Assuming $C^{1,1}$ regularity of $\partial\Omega$, in [89] it has been recently shown that $\lambda_1(\rho)$ and $\lambda_{-1}(\rho)$ are simple and any associated eigenfunction is one signed in Ω. We call *first eigenfunction* any eigenfunction relative to $\lambda_1(\rho)$. The variational characterization (4.1.9) for $k = 1$ becomes

$$\mu_1(\rho) = \max_{\substack{f \in H^s_0(\Omega) \\ f \neq 0}} \frac{\int_\Omega \rho f^2\, dx}{\|f\|^2_{H^s_0(\Omega)}} \tag{4.1.11}$$

and, thus, for $\lambda_1(\rho)$ we have

$$\lambda_1(\rho) = \min_{\substack{u \in H^s_0(\Omega) \\ u \neq 0 \\ \int_\Omega \rho u^2 dx > 0}} \frac{\|u\|^2_{H^s_0(\Omega)}}{\int_\Omega \rho u^2\, dx}. \tag{4.1.12}$$

The maximum in (4.1.11) (respectively the minimum in (4.1.12)) is obtained if and only if f (respectively u) is a first eigenfunction. Throughout this chapter we will denote by u_ρ the first positive eigenfunction of problem (4.0.1) normalized by

$$\|u_\rho\|_{H^s_0(\Omega)} = 1, \tag{4.1.13}$$

which is equivalent to

$$\int_\Omega \rho u_\rho^2\, dx = \frac{1}{\lambda_1(\rho)}. \tag{4.1.14}$$

As last comment, we observe that $\mu_1(\rho)$ is homogeneous of degree 1, i.e.

$$\mu_1(\alpha\rho) = \alpha\mu_1(\rho) \quad \forall\, \alpha > 0. \tag{4.1.15}$$

This follows immediately from (4.1.11).

4.2 Rearrangements of measurable functions

In this section we introduce the concept of rearrangement of a measurable function and summarize some related results we will use in next section. The idea of rearranging a function dates back to the book [102] of Hardy, Littlewood and Pólya, since then many authors have investigated both extensions and applications of this notion. Here we rely on the results in [2, 37, 38, 63, 123, 179].
Let Ω be an open bounded set of $\mathbb{R}^N$.

Definition 4.2.1. *For every measurable function $f : \Omega \to \mathbb{R}$ the function $d_f : \mathbb{R} \to [0, |\Omega|]$ defined by*

$$d_f(t) = |\{x \in \Omega : f(x) > t\}|$$

is called distribution function of f.

The symbol μ_f is also used. It is easy to prove the following properties of d_f.

Lemma 4.2.2. *For each f the distribution function d_f is decreasing, right continuous and the following identities hold true*

$$\lim_{t\to-\infty} d_f(t) = |\Omega|, \qquad \lim_{t\to\infty} d_f(t) = 0.$$

Definition 4.2.3. *Two measurable functions $f, g : \Omega \to \mathbb{R}$ are called* equimeasurable *functions or* rearrangements *of one another if one of the following equivalent conditions is satisfied*

(i) $|\{x \in \Omega : f(x) > t\}| = |\{x \in \Omega : g(x) > t\}| \quad \forall\, t \in \mathbb{R}$;

(ii) $d_f = d_g$.

Equimeasurability of f and g is denoted by $f \sim g$. Equimeasurable functions share global extrema and integrals as it is stated precisely by the following proposition.

Proposition 4.2.4. *Suppose $f \sim g$ and let $F : \mathbb{R} \to \mathbb{R}$ be a Borel measurable function, then*

(i) $|f| \sim |g|$;

(ii) $\operatorname{ess\,sup} f = \operatorname{ess\,sup} g$ *and* $\operatorname{ess\,inf} f = \operatorname{ess\,inf} g$;

(iii) $F \circ f \sim F \circ g$;

(iv) $F \circ f \in L^1(\Omega)$ *implies* $F \circ g \in L^1(\Omega)$ *and* $\int_\Omega F \circ f\, dx = \int_\Omega F \circ g\, dx$.

For a proof see, for example, [63, Proposition 3.3] or [38, Lemma 2.1].
In particular, for each $1 \leq p \leq \infty$, if $f \in L^p(\Omega)$ and $f \sim g$ then $g \in L^p(\Omega)$ and

$$\|f\|_p = \|g\|_p. \tag{4.2.1}$$

Definition 4.2.5. *For every measurable function $f : \Omega \to \mathbb{R}$ the function $f^* : (0, |\Omega|) \to \mathbb{R}$ defined by*

$$f^*(s) = \sup\{t \in \mathbb{R} : d_f(t) > s\}$$

is called decreasing rearrangement of f.

An equivalent definition (used by some authors) is $f^*(s) = \inf\{t \in \mathbb{R} : d_f(t) \leq s\}$.

Proposition 4.2.6. *For each f its decreasing rearrangement f^* is decreasing, right continuous and we have*

$$\lim_{s\to 0} f^*(s) = \operatorname{ess\,sup} f \quad \textit{and} \quad \lim_{s\to|\Omega|} f^*(s) = \operatorname{ess\,inf} f.$$

Moreover, if $F : \mathbb{R} \to \mathbb{R}$ is a Borel measurable function then $F \circ f \in L^1(\Omega)$ implies $F \circ f^* \in L^1(0, |\Omega|)$ and

$$\int_\Omega F \circ f \, dx = \int_0^{|\Omega|} F \circ f^* \, ds.$$

Finally, $d_{f^*} = d_f$ and, for each measurable function g we have $f \sim g$ if and only if $f^* = g^*$.

Some of the previous claims are simple consequences of the definition of f^*, for more details see [63, Chapter 2].
As before, it follows that, for each $1 \leq p \leq \infty$, if $f \in L^p(\Omega)$ then $f^* \in L^p(0, |\Omega|)$ and $\|f\|_{L^p(\Omega)} = \|f^*\|_{L^p(0,|\Omega|)}$.

Definition 4.2.7. *Given two functions $f, g \in L^1(\Omega)$, we write $g \prec f$ if*

$$\int_0^t g^* \, ds \leq \int_0^t f^* \, ds \quad \forall\, 0 \leq t \leq |\Omega| \qquad \textit{and} \qquad \int_0^{|\Omega|} g^* \, ds = \int_0^{|\Omega|} f^* \, ds.$$

Note that $g \sim f$ if and only if $g \prec f$ and $f \prec g$. Among many properties of the relation $\prec$ we mention the following (a proof is in [63, Lemma 8.2]).

Lemma 4.2.8. *For any pair of functions $f, g \in L^1(\Omega)$ and real numbers α and β, if $\alpha \leq f \leq \beta$ a.e. in Ω and $g \prec f$ then $\alpha \leq g \leq \beta$ a.e. in Ω.*

Lemma 4.2.9. *For $f \in L^1(\Omega)$ let $g = \frac{1}{|\Omega|} \int_\Omega f \, dx$. Then we have $g \prec f$.*

Definition 4.2.10. *Let $f : \Omega \to \mathbb{R}$ be a measurable function. We call the set*

$$\mathcal{G}(f) = \{g : \Omega \to \mathbb{R} : g \textit{ is measurable and } g \sim f\}$$

class of rearrangements of f *or* set of rearrangements of f.

Note that, for $1 \leq p \leq \infty$, if f is in $L^p(\Omega)$ then $\mathcal{G}(f)$ is contained in $L^p(\Omega)$.
As we will see in the next section, we are interested in the optimization of a functional defined on a class of rearrangements $\mathcal{G}(\rho_0)$, where ρ_0 belongs to $L^\infty(\Omega)$. For this reason, although almost all of what follows holds in a much more general context, hereafter we restrict our attention to classes of rearrangements of functions in $L^\infty(\Omega)$. We need compactness properties of the set $\mathcal{G}(\rho_0)$, with a little effort it can be showed that this set is closed but in general it is not compact in the norm topology of $L^\infty(\Omega)$. Therefore, we consider $L^\infty(\Omega)$ as the dual space of $L^1(\Omega)$ and we focus our attention on the weak* compactness. By $\overline{\mathcal{G}(\rho_0)}$ we denote the closure of $\mathcal{G}(\rho_0)$ in the weak* topology of $L^\infty(\Omega)$.

Proposition 4.2.11. *Let ρ_0 be a function of $L^\infty(\Omega)$. Then $\overline{\mathcal{G}(\rho_0)}$ is*

(i) weakly compact;*

(ii) metrizable in the weak topology;*

(iii) sequentially weakly compact.*

Proof. (i) By (4.2.1) it follows that $\mathcal{G}(\rho_0)$ is contained in $B_{\|\rho_0\|_\infty} = \{f \in L^\infty(\Omega) : \|f\|_\infty \leq \|\rho_0\|_\infty\}$. $B_{\|\rho_0\|_\infty}$ is weakly* compact and then it is also weakly* closed because the weak* topology is Hausdorff. Hence $\overline{\mathcal{G}(\rho_0)}$ is a weakly* closed subset of $B_{\|\rho_0\|_\infty}$ and thus it is weakly* compact as well. (ii) Owing to the separability of $L^1(\Omega)$, $B_{\|\rho_0\|_\infty}$ is metrizable in the weak* topology and the claim follows. (iii) It is an immediate consequence of (i) and (ii). □

Moreover, the sets $\mathcal{G}(\rho_0)$ and $\overline{\mathcal{G}(\rho_0)}$ have further properties.

Definition 4.2.12. *Let C be a convex set of a real vector space. An element v in C is said an* extreme point *of C if for every u and w in C the identity $v = \frac{u}{2} + \frac{w}{2}$ implies $u = w$.*

A vertex of a polygon is an example of extreme point.

Proposition 4.2.13. *Let ρ_0 be a function of $L^\infty(\Omega)$, then*

(i) $\overline{\mathcal{G}(\rho_0)} = \{f \in L^\infty(\Omega) : f \prec \rho_0\}$;

(ii) $\overline{\mathcal{G}(\rho_0)}$ is convex;

(iii) $\mathcal{G}(\rho_0)$ is the set of the extreme points of $\overline{\mathcal{G}(\rho_0)}$.

Proof. The claims follow from [63, Theorems 22.13, 22.2, 17.4, 20.3]. □

An evident consequence of the previous theorem is that $\overline{\mathcal{G}(\rho_0)}$ is the weakly* closed convex hull of $\mathcal{G}(\rho_0)$.

The following is [63, Theorem 11.1] rephrased for our case.

Proposition 4.2.14. *Let $u \in L^1(\Omega)$ and $\rho_0 \in L^\infty(\Omega)$. Then*

$$\int_0^{|\Omega|} \rho_0^*(|\Omega| - s)u^*(s)\, ds \leq \int_\Omega \rho\, u\, dx \leq \int_0^{|\Omega|} \rho_0^*(s)u^*(s)\, ds \qquad \forall \rho \in \mathcal{G}(\rho_0), \tag{4.2.2}$$

moreover both sides of (4.2.2) are achieved.

The previous proposition implies that the linear optimization problems

$$\sup_{\rho \in \mathcal{G}(\rho_0)} \int_\Omega \rho u\, dx \tag{4.2.3}$$

and

$$\inf_{\rho \in \mathcal{G}(\rho_0)} \int_\Omega \rho u\, dx$$

admit solution.

Finally, we recall the following result proved in [37, Theorem 5].

Proposition 4.2.15. *Let $u \in L^1(\Omega)$ and $\rho_0 \in L^\infty(\Omega)$. If problem (4.2.3) has a unique solution ρ_M, then there exists an increasing function ψ such that $\rho_M = \psi \circ u$ a.e. in Ω.*

4.3 Existence of minimizers

Let $\rho_0 \in L^\infty(\Omega)$, $\mathcal{G}(\rho_0)$ be the class of rearrangements of ρ_0 and $\lambda_k(\rho)$, $\rho \in \mathcal{G}(\rho_0)$, be the k-th positive eigenvalue of problem (4.0.1). In this section we investigate the optimization problem

$$\inf_{\rho\in\mathcal{G}(\rho_0)} \lambda_1(\rho),$$

which can be expressed in terms of the eigenvalue $\mu_1(\rho)$ of the operator G_ρ, defined in (4.1.6), as

$$\sup_{\rho\in\mathcal{G}(\rho_0)} \mu_1(\rho).$$

Observe that, by Proposition 4.1.3, $\mu_k(\rho)$ and u_ρ (the positive first eigenfuction of problem (4.0.1) normalized as in (4.1.13)) are well defined only when $|\{\rho > 0\}| > 0$. We extend them to the whole space $L^\infty(\Omega)$ by putting

$$\widetilde{\mu}_k(\rho) = \begin{cases} \mu_k(\rho) & \text{if } |\{\rho > 0\}| > 0 \\ 0 & \text{if } |\{\rho > 0\}| = 0 \end{cases} \tag{4.3.1}$$

and

$$\widetilde{u}_\rho = \begin{cases} u_\rho & \text{if } |\{\rho > 0\}| > 0 \\ 0 & \text{if } |\{\rho > 0\}| = 0. \end{cases} \tag{4.3.2}$$

Remark 4.3.1. Note that $\widetilde{\mu}_k(\rho) = 0$ if and only if $\rho \le 0$ a.e. in Ω and, in this circumstance, the inequality

$$\sup_{F_k} \min_{\substack{f\in F_k\\ f\neq 0}} \frac{\langle G_\rho f, f\rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} \le 0 \tag{4.3.3}$$

holds, where F_k varies among all the k-dimensional subspaces of $H_0^s(\Omega)$.
Moreover, from (4.1.15), we have $\widetilde{\mu}_1(\alpha\rho) = \alpha\widetilde{\mu}_1(\rho)$ for every $\alpha \ge 0$.

Lemma 4.3.2. *Let $\rho \in L^\infty(\Omega)$, G_ρ be the linear operator (4.1.6), $\widetilde{\mu}_k(\rho)$ as defined in (4.3.1) for $k = 1, 2, 3, \ldots$ and $\widetilde{u}_\rho$ as in (4.3.2). Then*
(i) the map $\rho \mapsto G_\rho$ is sequentially weakly continuous from $L^\infty(\Omega)$ to $\mathcal{L}(H_0^s(\Omega), H_0^s(\Omega))$ endowed with the norm topology;*
(ii) the map $\rho \mapsto \widetilde{\mu}_k(\rho)$ is sequentially weakly continuous in $L^\infty(\Omega)$;*
(iii) the map $\rho \mapsto \widetilde{\mu}_1(\rho)\widetilde{u}_\rho$ is sequentially weakly continuous from $L^\infty(\Omega)$ to $H_0^s(\Omega)$ (endowed with the norm topology). In particular, for any sequence (ρ_n) weakly* convergent to $\eta \in L^\infty(\Omega)$, with $\widetilde{\mu}_1(\eta) > 0$, then $(\widetilde{u}_{\rho_n})$ converges to $\widetilde{u}_\eta$ in $H_0^s(\Omega)$.*

Proof. (i) Let (ρ_n) be a sequence which weakly* converges to ρ in $L^\infty(\Omega)$. Being (ρ_n) bounded in $L^\infty(\Omega)$, there exists a constant $M > 0$ such that

$$|\rho| \le M \quad \text{and} \quad |\rho_n| \le M \quad \forall\, n. \tag{4.3.4}$$

We begin by proving that $G_{\rho_n}(f)$ tends to $G_\rho(f)$ in $H_0^s(\Omega)$ for any fixed $f \in H_0^s(\Omega)$. Note that the sequence $(\rho_n f)$ is weakly convergent to ρf in $L^2(\Omega)$, then, exploiting the compactness of the embedding $L^2(\Omega) \hookrightarrow H^{-s}(\Omega)$, we conclude that this convergence is also strong in $H^{-s}(\Omega)$. Then

$$\begin{aligned}
&\|G_{\rho_n}(f) - G_\rho(f)\|_{H_0^s(\Omega)} = \|G(\rho_n f - \rho f)\|_{H_0^s(\Omega)} \\
&\leq \|G\|_{\mathcal{L}(H^{-s}(\Omega),H_0^s(\Omega))}\|\rho_n f - \rho f\|_{H^{-s}(\Omega)} = \|\rho_n f - \rho f\|_{H^{-s}(\Omega)},
\end{aligned}$$

where we used $G_\rho(f) = G(\rho f)$, with G defined by (4.1.3), and (4.1.4). Therefore $G_{\rho_n}(f)$ converges to $G_\rho(f)$ in $H_0^s(\Omega)$.
Now, for fixed n, let $(f_{n,k})$, $k = 1, 2, 3, \ldots$, be a maximizing sequence of

$$\sup_{\substack{g\in H_0^s(\Omega)\\ \|g\|_{H_0^s(\Omega)}\leq 1}} \|G_{\rho_n}(g) - G_\rho(g)\|_{H_0^s(\Omega)} = \|G_{\rho_n} - G_\rho\|_{\mathcal{L}(H_0^s(\Omega),H_0^s(\Omega))}.$$

Then, being $\|f_{n,k}\|_{H_0^s(\Omega)} \leq 1$, we can extract a subsequence (still denoted by $(f_{n,k})$) weakly convergent to some $f_n \in H_0^s(\Omega)$. Since G_{ρ_n} and G_ρ are compact operators (see Lemma 4.1.1), it follows that $G_{\rho_n}(f_{n,k})$ converges to $G_{\rho_n}(f_n)$ and $G_\rho(f_{n,k})$ converges to $G_\rho(f_n)$ strongly in $H_0^s(\Omega)$ as k goes to ∞. Thus we find

$$\|G_{\rho_n} - G_\rho\|_{\mathcal{L}(H_0^s(\Omega),H_0^s(\Omega))} = \|G_{\rho_n}(f_n) - G_\rho(f_n)\|_{H_0^s(\Omega)}.$$

This procedure yields a sequence (f_n) in $H_0^s(\Omega)$ such that $\|f_n\|_{H_0^s(\Omega)} \leq 1$ for all n. Then, up to a subsequence, we can assume that (f_n) weakly converges to a function $f \in H_0^s(\Omega)$ and (by compactness of the embedding $H_0^s(\Omega) \hookrightarrow L^2(\Omega)$) strongly in $L^2(\Omega)$. By using (2.1.4), (4.1.4) and (4.3.4) we find

$$\begin{aligned}
&\|G_{\rho_n} - G_\rho\|_{\mathcal{L}(H_0^s(\Omega),H_0^s(\Omega))} = \|G_{\rho_n}(f_n) - G_\rho(f_n)\|_{H_0^s(\Omega)} \\
&\leq \|G_{\rho_n}(f) - G_\rho(f)\|_{H_0^s(\Omega)} + \|G_{\rho_n}(f_n - f) - G_\rho(f_n - f)\|_{H_0^s(\Omega)} \\
&= \|G_{\rho_n}(f) - G_\rho(f)\|_{H_0^s(\Omega)} + \|G(\rho_n(f_n - f) - \rho(f_n - f))\|_{H_0^s(\Omega)} \\
&\leq \|G_{\rho_n}(f) - G_\rho(f)\|_{H_0^s(\Omega)} \\
&+ \|G\|_{\mathcal{L}(H^{-s}(\Omega),H_0^s(\Omega))}\left(\|\rho_n(f_n - f)\|_{H^{-s}(\Omega)} + \|\rho(f_n - f))\|_{H^{-s}(\Omega)}\right) \\
&\leq \|G_{\rho_n}(f) - G_\rho(f)\|_{H_0^s(\Omega)} + 2CM\|f_n - f\|_2.
\end{aligned}$$

Therefore G_{ρ_n} converges to G_ρ in the operator norm.

(ii) If we show that, for any $k = 1, 2, 3, \ldots$ and $\rho, \eta \in L^\infty(\Omega)$ the estimates

$$|\widetilde{\mu}_k(\rho) - \widetilde{\mu}_k(\eta)| \leq \|G_\rho - G_\eta\|_{\mathcal{L}(H_0^s(\Omega),H_0^s(\Omega))} \tag{4.3.5}$$

hold, then the claim follows immediately from (i).
We split the argument in three cases.

Case 1. $\widetilde{\mu}_k(\rho),\ \widetilde{\mu}_k(\eta) > 0$.
Following [103, Theorem 2.3.1] and by means of Fischer's Principle (4.1.9) we have

$$\begin{aligned}
\widetilde{\mu}_k(\rho) - \widetilde{\mu}_k(\eta) &= \max_{F_k} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} - \max_{F_k} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\eta f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} \\
&\leq \min_{\substack{f \in F_k(\rho) \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} - \min_{\substack{f \in F_k(\rho) \\ f \neq 0}} \frac{\langle G_\eta f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} \\
&\leq \frac{\langle G_\rho f_\eta, f_\eta \rangle_{H_0^s(\Omega)}}{\|f_\eta\|^2_{H_0^s(\Omega)}} - \frac{\langle G_\eta f_\eta, f_\eta \rangle_{H_0^s(\Omega)}}{\|f_\eta\|^2_{H_0^s(\Omega)}} \\
&= \frac{\langle (G_\rho - G_\eta) f_\eta, f_\eta \rangle_{H_0^s(\Omega)}}{\|f_\eta\|^2_{H_0^s(\Omega)}} \leq \|G_\rho - G_\eta\|_{\mathcal{L}(H_0^s(\Omega), H_0^s(\Omega))},
\end{aligned}$$

where $F_k(\rho)$ is a k-dimensional subspace of $H_0^s(\Omega)$ such that

$$\max_{F_k} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} = \min_{\substack{f \in F_k(\rho) \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}}$$

and f_η is a function in $F_k(\rho)$ such that

$$\min_{\substack{f \in F_k(\rho) \\ f \neq 0}} \frac{\langle G_\eta f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} = \frac{\langle G_\eta f_\eta, f_\eta \rangle_{H_0^s(\Omega)}}{\|f_\eta\|^2_{H_0^s(\Omega)}}.$$

Interchanging the role of ρ and η we find (4.3.5).

Case 2. $\widetilde{\mu}_k(\rho) > 0$, $\widetilde{\mu}_k(\eta) = 0$ (and similarly in the case $\widetilde{\mu}_k(\eta) > 0$, $\widetilde{\mu}_k(\rho) = 0$).
Note that in this case (4.3.3) holds for the weight function η. Then the previous argument still applies provided that we replace the first step of the inequality chain by

$$|\widetilde{\mu}_k(\rho) - \widetilde{\mu}_k(\eta)| = \widetilde{\mu}_k(\rho) \leq \max_{F_k} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} - \sup_{F_k} \min_{\substack{f \in F_k \\ f \neq 0}} \frac{\langle G_\eta f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}}.$$

Case 3. $\widetilde{\mu}_k(\rho) = \widetilde{\mu}_k(\eta) = 0$.
In this case (4.3.5) is obvious.

Therefore statement (ii) is proved.

(iii) Let $(\rho_n), \rho$ be such that ρ_n is weakly* convergent to ρ in $L^\infty(\Omega)$. Being $\|\widetilde{u}_{\rho_n}\|_{H_0^s(\Omega)} \leq 1$, up to a subsequence we can assume that $\widetilde{u}_{\rho_n}$ is weakly convergent to $z \in H_0^s(\Omega)$, strongly in $L^2(\Omega)$ and pointwisely a.e. in Ω.
First suppose $\widetilde{\mu}_1(\rho) = 0$. Then, by (ii) $\widetilde{\mu}_1(\rho_n)\widetilde{u}_{\rho_n}$ weakly converges in $H_0^s(\Omega)$ to $\widetilde{\mu}_1(\rho)z = 0 = \widetilde{\mu}_1(\rho)\widetilde{u}_\rho$. Moreover, $\|\widetilde{\mu}_1(\rho_n)\widetilde{u}_{\rho_n}\|_{H_0^s(\Omega)} = \widetilde{\mu}_1(\rho_n)\|\widetilde{u}_{\rho_n}\|_{H_0^s(\Omega)}$ tends to $0 = \|\widetilde{\mu}_1(\rho)\widetilde{u}_\rho\|_{H_0^s(\Omega)}$. Therefore $\widetilde{\mu}_1(\rho_n)\widetilde{u}_{\rho_n}$ strongly converges to $\widetilde{\mu}_1(\rho)\widetilde{u}_\rho$ in $H_0^s(\Omega)$.
Next, consider the case $\widetilde{\mu}_1(\rho) > 0$. By (ii) we have $\widetilde{\mu}_1(\rho_n) > 0$ for all n large enough. This

implies $\widetilde{\mu}_1(\rho_n) = \frac{1}{\lambda_1(\rho_n)}$ and $\widetilde{u}_{\rho_n} = u_{\rho_n}$. Positiveness and pointwise convergence of u_{ρ_n} to z imply $z \geq 0$ a.e. in Ω. Moreover, by (4.1.14) we have

$$\int_\Omega \rho_n u_{\rho_n}^2 \, dx = \frac{1}{\lambda_1(\rho_n)}$$

and by (ii), passing to the limit, we find

$$\int_\Omega \rho z^2 \, dx = \frac{1}{\lambda_1(\rho)},$$

which implies $z \neq 0$. By using the weak form of problem (4.0.1) for u_{ρ_n} we have

$$\langle u_{\rho_n}, \varphi \rangle_{H_0^s(\Omega)} = \lambda_1(\rho_n) \langle \rho_n u_{\rho_n}, \varphi \rangle_{L^2(\Omega)} = \lambda_1(\rho_n) \int_\Omega \rho_n u_{\rho_n} \varphi \, dx \quad \forall \varphi \in H_0^s(\Omega)$$

and, letting n to infinity, we deduce $z = u_\rho$.
By (ii) $\mu_1(\rho_n) u_{\rho_n}$ weakly converges in $H_0^s(\Omega)$ to $\mu_1(\rho) u_\rho$ and $\|\mu_1(\rho_n) u_{\rho_n}\|_{H_0^s(\Omega)} = \mu_1(\rho_n)$ tends to $\mu_1(\rho) = \|\mu_1(\rho) u_\rho\|_{H_0^s(\Omega)}$. Hence $\mu_1(\rho_n) u_{\rho_n}$ strongly converges to $\mu_1(\rho) u_\rho$ in $H_0^s(\Omega)$. The last claim is immediate provided one observes that $\widetilde{\mu}_1(\eta) > 0$ implies $\widetilde{\mu}_1(\rho_n) > 0$ for all n large enough. □

Lemma 4.3.3. *Let $\rho, \eta, \rho_0 \in L^\infty(\Omega)$, $\widetilde{\mu}_1(\rho)$ be defined as in (4.3.1) for $k = 1$ and $\overline{\mathcal{G}(\rho_0)}$ the weak* closure in $L^\infty(\Omega)$ of the class of rearrangements $\mathcal{G}(\rho_0)$ introduced in Definition 4.2.10. Then*
(i) the map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is convex on $L^\infty(\Omega)$;
(ii) if ρ and η are linearly independent and $\widetilde{\mu}_1(\rho), \widetilde{\mu}_1(\eta) > 0$, then

$$\widetilde{\mu}_1(t\rho + (1-t)\mu) < t\widetilde{\mu}_1(\rho) + (1-t)\widetilde{\mu}_1(\eta)$$

for all $0 < t < 1$;
(iii) if $\int_\Omega \rho_0 \, dx > 0$, then the map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is strictly convex on $\overline{\mathcal{G}(\rho_0)}$.

Proof. (i) The Fischer Principle (4.1.9) and (4.3.3) both for $k = 1$ yield

$$\sup_{\substack{f \in H_0^s(\Omega) \\ f \neq 0}} \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} \leq \widetilde{\mu}_1(\rho) \tag{4.3.6}$$

for every $\rho \in L^\infty(\Omega)$. Moreover, if $\widetilde{\mu}_1(\rho) > 0$, then equality sign holds and the supremum is attained when f is an eigenfunction of $\mu_1(\rho) = \widetilde{\mu}_1(\rho)$. Let $\rho, \mu \in L^\infty(\Omega)$, $0 \leq t \leq 1$. We show that

$$\widetilde{\mu}_1(t\rho + (1-t)\eta) \leq t\widetilde{\mu}_1(\rho) + (1-t)\widetilde{\mu}_1(\eta). \tag{4.3.7}$$

If $\widetilde{\mu}_1(t\rho + (1-t)\eta) = 0$ (4.3.7) is obvious. Suppose $\widetilde{\mu}_1(t\rho + (1-t)\eta) > 0$. Then, for all $f \in H_0^s(\Omega)$ we have
(4.3.8)

$$\frac{\langle G_{t\rho+(1-t)\eta} f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} = t \frac{\langle G_\rho f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} + (1-t) \frac{\langle G_\eta f, f \rangle_{H_0^s(\Omega)}}{\|f\|^2_{H_0^s(\Omega)}} \leq t\widetilde{\mu}_1(\rho) + (1-t)\widetilde{\mu}_1(\eta),$$

where we used (4.1.8) and (4.3.6). Taking the supremum in the left-hand term and using (4.3.6) again with equality sign we find (4.3.7).

(ii) Arguing by contradiction, we suppose that equality holds in (4.3.7). We find out that ρ and μ are linearly dependent. Equality sign in (4.3.7) implies $\widetilde{\mu}_1(t\rho+(1-t)\eta) > 0$, then (by (4.3.6)) the equality also holds in (4.3.8) with $f = u = u_{t\rho+(1-t)\eta}$. We get

$$\frac{\langle G_\rho u, u\rangle_{H_0^s(\Omega)}}{\|u\|^2_{H_0^s(\Omega)}} = \widetilde{\mu}_1(\rho) \quad \text{ and } \quad \frac{\langle G_\eta u, u\rangle_{H_0^s(\Omega)}}{\|u\|^2_{H_0^s(\Omega)}} = \widetilde{\mu}_1(\eta).$$

The simplicity of the first eigenvalue, the positiveness of u and the normalization (4.1.13) imply that $u = u_\rho = u_\eta$. Writing the problem (4.0.1) in weak form for both weigths ρ and η we have

$$\langle u, \varphi\rangle_{H_0^s(\Omega)} = \frac{1}{\widetilde{\mu}_1(\rho)}\langle \rho u, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H_0^s(\Omega)$$

and

$$\langle u, \varphi\rangle_{H_0^s(\Omega)} = \frac{1}{\widetilde{\mu}_1(\eta)}\langle \eta u, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H_0^s(\Omega).$$

Taking the difference of these identities we find

$$\left\langle \left(\frac{\rho}{\widetilde{\mu}_1(\rho)} - \frac{\eta}{\widetilde{\mu}_1(\eta)}\right) u, \varphi \right\rangle_{L^2(\Omega)} = 0 \quad \forall\, \varphi \in H_0^s(\Omega),$$

which gives $\rho\widetilde{\mu}_1(\eta) - \eta\widetilde{\mu}_1(\rho) = 0$, i.e. ρ and η are linearly dependent.

(iii) First, note that $\int_\Omega \rho\, dx = \int_\Omega \rho_0\, dx > 0$ for any $\rho \in \overline{\mathcal{G}(\rho_0)}$. This follows easily by (i) of Proposition 4.2.13, Definition 4.2.7 and Proposition 4.2.6. Therefore, we have $|\{\rho > 0\}| > 0$ and thus $\widetilde{\mu}_1(\rho) > 0$ for all $\rho \in \overline{\mathcal{G}(\rho_0)}$. Next, we show that any distinct functions ρ and η in $\overline{\mathcal{G}(\rho_0)}$ are linearly independent. Indeed, let $\alpha\rho + \beta\eta = 0$ with $\alpha, \beta \in \mathbb{R}$. Integrating over Ω we obtain $(\alpha+\beta)\int_\Omega \rho_0\, dx = 0$, which implies $\beta = -\alpha$ and, in turn, $\alpha(\rho - \eta) = 0$ and $\alpha = 0$. Hence, ρ and η are linearly independent. The statement is now an immediate consequence of (ii). □

Remark 4.3.4. If $\int_\Omega \rho_0\, dx \leq 0$, $\rho_0 \neq 0$, the map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is not strictly convex on $\overline{\mathcal{G}(\rho_0)}$. Let us show this claim.
Applying Lemma 4.2.9, we find that the constant function $c = \frac{1}{|\Omega|}\int_\Omega \rho_0\, dx$ is in $\overline{\mathcal{G}(\rho_0)}$. By convexity of $\overline{\mathcal{G}(\rho_0)}$ (see Proposition 4.2.13), $t\rho_0 + (1-t)c \in \overline{\mathcal{G}(\rho_0)}$ for every $t \in [0,1]$. We discuss the two cases $\int_\Omega \rho_0\, dx = 0$, $\rho_0 \neq 0$ and $\int_\Omega \rho_0\, dx < 0$ separately.
If $\int_\Omega \rho_0\, dx = 0$, $\rho_0 \neq 0$, then $c = 0$. Thus, $t\rho_0 \in \overline{\mathcal{G}(\rho_0)}$ for every $t \in [0,1]$ and, by Remark 4.3.1, we have $\widetilde{\mu}_1(t\rho_0) = t\widetilde{\mu}_1(\rho_0)$, which excludes strict convexity.
We now turn to the case $\int_\Omega \rho_0\, dx < 0$, i.e. $c < 0$. From the inequality

$$t\rho_0 + (1-t)c \leq t\|\rho_0\|_\infty + (1-t)c,$$

we obtain

$$t\rho_0 + (1-t)c \le 0 \text{ in } \Omega \quad \forall\, t \le \frac{c}{c - \|\rho_0\|_\infty}.$$

Note that $c/(c - \|\rho_0\|_\infty) \in (0,1)$. Therefore, by (4.3.1), we conclude that $\widetilde{\mu}_1(\rho) = 0$ for any ρ in the line segment, contained in $\overline{\mathcal{G}(\rho_0)}$, that joins c and

$$\frac{c}{c - \|\rho_0\|_\infty}\rho_0 + \left(1 - \frac{c}{c - \|\rho_0\|_\infty}\right) c = \frac{\|\rho_0\|_\infty - \rho_0}{\|\rho_0\|_\infty - c} c.$$

This shows that the map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is not strictly convex also in this case.

Lemma 4.3.5. *Let $\rho \in L^\infty(\Omega)$, $\widetilde{\mu}_1(\rho)$ be defined as in (4.3.1) for $k = 1$ and u_ρ denote the first positive eigenfunction of problem (4.0.1) normalized as in (4.1.13). The map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is Gâteaux differentiable at any ρ such that $\widetilde{\mu}_1(\rho) > 0$ with Gâteaux differential equal to u_ρ^2. In other words, for every direction $v \in L^\infty(\Omega)$ we have*

$$\widetilde{\mu}_1'(\rho; v) = \int_\Omega u_\rho^2 v \, dx. \tag{4.3.9}$$

Proof. Let us compute

$$\lim_{t \to 0} \frac{\widetilde{\mu}_1(\rho + tv) - \widetilde{\mu}_1(\rho)}{t}.$$

Note that, by (ii) of Lemma 4.3.2, $\widetilde{\mu}_1(\rho + tv)$ converges to $\widetilde{\mu}_1(\rho)$ as t goes to zero for any $\rho, v \in L^\infty(\Omega)$. Therefore, $\widetilde{\mu}_1(\rho + tv) > 0$ for t small enough.
The eigenfunctions u_ρ and $u_{\rho+tv}$ satisfy

$$\widetilde{\mu}_1(\rho)\langle u_\rho, \varphi\rangle_{H_0^s(\Omega)} = \langle \rho u_\rho, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H_0^s(\Omega)$$

and

$$\widetilde{\mu}_1(\rho + tv)\langle u_{\rho+tv}, \varphi\rangle_{H_0^s(\Omega)} = \langle (\rho + tv) u_{\rho+tv}, \varphi\rangle_{L^2(\Omega)} \quad \forall\, \varphi \in H_0^s(\Omega).$$

By choosing $\varphi = u_{\rho+tv}$ in the former equation, $\varphi = u_\rho$ in the latter and comparing we get

$$\widetilde{\mu}_1(\rho + tv)\langle \rho u_\rho, u_{\rho+tv}\rangle_{L^2(\Omega)} = \widetilde{\mu}_1(\rho)\langle (\rho + tv) u_{\rho+tv}, u_\rho\rangle_{L^2(\Omega)}.$$

Rearranging we find

$$\frac{\widetilde{\mu}_1(\rho + tv) - \widetilde{\mu}_1(\rho)}{t} \int_\Omega \rho\, u_\rho u_{\rho+tv} \, dx = \widetilde{\mu}_1(\rho) \int_\Omega u_\rho u_{\rho+tv} v \, dx. \tag{4.3.10}$$

If t goes to zero, then by (iii) of Lemma 4.3.2 it follows that $u_{\rho+tv}$ converges to u_ρ in $H_0^s(\Omega)$ and therefore in $L^2(\Omega)$. Passing to the limit in (4.3.10) and using (4.1.14) we conclude

$$\lim_{t \to 0} \frac{\widetilde{\mu}_1(\rho + tv) - \widetilde{\mu}_1(\rho)}{t} = \int_\Omega u_\rho^2 v \, dx,$$

i.e. (4.3.9) holds. □

We are now able to prove our main result.

Theorem 4.3.6. *Let $\lambda_1(\rho)$ be the first positive eigenvalue of problem (4.0.1), $\rho_0 \in L^\infty(\Omega)$ such that $|\{x \in \Omega : \rho_0(x) > 0\}| > 0$ and $\mathcal{G}(\rho_0)$ the class of rearrangements of ρ_0 introduced in Definition 4.2.10. Then*
(i) there exists $\check{\rho}_1 \in \mathcal{G}(\rho_0)$ such that

$$\lambda_1(\check{\rho}_1) = \min_{\rho \in \mathcal{G}(\rho_0)} \lambda_1(\rho); \tag{4.3.11}$$

(ii) there exists an increasing function ψ such that

$$\check{\rho}_1 = \psi(u_{\check{\rho}_1}) \quad a.e. \ in \ \Omega, \tag{4.3.12}$$

where $u_{\check{\rho}_1}$ is the positive first eigenfunction relative to $\lambda_1(\check{\rho}_1)$ normalized as in (4.1.13).

Proof. (i) By (iii) of Proposition 4.2.11 and (ii) of Lemma 4.3.2, $\overline{\mathcal{G}(\rho_0)}$ is sequentially weakly* compact and the map $\rho \mapsto \widetilde{\mu}_1(\rho)$ is sequentially weakly* continuous. Therefore, there exists $\check{\rho}_1 \in \overline{\mathcal{G}(\rho_0)}$ such that

$$\widetilde{\mu}_1(\check{\rho}_1) = \max_{\rho \in \overline{\mathcal{G}(\rho_0)}} \widetilde{\mu}_1(\rho).$$

Note that, by Proposition 4.1.3, the condition $|\{\rho_0 > 0\}| > 0$ guarantees $\widetilde{\mu}_1(\check{\rho}_1) > 0$.
Let us show that $\check{\rho}_1$ actually belongs to $\mathcal{G}(\rho_0)$ (in fact, the following argument shows that there are not maximizers of $\widetilde{\mu}_1(\rho)$ in $\overline{\mathcal{G}(\rho_0)} \smallsetminus \mathcal{G}(\rho_0)$). Proceeding by contradiction, suppose that $\check{\rho}_1 \notin \mathcal{G}(\rho_0)$. Then, by (iii) of Proposition 4.2.13 and by Definition 4.2.12, $\check{\rho}_1$ is not an extreme point of $\overline{\mathcal{G}(\rho_0)}$ and thus there exist $\rho, \eta \in \overline{\mathcal{G}(\rho_0)}$ such that $\rho \neq \eta$ and $\check{\rho}_1 = \frac{\rho+\eta}{2}$. By (i) of Lemma 4.3.3 and being $\check{\rho}_1$ a maximizer, we have

$$\widetilde{\mu}_1(\check{\rho}_1) \leq \frac{\widetilde{\mu}_1(\rho) + \widetilde{\mu}_1(\eta)}{2} \leq \widetilde{\mu}_1(\check{\rho}_1)$$

and then, equality sign holds. This implies $\widetilde{\mu}_1(\rho) = \widetilde{\mu}_1(\eta) = \widetilde{\mu}_1(\check{\rho}_1) > 0$, that is ρ and η are maximizers as well. Now, applying (ii) of Lemma 4.3.3 to ρ and η with $t = \frac{1}{2}$, we conclude that ρ and η are linearly dependent. Without loss of generality, we can assume that there exists $\alpha \in \mathbb{R}$ such that $\eta = \alpha\rho$, moreover α is nonzero since η is a maximizer. Combining $\eta = \alpha\rho$ with $\check{\rho}_1 = \frac{\rho+\eta}{2}$ we get $\check{\rho}_1 = \frac{1+\alpha}{2}\rho = \frac{1+\alpha}{2\alpha}\eta$. It is immediate to show that at least one of the coefficients $\frac{1+\alpha}{2}$ and $\frac{1+\alpha}{2\alpha}$ must be non-negative. In either cases we find a contradiction. For instance, if $\frac{1+\alpha}{2\alpha} \geq 0$, by Remark 4.3.1 and maximality of η we obtain

$$\widetilde{\mu}_1(\check{\rho}_1) = \frac{1+\alpha}{2\alpha}\widetilde{\mu}_1(\eta) = \frac{1+\alpha}{2\alpha}\widetilde{\mu}_1(\check{\rho}_1),$$

which implies $\alpha = 1$ and yields the contradiction $\rho = \eta$. The other case is analogous. Thus, we conclude that $\check{\rho}_1 \in \mathcal{G}(\rho_0)$ and

$$\widetilde{\mu}_1(\check{\rho}_1) = \max_{\rho \in \mathcal{G}(\rho_0)} \widetilde{\mu}_1(\rho). \tag{4.3.13}$$

Being $|\{\rho_0 > 0\}| > 0$, we have

$$\lambda_1(\rho) = \frac{1}{\mu_1(\rho)} = \frac{1}{\widetilde{\mu}_1(\rho)}$$

for all $\rho \in \mathcal{G}(\rho_0)$. Therefore, (4.3.13) is equivalent to (4.3.11) and (i) is proved.

(ii) We prove the claim by using Proposition 4.2.15; more precisely, we show that

$$\int_\Omega \check{\rho}_1 u_{\check{\rho}_1}^2 \, dx > \int_\Omega \rho \, u_{\check{\rho}_1}^2 \, dx \tag{4.3.14}$$

for every $\rho \in \overline{\mathcal{G}(\rho_0)} \setminus \{\check{\rho}_1\}$. By exploiting the convexity of $\widetilde{\mu}_1(\rho)$ (see Lemma 4.3.3) and its Gâteaux differentiability in $\check{\rho}_1$ (see Lemma 4.3.5) we have (for details see [77])

$$\widetilde{\mu}_1(\rho) \geq \widetilde{\mu}_1(\check{\rho}_1) + \int_\Omega (\rho - \check{\rho}_1) u_{\check{\rho}_1}^2 \, dx \tag{4.3.15}$$

for all $\rho \in \overline{\mathcal{G}(\rho_0)}$. First, let us suppose $\widetilde{\mu}_1(\rho) < \widetilde{\mu}_1(\check{\rho}_1)$. Comparing with (4.3.15) we find

$$\int_\Omega (\rho - \check{\rho}_1) u_{\check{\rho}_1}^2 \, dx < 0,$$

that is (4.3.14).
Next, let us consider the case $\widetilde{\mu}_1(\rho) = \widetilde{\mu}_1(\check{\rho}_1)$, $\rho \in \overline{\mathcal{G}(\rho_0)} \setminus \{\check{\rho}_1\}$. By the argument used in part (i) there are not maximizers of $\widetilde{\mu}_1$ in $\overline{\mathcal{G}(\rho_0)} \setminus \mathcal{G}(\rho_0)$, therefore $\rho \in \mathcal{G}(\rho_0)$.
If $\check{\rho}_1$ and ρ are linearly independent, then, (ii) of Lemma 4.3.3 implies

$$\widetilde{\mu}_1\left(\frac{\check{\rho}_1 + \rho}{2}\right) < \frac{\widetilde{\mu}_1(\check{\rho}_1) + \widetilde{\mu}_1(\rho)}{2} = \widetilde{\mu}_1(\check{\rho}_1).$$

Then, as in the previous step, (4.3.15) with $\frac{\check{\rho}_1+\rho}{2}$ in place of ρ yields (4.3.14).
Finally, let $\check{\rho}_1$ and ρ be linearly dependent. Being $\check{\rho}_1$ and ρ both nonzero, we can assume $\rho = \alpha\check{\rho}_1$ for a constant $\alpha \in \mathbb{R}$. Therefore $|\rho| = |\alpha|\,|\check{\rho}_1|$. Now, by (i) and (ii) of Proposition 4.2.4, the functions $|\rho|$ and $|\check{\rho}_1|$ are equimeasurable and $\operatorname{ess\,sup}|\rho| = \operatorname{ess\,sup}|\check{\rho}_1| > 0$. This leads to $|\alpha| = 1$ and, being ρ and $\check{\rho}_1$ distinct, $\alpha = -1$. Thus $\rho = -\check{\rho}_1$, which by (4.1.14) gives

$$\int_\Omega \rho \, u_{\check{\rho}_1}^2 \, dx = -\int_\Omega \check{\rho}_1 u_{\check{\rho}_1}^2 \, dx = -\widetilde{\mu}_1(\check{\rho}_1) < \widetilde{\mu}_1(\check{\rho}_1) = \int_\Omega \check{\rho}_1 \, u_{\check{\rho}_1}^2 \, dx,$$

i.e. (4.3.14). This completes the proof. □

Remark 4.3.7. If ρ_0 satisfies the stronger condition $\int_\Omega \rho_0 \, dx > 0$, then the proof simplifies as one can rely on (iii) of Lemma 4.3.3 (strict convexity of $\rho \mapsto \widetilde{\mu}_1(\rho)$). Indeed, from $\check{\rho}_1 = \frac{\rho+\eta}{2}$, $\rho \neq \eta$, it follows immediately the contradiction

$$\widetilde{\mu}_1(\check{\rho}_1) < \frac{\widetilde{\mu}_1(\rho) + \widetilde{\mu}_1(\eta)}{2} \leq \widetilde{\mu}_1(\check{\rho}_1).$$

Further, note that in this case $\lambda_1(\rho)$ is well defined for all $\rho \in \overline{\mathcal{G}(\rho_0)}$ (it follows by (i) of Proposition 4.2.13, Definition 4.2.7 and Proposition 4.2.6 with F equal to the identity function). However, the previous proof shows that no minimizer of $\lambda_1(\rho)$ belongs to

$\overline{\mathcal{G}(\rho_0)} \smallsetminus \mathcal{G}(\rho_0)$.
Finally, in this case the estimate

$$\lambda_1(\check{\rho}_1) \leq \frac{\lambda_1\,|\Omega|}{\int_\Omega \rho_0\,dx} \tag{4.3.16}$$

holds, where λ_1 denotes the first eigenvalue of problem (4.0.1) with $\rho \equiv 1$. The estimate (4.3.16) follows by the fact that the constant function $c = \frac{1}{|\Omega|}\int_\Omega \rho_0\,dx$ belongs to $\overline{\mathcal{G}(\rho_0)}$ (see Lemma 4.2.9 and Proposition 4.2.13), the minimality of $\lambda_1(\check{\rho}_1)$ and the identity $\lambda_1 = c\lambda_1(c)$ (which is a straightforward consequence of the variational characterization (4.1.12)).

Remark 4.3.8. The study of the maximization of $\lambda_1(\rho)$ on $\mathcal{G}(\rho_0)$ seems to be rather different. We list here some partial results. Assume $|\{\rho_0 > 0\}| > 0$. If $\int_\Omega \rho_0\,dx \leq 0$, then, by Lemma 4.2.9 and Proposition 4.2.13, the non-positive constant function $c = \frac{1}{|\Omega|}\int_\Omega \rho_0\,dx$ belongs to $\overline{\mathcal{G}(\rho_0)}$. Therefore, by definition of $\widetilde{\mu}_1(\rho)$, $\min_{\rho\in\overline{\mathcal{G}(\rho_0)}} \widetilde{\mu}_1(\rho) = 0$ which, in turns, being $\mathcal{G}(\rho_0)$ dense in $\overline{\mathcal{G}(\rho_0)}$ and $\widetilde{\mu}_1(\rho)$ sequentially weak* continuous, implies $\inf_{\rho\in\mathcal{G}(\rho_0)} \widetilde{\mu}_1(\rho) = 0$ and, finally, $\sup_{\rho\in\mathcal{G}(\rho_0)} \lambda_1(\rho) = +\infty$.
If, instead $\int_\Omega \rho_0\,dx > 0$, then by proceeding as in the first part of the previous proof and using (iii) of Lemma 4.3.3, one immediately concludes that there is a unique $\widehat{\rho}_1 \in \overline{\mathcal{G}(\rho_0)}$ such that

$$\widetilde{\mu}_1(\widehat{\rho}_1) = \min_{\rho\in\overline{\mathcal{G}(\rho_0)}} \widetilde{\mu}_1(\rho),$$

which, in this case, is equivalent to

$$\lambda_1(\widehat{\rho}_1) = \max_{\rho\in\overline{\mathcal{G}(\rho_0)}} \lambda_1(\rho).$$

Moreover, by Lemma 4.3.5, for all $\rho \in \overline{\mathcal{G}(\rho_0)}$ and $t \in (0,1]$ we can write

$$\widetilde{\mu}_1(\widehat{\rho}_1) \leq \widetilde{\mu}_1(\widehat{\rho}_1 + t(\rho - \widehat{\rho}_1)) = \widetilde{\mu}_1(\widehat{\rho}_1) + t\int_\Omega (\rho - \widehat{\rho}_1)u_{\widehat{\rho}_1}^2\,dx + o(t)$$

for t that goes to zero. Finally, after some easy algebraic manipulations and passing to the limit we find

$$\int_\Omega \hat{\rho}_1 u_{\hat{\rho}_1}^2\,dx \leq \int_\Omega \rho u_{\hat{\rho}_1}^2\,dx \quad \forall\,\rho \in \overline{\mathcal{G}(\rho_0)}.$$

Remark 4.3.9. As already noted in Remark 4.1.2, we have $\lambda_{-1}(\rho) = -\lambda_1(-\rho)$ for all $\rho \in L^\infty(\Omega)$ such that $|\{\rho < 0\}| > 0$. Furthermore, it is easy to see from (4.1.10) that the eigenspaces relative to $\lambda_{-1}(\rho)$ and $\lambda_1(-\rho)$ coincide. Finally, observe that by (iii) of Proposition 4.2.4 with $F(t) = -t$, it follows that $\mathcal{G}(-\rho_0) = -\mathcal{G}(\rho_0) = \{\rho \in L^\infty(\Omega) : -\rho \in \mathcal{G}(\rho_0)\}$ and then $\overline{\mathcal{G}(-\rho_0)} = -\overline{\mathcal{G}(\rho_0)}$. Thus, Theorem 4.3.6 can be reformulated in terms of the first negative eigenvalue $\lambda_{-1}(\rho)$ as follows.

Theorem 4.3.10. *Let $\lambda_{-1}(\rho)$ be the first negative eigenvalue of problem (4.0.1), $\rho_0 \in L^\infty(\Omega)$ such that $|\{\rho_0 < 0\}| > 0$ and $\mathcal{G}(\rho_0)$ the class of rearrangements of ρ_0 introduced in*

Definition 4.2.10. Then

(i) there exist $\check{\rho}_{-1} \in \mathcal{G}(\rho_0)$ such that

$$\lambda_{-1}(\check{\rho}_{-1}) = \max_{\rho \in \mathcal{G}(\rho_0)} \lambda_{-1}(\rho); \tag{4.3.17}$$

(ii) there exists a decreasing function ϕ such that

$$\check{\rho}_{-1} = \phi(u_{-\check{\rho}_{-1}}) \quad a.e. \ in\ \Omega,$$

where $u_{-\check{\rho}_{-1}}$ is the first positive eigenfunction relative to $\lambda_1(-\check{\rho}_{-1})$ normalized as in (4.1.13).

4.4 Steiner symmetry

We introduce first the definitions and some results about the Steiner symmetrization of sets and functions. For a thorough treatment we refer the reader to [34]. Then, we prove our symmetry result.

Let $l(x') = \{x = (x_1, x') \in \mathbb{R}^N : x_1 \in \mathbb{R}\}$ for any $x' \in \mathbb{R}^{N-1}$ fixed and let T be the hyperplane $\{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$.

Definition 4.4.1. *Let $\Omega \subset \mathbb{R}^N$ be a measurable set. Then*

(i) the set

$$\Omega^\sharp = \left\{x = (x_1, x') \in \mathbb{R}^N : 2|x_1| < |\Omega \cap l(x')|_1, x' \in \mathbb{R}^{N-1}\right\},$$

where $|\cdot|_1$ denotes the one dimensional Lebesgue measure, is said Steiner symmetrization *of Ω with respect to the hyperplane T;*

(ii) the set Ω is said Steiner symmetric *if $\Omega^\sharp = \Omega$.*

It can be shown that $|\Omega| = |\Omega^\sharp|$.

Definition 4.4.2. *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite measure and $u : \Omega \to \mathbb{R}$ a measurable function bounded from below. Then*

(i) the function $u^\sharp : \Omega^\sharp \to \mathbb{R}$, defined by

$$u^\sharp(x) = \sup\{c \in \mathbb{R} : x \in \{u > c\}^\sharp\},$$

is said Steiner symmetrization *of u with respect to the hyperplane T;*

(ii) the function u is said Steiner symmetric *if $u^\sharp = u$.*

It can be proved that

$$|\{x \in \Omega^\sharp : u^\sharp(x) > t\}| = |\{x \in \Omega : u(x) > t\}| \quad \forall t \in \mathbb{R}. \tag{4.4.1}$$

Proposition 4.4.3. *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite measure, $u : \Omega \to \mathbb{R}$ a measurable function bounded from below and $\psi : \mathbb{R} \to \mathbb{R}$ an increasing function. Then $\psi(u^\sharp) = (\psi(u))^\sharp$ a.e. in Ω.*

For the proof see [34, Lemma 3.2].

Proposition 4.4.4 (Hardy-Littlewood's inequality). *Let $\Omega \subset \mathbb{R}^N$ be a measurable set of finite measure, $u, v : \Omega \to \mathbb{R}$ two measurable functions bounded from below such that $uv \in L^1(\Omega)$. Then*

$$\int_\Omega u(x)v(x)\,dx \leq \int_{\Omega^\sharp} u^\sharp(x)v^\sharp(x)\,dx.$$

This proposition follows easily from [34, Lemma 3.3].

Proposition 4.4.5 (nonlocal Pòlya-Szegö's inequality). *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set, $s \in (0,1)$ and $u \in H_0^s(\Omega)$. Then*

$$\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u^\sharp(x) - u^\sharp(y)|^2}{|x-y|^{N+2s}}\,dx\,dy \leq \iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x-y|^{N+2s}}\,dx\,dy;$$

moreover, the equality holds if and only if u is proportional to a translate of a function which is symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$.

For the proof we refer the reader to [156]. Integral inequalities of this type in a more general context can be found in [12].
We now prove the symmetry result.

Theorem 4.4.6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$ Steiner symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$ and $\rho_0 \in L^\infty(\Omega)$ such that $|\{x \in \Omega : \rho_0(x) > 0\}| > 0$. Then, every minimizer $\check{\rho}_1$ of the problem (4.3.11) is Steiner symmetric relative to T.*

Proof. Let $\check{\rho}_1$ be as in (4.3.11) and let $u_{\check{\rho}_1}$ be the positive first eigenfunction of the problem (4.0.1) normalized as in (4.1.13).
By (4.3.12) and Proposition 4.4.3, the Steiner symmetry of $\check{\rho}_1$ is a consequence of the analogous symmetry of $u_{\check{\rho}_1}$; hence it suffices to show that $u^\sharp_{\check{\rho}_1} = u_{\check{\rho}_1}$. By (4.1.12) we have

$$\lambda_1(\check{\rho}_1) = \frac{\|u_{\check{\rho}_1}\|^2_{H_0^s(\Omega)}}{\int_\Omega \check{\rho}_1 u^2_{\check{\rho}_1}\,dx}.$$

Propositions 4.4.3, 4.4.4 and 4.4.5 yield

$$\int_\Omega \check{\rho}_1 u^2_{\check{\rho}_1}\,dx \leq \int_\Omega \check{\rho}_1^\sharp (u^2_{\check{\rho}_1})^\sharp\,dx = \int_\Omega \check{\rho}_1^\sharp (u^\sharp_{\check{\rho}_1})^2\,dx$$

and

$$\|u_{\check{\rho}_1}\|^2_{H_0^s(\Omega)} \geq \|u^\sharp_{\check{\rho}_1}\|^2_{H_0^s(\Omega)}. \tag{4.4.2}$$

Consequently we find

$$\lambda_1(\check{\rho}_1) = \frac{\|u_{\check{\rho}_1}\|^2_{H_0^s(\Omega)}}{\int_\Omega \check{\rho}_1 u^2_{\check{\rho}_1}\,dx} \geq \frac{\|u^\sharp_{\check{\rho}_1}\|^2_{H_0^s(\Omega)}}{\int_\Omega \check{\rho}_1^\sharp (u^\sharp_{\check{\rho}_1})^2\,dx} \geq \frac{\|u_{\check{\rho}_1^\sharp}\|^2_{H_0^s(\Omega)}}{\int_\Omega \check{\rho}_1^\sharp u^2_{\check{\rho}_1^\sharp}\,dx} = \lambda_1(\check{\rho}_1^\sharp) \geq \lambda_1(\check{\rho}_1),$$

where $u_{\check{\rho}_1^\sharp}$ is the normalized positive first eigenfunction corresponding to $\check{\rho}_1^\sharp$ and the last inequality comes from $\check{\rho}_1^\sharp \in \mathcal{G}(\rho_0)$ (a straightforward consequence of (4.4.1)) and the minimality of $\check{\rho}_1$. Therefore, all the inequalities are actually equalities and this implies the equality sign also in (4.4.2). Then, by Proposition 4.4.5 it follows that

$$u_{\check{\rho}_1}(x) = \widetilde{u}(x),$$

where $\widetilde{u}$ is Steiner symmetric with respect to a hyperplane $T_v = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = v\}$, $v \in \mathbb{R}$. Therefore $\Omega = \{u_{\check{\rho}_1} > 0\} = \{\widetilde{u} > 0\}$ is symmetric with respect to both hyperplanes T and T_v. Being Ω bounded, it follows that $v = 0$ and then $u_{\check{\rho}_1}$ is Steiner symmetric relative to T, i.e.

$$u_{\check{\rho}_1}^\sharp = u_{\check{\rho}_1}.$$

This completes the proof. □

In particular, when Ω is a ball we find the following assertion.

Corollary 4.4.7. *Let Ω be a ball in $\mathbb{R}^N$ and $\rho_0 \in L^\infty(\Omega)$ such that $|\{\rho_0 > 0\}| > 0$. Then every minimizer $\check{\rho}_1$ of problem (4.3.11) is decreasing radially symmetric.*

Remark 4.4.8. Note that, in this case, $\check{\rho}_1$ is unique and explicitly determined by the class of rearrangements of ρ_0. Indeed, we have $\check{\rho}_1(x) = \rho_0^*(\omega_N |x|^N)$ for any $x \in \Omega$, where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$.

Recalling Remark 4.3.9, we can immediately state the symmetry results for $\lambda_{-1}(\rho)$.

Theorem 4.4.9. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain of class $C^{1,1}$ Steiner symmetric with respect to the hyperplane $T = \{x = (x_1, x') \in \mathbb{R}^N : x_1 = 0\}$ and $\rho_0 \in L^\infty(\Omega)$ such that $|\{\rho_0 < 0\}| > 0$. Then, every maximizer $\check{\rho}_{-1}$ of the problem (4.3.17) is such that $-\check{\rho}_{-1}$ is Steiner symmetric relative to T.*

Corollary 4.4.10. *Let Ω be a ball in $\mathbb{R}^N$ and $\rho_0 \in L^\infty(\Omega)$ such that $|\{\rho_0 < 0\}| > 0$. Then every maximizer $\check{\rho}_{-1}$ of problem (4.3.17) is increasing radially symmetric. More precisely, we have the unique maximizer $\check{\rho}_{-1}(x) = -(-\rho_0)^*(\omega_N |x|^N)$ for any $x \in \Omega$.*

Chapter 5

Nonlinear Dirichlet problem for the nonlocal anisotropic operator L_K

In this chapter we study the nonlinear Dirichlet problem driven by the nonlocal anisotropic operator L_K, defined as in (2.2.3),

$$\begin{cases} L_K u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1,1}$ boundary, $N > 2s$, $s \in (0,1)$, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.
As application of Theorem 2.3.8 of equivalence of minimizers, we show that, under suitable assumptions on the reaction term, problem (2.3.1) admits at least three nontrivial solutions: one positive, one negative and one of unknown sign, using variational methods and, in particular Morse theory (see Chapters 1-2).
In order to prove our result, it is crucial and extremely delicate the choice of the functional setting $X_K(\Omega)$, in particular the hypotheses on the function a in the kernel K. By the results of Ros Oton in [173], if a is non-negative the Poincaré inequality and regularity results still hold, therefore they are used to solve linear problems; on the other hand, by results of Servadei and Valdinoci in [183], if a is positive we know that the embedding properties (stated in Chapter 2) still hold, and these tools are necessary to solve nonlinear problems.
An alternative to preserve regularity results is taking kernels between two positive constants, for instance considering $a \in L^\infty(\Omega)$, but in this way the operator L_K behaves exactly as the fractional Laplacian and, in particular $X_K(\Omega)$ coincides with the Sobolev space $H_0^s(\Omega)$, consequently there is not any real novelty. These reasons explain our assumptions on the kernel K, in particular the choice of a belonging to $L^1(\Omega)$.

The chapter is organized as follows: in Section 5.1 we prove a multiplicity result and in Section 5.2 we study a general Hopf's lemma where the nonlinearity is slightly negative.

5.1 A multiplicity result

In this section we present an existence and multiplicity result for the solutions of problem (2.3.1), under condition (2.3.2) plus some further conditions; in the proof Theorem 2.3.8 will play an essential part. This result is an extension to the anisotropic case of a result on the fractional Laplacian [113, Theorem 5.2]. By a truncation argument and minimization, we show the existence of two constant sign solutions, then we apply Morse theory to find a third nontrivial solution. In order to do this, the nonlinearity f satisfies the following:

($\mathbf{H}_{5.1}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

- *(i)* $|f(x,t)| \leq a(1+|t|^{q-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ $(a > 0, 1 < q < 2_s^*)$;
- *(ii)* $f(x,t)t \geq 0$ a.e. in Ω and for all $t \in \mathbb{R}$;
- *(iii)* $\lim_{t\to 0} \frac{f(x,t)-b|t|^{r-2}t}{t} = 0$ uniformly a.e. in Ω $(b > 0, 1 < r < 2)$;
- *(iv)* $\limsup_{|t|\to\infty} \frac{2F(x,t)}{t^2} < \lambda_1$ uniformly a.e. in Ω.

Example 5.1.1. As a model for f we can take the function

$$f(t) := \begin{cases} b|t|^{r-2}t + a_1|t|^{q-2}t, & \text{if } |t| \leq 1, \\ \beta_1 t, & \text{if } |t| > 1, \end{cases}$$

with $1 < r < 2 < q < 2_s^*$, $a_1, b > 0$, $\beta_1 \in (0, \lambda_1)$ such that $a_1 + b = \beta_1$.

Theorem 5.1.2. *Let* ($\mathbf{H}_{5.1}$) *hold. Then problem* (2.3.1) *admits at least three nontrivial solutions* $u^{\pm} \in \pm\, \mathrm{int}(C_s^0(\overline{\Omega})_+)$, $\tilde{u} \in C_s^0(\overline{\Omega}) \setminus \{0\}$.

Proof. We define $J \in C^1(X_K(\Omega))$ as

$$J(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F(x,u(x))\,dx.$$

Without loss of generality, we assume $q > 2$ and $\epsilon, \epsilon_1, b_1, a_1, a_2$ are positive constants. From ($\mathbf{H}_{5.1}$) *(ii)* we have immediately that $0 \in K_J$, but from ($\mathbf{H}_{5.1}$) *(iii)* 0 is not a local minimizer. Indeed, by ($\mathbf{H}_{5.1}$) *(iii)*, there exists $\delta > 0$ such that for $t \in (0, \delta)$ we have

$$\frac{f(x,t) - bt^{r-1}}{t} \geq -\epsilon,$$

by integrating $F(x,t) \geq b_1 t^r - \epsilon_1 t^2$ $(\epsilon_1 < b_1)$, but by ($\mathbf{H}_{5.1}$) *(i)* $F(x,t) \geq -a_1 t - a_2 t^q$, hence, in the end, we obtain a.e. in Ω and for all $t \in \mathbb{R}$

$$F(x,t) \geq c_0|t|^r - c_1|t|^q \quad (c_0, c_1 > 0). \tag{5.1.1}$$

We consider a function $u \in X_K(\Omega)$, $u(x) > 0$ a.e. in Ω, for all $\tau > 0$ we have

$$J(\tau u) = \frac{\tau^2\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F(x,\tau u)\,dx \leq \frac{\tau^2\|u\|^2_{X_K(\Omega)}}{2} - c_0\tau^r\|u\|_r^r + c_1\tau^q\|u\|_q^q,$$

and the latter is negative for $\tau > 0$ close enough to 0, therefore, 0 is not a local minimizer of J.
We define two truncated energy functionals

$$J_{\pm}(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_{\Omega} F_{\pm}(x,u)\,dx \quad \forall u \in X_K(\Omega),$$

setting for all $(x,t) \in \Omega \times \mathbb{R}$

$$f_{\pm}(x,t) = f(x, \pm t_{\pm}),\ F_{\pm}(x,t) = \int_0^t f_{\pm}(x,\tau)\,d\tau,\ t_{\pm} = \max\{\pm t, 0\}\ \forall t \in \mathbb{R}.$$

In a similar way, by (5.1.1), we obtain that 0 is not a local minimizer for the truncated functionals $J_{\pm}$.
We focus on the functional J_+, clearly $J_+ \in C^1(X_K(\Omega))$ and f_+ satisfies (2.3.2). We now prove that J_+ is coercive in $X_K(\Omega)$, i.e.,

$$\lim_{\|u\|_{X_K(\Omega)} \to \infty} J_+(u) = \infty.$$

Indeed, by $(\mathbf{H}_{5.1})$ *(iv)*, for all $\epsilon > 0$ small enough, we have a.e. in Ω and for all $t \in \mathbb{R}$

$$F_+(x,t) \leq \frac{\lambda_1 - \epsilon}{2} t^2 + C.$$

By the definition of λ_1, we have for all $u \in X_K(\Omega)$

$$J_+(u) \geq \frac{\|u\|^2_{X_K(\Omega)}}{2} - \frac{\lambda_1 - \epsilon}{2}\|u\|_2^2 - C \geq \frac{\epsilon}{2\lambda_1}\|u\|^2_{X_K(\Omega)} - C,$$

and the latter goes to ∞ as $\|u\|_{X_K(\Omega)} \to \infty$. Consequently, J_+ is coercive in $X_K(\Omega)$.
Moreover, J_+ is sequentially weakly lower semicontinuous in $X_K(\Omega)$. Indeed, let $u_n \rightharpoonup u$ in $X_K(\Omega)$, passing to a subsequence, we may assume $u_n \to u$ in $L^q(\Omega)$ and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, moreover, there exists $g \in L^q(\Omega)$ such that $|u_n(x)| \leq g(x)$ for a.e. $x \in \Omega$ and all $n \in \mathbb{N}$ [30, Theorem 4.9]. Hence,

$$\lim_n \int_{\Omega} F_+(x,u_n)\,dx = \int_{\Omega} F_+(x,u)\,dx.$$

Besides, by convexity we have

$$\liminf_n \frac{\|u_n\|^2_{X_K(\Omega)}}{2} \geq \frac{\|u\|^2_{X_K(\Omega)}}{2},$$

as a result

$$\liminf_n J_+(u_n) \geq J_+(u).$$

Thus, there exists $u^+ \in X_K(\Omega) \setminus \{0\}$ such that

$$J_+(u^+) = \inf_{u \in X_K(\Omega)} J_+(u).$$

By Proposition 2.3.2 and by $(\mathbf{H}_{5.1})$ *(ii)* we have that u^+ is a non-negative weak solution to (2.3.1). By Theorem 2.3.1, we obtain $u^+ \in L^\infty(\Omega)$, hence by Proposition 2.3.4 and Theorem 2.3.6 we deduce $u^+ \in C^0_s(\overline{\Omega})$. Furthermore, by Hopf's lemma $\frac{u^+}{\mathrm{d}^s_\Omega} > 0$ in $\overline{\Omega}$, and by (2.3.10) $u^+ \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$.
Let $\rho > 0$ be such that $B^s_\rho(u^+) \subset C^0_s(\overline{\Omega})_+$, $u^+ + v \in B^s_\rho(u^+)$, $\forall v \in C^0_s(\overline{\Omega})$ with $||v||_{0,s} \le \rho$, since J and J_+ agree on $C^0_s(\overline{\Omega})_+ \cap X_K(\Omega)$,

$$J(u^+ + v) \ge J(u^+), \qquad v \in B_\rho(0) \cap X_K(\Omega)$$

and by Theorem 2.3.8, u^+ is a strictly positive local minimizer for J in $X_K(\Omega)$. Similarly, looking at J_-, we can detect another strictly negative local minimizer $u^- \in -\mathrm{int}(C^0_s(\overline{\Omega})_+)$ of J. Now, by Theorem 1.1.3 there exists, besides the two points of minimum, a third critical point $\tilde{u}$, such point is of mountain pass type. We only have to show that $\tilde{u} \neq 0$, to do this we use a Morse-theoretic argument. First of all, we prove that J satisfies Cerami condition (which in this case is equivalent to the Palais-Smale condition) to apply Morse theory.
Let (u_n) be a sequence in $X_K(\Omega)$ such that $|J(u_n)| \le C$ for all $n \in \mathbb{N}$ and $(1 + \|u_n\|_{X_K(\Omega)})J'(u_n) \to 0$ in $X_K(\Omega)^*$. Since J is coercive, the sequence (u_n) is bounded in $X_K(\Omega)$, hence, passing to a subsequence, we may assume $u_n \rightharpoonup u$ in $X_K(\Omega)$, $u_n \to u$ in $L^q(\Omega)$ and $L^1(\Omega)$, and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, with some $u \in X_K(\Omega)$. Moreover, by [30, Theorem 4.9] there exists $g \in L^q(\Omega)$ such that $|u_n(x)| \le g(x)$ for all $n \in \mathbb{N}$ and a.e. $x \in \Omega$. Using such relations along with $(\mathbf{H}_{5.1})$ *(i)*, we obtain

$$\begin{aligned}
\|u_n - u\|^2_{X_K(\Omega)} &= \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \\
&= J'(u_n)(u_n - u) + \int_\Omega f(x, u_n)(u_n - u)\,dx - \langle u, u_n - u\rangle \\
&\le \|J'(u_n)\|_*\|u_n - u\|_{X_K(\Omega)} + \int_\Omega a(1 + |u_n|^{q-1})|u_n - u|\,dx \\
&- \langle u, u_n - u\rangle \\
&\le \|J'(u_n)\|_*\|u_n - u\|_{X_K(\Omega)} + a(\|u_n - u\|_1 + \|u_n\|^{q-1}_q\|u_n - u\|_q) \\
&- \langle u, u_n - u\rangle
\end{aligned}$$

for all $n \in \mathbb{N}$ and the latter tends to 0 as $n \to \infty$. Thus, $u_n \to u$ in $X_K(\Omega)$.
Without loss of generality, we assume that 0 is an isolated critical point, therefore we can determine the corresponding critical group.
Claim: $C_k(J, 0) = 0 \quad \forall k \in \mathbb{N}_0$. By $(\mathbf{H}_{5.1})$ *(iii)*, we have

$$\lim_{t\to 0} \frac{rF(x,t) - f(x,t)t}{t^2} = 0,$$

hence, for all $\epsilon > 0$ we can find $C_\epsilon > 0$ such that a.e. in Ω and for all $t \in \mathbb{R}$

$$\left| F(x,t) - \frac{f(x,t)t}{r} \right| \le \epsilon t^2 + C_\epsilon |t|^q.$$

By the relations above we obtain

$$\int_\Omega \left(F(x,u) - \frac{f(x,u)u}{r} \right) dx = o(\|u\|^2_{X_K(\Omega)}) \qquad \text{as } \|u\|_{X_K(\Omega)} \to 0.$$

For all $u \in X_K(\Omega) \setminus \{0\}$ such that $J(u) > 0$ we have

$$\frac{1}{r}\frac{d}{d\tau}J(\tau u)|_{\tau=1} = \frac{\|u\|^2_{X_K(\Omega)}}{r} - \int_\Omega \frac{f(x,u)u}{r}\, dx = J(u) + \left(\frac{1}{r} - \frac{1}{2}\right) \|u\|^2_{X_K(\Omega)} + o(\|u\|^2_{X_K(\Omega)})$$

as $\|u\|_{X_K(\Omega)} \to 0$. Therefore we can find some $\rho > 0$ such that, for all $u \in B_\rho(0) \setminus \{0\}$ with $J(u) > 0$,

$$\frac{d}{d\tau}J(\tau u)|_{\tau=1} > 0. \tag{5.1.2}$$

Using again (5.1.1), there exists $\tau(u) \in (0,1)$ such that $J(\tau u) < 0$ for all $0 < \tau < \tau(u)$ and $J(\tau(u)u) = 0$. This assures uniqueness of $\tau(u)$ defined as above, for all $u \in B_\rho(0)$ with $J(u) > 0$. We set $\tau(u) = 1$ for all $u \in B_\rho(0)$ with $J(u) \leq 0$, hence we have defined a map $\tau : B_\rho(0) \to (0,1]$ such that for $\tau \in (0,1)$ and for all $u \in B_\rho(0)$ we have

$$\begin{cases} J(\tau u) < 0 & \text{if } \tau < \tau(u) \\ J(\tau u) = 0 & \text{if } \tau = \tau(u) \\ J(\tau u) > 0 & \text{if } \tau > \tau(u). \end{cases}$$

By (5.1.2) and the Implicit Function Theorem, τ turns out to be continuous. We set for all $(t,u) \in [0,1] \times B_\rho(0)$

$$h(t,u) = (1-t)u + t\tau(u)u,$$

hence $h : [0,1] \times B_\rho(0) \to B_\rho(0)$ is a continuous deformation and the set $B_\rho(0) \cap J^0 = \{\tau u : u \in B_\rho(0), \tau \in [0, \tau(u)]\}$ is a deformation retract of $B_\rho(0)$. Similarly we deduce that the set $B_\rho(0) \cap J^0 \setminus \{0\}$ is a deformation retract of $B_\rho(0) \setminus \{0\}$. Consequently, we have for all $k \in \mathbb{N}_0$

$$C_k(J,0) = H_k(J^0 \cap B_\rho(0), J^0 \cap B_\rho(0) \setminus \{0\}) = H_k(B_\rho(0), B_\rho(0) \setminus \{0\}) = 0,$$

the last passage following from contractibility of $B_\rho(0) \setminus \{0\}$, recalling that $\dim(X_K(\Omega)) = \infty$. Since by Proposition 1.1.4 $C_1(J, \tilde{u}) \neq 0$ and $C_k(J,0) = 0$ $\forall k \in \mathbb{N}_0$, then $\tilde{u}$ is a nontrivial solution. □

Remark 5.1.3. We remark that we can use Morse identity (Proposition 1.1.5) to conclude the proof. Indeed, we note that $J(u_\pm) < J(0) = 0$, in particular 0 and $u_\pm$ are isolated critical points, hence we can compute the corresponding critical groups. By Proposition 1.1.1, since $u_\pm$ are strict local minimizers of J, we have $C_k(J, u_\pm) = \delta_{k,0}\mathbb{R}$ for all $k \in \mathbb{N}_0$. We have already determined $C_k(J,0) = 0$ for all $k \in \mathbb{N}_0$, and we already know the k-th critical group at infinity of J.
Since J is coercive and sequentially weakly lower semicontinuous, J is bounded below in $X_K(\Omega)$, then, by [150, Proposition 6.64 (a)], $C_k(J, \infty) = \delta_{k,0}\mathbb{R}$ for all $k \in \mathbb{N}_0$. Applying

Morse identity and choosing, for instance, $t = -1$, we obtain a contradiction, therefore there exists another critical point $\tilde{u} \in K_J \setminus \{0, u_\pm\}$.
But in this way we lose the information that $\tilde{u}$ is of mountain pass type.

5.2 General Hopf's lemma

Inspired by the work of Greco and Servadei [101], in this section we show that weak and strong maximum principle, Hopf's lemma can be generalized to the case in which the sign of f is unknown. Now we focus on the following problem

$$\begin{cases} L_K u = f(x,u) & \text{in } \Omega \\ u = h & \text{in } \Omega^c, \end{cases} \tag{5.2.1}$$

where $h \in C^s(\Omega^c)$, and we have the same assumptions (2.3.2) on the function f, in addition we assume

$$f(x,t) \geq -ct \quad \forall (x,t) \in \overline{\Omega} \times \mathbb{R}_+ \quad (c > 0). \tag{5.2.2}$$

Remark 5.2.1. Since Dirichlet data is not homogeneous in (5.2.1), the energy functional associated to the problem (5.2.1) is

$$J(u) = \frac{1}{2} \iint_{\mathbb{R}^{2N} \setminus \mathcal{O}} |u(x) - u(y)|^2 K(x-y)\, dxdy - \int_\Omega F(x, u(x))\, dx, \tag{5.2.3}$$

for all $u \in \tilde{X} := \{u \in L^2(\mathbb{R}^N) : [u]_K < \infty\}$ with $u = h$ a.e. in Ω^c, where $\mathcal{O} = \Omega^c \times \Omega^c$. When h is not zero, the term $\iint_{\mathcal{O}} |h(x) - h(y)|^2 K(x-y)\, dxdy$ could be infinite, this is the reason why one has to take (5.2.3), see [173].

We begin with a weak maximum principle for (5.2.1).

Proposition 5.2.2 (Weak maximum principle)**.** *Let (5.2.2) hold and let u be a weak solution of (5.2.1) with $h \geq 0$ in Ω^c. Then, $u \geq 0$ in Ω.*

Proof. Let u be a weak solution of (5.2.1), i.e.

$$\iint_{\mathbb{R}^{2N} \setminus \mathcal{O}} (u(x) - u(y))(\varphi(x) - \varphi(y)) K(x-y)\, dxdy = \int_\Omega f(x, u(x))\varphi(x)\, dx \tag{5.2.4}$$

for all $\varphi \in X_K(\Omega)$. We write $u = u^+ - u^-$ in Ω, where u^+ and u^- stand for the positive and the negative part of u, respectively. We take $\varphi = u^-$, we assume that u^- is not identically zero, and we argue by contradiction.
From hypotheses we have

$$\begin{aligned} \int_\Omega f(x, u(x))\varphi(x)\, dx &= \int_\Omega f(x, u(x)) u^-(x)\, dx \\ &\geq - \int_\Omega c u(x) u^-(x)\, dx = \int_{\Omega^-} c u(x)^2\, dx > 0, \end{aligned} \tag{5.2.5}$$

where $\Omega^- := \{x \in \Omega : u(x) < 0\}$.
On the other hand, we obtain that

$$\begin{aligned}
&\iint_{\mathbb{R}^{2N}\setminus\mathcal{O}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy \\
&= \iint_{\Omega\times\Omega} (u(x)-u(y))(u^-(x)-u^-(y))K(x-y)\,dxdy \\
&+ 2\iint_{\Omega\times\Omega^c} (u(x)-h(y))u^-(x)K(x-y)\,dxdy.
\end{aligned}$$

Moreover, $(u^+(x)-u^+(y))(u^-(x)-u^-(y)) \leq 0$, and thus

$$\begin{aligned}
&\iint_{\Omega\times\Omega} (u(x)-u(y))(u^-(x)-u^-(y))K(x-y)\,dxdy \\
&\leq -\iint_{\Omega\times\Omega} (u^-(x)-u^-(y))^2K(x-y)\,dxdy < 0.
\end{aligned}$$

Since $h \geq 0$, then

$$\iint_{\Omega\times\Omega^c} (u(x)-h(y))u^-(x)K(x-y)\,dxdy \leq 0.$$

Therefore, we have obtained that

$$\iint_{\mathbb{R}^{2N}\setminus\mathcal{O}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dxdy < 0,$$

and this contradicts (5.2.4)-(5.2.5). □

The next step consists in proving a strong maximum principle for (5.2.1). To do so we will need a slightly more restrictive notion of solution, namely a pointwise solution, which is equivalent to that of weak solution under further regularity assumptions on the reaction f. Therefore we add extra hypotheses on f to obtain a better interior regularity of the solutions, as we have seen previously, as a consequence we can show a strong maximum principle and Hopf's Lemma in a more general case.

Proposition 5.2.3 (Strong maximum principle). *Let* (2.3.2) *and* (5.2.2) *hold,* $f(.,t) \in C^s(\overline{\Omega})$ *for all* $t \in \mathbb{R}$*,* $f(x,.) \in C^{0,1}_{loc}(\mathbb{R})$ *for all* $x \in \overline{\Omega}$*,* $a \in L^\infty(S^{N-1})$*, let* u *be a weak solution of* (5.2.1) *with* $h \geq 0$ *in* Ω^c*. Then either* $u(x) = 0$ *for all* $x \in \Omega$ *or* $u > 0$ *in* Ω*.*

Proof. The assumptions $f(.,t) \in C^s(\overline{\Omega})$ for all $t \in \mathbb{R}$ and $f(x,.) \in C^{0,1}_{loc}(\mathbb{R})$ for all $x \in \overline{\Omega}$ imply that $f(x,u(x)) \in C^s(\overline{\Omega}\times\mathbb{R})$.
We fix $x \in \Omega$, since Ω is an open set, there exists a ball $B_R(x)$ such that u satisfies $L_K(u) = f(x,u)$ weakly in $B_R(x)$, hence by Theorem 2.3.3 $u \in C^{3s}\big(B_{\frac{R}{2}}\big)$ and by Proposition 2.3.4 $u \in C^s(\mathbb{R}^N)$, then u is a pointwise solution, namely the operator L_K can be evaluated

pointwisely:

$$\begin{aligned}&\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|}{|x-y|^{N+2s}} a\Big(\frac{x-y}{|x-y|}\Big)\, dy\\&\leq C||a||_\infty \int_{B_{\frac{R}{2}}} \frac{|x-y|^{3s}}{|x-y|^{N+2s}}\, dy + C||a||_\infty \int_{B^c_{\frac{R}{2}}} \frac{|x-y|^{s}}{|x-y|^{N+2s}}\, dy\\&= C\left(\int_{B_{\frac{R}{2}}} \frac{1}{|x-y|^{N-s}}\, dy + \int_{B^c_{\frac{R}{2}}} \frac{1}{|x-y|^{N+s}}\, dy\right) < \infty.\end{aligned}$$

Therefore, if u is a weak solution of problem (5.2.1), under these hypotheses, u becomes a pointwise solution of this problem.
By weak maximum principle, $u \geq 0$ in $\mathbb{R}^N$. We assume that u does not vanish identically. Now, we argue by contradiction. We suppose that there exists a point $x_0 \in \Omega$ such that $u(x_0) = 0$, hence x_0 is a minimum of u in $\mathbb{R}^N$, then

$$0 = -cu(x_0) \leq L_K u(x_0) = \int_{\mathbb{R}^N} (u(x_0) - u(y))K(x_0 - y)\, \mathrm{d}y < 0,$$

a contradiction. □

Finally, by using the previous results, we can prove a generalized Hopf's Lemma for (5.2.1) with possibly negative reaction.

Lemma 5.2.4 (Hopf's Lemma). *Let (2.3.2) and (5.2.2) hold, $f(.,t) \in C^s(\overline{\Omega})$ for all $t \in \mathbb{R}$, $f(x,.) \in C^{0,1}_{loc}(\mathbb{R})$ for all $x \in \overline{\Omega}$, $a \in L^\infty(S^{N-1})$. If u is a solution of (5.2.1) and $h \geq 0$ in Ω^c, then either $u(x) = 0$ for all $x \in \Omega$ or*

$$\liminf_{\Omega \ni x \to x_0} \frac{u(x)}{\mathrm{d}^s_\Omega(x)} > 0 \quad \forall x_0 \in \partial\Omega.$$

Proof. The proof is divided in two parts, firstly we show this result in a ball B_R, $R > 0$, and secondly in a general Ω satisfying an interior ball condition. (We assume that B_R is centered at the origin without loss of generality).
We argue as in [101, Lemma 1.2].
Case $\Omega = B_R$. We suppose that u does not vanish identically in B_R. By Proposition 5.2.3 $u > 0$ in B_R, hence for every compact set $K \subset B_R$ we have $\min_K u > 0$. We recall [173, Lemma 5.4] that $u_R(x) = C(R^2 - |x|^2)^s_+$ is a solution of

$$\begin{cases} L_K u_R = 1 & \text{in } B_R\\ u_R = 0 & \text{in } B^c_R,\end{cases}$$

we define $v_m(x) = \frac{1}{m} u_R(x)$ for $x \in \mathbb{R}^N$ and $\forall m \in \mathbb{N}$, consequently $L_K v_m = \frac{1}{m}$.
Claim: There exists some $\bar{m} \in \mathbb{N}$ such that $u \geq v_{\bar{m}}$ in $\mathbb{R}^N$.
We argue by contradiction, we define $w_m = v_m - u$ $\forall m \in \mathbb{N}$, and we suppose that $w_m > 0$ in $\mathbb{R}^N$. Since $v_m = 0 \leq u$ in B^c_R, there exists $x_m \in B_R$ such that $w_m(x_m) = \max_{B_R} w_m > 0$,

hence we may write $0 < u(x_m) < v_m(x_m)$. As a consequence of this and of the fact that

$$v_m \to 0 \text{ uniformly in } \mathbb{R}^N, \tag{5.2.6}$$

we obtain

$$\lim_{m\to+\infty} u(x_m) = 0. \tag{5.2.7}$$

This and the fact of $\min_K u > 0$ imply $|x_m| \to R$ as $m \to +\infty$. Consequently, as long as y ranges in the ball $\overline{B}_{\frac{R}{2}} \subset B_R$, the difference $x_m - y$ keeps far from zero when m is large. Therefore, recalling also Remark 2.2.2, there exist a positive constant $C > 1$, independent of m, such that

$$\frac{1}{C} \le \int_{B_{\frac{R}{2}}} a\Big(\frac{x_m - y}{|x_m - y|}\Big)\frac{1}{|x_m - y|^{N+2s}}\, dy \le C. \tag{5.2.8}$$

By assumption and arguing as in the previous proof, the operator L_K can be evaluated pointwise, hence we obtain

$$\begin{aligned}
-cu(x_m) \le L_K u(x_m) &= \int_{\mathbb{R}^N} \frac{u(x_m) - u(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \\
&= \int_{B_{\frac{R}{2}}} \frac{u(x_m) - u(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy + \int_{B^c_{\frac{R}{2}}} \frac{u(x_m) - u(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \\
&= A_m + B_m. \qquad (5.2.9)
\end{aligned}$$

We focus on the first integral, since there exists a positive constant b such that $\min_{B_{\frac{R}{2}}} u = b$, and by previous estimates and by Fatou's lemma we have

$$\limsup_m A_m = \limsup_m \int_{B_{\frac{R}{2}}} \frac{u(x_m) - u(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \le -\frac{b}{C} < 0,$$

where we used (5.2.7) and (5.2.8).
For the second integral we observe $u(x_m) - u(y) \le v_m(x_m) - v_m(y)$, indeed we recall that $w_m(y) \le w_m(x_m)$ for all $y \in \mathbb{R}^N$ (being x_m the maximum of w_m in $\mathbb{R}^N$), hence, passing to the limit, by (5.2.6) and (5.2.8) we obtain

$$\begin{aligned}
B_m &\le \int_{B^c_{\frac{R}{2}}} \frac{v(x_m) - v(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \\
&= L_K v_m(x_m) - \int_{B_{\frac{R}{2}}} \frac{v(x_m) - v(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \\
&= \frac{1}{m} - \int_{B_{\frac{R}{2}}} \frac{v(x_m) - v(y)}{|x_m - y|^{N+2s}}\, a\Big(\frac{x_m - y}{|x_m - y|}\Big)\, dy \to 0 \quad m \to \infty.
\end{aligned}$$

Therefore, inserting these in (5.2.9), we obtain $0 \le -\frac{b}{C}$, a contradiction.

Then $u \geq v_{\bar{m}}$ for some $\bar{m}$, therefore

$$u(x) \geq \frac{1}{\bar{m}}(R^2 - |x|^2)^s = \frac{1}{\bar{m}}(R + |x|)^s(R - |x|)^s \geq \frac{2^s R^s}{\bar{m}}(\operatorname{dist}(x, B_R^c))^s,$$

then

$$\liminf_{B_R \ni x \to x_0} \frac{u(x)}{\mathrm{d}_\Omega^s(x)} \geq \frac{1}{\bar{m}} 2^s R^s > 0.$$

Case of a general domain Ω. We define $\Omega_\rho = \{x \in \Omega : \mathrm{d}_\Omega(x) < \rho\}$ with $\rho > 0$, for all $x \in \Omega_\rho$ there exists $x_0 \in \partial\Omega$ such that $|x - x_0| = \mathrm{d}_\Omega(x)$. Since Ω satisfies an interior ball condition, there exists $x_1 \in \Omega$ such that $B_\rho(x_1) \subseteq \Omega$, tangent to $\partial\Omega$ at x_0. Then we have that $x \in [x_0, x_1]$ and $\mathrm{d}_\Omega(x) = \mathrm{d}_{B_\rho(x_1)}(x)$.
Since u is a solution of (5.2.1) and by Proposition 5.2.3 we observe that either $u \equiv 0$ in Ω, or $u > 0$ in Ω. If $u > 0$ in Ω, in particular $u > 0$ in $B_\rho(x_1)$ and $u \geq 0$ in $B_\rho^c(x_1)$, then u is a solution of

$$\begin{cases} L_K u = f(x, u) & \text{in } B_\rho(x_1) \\ u = \tilde{h} & \text{in } B_\rho^c(x_1), \end{cases}$$

with

$$\tilde{h}(y) = \begin{cases} u(y), & \text{if } y \in \Omega, \\ h(y), & \text{if } y \in \Omega^c. \end{cases}$$

Therefore, by the first case there exists $C = C(\rho, m, s) > 0$ such that $u(y) \geq C\mathrm{d}^s_{B_\rho(x_1)}(y)$ for all $y \in \mathbb{R}^N$, in particular we obtain $u(x) \geq C\mathrm{d}^s_{B_\rho(x_1)}(x)$.
Then, by $\mathrm{d}_\Omega(x) = \mathrm{d}_{B_\rho(x_1)}(x)$, we have

$$\liminf_{\Omega \ni x \to x_0} \frac{u(x)}{\mathrm{d}_\Omega^s(x)} \geq \liminf_{\Omega_\rho \ni x \to x_0} \frac{C\mathrm{d}_\Omega^s(x)}{\mathrm{d}_\Omega^s(x)} = C > 0 \quad \forall x_0 \in \partial\Omega.$$

□

Remark 5.2.5. We stress that in Lemma 2.3.5 we consider only weak solutions, while in Lemma 5.2.4 pointwise solutions. Moreover, the regularity of u/d_Ω^s yields in particular the existence of the limit

$$\lim_{\Omega \ni x \to x_0} \frac{u(x)}{\mathrm{d}_\Omega^s(x)}$$

for all $x_0 \in \partial\Omega$.

Chapter 6

Existence and multiplicity of positive solutions for the fractional Laplacian under subcritical or critical growth

This chapter is devoted to the following Dirichlet problem for a pseudo-differential equation of fractional order:

$$\begin{cases} (-\Delta)^s\, u = \lambda f(u) & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c. \end{cases} \tag{6.0.1}$$

Here $s \in (0,1)$, $\Omega \subset \mathbb{R}^N$ $(N > 2s)$ is a bounded domain with $C^{1,1}$ boundary, and the leading operator is the fractional Laplacian defined for all $u \in \mathcal{S}(\mathbb{R}^N)$ by

$$(-\Delta)^s\, u(x) = 2\,\mathrm{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy. \tag{6.0.2}$$

We note that some results here are affected by the definition (6.0.2), which is the same adopted in [17]. Other works on the subject, for instance [36, 58, 72], define the fractional Laplacian as in Chapter 2, where the multiplicative constant $C(N,s)$ is required to equivalently define $(-\Delta)^s$ by means of the Fourier transform. In this chapter, explicit constants are one of the main issues, so we decide to follow the standard of [17] in order to easily compare similar results. Moreover, the fractional Talenti constant (see Lemma 2.1.5) is known only for the fractional Laplacian, so these reasons justify our choice of such operator.

The autonomous reaction $f \in C(\mathbb{R})$ is assumed to be non-negative and dominated at infinity by a power of u, namely, for all $t \in \mathbb{R}$

$$0 \leqslant f(t) \leqslant a_0(1 + |t|^{q-1}) \; (a_0 > 0,\, q \leqslant 2^*_s). \tag{6.0.3}$$

Finally, $\lambda > 0$ is a parameter.

Problem (6.0.1) admits a variational formulation by means of the energy functional

$$J_\lambda(u) = \frac{\|u\|^2_{H^s_0(\Omega)}}{2} - \lambda \int_\Omega F(u)\, dx,$$

where F is the primitive of f, i.e., weak solutions of (6.0.1) coincide with critical points of J_λ in $H^s_0(\Omega)$.
We note that, for $\lambda = 1$, problem (6.0.1) embraces the following Dirichlet problem with pure power nonlinearities:

$$
\begin{cases} (-\Delta)^s\, u = \mu u^{p-1} + u^{q-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{6.0.4}
$$

with $1 < p < q \leqslant 2^*_s$ and $\mu > 0$.
Here, in the study of (6.0.1) we distinguish between the *subcritical* ($q < 2^*_s$ in (6.0.3)) and *critical* ($q = 2^*_s$) cases. For the subcritical case, we refer the reader to Chapters 2 - 5. In the critical case, the main difficulty lies in the fact that J_λ does not satisfy the (usual in variational methods) Palais-Smale compactness condition. In particular, problem (6.0.4) with $p = 2$, $q = 2^*_s$ represents a fractional counterpart of the famous Brezis-Nirenberg problem [32]. Again, the first result in this direction is due to Servadei and Valdinoci [187] (see also [11, 16, 146]). Later, Barrios *et al.* [17] studied (6.0.4) with $1 < p < q = 2^*_s$, which for $s = 1$ reduces to the problem with concave-convex nonlinearities studied by Ambrosetti, Brezis and Cerami in [3]. In particular, they proved that in the concave case $1 < p < 2$, for $\mu > 0$ small enough, such problem has at least two positive solutions $u_\mu < w_\mu$, employing both topological (sub-supersolutions) and variational methods.
Our approach to problem (6.0.1) is purely variational, mainly based on a critical point theorem of Bonanno [23] and some of its consequences, presented in [24–26] (see Subsection 1.1.1). The main feature of such method is a strategy to find a local minimizer of a J_λ-type functional, which only requires a *local* Palais-Smale condition. Our results are the following:

(a) In the subcritical case ($q < 2^*_s$) we apply an abstract result of [25] and explicitly compute a real number $\lambda^* > 0$ such that problem (6.0.1) admits at least two positive solutions u_λ, v_λ for all $\lambda \in (0, \lambda^*)$.

(b) In the critical case ($q = 2^*_s$) we first study a generalization of problem (6.0.4), explicitly determining a real number $\mu^* > 0$ such that there exist at least one positive solution u_μ for all $\mu \in (0, \mu^*)$. Then, we focus on (6.0.4) with $1 < p < 2 < q = 2^*_s$ and, applying the mountain pass theorem, we produce a second positive solution $w_\mu > u_\mu$ for all $\mu \in (0, \mu^*)$ (here we mainly follow [26]).

A noteworthy difference with respect to the classical elliptic case is the following: in this approach, it is essential to explicitly compute $J_\lambda(\bar{u})$ at some Sobolev-type function $\bar{u} : \Omega \to \mathbb{R}$, which is usually chosen in such a way to have a piecewise constant $|\nabla \bar{u}|$. In the fractional framework, functions may have no gradient at all, and the computation of the Gagliardo seminorm is often prohibitive, so $\bar{u}$ will be chosen as (a multiple of) the solution of a fractional torsion equation in a ball (see (6.1.2)).

We also remark that our main result in part (b) is formally equivalent to the main result of [17], but with two substantial differences: the first solution u_μ is found as a local minimizer of J_λ (instead of being detected via sub-supersolutions, and *a posteriori* proved to be a minimizer), and moreover the interval $(0,\mu^*)$ is *explicitly* determined (though possibly not optimal).

This chapter has the following structure: in Section 6.1 we collect the necessary preliminaries; in Section 6.2 we develop part (a) of our study; in Sections 6.3 and 6.4 we focus on part (b).

6.1 Preliminaries

In order to deal with problem (6.0.1) variationally, we assume the following hypotheses on the reaction f:

($\mathbf{H}_{6.1}$) $f \in C(\mathbb{R})$, $F(t) = \int_0^t f(\tau)\,d\tau$, and

- (i) $f(t) \geqslant 0$ for all $t \in \mathbb{R}$;
- (ii) $f(t) \leqslant a_0(1+|t|^{2_s^*-1})$ for all $t \in \mathbb{R}$ ($a_0 > 0$).

We set for all $u \in H_0^s(\Omega)$, $\lambda > 0$

$$\Phi(u) = \frac{\|u\|^2_{H_0^s(\Omega)}}{2}, \quad \Psi(u) = \int_\Omega F(u)\,dx, \quad J_\lambda(u) = \Phi(u) - \lambda\Psi(u)$$

(Ψ is well defined by virtue of hypothesis ($\mathbf{H}_{6.1}$) (i) (ii)). Then $\Phi, \Psi, J_\lambda \in C^1(H_0^s(\Omega))$ with

$$\langle J_\lambda'(u), \varphi\rangle = \langle u, \varphi\rangle - \lambda\int_\Omega f(u)\varphi\,dx$$

for all $u, \varphi \in H_0^s(\Omega)$. We say that u is a (weak) solution of problem (6.0.1) if $J_\lambda'(u) = 0$ in $H^{-s}(\Omega)$, that is, for all $\varphi \in H_0^s(\Omega)$ we have

$$\langle u, \varphi\rangle = \lambda\int_\Omega f(u)\varphi\,dx. \tag{6.1.1}$$

For the reader's convenience we summarize from Chapter 2 the main properties of weak solutions.

Proposition 6.1.1. *Let* ($\mathbf{H}_{6.1}$) *hold,* $u \in H_0^s(\Omega)$ *be a weak solution of* (6.0.1). *Then:*

(i) *(a priori bound)* $u \in L^\infty(\Omega)$*;*

(ii) *(regularity)* $u \in C_s^\alpha(\overline{\Omega})$ *with* $\alpha \in (0, s]$ *depending only on* s *and* Ω*;*

(iii) *(Hopf's lemma) if* $u \neq 0$*, then* $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

By Proposition 6.1.1 *(iii)* we see that, whenever $u \in H^s_0(\Omega) \setminus \{0\}$ satisfies (6.1.1), then in particular $u > 0$ in Ω. Moreover, assuming further that f is locally Lipschitz in $\mathbb{R}$, from [174, Corollary 1.6] we deduce that $u \in C^\beta(\Omega)$ for any $\beta \in [1, 1+2s)$, which along with Proposition 6.1.1 *(ii)* implies that for all $x \in \mathbb{R}^N$ the mapping

$$x \mapsto \frac{u(x) - u(y)}{|x-y|^{N+2s}}$$

lies in $L^1(\mathbb{R}^N)$. Then, testing (6.1.1) with any $\varphi \in C^\infty_0(\Omega)$ and applying (6.0.2), we have

$$\int_\Omega (-\Delta)^s\, u\varphi\, dx = \langle u, \varphi \rangle = \int_\Omega f(u)\varphi\, dx,$$

i.e., u solves (6.0.1) pointwisely.

As pointed out previously, we will make use of the following fractional torsion equation on a ball:

$$\begin{cases} (-\Delta)^s\, u_R = 1 & \text{in } B_R(x_0) \\ u_R = 0 & \text{in } B_R(x_0)^c, \end{cases} \tag{6.1.2}$$

where $x_0 \in \mathbb{R}^N$, $R > 0$. The solution of (6.1.2) (defined as in (6.1.1)) is unique, given by

$$u_R(x) = A(N,s)(R^2 - |x-x_0|^2)^s_+, \;\; A(N,s) = \frac{s\Gamma(\frac{N}{2})}{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)}$$

(see [36, p. 33] or [174, equation (1.4)]). We already recalled this problem in Chapter 2. For future use we compute some norms of u_R.

Lemma 6.1.2. *For all $x_0 \in \mathbb{R}^N$, $R > 0$ we have*

(i) $\|u_R\|_\nu = A(N,s)\Big[\dfrac{\pi^{\frac{N}{2}}\Gamma(1+\nu s)R^{N+2\nu s}}{\Gamma(\frac{N+2\nu s+2}{2})}\Big]^{\frac{1}{\nu}}$ *for all $\nu \geqslant 1$;*

(ii) $[u_R]_s = \Big[\dfrac{s\Gamma(\frac{N}{2})R^{N+2s}}{2\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})}\Big]^{\frac{1}{2}}$.

Proof. First we recall the well-known formulas

$$|\partial B_1(0)| = \frac{2\pi^{\frac{N}{2}}}{\Gamma(\frac{N}{2})}, \quad \int_0^1 (1-\rho^2)^\alpha \rho^{N-1}\, d\rho = \frac{\Gamma(\frac{N}{2})\Gamma(1+\alpha)}{2\Gamma(\frac{N+2\alpha+2}{2})} \quad (\alpha > 0),$$

then for all $\nu \geqslant 1$ we compute

$$\begin{aligned} \int_{B_R(x_0)} u^\nu_R(x)\, dx &= A(N,s)^\nu \int_{B_R(x_0)} (R^2 - |x-x_0|^2)^{\nu s}\, dx \\ &= A(N,s)^\nu R^{N+2\nu s}|\partial B_1(0)| \int_0^1 (1-\rho^2)^{\nu s}\rho^{N-1}\, d\rho \\ &= A(N,s)^\nu \frac{\pi^{\frac{N}{2}}\Gamma(1+\nu s)R^{N+2\nu s}}{\Gamma(\frac{N+2\nu s+2}{2})}, \end{aligned}$$

which implies (i). Further, testing (6.1.2) with $u_R \in H_0^s(B_R(x_0))$ and applying (i) with $\nu = 1$, we have

$$\begin{aligned}[u_R]_s^2 &= \int_{B_R(x_0)} u_R \, dx \\ &= A(N,s) \frac{\pi^{\frac{N}{2}} \Gamma(1+s) R^{N+2s}}{\Gamma(\frac{N+2s+2}{2})} \\ &= \frac{s\Gamma(\frac{N}{2}) R^{N+2s}}{2\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})},\end{aligned}$$

which gives (ii). $\square$

6.2 Two positive solutions under subcritical growth

In this section, following [25] as a model, we study (6.0.1) under the following hypotheses:

$(\mathbf{H}_{6.2})$ $f \in C(\mathbb{R})$, $F(t) = \displaystyle\int_0^t f(\tau)\, d\tau$ satisfy

- (i) $f(t) \geqslant 0$ for all $t \in \mathbb{R}$;
- (ii) $f(t) \leqslant a_p|t|^{p-1} + a_q|t|^{q-1}$ for all $t \in \mathbb{R}$ $(1 \leqslant p < 2 < q < 2_s^*,\ a_p, a_q > 0)$;
- (iii) $\displaystyle\lim_{t \to 0^+} \frac{F(t)}{t^2} = \infty$
- (iv) $0 < \rho F(t) \leqslant f(t)t$ for all $t \geqslant M$ $(\rho > 2,\ M > 0)$.

Hypotheses $(\mathbf{H}_{6.2})$ conjure for f a subcritical, superlinear growth at infinity, as well as a sublinear growth near the origin, while $(\mathbf{H}_{6.2})$ (iv) is an Ambrosetti-Rabinowitz condition. Let $T(N,s) > 0$ be defined by Lemma 2.1.5, set

$$\lambda^* = \frac{1}{2T(N,s)^2 |\Omega|^{\frac{2_s^*-2}{2_s^*}}} \Big(\frac{a_p}{p}\Big)^{\frac{2-q}{q-p}} \Big(\frac{a_q}{q}\Big)^{\frac{p-2}{q-p}} \Big(\frac{2-p}{q-2}\Big)^{\frac{2-p}{q-p}} \frac{q-2}{q-p} > 0. \tag{6.2.1}$$

We have the following multiplicity result.

Theorem 6.2.1. *Let $(\mathbf{H}_{6.2})$ hold, $\lambda^* > 0$ be defined by (6.2.1). Then, for all $\lambda \in (0, \lambda^*)$, (6.0.1) has at least two solutions $u_\lambda, v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. Without loss of generality we may assume $f(t) = 0$ for all $t \leqslant 0$. We are going to apply Theorem 1.1.6. Set $X = H_0^s(\Omega)$ and define Φ, Ψ, J_λ as in Section 6.1, then clearly $\Phi, \Psi \in C^1(H_0^s(\Omega))$ and

$$\inf_{u \in H_0^s(\Omega)} \Phi(u) = \Phi(0) = \Psi(0) = 0,$$

hence hypothesis (i) holds. Set

$$r = \frac{|\Omega|^{\frac{2}{2_s^*}}}{2T(N,s)^2} \Big[\frac{a_p q(2-p)}{a_q p(q-2)}\Big]^{\frac{2}{q-p}} > 0. \tag{6.2.2}$$

For all $u \in H^s_0(\Omega)$, $\Phi(u) \leqslant r$, we have $\|u\|_{H^s_0(\Omega)} \leqslant (2r)^{\frac{1}{2}}$. So, by hypotheses $(\mathbf{H}_{6.2})$ *(i)* *(ii)*, along with (2.1.2), (6.2.1) and (6.2.2), we obtain

$$\begin{aligned}
\frac{\Psi(u)}{r} &\leqslant \frac{a_p}{pr}\|u\|_p^p + \frac{a_q}{qr}\|u\|_q^q \\
&\leqslant \frac{a_p}{pr}T(N,s)^p|\Omega|^{\frac{2^*_s-p}{2^*_s}}(2r)^{\frac{p}{2}} + \frac{a_q}{qr}T(N,s)^q|\Omega|^{\frac{2^*_s-q}{2^*_s}}(2r)^{\frac{q}{2}} \\
&= 2T(N,s)^2|\Omega|^{\frac{2^*_s-2}{2^*_s}}\Big(\frac{a_p}{p}\Big)^{\frac{q-2}{q-p}}\Big(\frac{a_q}{q}\Big)^{\frac{2-p}{q-p}}\Big(\frac{2-p}{q-2}\Big)^{\frac{p-2}{q-p}} \\
&+ 2T(N,s)^2|\Omega|^{\frac{2^*_s-2}{2^*_s}}\Big(\frac{a_p}{p}\Big)^{\frac{q-2}{q-p}}\Big(\frac{a_q}{q}\Big)^{\frac{2-p}{q-p}}\Big(\frac{2-p}{q-2}\Big)^{\frac{q-2}{q-p}} \\
&= 2T(N,s)^2|\Omega|^{\frac{2^*_s-2}{2^*_s}}\Big(\frac{a_p}{p}\Big)^{\frac{q-2}{q-p}}\Big(\frac{a_q}{q}\Big)^{\frac{2-p}{q-p}}\Big(\frac{2-p}{q-2}\Big)^{\frac{p-2}{q-p}}\frac{q-p}{q-2} = \frac{1}{\lambda^*}.
\end{aligned}$$

Summarizing,

$$\sup_{\Phi(u)\leqslant r}\frac{\Psi(u)}{r} \leqslant \frac{1}{\lambda^*}. \tag{6.2.3}$$

Now fix $\lambda \in (0, \lambda^*)$. Since $\partial\Omega$ is $C^{1,1}$, we can find $x_0 \in \mathbb{R}^N$, $R > 0$ largest such that $B_R(x_0) \subseteq \Omega$. Let $K > 0$ be such that

$$K\frac{s\Gamma(\frac{N}{2})\Gamma(1+2s)\Gamma(\frac{N+2s+2}{2})R^{2s}}{\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)\Gamma(\frac{N+4s+2}{2})} > \frac{1}{\lambda}. \tag{6.2.4}$$

By $(\mathbf{H}_{6.2})$ *(iii)*, we can find $\varepsilon > 0$ such that for all $t \in [0, \varepsilon]$

$$F(t) \geqslant Kt^2. \tag{6.2.5}$$

Finally, fix

$$0 < \delta < \min\Big\{\Big[\frac{4\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})r}{s\Gamma(\frac{N}{2})R^{N+2s}}\Big]^{\frac{1}{2}}, \frac{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)\varepsilon}{s\Gamma(\frac{N}{2})R^{2s}}\Big\}. \tag{6.2.6}$$

Now let u_R be the solution of (6.1.2) in $B_R(x_0)$, and set $\bar{u} = \delta u_R \in H^s_0(\Omega)$. Then we have by Lemma 6.1.2 *(ii)* and (6.2.6)

$$\Phi(\bar{u}) = \frac{s\Gamma(\frac{N}{2})R^{N+2s}\delta^2}{4\Gamma(1-s)\Gamma(\frac{N+2s+2}{2})} < r,$$

which implies *(ii)*. Besides, by (6.2.6) we have for all $x \in \Omega$

$$0 \leqslant \bar{u}(x) \leqslant \frac{s\Gamma(\frac{N}{2})R^{2s}\delta}{2\pi^{\frac{N}{2}}\Gamma(1+s)\Gamma(1-s)} < \varepsilon,$$

hence by (6.2.5) and Lemma 6.1.2 *(i)*

$$\Psi(\bar{u}) \geqslant \int_\Omega K\bar{u}^2\,dx = K\delta^2\|u_R\|_2^2 = K\frac{s^2\Gamma(\frac{N}{2})^2\Gamma(1+2s)R^{N+4s}}{4\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)^2\Gamma(\frac{N+4s+2}{2})}\delta^2.$$

The relations above and (6.2.4) imply

$$\frac{\Psi(\bar u)}{\Phi(\bar u)} \geqslant K\frac{s\Gamma(\frac{N}{2})\Gamma(1+2s)\Gamma(\frac{N+2s+2}{2})R^{2s}}{\pi^{\frac{N}{2}}\Gamma(1+s)^2\Gamma(1-s)\Gamma(\frac{N+4s+2}{2})} > \frac{1}{\lambda}.$$

Recalling that $\lambda < \lambda^*$, by (6.2.3) we have

$$\sup_{\Phi(u)\leqslant r}\frac{\Psi(u)}{r} < \frac{1}{\lambda} < \frac{\Psi(\bar u)}{\Phi(\bar u)},$$

which yields at once *(iii)* and $\lambda \in I_r$. By $(\mathbf{H}_{6.2})$ *(iv)* we can find $C > 0$ such that for all $t \geqslant M$

$$F(t) \geqslant Ct^\rho. \tag{6.2.7}$$

Now pick $w \in C_0^\infty(\Omega)\setminus\{0\}$. By (6.2.7), and recalling that $F(t) \geqslant 0$ for all $t \in \mathbb{R}$, we have for all $\tau > 0$

$$\begin{aligned}
J_\lambda(\tau w) &\leqslant \frac{\|w\|^2_{H^s_0(\Omega)}}{2}\tau^2 - \lambda\int_{\{w\leqslant M/\tau\}} F(\tau w)\,dx - \lambda\int_{\{w>M/\tau\}} C(\tau w)^\rho\,dx \\
&\leqslant \frac{\|w\|^2_{H^s_0(\Omega)}}{2}\tau^2 - \lambda\int_\Omega C(\tau w)^\rho\,dx + \lambda\int_{\{w\leqslant M/\tau\}} CM^\rho\,dx \\
&\leqslant \frac{\|w\|^2_{H^s_0(\Omega)}}{2}\tau^2 - \lambda C\|w\|^\rho_\infty|\Omega|\tau^\rho + \lambda CM^\rho|\Omega|
\end{aligned}$$

and the latter tends to $-\infty$ as $\tau \to \infty$ (since $\rho > 2$). So we see that *(iv)* holds as well.

Finally, we prove that J_λ satisfies (PS). Let (u_n) be a sequence in $H^s_0(\Omega)$ such that $|J_\lambda(u_n)| \leqslant C$, $J'_\lambda(u_n) \to 0$ in $H^{-s}(\Omega)$. Then, for all $n \in \mathbb{N}$ we have

$$\frac{\|u_n\|^2_{H^s_0(\Omega)}}{2} - \lambda\int_\Omega F(u_n)\,dx \leqslant C \tag{6.2.8}$$

and for all $\varphi \in H^s_0(\Omega)$

$$\Big|\langle u_n,\varphi\rangle - \lambda\int_\Omega f(u_n)\varphi\,dx\Big| \leqslant \|J'_\lambda(u_n)\|_*\,\|\varphi\|_{H^s_0(\Omega)} \tag{6.2.9}$$

Multiplying (6.2.8) by $\rho > 2$ (as in $(\mathbf{H}_{6.2})$ *(iv)*), testing (6.2.9) with u_n, and subtracting,

$$\begin{aligned}
\frac{\rho-2}{2}\|u_n\|^2_{H^s_0(\Omega)} &\leqslant \lambda\int_\Omega\big(\rho F(u_n) - f(u_n)u_n\big)\,dx + \|J'_\lambda(u_n)\|_*\,\|u_n\|_{H^s_0(\Omega)} + C \\
&\leqslant \lambda\int_{\{0\leqslant u_n\leqslant M\}} C\big(|u_n|^p + |u_n|^q\big)\,dx + \|J'_\lambda(u_n)\|_*\,\|u_n\|_{H^s_0(\Omega)} + C \\
&\leqslant \lambda C(M^p + M^q)|\Omega| + \|J'_\lambda(u_n)\|_*\,\|u_n\|_{H^s_0(\Omega)} + C.
\end{aligned}$$

So (u_n) is bounded in $H^s_0(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, $u_n \to u$ in $L^p(\Omega), L^q(\Omega)$, and $u_n(x) \to u(x)$ for a.e. $x \in \Omega$. Testing (6.2.9) this time with $u_n - u \in H^s_0(\Omega)$, we have for all $n \in \mathbb{N}$

$$\begin{aligned}
&\|u_n - u\|^2_{H^s_0(\Omega)} \leqslant \langle u, u_n - u\rangle + \lambda\int_\Omega\big(a_p|u_n|^{p-1} + a_q|u_n|^{q-1}\big)|u_n - u|\,dx + \|J'_\lambda(u_n)\|_*\,\|u_n - u\|_{H^s_0(\Omega)} \\
&\leqslant \langle u, u_n - u\rangle + \lambda\big(a_p\|u_n\|_p^{p-1}\|u_n - u\|_p + a_q\|u_n\|_q^{q-1}\|u_n - u\|_q\big) + \|J'_\lambda(u_n)\|_*\,\|u_n - u\|_{H^s_0(\Omega)},
\end{aligned}$$

(where we used $(\mathbf{H}_{6.2})$ *(ii)* and Hölder's inequality), and the latter tends to 0 as $n \to \infty$. So, $u_n \to u$ in $H_0^s(\Omega)$. (Note that we actually proved that J_λ is unbounded from below and satisfies (PS) for *all* $\lambda > 0$.)

By Theorem 1.1.6, there exist $u_\lambda, v_\lambda \in H_0^s(\Omega)$ such that

$$J'_\lambda(u_\lambda) = J'_\lambda(v_\lambda) = 0, \quad J_\lambda(u_\lambda) < 0 < J_\lambda(v_\lambda).$$

Therefore, $u_\lambda, v_\lambda \not\equiv 0$ solve (6.0.1). By $(\mathbf{H}_{6.1})$ *(i)* and Proposition 6.1.1, finally, we have $u_\lambda, v_\lambda \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. $\square$

We focus now on problem (6.0.4), with $1 < p < 2 < q < 2_s^*$ (subcritical case) and $\mu > 0$. Set

$$\mu^* = \Big[2T(N,s)^2|\Omega|^{\frac{2_s^*-2}{2_s^*}}\Big]^{\frac{p-q}{q-2}} p\, q^{\frac{2-p}{q-2}} \Big(\frac{2-p}{q-2}\Big)^{\frac{2-p}{q-2}} \Big(\frac{q-2}{q-p}\Big)^{\frac{q-p}{q-2}} > 0. \tag{6.2.10}$$

We have the following multiplicity result.

Corollary 6.2.2. *Let $1 < p < 2 < q < 2_s^*$, $\mu^* > 0$ be defined by (6.2.10). Then, for all $\mu \in (0, \mu^*)$ (6.0.4) has at least two solutions $u_\mu, v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.*

Proof. Set for all $t \in \mathbb{R}$, $\mu \in (0, \mu^*)$

$$f(t) = \mu(t^+)^{p-1} + (t^+)^{q-1}.$$

Then f satisfies $(\mathbf{H}_{6.2})$ with $a_p = \mu$, $a_q = 1$, and any $\rho \in (2, q)$. In view of (6.2.10), here (6.2.1) rephrases as

$$\lambda^* = \frac{1}{2T(N,s)^2|\Omega|^{\frac{2_s^*-2}{2_s^*}}} p^{\frac{q-2}{q-p}} q^{\frac{2-p}{q-p}} \Big(\frac{2-p}{q-2}\Big)^{\frac{2-p}{q-p}} \frac{q-2}{q-p} \mu^{\frac{2-q}{q-p}} > 1.$$

Hence we can apply Theorem 6.2.1 with $\lambda = 1$ and find $u_\mu, v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ solutions to (6.0.4). $\square$

We present an example for Corollary 6.2.2.

Example 6.2.3. Set $s = \frac{1}{2}$, $p = \frac{3}{2}$, $q = 3$, $N = 2$ and

$$\Omega = \Big\{(x,y) \in \mathbb{R}^2 : \frac{x^2}{4} + \frac{y^2}{9} \leqslant 1\Big\}.$$

Then we have $2_{1/2}^* = 4 > 3$, $|\Omega| = 6\pi$, while Lemma 2.1.5 gives

$$T\Big(2, \frac{1}{2}\Big) = \frac{(\frac{1}{2})^{\frac{1}{2}}\Gamma(\frac{1}{2})^{\frac{1}{2}}\Gamma(2)^{\frac{1}{4}}}{2^{\frac{1}{2}}\pi^{\frac{3}{4}}\Gamma(\frac{1}{2})^{\frac{1}{2}}\Gamma(1)^{\frac{1}{4}}} = \frac{1}{2\pi^{\frac{3}{4}}}.$$

Therefore, (6.2.10) becomes

$$\mu^* = \Big[2\Big(\frac{1}{2\pi^{\frac{3}{4}}}\Big)^2(6\pi)^{\frac{1}{2}}\Big]^{-\frac{3}{2}} \frac{3}{2} 3^{\frac{1}{2}} \Big(\frac{1}{2}\Big)^{\frac{1}{2}} \Big(\frac{2}{3}\Big)^{\frac{3}{2}} = \frac{2^{\frac{3}{4}}\pi^{\frac{3}{2}}}{3^{\frac{3}{4}}}.$$

By Corollary 6.2.2, for all $\mu \in (0, \mu^*)$ (6.0.4) has at least two positive solutions.

6.3 One positive solution under critical growth

In this section, we study the following slight generalization of problem (6.0.4):

(6.3.1)
$$\begin{cases} (-\Delta)^s\, u = \mu g(u) + u^{2^*_s-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

with $\mu > 0$ and assuming the following hypotheses on g:

$(\mathbf{H}_{6.3})$ $g \in C(\mathbb{R})$, $G(t) = \displaystyle\int_0^t g(\tau)\, d\tau$ satisfy

- (i) $g(t) \geqslant 0$ for all $t \in \mathbb{R}$;
- (ii) $g(t) \leqslant a_p|t|^{p-1}$ for all $t \in \mathbb{R}$ ($p \in (1, 2^*_s)$, $a_p > 0$);
- (iii) $\displaystyle\lim_{t\to 0^+} \frac{G(t)}{t^2} = \infty$.

Note that, due to hypothesis $(\mathbf{H}_{6.2})$ (iii), problem (6.3.1) reduces to (6.0.4) with $g(t) = t^{p-1}$ only for $p \in (1, 2)$ (concave case). Although, the results of this section also embrace the case $p \in [2, 2^*_s)$ (linear/convex case).

Due to the presence of the critical term $u^{2^*_s-1}$ in (6.3.1), we cannot apply Theorem 1.1.6, as the associated energy functional does not satisfy (PS) in general. Hence, our main tool will be Theorem 1.1.7, which requires the truncated Palais-Smale condition for J_λ, as stated in Chapter 1.

Set for all $\mu > 0$, $t \in \mathbb{R}$

$$f(t) = \mu g(t) + (t^+)^{2^*-1},\ F(t) = \int_0^t f(\tau)\, d\tau,$$

then define $\Phi, \Psi \in C^1(H^s_0(\Omega))$ as in Section 6.1. Further, for all $\lambda > 0$ set $J_\lambda = \Phi - \lambda\Psi$. Set for all $r, \mu > 0$

(6.3.2)
$$\lambda^*_r = \min\left\{\left[\frac{2^{\frac{2^*_s}{2}} T(N,s)^{2^*_s} r^{\frac{2^*_s-2}{2}}}{2^*_s} + \mu\frac{2^{\frac{p}{2}} a_p T(N,s)^p |\Omega|^{\frac{2^*_s-p}{2^*_s}} r^{\frac{p-2}{2}}}{p}\right]^{-1}, \frac{1}{T(N,s)^{2^*_s}}\left[\frac{s}{2Nr}\right]^{\frac{2s}{N-2s}}\right\}$$

We prove now that J_λ satisfies $(PS)^r$ for all $r > 0$ and all $\lambda > 0$ small enough.

Lemma 6.3.1. *Let $r, \mu > 0$, $\lambda^*_r > 0$ be defined by (6.3.2). Then J_λ satisfies $(PS)^r$ for all $\lambda \in (0, \lambda^*_r)$.*

Proof. Let (u_n) be a sequence in $H^s_0(\Omega)$ such that $(J_\lambda(u_n))$ is bounded, $J'_\lambda(u_n) \to 0$ in $H^{-s}(\Omega)$, and $\Phi(u_n) \leqslant r$ for all $n \in \mathbb{N}$. Then (u_n) is bounded in $H^s_0(\Omega)$, hence in $L^{2^*_s}(\Omega)$ (Lemma 2.1.5). Passing to a subsequence we have $u_n \rightharpoonup u$ in $H^s_0(\Omega)$, $L^{2^*_s}(\Omega)$, $u_n \to u$ in $L^p(\Omega)$, $u_n(x) \to u(x)$ for a.e. $x \in \Omega$, and $J_\lambda(u_n) \to c$.

First we see that

(6.3.3)
$$J'_\lambda(u) = 0.$$

Indeed, since $(u_n^{2_s^*-1})$ is bounded in $L^{(2_s^*)'}(\Omega)$, up to a further subsequence we have $u_n^{2_s^*-1} \rightharpoonup u^{2_s^*-1}$ in $L^{(2_s^*)'}(\Omega)$, while by $(\mathbf{H}_{6.3})$ *(i)* *(ii)* we have $g(u_n) \to g(u)$ in $L^{p'}(\Omega)$. So, for all $\varphi \in H_0^s(\Omega)$ we have

$$
\begin{aligned}
\langle J_\lambda'(u_n), \varphi\rangle &= \langle u_n, \varphi\rangle - \lambda \int_\Omega u_n^{2_s^*-1}\varphi\, dx - \lambda\mu \int_\Omega g(u_n)\varphi\, dx \\
&\to \langle u, \varphi\rangle - \lambda \int_\Omega u^{2_s^*-1}\varphi\, dx - \lambda\mu \int_\Omega g(u)\varphi\, dx = \langle J_\lambda'(u), \varphi\rangle,
\end{aligned}
$$

which along with $J_\lambda'(u_n) \to 0$ gives (6.3.3). Besides,

$$
J_\lambda(u) > -r. \tag{6.3.4}
$$

Indeed, since $u_n \rightharpoonup u$ in $H_0^s(\Omega)$ and Φ is convex, we have $\Phi(u) \leqslant r$, i.e., $\|u\|_{H_0^s(\Omega)} \leqslant (2r)^{\frac{1}{2}}$. So using Lemma 2.1.5, (2.1.2) with $\nu = p$, (6.3.2), and $\lambda < \lambda_r^*$, we have

$$
\begin{aligned}
J_\lambda(u) &\geqslant -\lambda\Psi(u) \\
&\geqslant -\frac{\lambda}{2_s^*}\|u\|_{2_s^*}^{2_s^*} - \frac{\lambda\mu a_p}{p}\|u\|_p^p \\
&\geqslant -\frac{\lambda}{2_s^*}T(N,s)^{2_s^*}(2r)^{\frac{2_s^*}{2}} - \frac{\lambda\mu a_p}{p}T(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}(2r)^{\frac{p}{2}} \\
&\geqslant -\lambda r\Big[\frac{2^{\frac{2_s^*}{2}}T(N,s)^{2_s^*}r^{\frac{2_s^*-2}{2}}}{2_s^*} + \mu\frac{2^{\frac{p}{2}}a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}r^{\frac{p-2}{2}}}{p}\Big] \\
&\geqslant -\frac{\lambda r}{\lambda_r^*},
\end{aligned}
$$

and the latter gives (6.3.4) since $\lambda > \lambda_r^*$. Now set $v_n = u_n - u$. We have

$$
\lim_n \Big[\Phi(v_n) - \frac{\lambda}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*}\Big] = c - J_\lambda(u). \tag{6.3.5}
$$

Indeed, since $v_n \rightharpoonup 0$ in $H_0^s(\Omega)$, we have

$$
\|v_n\|_{H_0^s(\Omega)}^2 = \|u_n\|_{H_0^s(\Omega)}^2 - 2\langle u_n, u\rangle + \|u\|_{H_0^s(\Omega)}^2 = \|u_n\|_{H_0^s(\Omega)}^2 - \|u\|_{H_0^s(\Omega)}^2 + o(1)
$$

(as $n \to \infty$). Since $v_n \rightharpoonup 0$ in $L^{2_s^*}(\Omega)$, by the Brezis-Lieb Lemma [31, Theorem 1] we have

$$
\|v_n\|_{2_s^*}^{2_s^*} = \|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*} + o(1).
$$

Since $u_n \to u$ in $L^p(\Omega)$, we have $G(u_n) \to G(u)$ in $L^1(\Omega)$. So,

$$
\begin{aligned}
&\Phi(v_n) - \frac{\lambda}{2_s^*}\|v_n\|_{2_s^*}^{2_s^*} \\
&= [\Phi(u_n) - \Phi(u)] - \frac{\lambda}{2_s^*}\big[\|u_n\|_{2_s^*}^{2_s^*} - \|u\|_{2_s^*}^{2_s^*}\big] - \lambda\mu\int_\Omega [G(u_n) - G(u)]\, dx + o(1) \\
&= J_\lambda(u_n) - J_\lambda(u) + o(1) \to c - J_\lambda(u).
\end{aligned}
$$

On the other hand,

$$
\lim_n \big[\|v_n\|_{H_0^s(\Omega)}^2 - \lambda\|v_n\|_{2_s^*}^{2_s^*}\big] = 0. \tag{6.3.6}
$$

Indeed, arguing as above and recalling that $g(u_n)u_n \to g(u)u$ in $L^1(\Omega)$, we have

$$\begin{aligned}
&\|v_n\|^2_{H^s_0(\Omega)} - \lambda\|v_n\|^{2^*_s}_{2^*_s} \\
&= [\|u_n\|^2_{H^s_0(\Omega)} - \|u\|^2_{H^s_0(\Omega)}] - \lambda\big[\|u_n\|^{2^*_s}_{2^*_s} - \|u\|^{2^*_s}_{2^*_s}\big] - \lambda\mu\int_\Omega [g(u_n)u_n - g(u)u]\,dx + o(1) \\
&= \langle J'_\lambda(u_n), u_n\rangle - \langle J'_\lambda(u), u\rangle + o(1),
\end{aligned}$$

and the latter tends to 0 as $n \to \infty$, by $J'_\lambda(u_n) \to 0$, boundedness of (u_n), and (6.3.3). Recalling that (v_n) is bounded in $H^s_0(\Omega)$, up to a subsequence we have $\|v_n\|_{H^s_0(\Omega)} \to \beta \geqslant 0$. We prove that

$$\beta = 0, \tag{6.3.7}$$

arguing by contradiction. Assume $\beta > 0$. Then, by (6.3.6) we have

$$\beta^2 = \lim_n \lambda\|v_n\|^{2^*_s}_{2^*_s} \leqslant \lambda T(N,s)^{2^*_s}\beta^{2^*_s},$$

hence

$$\beta \geqslant \Big[\frac{1}{\lambda T(N,s)^{2^*_s}}\Big]^{\frac{1}{2^*_s-2}}.$$

By (6.3.4) and (6.3.5), we also have

$$\Big(\frac{1}{2} - \frac{1}{2^*_s}\Big)\beta^2 = c - J_\lambda(u) < 2r,$$

hence

$$\beta < \Big[\frac{2Nr}{s}\Big]^{\frac{1}{2}}.$$

Comparing the last inequalities and recalling (6.3.2), we get

$$\lambda > \frac{1}{T(N,s)^{2^*_s}}\Big[\frac{s}{2Nr}\Big]^{\frac{2s}{N-2s}} \geqslant \lambda^*_r,$$

a contradiction. So (6.3.7) is proved, which means $u_n \to u$ in $H^s_0(\Omega)$. Thus, J_λ satisfies $(PS)^r$. □

Set

$$\mu^* = \min\Big\{\Big[\frac{2^*_s}{2^{\frac{2^*_s+2}{2}}T(N,s)^{2^*_s}}\Big]^{\frac{2}{2^*_s-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\Big\}^{\frac{2-p}{2}} \frac{p}{2^{\frac{p+2}{2}}a_pT(N,s)^p|\Omega|^{\frac{2^*_s-p}{2^*_s}}} > 0. \tag{6.3.8}$$

We have the following existence result for problem (6.3.1).

Theorem 6.3.2. *Let* $(\mathbf{H}_{6.3})$ *hold,* $\mu^* > 0$ *be defined by* (6.3.8)*. Then, for all* $\mu \in (0,\mu^*)$*,* (6.3.1) *has at least one solution* $u_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$*.*

Proof. Fix $\mu \in (0,\mu^*)$ and set

$$r = \min\Big\{\Big[\frac{2^*_s}{2^{\frac{2^*_s+2}{2}}T(N,s)^{2^*_s}}\Big]^{\frac{2}{2^*_s-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\Big\} > 0. \tag{6.3.9}$$

By (6.3.8), (6.3.9) we have

$$\frac{2^{\frac{2_s^*}{2}}T(N,s)^{2_s^*}r^{\frac{2_s^*-2}{2}}}{2_s^*}+\mu\frac{2^{\frac{p}{2}}a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}r^{\frac{p-2}{2}}}{p}\leqslant\frac{1}{2}+\frac{\mu}{2\mu^*}<1,$$

as well as

$$\frac{1}{T(N,s)^{2_s^*}}\left[\frac{s}{2Nr}\right]^{\frac{2s}{N-2s}}\geqslant\frac{1}{T(N,s)^{2_s^*}}\left[\frac{s}{2N}\frac{3NT(N,s)^{\frac{N}{s}}}{s}\right]^{\frac{2s}{N-2s}}=\left(\frac{3}{2}\right)^{\frac{2s}{N-2s}}>1,$$

hence by (6.3.2) we have $\lambda_r^*>1$.

We intend to apply Theorem 1.1.7. First, we see that hypothesis *(i)* holds. Then, for all $u\in H_0^s(\Omega)$, $\Phi(u)\leqslant r$ we have by $(\mathbf{H}_{6.3})$ *(i)* *(ii)*, Lemma 2.1.5, and (2.1.2)

$$\begin{aligned}\frac{\Psi(u)}{r}&\leqslant\frac{\|u\|_{2_s^*}^{2_s^*}}{2_s^*r}+\mu\frac{a_p\|u\|_p^p}{pr}\\&\leqslant\frac{T(N,s)^{2_s^*}(2r)^{\frac{2_s^*}{2}}}{2_s^*r}+\mu\frac{a_pT(N,s)^p|\Omega|^{\frac{2_s^*-p}{2_s^*}}(2r)^{\frac{p}{2}}}{pr}\\&\leqslant\frac{1}{\lambda_r^*}.\end{aligned}$$

On the other hand, by $(\mathbf{H}_{6.3})$ *(iii)* we have

$$\lim_{t\to0^+}\frac{F(t)}{t^2}=\infty.$$

So, arguing as in the proof of Theorem 6.2.1, we can find $\bar u\in H_0^s(\Omega)$ such that

$$0<\Phi(\bar u)<r,\quad\frac{\Psi(\bar u)}{\Phi(\bar u)}>\frac{1}{\lambda_r^*},$$

which ensures *(ii)* and *(iii)*. Finally, since $\lambda_r^*>1$, by Lemma 6.3.1 the functional J_1 satisfies $(PS)^r$.

Since $1\in I_r$, from Theorem 1.1.7 we deduce the existence of a (relabeled) function $u_\mu\in H_0^s(\Omega)$ such that

$$0<\Phi(u_\mu)<r,\ J_1(u_\mu)=\min_{0<\Phi(u_\mu)<r}J_1(u).$$

In particular, we have $J_1'(u_\mu)=0$ in $H^{-s}(\Omega)$. Thus, by Proposition 6.1.1, $u_\mu\in\mathrm{int}(C_s^0(\overline{\Omega})_+)$ is a solution of (6.3.1). □

Remark 6.3.3. The proof of Theorem 6.3.2 gives additional information: u_μ is a local minimizer of J_1 in $H_0^s(\Omega)$, satisfies the bound $\|u_\mu\|_{H_0^s(\Omega)}<(2r)^{\frac{1}{2}}$, and the mapping $\mu\mapsto J_1(u_\mu)$ is decreasing in $(0,\mu^*)$.

6.4 Two positive solutions under critical growth

Finally, we turn to problem (6.0.4) with $q = 2^*_s$, namely, the Ambrosetti-Brezis-Cerami problem for the fractional Laplacian:

$$\begin{cases} (-\Delta)^s\, u = \mu u^{p-1} + u^{2^*_s-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{6.4.1}$$

with $p \in (1,2)$, $\mu > 0$. This is a special case of (6.3.1) with $g(t) = (t^+)^{p-1}$, which satisfies $(\mathbf{H}_{6.3})$ with $a_p = 1$. We know from [17, Theorem 1.1] that (6.4.1) has at least two positive solutions for all $\mu > 0$ small enough. Our last result yields an explicitly estimate of 'how small' μ should be, given by (6.3.8) which in the present case rephrases as

$$\mu^* = \min\left\{\left[\frac{2^*_s}{2^{\frac{2^*_s+2}{2}}T(N,s)^{2^*_s}}\right]^{\frac{2}{2^*_s-2}}, \frac{s}{3NT(N,s)^{\frac{N}{s}}}\right\}^{\frac{2-p}{2}} \frac{p}{2^{\frac{p+2}{2}}T(N,s)^p|\Omega|^{\frac{2^*_s-p}{2^*_s}}} > 0. \tag{6.4.2}$$

Indeed, we have the following multiplicity result.

Theorem 6.4.1. *Let $p \in (1,2)$, $\mu^* > 0$ be defined by (6.4.2). Then, for all $\mu \in (0,\mu^*)$, (6.4.1) has at least two solutions $u_\mu, w_\mu \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, $u_\mu < w_\mu$ in Ω.*

Proof. Fix $\mu \in (0,\mu^*)$, define $f \in C(\mathbb{R})$, $\Phi, \Psi \in C^1(H^s_0(\Omega))$ as in Section 6.3, and set for brevity $J = J_1 = \Phi - \Psi$. From Theorem 6.3.2 and Remark 6.3.3 we know that there exists $u_\mu \in H^s_0(\Omega) \cap \mathrm{int}(C^0_s(\overline{\Omega})_+)$ which solves (6.4.1) and is a local minimizer of J. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x,t) = f(u_\mu(x) + t^+) - f(u_\mu(x)),$$
$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,\tau)\, d\tau = F(u_\mu(x) + t^+) - F(u_\mu(x)) - f(u_\mu(x))t^+.$$

For all $v \in H^s_0(\Omega)$ set

$$\tilde{\Psi}(v) = \int_\Omega \tilde{F}(x,v)\, dx, \quad \tilde{J}(v) = \Phi(v) - \tilde{\Psi}(v).$$

As in Section 6.1, it is easily seen that $\tilde{J} \in C^1(H^s_0(\Omega))$ and all its critical points solve the (nonautonomous) auxiliary problem

$$\begin{cases} (-\Delta)^s\, v = \tilde{f}(x,v) & \text{in } \Omega \\ v = 0 & \text{in } \Omega^c. \end{cases} \tag{6.4.3}$$

The functionals $\tilde{J}$ and J are related to each other by the following inequality for all $v \in H^s_0(\Omega)$:

$$\tilde{J}(v) \geqslant J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2_{H^s_0(\Omega)}}{2}. \tag{6.4.4}$$

Indeed, we have $v^\pm \in H^s_0(\Omega)$ and, setting

$$\Omega_+ = \big\{x \in \Omega : v(x) > 0\big\}, \quad \Omega_- = \Omega \setminus \Omega_+,$$

from $v = v^+ - v^-$ we have

$$\begin{aligned}\|v\|^2_{H^s_0(\Omega)} &= \|v^+\|^2_{H^s_0(\Omega)} + \|v^-\|^2_{H^s_0(\Omega)} - 2\iint_{\mathbb{R}^N\times\mathbb{R}^N} \frac{(v^+(x)-v^+(y))(v^-(x)-v^-(y))}{|x-y|^{N+2s}}\,dx\,dy \\ &\geqslant \|v^+\|^2_{H^s_0(\Omega)} + \|v^-\|^2_{H^s_0(\Omega)},\end{aligned}$$

as the integrand vanishes everywhere but in $\Omega_+ \times \Omega_-$ and in $\Omega_- \times \Omega_+$, where it is negative. So we have

$$\begin{aligned}\tilde{J}(v) &= \frac{\|v\|^2_{H^s_0(\Omega)}}{2} - \int_\Omega \tilde{F}(x,v)\,dx \\ &\geqslant \frac{\|v^+\|^2_{H^s_0(\Omega)}}{2} + \frac{\|v^-\|^2_{H^s_0(\Omega)}}{2} - \int_\Omega \big[F(u_\mu + v^+) - F(u_\mu) - f(u_\mu)v^+\big]\,dx \\ &= \frac{\|u_\mu + v^+\|^2_{H^s_0(\Omega)}}{2} - \frac{\|u_\mu\|^2_{H^s_0(\Omega)}}{2} - \langle u_\mu, v^+\rangle + \frac{\|v^-\|^2_{H^s_0(\Omega)}}{2} \\ &\quad - \int_\Omega \big[F(u_\mu + v^+) - F(u_\mu) - f(u_\mu)v^+\big]\,dx \\ &= J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2_{H^s_0(\Omega)}}{2}\end{aligned}$$

(where we used that u_μ solves (6.4.1)).

We claim that 0 is a local minimizer of $\tilde{J}$. Indeed, by Theorem 2.3.8 there exists $\rho > 0$ such that for all $v \in H^s_0(\Omega) \cap C^0_s(\overline{\Omega})$, $\|v\|_{0,s} \leqslant \rho$ we have $J(u_\mu + v) \geqslant J(u_\mu)$. Then, for any such v we have as well $\|v^+\|_{0,s} \leqslant \rho$, which along with (6.4.4) implies

$$\tilde{J}(v) \geqslant J(u_\mu + v^+) - J(u_\mu) + \frac{\|v^-\|^2_{H^s_0(\Omega)}}{2} \geqslant 0 = \tilde{J}(0).$$

So, 0 is a local minimizer of $\tilde{J}$ in $C^0_s(\overline{\Omega})$ and hence, by Theorem 2.3.8 again, it is such also in $H^s_0(\Omega)$. In particular, $\tilde{J}'(0) = 0$ in $H^{-s}(\Omega)$, i.e., 0 solves (6.4.3).

From now on we closely follow [17]. Arguing by contradiction, assume that 0 is the *only* critical point of $\tilde{J}$ in $H^s_0(\Omega)$. Under such assumption, by [17, Lemma 2.10] $\tilde{J}$ satisfies $(PS)_c$ at any level $c < c^*$, where

$$c^* = \frac{s}{NT(N,s)^{\frac{N}{s}}}. \tag{6.4.5}$$

Fix $x_0 \in \Omega$, and for all $\varepsilon > 0$, define $v_\varepsilon \in H^s(\mathbb{R}^N)$ by setting for all $x \in \mathbb{R}^N$

$$v_\varepsilon(x) = \frac{\varepsilon^{\frac{N-2s}{2}}}{(\varepsilon^2 + |x - x_0|^2)^{\frac{N-2s}{2}}}.$$

By Lemma 2.1.5 we have

$$\|v_\varepsilon\|_{2^*_s} = T(N,s)[v_\varepsilon]_s. \tag{6.4.6}$$

Now fix $r > 0$ such that $\overline{B}_r(x_0) \subset \Omega$, $\eta \in C^\infty(\mathbb{R}^N)$ such that $\eta = 1$ in $B_{\frac{r}{2}}(x_0)$, $\eta = 0$ in $B_1^c(x_0)$, and $0 \leqslant \eta \leqslant 1$ in $\mathbb{R}^N$, then define $w_\varepsilon \in H_0^s(\Omega)$ by setting for all $x \in \mathbb{R}^N$

$$w_\varepsilon(x) = \frac{\eta(x) v_\varepsilon(x)}{\|\eta v_\varepsilon\|_{2_s^*}}.$$

Clearly $\|w_\varepsilon\|_{2_s^*} = 1$. Besides, we will prove that for all $\varepsilon > 0$ small enough

$$\max_{\tau \geqslant 0} \tilde{J}(\tau w_\varepsilon) < c^*. \tag{6.4.7}$$

Assume $N > 4s$. Then, by [187, Propositions 21, 22] we find for all $\varepsilon > 0$ small enough

$$\begin{aligned} \|w_\varepsilon\|^2_{H_0^s(\Omega)} &\leqslant \frac{1}{T(N,s)^2} + C\varepsilon^{N-2s} \\ \|w_\varepsilon\|_2^2 &\geqslant C\varepsilon^{2s} - C\varepsilon^{N-2s} \end{aligned}$$

($C > 0$ denotes several constants, independent of ε). By convexity we have for all $x \in \Omega$, $t \geqslant 0$

$$\tilde{F}(x,t) \geqslant \frac{t^{2_s^*}}{2_s^*} + \frac{C}{2} u_\mu(x)^{2_s^*-2} t^2.$$

Using (6.4.6) and the relations above, we see that for all $\varepsilon > 0$ small enough and all $\tau \geqslant 0$

$$\begin{aligned} \tilde{J}(\tau w_\varepsilon) &\leqslant \frac{\tau^2}{2}\|w_\varepsilon\|^2_{H_0^s(\Omega)} - \frac{\tau^{2_s^*}}{2_s^*}\|w_\varepsilon\|^{2_s^*}_{2_s^*} - \frac{C\tau^2}{2}\int_\Omega u_\mu^{2_s^*-2} w_\varepsilon^2 \, dx \\ &\leqslant \frac{\tau^2}{2}\Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s} - C'\varepsilon^{2s}\Big] - \frac{\tau^{2_s^*}}{2_s^*} =: h_\varepsilon(\tau) \end{aligned} \tag{6.4.8}$$

($C, C' > 0$ independent of ε). Now we focus on the mapping $h_\varepsilon \in C^1(\mathbb{R}_+)$. First we note that

$$\lim_{\tau \to \infty} h_\varepsilon(\tau) = -\infty,$$

so there exists $\tau_\varepsilon \geqslant 0$ such that

$$h_\varepsilon(\tau_\varepsilon) = \max_{\tau \geqslant 0} h_\varepsilon(\tau).$$

If $\tau_\varepsilon = 0$, from (6.4.8) we immediately deduce (6.4.7). So, let $\tau_\varepsilon > 0$. Differentiating h_ε, we get

$$\tau_\varepsilon = \Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s} - C'\varepsilon^{2s}\Big]^{\frac{1}{2_s^*-2}},$$

which tends to $T(N,s)^{-\frac{2}{2_s^*-2}} > 0$ as $\varepsilon \to 0^+$. So, taking $\varepsilon > 0$ small enough, we have $\tau_\varepsilon \geqslant \tau_0 > 0$. Set

$$\tilde{\tau}_\varepsilon = \Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\Big]^{\frac{1}{2_s^*-2}},$$

and note that the mapping

$$\tau \mapsto \frac{\tau^2}{2}\Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\Big] - \frac{\tau^{2_s^*}}{2_s^*}$$

is increasing in $[0, \tilde{\tau}_\varepsilon]$. So we have

$$\begin{aligned} h_\varepsilon(\tau_\varepsilon) &= \frac{\tau_\varepsilon^2}{2}\Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\Big] - \frac{\tau_\varepsilon^{2_s^*}}{2_s^*} - \frac{C'\varepsilon^{2s}\tau_\varepsilon^2}{2} \\ &\leqslant \frac{\tilde{\tau}_\varepsilon^2}{2}\Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\Big] - \frac{\tilde{\tau}_\varepsilon^{2_s^*}}{2_s^*} - C''\varepsilon^{2s} \\ &= \frac{s}{N}\Big[\frac{1}{T(N,s)^2} + C\varepsilon^{N-2s}\Big]^{\frac{N}{2s}} - C''\varepsilon^{2s}. \end{aligned}$$

Since $N - 2s > 2s$, for all $\varepsilon > 0$ small enough we have by (6.4.5)

$$h_\varepsilon(\tau_\varepsilon) < \frac{s}{NT(N,s)^{\frac{N}{s}}} = c^*.$$

Then, by (6.4.8) we obtain (6.4.7). The cases $2s < N \leqslant 4s$ are treated in similar ways, see [17, Lemma 2.11].

As a byproduct of (6.4.8) we have that $\tilde{J}(\tau w_\varepsilon) \to -\infty$ as $\tau \to \infty$, so we can find $\bar{\tau} > 0$ such that

$$\tilde{J}(\bar{\tau} w_\varepsilon) < 0.$$

Since $\tilde{J}$ has a local minimum at 0 and no other critical point, we can find $\sigma \in (0, \|\bar{\tau} w_\varepsilon\|_{H_0^s(\Omega)})$ such that $\tilde{J}(v) > 0$ for all $v \in H_0^s(\Omega)$, $\|v\|_{H_0^s(\Omega)} = \sigma$. That is, $\tilde{J}$ exhibits a mountain pass geometry around 0. Set

$$\Gamma = \big\{\gamma \in C([0,1], H_0^s(\Omega)) : \gamma(0) = 0, \gamma(1) = \bar{\tau} w_\varepsilon\big\}, \quad c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} \tilde{J}(\gamma(t)).$$

Clearly, $\gamma(t) = t\bar{\tau} w_\varepsilon$ define a path of the family Γ, so by (6.4.7) we have

$$c \leqslant \max_{t\in[0,1]} \tilde{J}(t\bar{\tau} w_\varepsilon) < c^*.$$

Thus, $\tilde{J}$ satisfies $(PS)_c$. By the mountain pass Theorem given in [5, 99], there exists $v \in H_0^s(\Omega) \setminus \{0\}$ such that $\tilde{J}'(v) = 0$ in $H^{-s}(\Omega)$, a contradiction.

So we have proved the existence of $v_\mu \in H_0^s(\Omega) \setminus \{0\}$ such that $\tilde{J}'(v_\mu) = 0$ in $H^{-s}(\Omega)$. Such v_μ solves (6.4.3), and by monotonicity of f we have for a.e. $x \in \Omega$

$$\tilde{f}(x, v_\mu(x)) = f(u_\mu(x) + v_\mu^+(x)) - f(u_\mu(x)) \geqslant 0,$$

so by the fractional Hopf lemma (see for instance [113, Lemma 2.7], as Proposition 6.1.1 here does not apply) we have $v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Now set

$$w_\mu = u_\mu + v_\mu \in \mathrm{int}(C_s^0(\overline{\Omega})_+).$$

Clearly $w_\mu > u_\mu$ in Ω, and for all $\varphi \in H_0^s(\Omega)$ we have

$$\begin{aligned} \langle J'(w_\mu), \varphi\rangle &= \langle u_\mu + v_\mu, \varphi\rangle - \int_\Omega f(u_\mu + v_\mu)\varphi\,dx \\ &= \Big[\langle u_\mu, \varphi\rangle - \int_\Omega f(u_\mu)\varphi\,dx\Big] + \Big[\langle v_\mu, \varphi\rangle - \int_\Omega \tilde{f}(x, v_\mu)\varphi\,dx\Big] \\ &= \langle J'(u_\mu), \varphi\rangle + \langle \tilde{J}'(v_\mu), \varphi\rangle = 0, \end{aligned}$$

so w_μ solves (6.4.1), which concludes the proof. □

Finally, we present an example.

Example 6.4.2. Let $s = \frac{1}{2}$, $N = 2$, $p = \frac{3}{2}$ and $\Omega \subset \mathbb{R}^2$ be as in Example 6.2.3, but set this time $q = 2^*_{1/2} = 4$. Recall that in such case

$$T\left(2, \frac{1}{2}\right) = \frac{1}{2\pi^{\frac{3}{4}}}.$$

So, (6.4.2) yields

$$\mu^* = \frac{3^{\frac{1}{8}}\pi^{\frac{5}{4}}}{2^{\frac{11}{8}}}.$$

By Theorem 6.4.1, for all $\mu \in (0, \mu^*)$ problem (6.4.1) has at least two positive solutions $u_\mu < w_\mu$.

Chapter 7

Extremal constant sign solutions and nodal solutions for the fractional p-Laplacian

The present chapter is devoted to the study of the following Dirichlet-type problem for a nonlinear fractional equation:

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N > 1)$ is a bounded domain with $C^{1,1}$ boundary, $p \geqslant 2$, $s \in (0,1)$, $N > ps$, and $(-\Delta)_p^s$ denotes the fractional p-Laplacian. The reaction $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping subject to a subcritical growth condition.

In the study of nonlinear boundary value problems, one classical issue is that about the sign of solutions, especially in the case of multiple solutions. Typically, constant sign solutions can be detected as critical points of a truncated energy functional by direct minimization or min-max methods, while the existence of a *nodal* (i.e., sign-changing) solution is a more delicate question (some classical results, based on Morse theory, can be found in [4, 18, 201]). An interesting approach was proposed in [62] for the Dirichlet problem driven by the Laplacian operator: it consists in proving that the problem admits a *smallest* positive solution and a *biggest* negative one, plus a third nontrivial solution lying between the two, which must then be nodal. The method used for finding the nodal solution is based on the Fučik spectrum. Such approach was then extended to the p-Laplacian in [46], and then combined with a variational characterization of the second eigenvalue to detect a nodal solution under more general assumptions in [82] (see also [96, 149] and the monograph [150]).

Here in the first part of this chapter we focus on the structure of the set $\mathcal{S}(\underline{u}, \overline{u})$, namely the set of solutions of (2.3.14) lying within the interval $[\underline{u}, \overline{u}]$ where $\underline{u}$ and $\overline{u}$ are a subsolution and a supersolution of (2.3.14), respectively, with $\underline{u} \leqslant \overline{u}$ in Ω. By applying topological methods, we shall prove that $\mathcal{S}(\underline{u}, \overline{u})$ is nonempty, directed, and compact in $W_0^{s,p}(\Omega)$, hence endowed with extremal elements. Since $(-\Delta)_p^s$ is nonlinear, a major role is played by its monotonicity and continuity properties.

Then, in the second part we will follow a variational approach to show the existence of extremal constant sign solutions and nodal solutions. More precisely, we will assume that $f(x,\cdot)$ is $(p-1)$-sublinear at infinity and asymptotically linear near the origin without resonance on the first eigenvalue, and prove that (2.3.14) has a smallest positive solution u_+ and a biggest negative solution u_-. Finally, under more restrictive assumptions on the behavior of $f(x,\cdot)$ near the origin, we will prove existence of a nodal solution $\tilde{u}$ such that $u_- \leqslant \tilde{u} \leqslant u_+$ in Ω, thus extending some results of [46, 82] to the fractional p-Laplacian.

We remark that our results are new (to our knowledge) even in the semilinear case $p = 2$, and that the structure of the set $\mathcal{S}(\underline{u}, \overline{u})$ can provide valuable information about extremal solutions also in different frameworks.

The chapter has the following structure: in Section 7.1 we focus on the properties of the solution set; in Section 7.2 we show existence of extremal constant sign solutions; and in Section 7.3 we prove existence of a nontrivial nodal solution.

7.1 Solutions in a sub-supersolution interval

In this section we study the properties of the solution set. First of all, we introduce basic hypothesis on the reaction f:

($\mathbf{H}_{7.1}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|f(x,t)| \le c_0(1 + |t|^{q-1}) \qquad (c_0 > 0, q \in (p, p_s^*))$$

Now, we recall some definitions.

Definition 7.1.1. *Let $u \in \widetilde{W}^{s,p}(\Omega)$ (defined in Subsection 2.1.1).*

(i) u is a supersolution of (2.3.14) if $u \geqslant 0$ in Ω^c and for all $v \in W_0^{s,p}(\Omega)_+$

$$\langle (-\Delta)_p^s u, v \rangle \geqslant \int_\Omega f(x,u) v \, dx;$$

(ii) u is a subsolution of (2.3.14) if $u \leqslant 0$ in Ω^c and for all $v \in W_0^{s,p}(\Omega)_+$

$$\langle (-\Delta)_p^s u, v \rangle \leqslant \int_\Omega f(x,u) v \, dx.$$

Clearly, $u \in W_0^{s,p}(\Omega)$ is a solution of (2.3.14) iff it is both a supersolution and a subsolution. Sub-, supersolutions, and solutions of similar problems will be meant in the same sense as in Definitions 7.1.1, 2.3.9 above.

We say that $(\underline{u}, \overline{u}) \in \widetilde{W}^{s,p}(\Omega) \times \widetilde{W}^{s,p}(\Omega)$ is a *sub-supersolution pair* of (2.3.14), if $\underline{u}$ is a subsolution, $\overline{u}$ is a supersolution, and $\underline{u} \leqslant \overline{u}$ in Ω.

On spaces $W^{s,p}_0(\Omega), \widetilde{W}^{s,p}(\Omega)$ we consider the pointwise partial ordering, inducing a lattice structure. We set $u \wedge v = \min\{u, v\}$ and $u \vee v = \max\{u, v\}$.

The first result shows that the pointwise minimum of supersolutions is a supersolution, as well as the maximum of subsolutions is a subsolution. A similar result was proved in [125] for a homogeneous problem, under a different definition of super- and subsolutions. We give the proof in full detail, as it requires some careful calculations.

Lemma 7.1.2. *Let* $(\mathbf{H}_{7.1})$ *hold and* $u_1, u_2 \in \widetilde{W}^{s,p}(\Omega)$*:*

(i) *if* u_1, u_2 *are supersolutions of* (2.3.14)*, then so is* $u_1 \wedge u_2$*;*

(ii) *if* u_1, u_2 *are subsolutions of* (2.3.14) *then so is* $u_1 \vee u_2$.

Proof. We prove (i). We have for $i = 1, 2$

$$\begin{cases} \langle (-\Delta)^s_p u_i, v\rangle \geqslant \int_\Omega f(x, u_i)v\,dx & \text{for all } v \in W^{s,p}_0(\Omega)_+ \\ u_i \geqslant 0 & \text{in } \Omega^c. \end{cases} \tag{7.1.1}$$

Set $u = u_1 \wedge u_2 \in \widetilde{W}^{s,p}(\Omega)$ (by the lattice structure of $\widetilde{W}^{s,p}(\Omega)$), then $u \geqslant 0$ in Ω^c. Set also

$$A_1 = \{x \in \mathbb{R}^N : u_1(x) < u_2(x)\}, \quad A_2 = A_1^c.$$

Now fix $\varphi \in C^\infty_0(\Omega)_+$, $\varepsilon > 0$, and set for all $t \in \mathbb{R}$

$$\tau_\varepsilon(t) = \begin{cases} 0 & \text{if } t \leqslant 0 \\ \dfrac{t}{\varepsilon} & \text{if } 0 < t < \varepsilon \\ 1 & \text{if } t \geqslant \varepsilon. \end{cases}$$

The mapping $\tau_\varepsilon : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous, nondecreasing, and $0 \leqslant \tau_\varepsilon(t) \leqslant 1$ for all $t \in \mathbb{R}$, and clearly

$$\tau_\varepsilon(u_2 - u_1) \to \chi_{A_1}, \qquad 1 - \tau_\varepsilon(u_2 - u_1) \to \chi_{A_2}$$

a.e. in $\mathbb{R}^N$, as $\varepsilon \to 0^+$, with dominated convergence. Testing (7.1.1) with $\tau_\varepsilon(u_2 - u_1)\varphi, (1 - \tau_\varepsilon(u_2 - u_1))\varphi \in W^{s,p}_0(\Omega)_+$ for $i = 1, 2$ respectively, we get

$$\begin{aligned} &\langle (-\Delta)^s_p u_1, \tau_\varepsilon(u_2 - u_1)\varphi\rangle + \langle (-\Delta)^s_p u_2, (1 - \tau_\varepsilon(u_2 - u_1))\varphi\rangle \\ &\geqslant \int_\Omega f(x, u_1)\tau_\varepsilon(u_2 - u_1)\varphi\,dx + \int_\Omega f(x, u_2)(1 - \tau_\varepsilon(u_2 - u_1))\varphi\,dx. \end{aligned} \tag{7.1.2}$$

We focus on the left-hand side of (7.1.2). Setting for brevity $\tau_\varepsilon = \tau_\varepsilon(u_2 - u_1)$, and recalling that $\tau_\varepsilon = 0$ in A_2, while $\tau_\varepsilon \to 1$ in A_1 as $\varepsilon \to 0^+$, we get

$$\begin{aligned} &\langle (-\Delta)^s_p u_1, \tau_\varepsilon\varphi\rangle + \langle (-\Delta)^s_p u_2, (1 - \tau_\varepsilon)\varphi\rangle \\ &= \iint_{\mathbb{R}^N\times\mathbb{R}^N} (u_1(x) - u_1(y))^{p-1}(\tau_\varepsilon(x)\varphi(x) - \tau_\varepsilon(y)\varphi(y))\,d\mu \\ &+ \iint_{\mathbb{R}^N\times\mathbb{R}^N} (u_2(x) - u_2(y))^{p-1}[(1 - \tau_\varepsilon(x))\varphi(x) - (1 - \tau_\varepsilon(y))\varphi(y)]\,d\mu \\ &=: I. \end{aligned}$$

Using the definition of A_1 and A_2, we obtain

$$I = \iint_{A_1\times A_1} (u_1(x)-u_1(y))^{p-1}(\varphi(x)-\varphi(y))\tau_\varepsilon(x)\,d\mu \tag{A}$$

$$+ \iint_{A_1\times A_1} (u_1(x)-u_1(y))^{p-1}\varphi(y)(\tau_\varepsilon(x)-\tau_\varepsilon(y))\,d\mu \tag{B}$$

$$+ \iint_{A_1\times A_2} (u_1(x)-u_1(y))^{p-1}\varphi(x)\tau_\varepsilon(x)\,d\mu \tag{C}$$

$$- \iint_{A_2\times A_1} (u_1(x)-u_1(y))^{p-1}\varphi(y)\tau_\varepsilon(y)\,d\mu \tag{D}$$

$$+ \iint_{A_1\times A_1} (u_2(x)-u_2(y))^{p-1}(\varphi(x)-\varphi(y))(1-\tau_\varepsilon(x))\,d\mu \tag{E}$$

$$- \iint_{A_1\times A_1} (u_2(x)-u_2(y))^{p-1}\varphi(y)(\tau_\varepsilon(x)-\tau_\varepsilon(y))\,d\mu \tag{B}$$

$$+ \iint_{A_1\times A_2} (u_2(x)-u_2(y))^{p-1}(\varphi(x)-\varphi(y))(1-\tau_\varepsilon(x))\,d\mu \tag{F}$$

$$- \iint_{A_1\times A_2} (u_2(x)-u_2(y))^{p-1}\varphi(y)\tau_\varepsilon(x)\,d\mu \tag{C}$$

$$+ \iint_{A_2\times A_1} (u_2(x)-u_2(y))^{p-1}\varphi(x)\tau_\varepsilon(y)\,d\mu \tag{D}$$

$$+ \iint_{A_2\times A_1} (u_2(x)-u_2(y))^{p-1}(\varphi(x)-\varphi(y))(1-\tau_\varepsilon(y))\,d\mu \tag{G}$$

$$+ \iint_{A_2\times A_2} (u_2(x)-u_2(y))^{p-1}(\varphi(x)-\varphi(y))\,d\mu. \tag{H}$$

We then put together the integrals with the same letter and note that (E), (F), (G) $\to 0$ as $\varepsilon \to 0^+$. So, we have

$$I = \iint_{A_1\times A_1} (u_1(x)-u_1(y))^{p-1}(\varphi(x)-\varphi(y))\,d\mu \tag{A}$$

$$+ \iint_{A_1\times A_1} [(u_1(x)-u_1(y))^{p-1}-(u_2(x)-u_2(y))^{p-1}]\varphi(y)(\tau_\varepsilon(x)-\tau_\varepsilon(y))\,d\mu \tag{B}$$

$$+ \iint_{A_1\times A_2} [(u_1(x)-u_1(y))^{p-1}\varphi(x)-(u_2(x)-u_2(y))^{p-1}\varphi(y)]\tau_\varepsilon(x)\,d\mu \tag{C}$$

$$+ \iint_{A_2\times A_1} [(u_2(x)-u_2(y))^{p-1}\varphi(x)-(u_1(x)-u_1(y))^{p-1}\varphi(y)]\tau_\varepsilon(y)\,d\mu \tag{D}$$

$$+ \iint_{A_2\times A_2} (u_2(x)-u_2(y))^{p-1}(\varphi(x)-\varphi(y))\,d\mu \tag{H}$$

$$+ o(1).$$

Now we note that for all $x, y \in A_1$

$$u_1(x)-u_1(y) \geqslant u_2(x)-u_2(y) \;\Leftrightarrow\; u_2(y)-u_1(y) \geqslant u_2(x)-u_1(x) \;\Leftrightarrow\; \tau_\varepsilon(y) \geqslant \tau_\varepsilon(x),$$

hence the integrand in (B) is negative. Besides, for all $x \in A_1$, $y \in A_2$

$$u_1(x)-u_1(y) \leqslant u_1(x)-u_2(y) \leqslant u_2(x)-u_2(y),$$

and for all $x \in A_2$, $y \in A_1$

$$u_2(x) - u_2(y) \leqslant u_2(x) - u_1(y) \leqslant u_1(x) - u_1(y),$$

so we can estimate the integrands in (C), (D) respectively and get

$$\begin{aligned} I &\leqslant \iint_{A_1 \times A_1} (u_1(x) - u_1(y))^{p-1}(\varphi(x) - \varphi(y))\, d\mu \\ &+ \iint_{A_1 \times A_2} (u_1(x) - u_2(y))^{p-1}(\varphi(x) - \varphi(y))\, d\mu \\ &+ \iint_{A_2 \times A_1} [(u_2(x) - u_1(y))^{p-1}(\varphi(x) - \varphi(y))\, d\mu \\ &+ \iint_{A_2 \times A_2} (u_2(x) - u_2(y))^{p-1}(\varphi(x) - \varphi(y))\, d\mu + o(1) \\ &= \langle (-\Delta)_p^s\, u, \varphi \rangle + o(1). \end{aligned}$$

All in all, we have

$$\langle (-\Delta)_p^s\, u_1, \tau_\varepsilon(u_2 - u_1)\varphi \rangle + \langle (-\Delta)_p^s\, u_2, (1 - \tau_\varepsilon(u_2 - u_1))\varphi \rangle \leqslant \langle (-\Delta)_p^s\, u, \varphi \rangle + o(1), \tag{7.1.3}$$

as $\varepsilon \to 0^+$. Regarding the right-hand side of (7.1.2), we use the bounds from $(\mathbf{H}_{7.1})$ and the definition of τ_ε to get

$$|f(\cdot, u_1)\tau_\varepsilon^+(u_2 - u_1)\varphi| \leqslant c_0(1 + |u_1|^{q-1})\varphi,$$

$$|f(\cdot, u_2)(1 - \tau_\varepsilon^+(u_2 - u_1))\varphi| \leqslant c_0(1 + |u_2|^{q-1})\varphi,$$

and pass to the limit as $\varepsilon \to 0^+$:

$$\begin{aligned} &\int_\Omega f(x, u_1)\tau_\varepsilon(u_2 - u_1)\varphi\, dx + \int_\Omega f(x, u_2)(1 - \tau_\varepsilon(u_2 - u_1))\varphi\, dx \\ &= \int_\Omega f(x, u_1)\chi_{A_1}\varphi\, dx + \int_\Omega f(x, u_2)\chi_{A_2}\varphi\, dx + o(1) \\ &= \int_\Omega f(x, u)\varphi\, dx + o(1). \end{aligned} \tag{7.1.4}$$

Plugging (7.1.3), (7.1.4) into (7.1.2) we have for all $\varphi \in C_0^\infty(\overline{\Omega})_+$

$$\langle (-\Delta)_p^s\, u, \varphi \rangle \geqslant \int_\Omega f(x, u)\varphi\, dx.$$

By density, the same holds with test functions in $W_0^{s,p}(\Omega)_+$, hence u is a supersolution of (2.3.14), which proves (i). Similarly we prove (ii). □

Now we consider a sub-supersolution pair $(\underline{u}, \overline{u})$ and we study the solution set

$$\mathcal{S}(\underline{u}, \overline{u}) = \{u \in W_0^{s,p}(\Omega) : u \text{ solves } (2.3.14), \underline{u} \leqslant u \leqslant \overline{u}\}.$$

We begin with a sub-supersolution principle, showing that $\mathcal{S}(\underline{u}, \overline{u}) \neq \emptyset$.

Lemma 7.1.3. *Let* $(\mathbf{H}_{7.1})$ *hold and* $(\underline{u}, \overline{u})$ *be a sub-supersolution pair of* (2.3.14)*. Then, there exists* $u \in \mathcal{S}(\underline{u}, \overline{u})$*.*

Proof. In this argument we use some nonlinear operator theory from [45]. First we define $A = (-\Delta)_p^s : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$. By Lemma 2.2.5 A is monotone and continuous, hence hemicontinuous (Definition 1.2.1 *(iii)*), therefore A is pseudomonotone [45, Lemma 2.98 (i)].

Besides, we set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x,t) = \begin{cases} f(x, \underline{u}(x)) & \text{if } t \leqslant \underline{u}(x) \\ f(x,t) & \text{if } \underline{u}(x) < t < \overline{u}(x) \\ f(x, \overline{u}(x)) & \text{if } t \geqslant \overline{u}(x). \end{cases}$$

In general, $\tilde{f}$ does not satisfy $(\mathbf{H}_{7.1})$, but still $\tilde{f} : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$|\tilde{f}(x,t)| \leqslant c_0(1 + |\underline{u}|^{q-1} + |\overline{u}|^{q-1}). \tag{7.1.5}$$

We define $B : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ by setting for all $u, v \in W_0^{s,p}(\Omega)$

$$\langle B(u), v\rangle = -\int_\Omega \tilde{f}(x,u)v\,dx,$$

well posed by (7.1.5), as $|\underline{u}|^{q-1}, |\overline{u}|^{q-1} \in L^{q'}(\Omega)$. We prove that B is strongly continuous (Definition 1.2.1 *(iv)*). Indeed, let (u_n) be a sequence such that $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, passing to a subsequence if necessary, we have $u_n \to u$ in $L^q(\Omega)$, $u_n(x) \to u(x)$ and $|u_n(x)| \leqslant h(x)$ for a.e. $x \in \Omega$, for some $h \in L^q(\Omega)$. Therefore, for all $n \in \mathbb{N}$, by (7.1.5) we have for a.e. $x \in \Omega$

$$|\tilde{f}(x,u_n) - \tilde{f}(x,u)| \leqslant 2c_0(1 + |\underline{u}|^{q-1} + |\overline{u}|^{q-1}) \in L^{q'}(\Omega),$$

while by continuity of $f(x,\cdot)$ we have $\tilde{f}(x,u_n) \to \tilde{f}(x,u)$. Hence, for all $v \in W_0^{s,p}(\Omega)$,

$$\begin{aligned}|\langle B(u_n) - B(u), v\rangle| &\leqslant \int_\Omega |\tilde{f}(x,u_n) - \tilde{f}(x,u)||v|\,dx \\ &\leqslant \|\tilde{f}(\cdot,u_n) - \tilde{f}(\cdot,u)\|_{q'}\|v\|_q\end{aligned}$$

and the latter tends to 0 as $n \to \infty$, uniformly with respect to v. Therefore $B(u_n) \to B(u)$ in $W^{-s,p'}(\Omega)$. By [45, Lemma 2.98 (ii)], B is pseudomonotone. Thus, $A + B$ is pseudomonotone.

Now we prove that $A+B$ is bounded. Indeed, for all $u \in W_0^{s,p}(\Omega)$ we have $\|A(u)\|_{W^{-s,p'}(\Omega)} \leqslant \|u\|_{W_0^{s,p}(\Omega)}^{p-1}$ and

$$\begin{aligned}\|B(u)\|_{W^{-s,p'}(\Omega)} &= \sup_{\|v\|_{W_0^{s,p}(\Omega)} \leqslant 1} \int_\Omega \tilde{f}(x,u)v\,dx \\ &\leqslant C\|\tilde{f}(\cdot,u)\|_{q'} \\ &\leqslant C(1 + \|\underline{u}\|_q^{q-1} + \|\overline{u}\|_q^{q-1}),\end{aligned}$$

where we have used (7.1.5) and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$.
Finally we prove that $A + B$ is coercive. Indeed, for all $u \in W_0^{s,p}(\Omega) \setminus \{0\}$ we have

$$\begin{aligned}
\frac{\langle A(u) + B(u), u\rangle}{\|u\|_{W_0^{s,p}(\Omega)}} &= \|u\|^{p-1}_{W_0^{s,p}(\Omega)} - \frac{1}{\|u\|_{W_0^{s,p}(\Omega)}} \int_\Omega \tilde{f}(x,u)u\,dx \\
&\geqslant \|u\|^{p-1}_{W_0^{s,p}(\Omega)} - \frac{C}{\|u\|_{W_0^{s,p}(\Omega)}} \int_\Omega (1 + |\underline{u}|^{q-1} + |\overline{u}|^{q-1})|u|\,dx \\
&\geqslant \|u\|^{p-1}_{W_0^{s,p}(\Omega)} - \frac{C}{\|u\|_{W_0^{s,p}(\Omega)}} \big(\|u\|_1 + \|\underline{u}\|_q^{q-1}\|u\|_q + \|\overline{u}\|_q^{q-1}\|u\|_q\big) \\
&\geqslant \|u\|^{p-1}_{W_0^{s,p}(\Omega)} - C,
\end{aligned}$$

and the latter tends to ∞ as $\|u\|_{W_0^{s,p}(\Omega)} \to \infty$ (here we have used the continuous embeddings $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$, $L^q(\Omega)$). By Theorem 1.2.4, the equation

$$A(u) + B(u) = 0 \text{ in } W^{-s,p'}(\Omega) \tag{7.1.6}$$

has a solution $u \in W_0^{s,p}(\Omega)$. Now we prove that in Ω

$$\underline{u} \leqslant u \leqslant \overline{u}. \tag{7.1.7}$$

Clearly (7.1.7) holds in Ω^c. Testing (7.1.6) with $(u - \overline{u})^+ \in W_0^{s,p}(\Omega)_+$ we have

$$\begin{aligned}
\langle (-\Delta)_p^s\, u, (u - \overline{u})^+\rangle &= \int_\Omega \tilde{f}(x,u)(u - \overline{u})^+\,dx \\
&= \int_\Omega f(x,\overline{u})(u - \overline{u})^+\,dx \\
&\leqslant \langle (-\Delta)_p^s\, \overline{u}, (u - \overline{u})^+\rangle,
\end{aligned}$$

where we also used that $\overline{u}$ is a supersolution of (2.3.14), so

$$\langle (-\Delta)_p^s\, u - (-\Delta)_p^s\, \overline{u}, (u - \overline{u})^+\rangle \leqslant 0.$$

By [29, Lemma A.2] and [116, Lemma 2.3] (with $g(t) = t^+$) we have for all $a, b \in \mathbb{R}$

$$|a^+ - b^+|^p \leqslant (a-b)^{p-1}(a^+ - b^+), \quad (a-b)^{p-1} \leqslant C(a^{p-1} - b^{p-1}),$$

hence

$$\begin{aligned}
&\|(u - \overline{u})^+\|^p_{W_0^{s,p}(\Omega)} = \iint_{\mathbb{R}^N\times\mathbb{R}^N} |(u(x) - \overline{u}(x))^+ - (u(y) - \overline{u}(y))^+|^p\,d\mu \\
&\leqslant \iint_{\mathbb{R}^N\times\mathbb{R}^N} [(u(x) - \overline{u}(x)) - (u(y) - \overline{u}(y))]^{p-1}[(u(x) - \overline{u}(x))^+ - (u(y) - \overline{u}(y))^+]\,d\mu \\
&\leqslant C \iint_{\mathbb{R}^N\times\mathbb{R}^N} [(u(x) - u(y))^{p-1} - (\overline{u}(x) - \overline{u}(y))^{p-1}][(u(x) - \overline{u}(x))^+ - (u(y) - \overline{u}(y))^+]\,d\mu \\
&= C\langle (-\Delta)_p^s\, u - (-\Delta)_p^s\, \overline{u}, (u - \overline{u})^+\rangle \leqslant 0,
\end{aligned}$$

so $(u - \overline{u})^+ = 0$, i.e., $u \leqslant \overline{u}$ in Ω. Similarly we prove $u \geqslant \underline{u}$ and achieve (7.1.7). Finally, using (7.1.7) in (7.1.6) we see that $u \in W_0^{s,p}(\Omega)$ solves (2.3.14). Thus $u \in \mathcal{S}(\underline{u}, \overline{u})$. □

We recall that a partially ordered set $(S, \leqslant)$ is *downward directed* (resp., *upward directed*) if for all $u_1, u_2 \in S$ there exists $u_3 \in S$ such that $u_3 \leqslant u_1, u_2$ (resp., $u_3 \geqslant u_1, u_2$), and that S is *directed* if it is both downward and upward directed.

Lemma 7.1.4. *Let* $(\mathbf{H}_{7.1})$ *hold,* $(\underline{u}, \overline{u})$ *be a sub-supersolution pair of* (2.3.14)*. Then,* $\mathcal{S}(\underline{u}, \overline{u})$ *is directed.*

Proof. We prove that $\mathcal{S}(\underline{u}, \overline{u})$ is downward directed. Let $u_1, u_2 \in \mathcal{S}(\underline{u}, \overline{u})$, then in particular u_1, u_2 are supersolutions of (2.3.14). Set $\hat{u} = u_1 \wedge u_2 \in W_0^{s,p}(\Omega)$, then by Lemma 7.1.2 $\hat{u}$ is a supersolution of (2.3.14) and $\underline{u} \leqslant \hat{u}$. By Lemma 7.1.3 there exists $u_3 \in \mathcal{S}(\underline{u}, \hat{u})$, in particular $u_3 \in \mathcal{S}(\underline{u}, \overline{u})$ and $u_3 \leqslant u_1 \wedge u_2$.
Similarly we see that $\mathcal{S}(\underline{u}, \overline{u})$ is upward directed. $\square$

Another important property of $\mathcal{S}(\underline{u}, \overline{u})$ is compactness.

Lemma 7.1.5. *Let* $(\mathbf{H}_{7.1})$ *hold,* $(\underline{u}, \overline{u})$ *be a sub-supersolution pair of* (2.3.14)*. Then,* $\mathcal{S}(\underline{u}, \overline{u})$ *is compact in* $W_0^{s,p}(\Omega)$*.*

Proof. Let (u_n) be a sequence in $\mathcal{S}(\underline{u}, \overline{u})$, then for all $n \in \mathbb{N}$, $v \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u_n, v \rangle = \int_\Omega f(x, u_n) v \, dx \tag{7.1.8}$$

and $\underline{u} \leqslant u_n \leqslant \overline{u}$. Testing (7.1.8) with $u_n \in W_0^{s,p}(\Omega)$, we have by $(\mathbf{H}_{7.1})$

$$\begin{aligned}
\|u_n\|^p_{W_0^{s,p}(\Omega)} &= \int_\Omega f(x, u_n) u_n \, dx \\
&\leqslant c_0 \int_\Omega (|u_n| + |u_n|^q) \, dx \\
&\leqslant c_0 (\|\underline{u}\|_1 + \|\overline{u}\|_1 + \|\underline{u}\|_q^q + \|\overline{u}\|_q^q) \leqslant C,
\end{aligned}$$

hence (u_n) is bounded in $W_0^{s,p}(\Omega)$. Passing to a subsequence, we have $u_n \rightharpoonup u$ in $W_0^{s,p}(\Omega)$, $u_n(x) \to u(x)$ and $|u_n(x)| \leqslant h(x)$ for a.e. $x \in \mathbb{N}$, with $h \in L^q(\Omega)$. Therefore,

$$\begin{aligned}
|f(x, u_n)(u_n - u)| &\leqslant c_0 (1 + |u_n|^{q-1}) |u_n - u| \\
&\leqslant 2c_0 (1 + g(x)^{q-1}) (|\underline{u}| + |\overline{u}|) \in L^1(\Omega).
\end{aligned}$$

Testing (7.1.8) with $u_n - u \in W_0^{s,p}(\Omega)$, we get

$$\langle (-\Delta)_p^s\, (u_n), u_n - u \rangle = \int_\Omega f(x, u_n)(u_n - u) \, dx,$$

and the latter tends to 0 as $n \to \infty$. By Lemma 2.2.5 we have $u_n \to u$ in $W_0^{s,p}(\Omega)$. Then, we can pass to the limit in (7.1.8) and conclude that $u \in \mathcal{S}(\underline{u}, \overline{u})$. $\square$

The main result of this section states that $\mathcal{S}(\underline{u}, \overline{u})$ contains extremal elements with respect to the pointwise ordering.

Theorem 7.1.6. *Let* $(\mathbf{H}_{7.1})$ *hold,* $(\underline{u}, \overline{u})$ *be a sub-supersolution pair of* (2.3.14). *Then* $\mathcal{S}(\underline{u}, \overline{u})$ *contains a smallest and a biggest element.*

Proof. The set $\mathcal{S}(\underline{u}, \overline{u})$ is bounded in both $W_0^{s,p}(\Omega)$ and $C_s^\alpha(\overline{\Omega})$. Indeed, for all $u \in \mathcal{S}(\underline{u}, \overline{u})$, testing (2.3.14) with $u \in W_0^{s,p}(\Omega)$ we have

$$\begin{aligned}\|u\|^p_{W_0^{s,p}(\Omega)} &= \int_\Omega f(x,u)u\,dx \\ &\leqslant c_0 \int_\Omega (|u| + |u|^q)\,dx \\ &\leqslant c_0(\|\underline{u}\|_1 + \|\overline{u}\|_1 + \|\underline{u}\|_q^q + \|\overline{u}\|_q^q),\end{aligned}$$

hence $\mathcal{S}(\underline{u}, \overline{u})$ is bounded in $W_0^{s,p}(\Omega)$. Further, by Lemma 2.3.10, for all $u \in \mathcal{S}(\underline{u}, \overline{u})$ we have $u \in L^\infty(\Omega)$, $\|u\|_\infty \leqslant C$ (with $C = C(\underline{u}, \overline{u}) > 0$, here and in the forthcoming bounds). In turn, this implies $\|f(\cdot, u)\|_\infty \leqslant C$. Then we apply Lemma 2.3.11 (with $g = f(\cdot, u)$) to see that $u \in C_s^\alpha(\overline{\Omega})$, $\|u\|_{\alpha,s} \leqslant C$. So, $\mathcal{S}(\underline{u}, \overline{u})$ is bounded in $C_s^\alpha(\overline{\Omega})$ as well (in particular, then, $\mathcal{S}(\underline{u}, \overline{u})$ is equibounded in Ω).

Now we prove that $\mathcal{S}(\underline{u}, \overline{u})$ has a minimum. Let (x_k) be a dense subset of Ω, and set

$$m_k = \inf_{u \in \mathcal{S}(\underline{u}, \overline{u})} u(x_k) > -\infty$$

for each $k \geqslant 1$ (recall $\mathcal{S}(\underline{u}, \overline{u})$ is equibounded). For all $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$ we can find $u_{n,k} \in \mathcal{S}(\underline{u}, \overline{u})$ such that

$$u_{n,k}(x_k) \leqslant m_k + \frac{1}{n}.$$

Since $\mathcal{S}(\underline{u}, \overline{u})$ is downward directed (Lemma 7.1.4), we can find $u_n \in \mathcal{S}(\underline{u}, \overline{u})$ such that $u_n \leqslant u_{n,k}$ for all $k \in \{1, \dots, n\}$. In particular, for all $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$ we have

$$u_n(x_k) \leqslant m_k + \frac{1}{n}. \tag{7.1.9}$$

Since $\mathcal{S}(\underline{u}, \overline{u})$ is compact (Lemma 7.1.5), passing to a subsequence we have $u_n \to u_0$ in $W_0^{s,p}(\Omega)$ for some $u_0 \in \mathcal{S}(\underline{u}, \overline{u})$. Besides, $(u_n) \subseteq \mathcal{S}(\underline{u}, \overline{u})$ is bounded in $C_s^\alpha(\overline{\Omega})$, hence up to a further subsequence $u_n \to u_0$ in $C_s^0(\overline{\Omega})$, in particular $u_n(x) \to u_0(x)$ for all $x \in \overline{\Omega}$. By (7.1.9) we have for all $k \in \mathbb{N}$

$$u_0(x_k) = \lim_n u_n(x_k) \leqslant \lim_n \left(m_k + \frac{1}{n}\right) = m_k.$$

Therefore, given $u \in \mathcal{S}(\underline{u}, \overline{u})$ we have $u_0(x_k) \leqslant u(x_k)$ for all $k \geqslant 1$, which by density of (x_k) implies $u_0 \leqslant u$. Hence,

$$u_0 = \min \mathcal{S}(\underline{u}, \overline{u}).$$

Similarly we prove the existence of $\max \mathcal{S}(\underline{u}, \overline{u})$. $\square$

Remark 7.1.7. For the sake of completeness, we recall that Theorem 7.1.6 can be proved following closely the proof of [45, Theorem 3.11], using Lemmas 7.1.4, 7.1.5, and the fact that $W_0^{s,p}(\Omega)$ is separable (another way consists in applying Zorn's Lemma, as in [45, Remark 3.12]). We also note that, as seen in the proof of Theorem 7.1.6, $\mathcal{S}(\underline{u},\overline{u})$ turns out to be compact in $C_s^0(\overline{\Omega})$.

7.2 Extremal constant sign solutions

In this section we prove that (2.3.14) has a smallest positive and a biggest negative solution (following the ideas of [46]), under the following hypotheses on f:

($\mathbf{H}_{7.2}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, for all $(x,t) \in \Omega \times \mathbb{R}$ we set

$$F(x,t) = \int_0^t f(x,\tau)\, d\tau,$$

and the following conditions hold:

(*i*) $|f(x,t)| \leq c_0(1+|t|^{q-1})$ for all a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ ($c_0 > 0$, $q \in (p, p_s^*)$);

(*ii*) $\limsup_{|t|\to\infty} \dfrac{F(x,t)}{|t|^p} < \dfrac{\lambda_1}{p}$ uniformly for a.e. $x \in \Omega$;

(*iii*) $\lambda_1 < \liminf_{t\to 0} \dfrac{f(x,t)}{|t|^{p-2}t} \leqslant \limsup_{t\to 0} \dfrac{f(x,t)}{|t|^{p-2}t} < \infty$ uniformly for a.e. $x \in \Omega$.

Clearly ($\mathbf{H}_{7.2}$) implies ($\mathbf{H}_{7.1}$). Here $\lambda_1 > 0$ denotes the principal eigenvalue of $(-\Delta)_p^s$ in $W_0^{s,p}(\Omega)$, with associated positive, $L^p(\Omega)$-normalized eigenfunction $\hat{u}_1 \in W_0^{s,p}(\Omega)$ (see Lemma 3.4.1 (*i*)). Note that by ($\mathbf{H}_{7.2}$) (*iii*) we have $f(\cdot,0) = 0$ in Ω, hence (2.3.14) has the trivial solution 0. Condition ($\mathbf{H}_{7.2}$) (*iii*) conjures a $(p-1)$-linear behavior of $f(x,\cdot)$ near the origin.

In this and the forthcoming section, our approach to problem (2.3.14) is purely variational. Our result is the following.

Theorem 7.2.1. *Let* ($\mathbf{H}_{7.2}$) *hold. Then,* (2.3.14) *has a smallest positive solution* $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ *and a biggest negative solution* $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Proof. We focus on positive solutions. Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$f_+(x,t) = f(x,t^+), \quad F_+(x,t) = \int_0^t f_+(x,\tau)\, d\tau,$$

and for all $u \in W_0^{s,p}(\Omega)$

$$J_+(u) = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega F_+(x,u)\, dx.$$

Since $f_+(x,t) = 0$ for all $(x,t) \in \Omega \times \mathbb{R}^-$, f_+ satisfies ($\mathbf{H}_{7.2}$) (with $t \to 0^+$ in (*iii*)). Therefore, $J_+ \in C^1(W_0^{s,p}(\Omega))$. By ($\mathbf{H}_{7.2}$) (*i*) and the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, it is easily seen that J_+ is sequentially weakly lower semicontinuous in $W_0^{s,p}(\Omega)$.

By $(\mathbf{H}_{7.2})$ *(ii)* there exist $\theta \in (0, \lambda_1)$, $K > 0$ such that for a.e. $x \in \Omega$ and all $|t| \geqslant K$

$$F_+(x,t) \leqslant \frac{\theta}{p}|t|^p.$$

Besides, by $(\mathbf{H}_{7.2})$ *(i)* we can find $C_K > 0$ such that for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$F_+(x,t) \leqslant \frac{\theta}{p}|t|^p + C_K.$$

So, for all $u \in W_0^{s,p}(\Omega)$ we have

$$\begin{aligned} J_+(u) &\geqslant \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega \Big(\frac{\theta}{p}|u|^p + C_K\Big)\,dx \\ &\geqslant \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \frac{\theta}{p}\|u\|_p^p - C_K|\Omega| \\ &\geqslant \Big(1 - \frac{\theta}{\lambda_1}\Big)\frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - C_K|\Omega| \end{aligned}$$

(where we used Lemma 3.4.1), and the latter tends to infinity as $\|u\|_{W_0^{s,p}(\Omega)} \to \infty$. Therefore J_+ is coercive. Thus, there is $\hat{u} \in W_0^{s,p}(\Omega)$ such that

$$J_+(\hat{u}) = \inf_{u \in W_0^{s,p}(\Omega)} J_+(u). \tag{7.2.1}$$

In particular, we have $J_+'(\hat{u}) = 0$, i.e.,

$$(-\Delta)_p^s\,\hat{u} = f_+(\cdot,\hat{u}) \text{ in } W^{-s,p'}(\Omega). \tag{7.2.2}$$

Testing (7.2.2) with $-\hat{u}^- \in W_0^{s,p}(\Omega)$, we get

$$\|\hat{u}^-\|^p_{W_0^{s,p}(\Omega)} \leqslant -\langle(-\Delta)_p^s\,\hat{u}, \hat{u}^-\rangle = -\int_\Omega f_+(x,\hat{u})\hat{u}^-\,dx = 0,$$

so $\hat{u} \geqslant 0$. Hence, $f_+(\cdot,\hat{u}) = f(\cdot,\hat{u})$, therefore (7.2.2) rephrases as

$$(-\Delta)_p^s\,(\hat{u}) = f(\cdot,\hat{u}) \text{ in } W^{-s,p'}(\Omega),$$

i.e., $\hat{u} \in W_0^{s,p}(\Omega)_+$ is a solution of (2.3.14). By Lemmas 2.3.10, 2.3.11 we have $\hat{u} \in C_s^0(\overline{\Omega})_+$. By $(\mathbf{H}_{7.2})$ *(iii)*, we can find $\lambda_1 < c_1 < c_2$, $\delta > 0$ such that for a.e. $x \in \Omega$ and all $t \in [0,\delta]$

$$c_1 t^{p-1} \leqslant f(x,t) \leqslant c_2 t^{p-1}. \tag{7.2.3}$$

Choose $\tau > 0$ such that $0 < \tau\hat{u}_1 \leqslant \delta$ in Ω. Then by (7.2.1), (7.2.3), and Lemma 3.4.1 we have

$$\begin{aligned} J_+(\hat{u}) &\leqslant J_+(\tau\hat{u}_1) \\ &= \frac{\tau^p}{p}\|\hat{u}_1\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega F_+(x,\tau\hat{u}_1)\,dx \\ &\leqslant \frac{\tau^p}{p}\|\hat{u}_1\|^p_{W_0^{s,p}(\Omega)} - \frac{\tau^p c_1}{p}\|\hat{u}_1\|_p^p \\ &= \frac{\tau^p}{p}(\lambda_1 - c_1) < 0, \end{aligned}$$

hence $\hat{u} \neq 0$. By (7.2.2), (7.2.3) we have for all $v \in W_0^{s,p}(\Omega)_+$

$$\begin{aligned}\langle(-\Delta)_p^s\,\hat{u}, v\rangle &\geqslant \int_{\{\hat{u}\leqslant\delta\}} c_1\hat{u}^{p-1}v\,dx - \int_{\{\hat{u}>\delta\}} c_0(1+\hat{u}^{q-1})v\,dx \\ &\geqslant \int_\Omega c_1\hat{u}^{p-1}v\,dx - c_0\int_{\{\hat{u}>\delta\}}\Big[\frac{1}{\delta^{p-1}} + \|\hat{u}\|_\infty^{q-p}\Big]\hat{u}^{p-1}v\,dx \\ &\geqslant -C\int_\Omega \hat{u}^{p-1}v\,dx\end{aligned}$$

for some $C > 0$. By Lemma 2.3.13 and (2.3.10) we have $\hat{u} \in \operatorname{int}(C_s^0(\overline{\Omega})_+)$, so there is $r > 0$ such that $u \in C_s^0(\overline{\Omega})_+$ for all $u \in C_s^0(\overline{\Omega})$ with $\|u - \hat{u}\|_{0,s} < r$. Now pick

$$0 < \varepsilon < \min\Big\{\frac{\delta}{\|\hat{u}\|_\infty}, \frac{r}{\|\hat{u}_1\|_{0,s}}\Big\}. \tag{7.2.4}$$

By (7.2.3) we have for all $v \in W_0^{s,p}(\Omega)_+$

$$\langle(-\Delta)_p^s\,(\varepsilon\hat{u}_1), v\rangle = \lambda_1\int_\Omega(\varepsilon\hat{u}_1)^{p-1}v\,dx \leqslant \int_\Omega f(x,\varepsilon\hat{u}_1)v\,dx,$$

hence $\varepsilon\hat{u}_1$ is a subsolution of (2.3.14). Besides,

$$\|(\hat{u} - \varepsilon\hat{u}_1) - \hat{u}\|_{0,s} = \varepsilon\|\hat{u}_1\|_{0,s} < r,$$

so $\hat{u} - \varepsilon\hat{u}_1 \in C_s^0(\overline{\Omega})_+$, in particular $\varepsilon\hat{u}_1 \leqslant \hat{u}$. Therefore $(\varepsilon\hat{u}_1, \hat{u})$ is a sub-supersolution pair of (2.3.14).

For all $n \in \mathbb{N}$ big enough, $\varepsilon = \frac{1}{n}$ satisfies (7.2.4). By Theorem 7.1.6, there exists

$$u_n = \min\mathcal{S}\Big(\frac{\hat{u}_1}{n}, \hat{u}\Big).$$

Clearly $(0, \hat{u})$ is a sub-supersolution pair of (2.3.14) and $u_n \in \mathcal{S}(0,\hat{u})$, so by Lemma 7.1.5, passing if necessary to a subsequence, we have $u_n \to u_+$ in $W_0^{s,p}(\Omega)$ for some $u_+ \in \mathcal{S}(0,\hat{u})$. On the other hand we have for all $n \in \mathbb{N}$

$$\mathcal{S}\Big(\frac{\hat{u}_1}{n}, \hat{u}\Big) \subseteq \mathcal{S}\Big(\frac{\hat{u}_1}{n+1}, \hat{u}\Big),$$

hence by minimality $u_{n+1} \leqslant u_n$. This in turn implies that $u_n(x) \to u_+(x)$ for a.e. $x \in \Omega$. Now, since $0 \leqslant u_n \leqslant \hat{u}$, we see that (u_n) is a bounded sequence in $L^\infty(\Omega)$, hence by $(\mathbf{H}_{7.2})$ *(i)* $(f(\cdot, u_n))$ is uniformly bounded as well. Then, since for all $n \in \mathbb{N}$

$$(-\Delta)_p^s\,u_n = f(\cdot, u_n) \text{ in } W^{-s,p'}(\Omega), \tag{7.2.5}$$

Lemmas 2.3.10, 2.3.11 imply that (u_n) is bounded in $C_s^\alpha(\overline{\Omega})$ as well. So, passing to a further subsequence, we have $u_n \to u_+$ in $C_s^0(\overline{\Omega})$.

We prove now that $u_+ \neq 0$, by contradiction. If $u_+ = 0$, then $u_n \to 0$ uniformly in $\overline{\Omega}$. Set

$$v_n = \frac{u_n}{\|u_n\|_{W_0^{s,p}(\Omega)}} \in W_0^{s,p}(\Omega)_+,$$

then by (7.2.5) we have for all $n \in \mathbb{N}$

$$(-\Delta)_p^s\, v_n = \frac{f(\cdot, u_n)}{\|u_n\|_{W_0^{s,p}(\Omega)}^{p-1}} = \frac{f(\cdot, u_n)}{u_n^{p-1}} v_n^{p-1} \text{ in } W^{-s,p'}(\Omega).$$

Set for all $n \in \mathbb{N}$

$$\rho_n = \frac{f(\cdot, u_n)}{u_n^{p-1}}.$$

By (7.2.3), for $n \in \mathbb{N}$ big enough we have $c_1 \leqslant \rho_n \leqslant c_2$ in Ω, in particular $\rho_n \in L^\infty(\Omega)$. Then $v_n \in W_0^{s,p}(\Omega) \setminus \{0\}$ is an eigenfunction of the (3.4.1)-type eigenvalue problem

$$(-\Delta)_p^s\, v_n = \lambda \rho_n v_n^{p-1} \text{ in } W^{-s,p'}(\Omega), \tag{7.2.6}$$

associated with the eigenvalue $\lambda = 1$. Since $\rho_n \geqslant c_1 > \lambda_1$, by Lemma 3.4.1 *(iii)* we have

$$\lambda_1(\rho_n) < \lambda_1(\lambda_1) = 1,$$

therefore v_n is a non-principal eigenfunction of (7.2.6). By Lemma 3.4.1 *(ii)* v_n is nodal, a contradiction. Hence, by Lemma 2.3.13 and (2.3.10) we have $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$.

Finally, we prove that u_+ is the smallest positive solution of (2.3.14). Let $u \in W_0^{s,p}(\Omega)_+ \setminus \{0\}$ be a solution of (2.3.14). Arguing as above we see that $u \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Set $w = u \wedge \hat{u} \in W_0^{s,p}(\Omega)_+$, then by Lemma 7.1.2 w is a supersolution of (2.3.14). As above, for all $n \in \mathbb{N}$ big enough we have that $\frac{\hat{u}_1}{n}$ is a subsolution of (2.3.14) and $\frac{\hat{u}_1}{n} \leqslant w$ in Ω, i.e., $(\hat{u}_1/n, w)$ is a sub-supersolution pair. Therefore, by Lemma 7.1.3 we can find

$$w_n \in \mathcal{S}\Big(\frac{\hat{u}_1}{n}, w\Big).$$

Since

$$\mathcal{S}\Big(\frac{\hat{u}_1}{n}, w\Big) \subseteq \mathcal{S}\Big(\frac{\hat{u}_1}{n}, \hat{u}\Big),$$

by minimality, for all $n \in \mathbb{N}$ big enough we have $u_n \leqslant w_n$, hence $u_n \leqslant u$. Passing to the limit as $n \to \infty$, we have $u_+ \leqslant u$.

Similarly we prove existence of the biggest negative solution $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$. $\square$

Remark 7.2.2. According to [105], most properties in Lemma 3.4.1 also hold if ρ lies in a special class $\widetilde{W}_p$ of *singular* weights, namely if $\rho\, \mathrm{d}_\Omega^{sa} \in L^r(\Omega)$ for some $a \in [0,1]$, $r > 1$ satisfying

$$\frac{1}{r} + \frac{a}{p} + \frac{p-a}{p_s^*} < 1.$$

So, in view of the proof of Theorem 7.2.1 above, a natural question is whether we may replace $(\mathbf{H}_{7.2})$ *(iii)* with the weaker condition

$$\liminf_{t \to 0} \frac{f(x,t)}{t^{p-1}} > \lambda_1 \text{ uniformly for a.e. } x \in \Omega.$$

Define $\rho_n = f(\cdot, u_n)/u_n^{p-1}$ as above, then recalling that $u_n \geqslant c\mathrm{d}_\Omega^s$ in $\overline{\Omega}$ we have

$$0 < \rho_n \leqslant C(1 + \mathrm{d}_\Omega^{-s(p+1)}).$$

Unfortunately, this *does not* ensure that $\rho_n \in \widetilde{W}_p$, in general. For instance, consider the case $\Omega = B_1(0)$, $\mathrm{d}_\Omega(x) = 1 - |x|$. Then we have $\mathrm{d}_\Omega^s \in L^\alpha(\Omega)$ iff $\alpha \in (0, 1)$. Therefore, $\rho_n \in \widetilde{W}_p$ implies

$$\begin{cases} sr(p-a-1) < 1 \\ \dfrac{1}{r} + \dfrac{a}{p} + \dfrac{p-a}{p_s^*} < 1, \end{cases}$$

in particular $(p-2)s < 1$. Yet, for special values of p, s, and a suitable domain Ω, analogues to Theorem 7.2.1 could be proved for reactions $f(x, \cdot)$ with a $(p-1)$-sublinear behavior near the origin.

7.3 Nodal solutions

In this section we present an application of our main result, following the ideas of [82] (see also [150, Theorem 11.26]). Applying Theorem 7.2.1, along with the mountain pass Theorem and spectral theory for $(-\Delta)_p^s$, we prove existence of a nodal solution of (2.3.14). Our hypotheses on the reaction f are the following:

($\mathbf{H}_{7.3}$) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, for all $(x, t) \in \Omega \times \mathbb{R}$ we set

$$F(x, t) = \int_0^t f(x, \tau)\, d\tau,$$

and the following conditions hold:

(*i*) $|f(x, t)| \le c_0(1 + |t|^{q-1})$ for all a.e. $x \in \Omega$ and all $t \in \mathbb{R}$ ($c_0 > 0$, $q \in (p, p_s^*)$);

(*ii*) $\displaystyle\limsup_{|t|\to\infty} \frac{F(x,t)}{|t|^p} < \frac{\lambda_1}{p}$ uniformly for a.e. $x \in \Omega$;

(*iii*) $\displaystyle\lambda_2 < \liminf_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} \leqslant \limsup_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} < \infty$ uniformly for a.e. $x \in \Omega$.

Here $\lambda_2 > \lambda_1$ denotes the second (variational) eigenvalue of $(-\Delta)_p^s$ in $W_0^{s,p}(\Omega)$, defined by (3.4.2). Again, we are assuming for $f(x, \cdot)$ a $(p-1)$-linear behavior near the origin.

We define the energy functional J as in Chapter 2. Now we show our following result via variational method.

Theorem 7.3.1. *Let* ($\mathbf{H}_{7.3}$) *hold. Then,* (2.3.14) *has a smallest positive solution* $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$*, a biggest negative solution* $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$*, and a nodal solution* $\tilde{u} \in C_s^0(\overline{\Omega})$ *such that* $u_- \leqslant \tilde{u} \leqslant u_+$ *in* Ω.

Proof. Clearly ($\mathbf{H}_{7.3}$) implies ($\mathbf{H}_{7.2}$). From Theorem 7.2.1, then, we know that (2.3.14) has a smallest positive solution $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ and a biggest negative solution $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$. Plus, by ($\mathbf{H}_{7.3}$) *(iii)*, 0 is a solution of (2.3.14). We are going to detect a fourth solution $\tilde{u} \in W_0^{s,p}(\Omega)$, and then show that it is nodal.

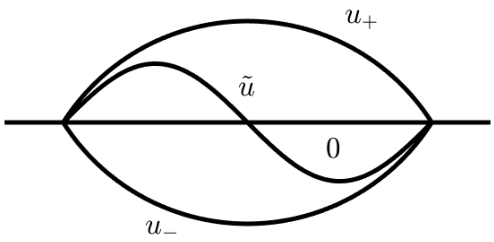

Figure 7.1: This figure represents graphically the statement of Theorem 7.3.1

Set for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}(x,t) = \begin{cases} f(x, u_-(x)) & \text{if } t < u_-(x) \\ f(x,t) & \text{if } u_-(x) \leqslant t \leqslant u_+(x) \\ f(x, u_+(x)) & \text{if } t > u_+ \end{cases}$$

and

$$\tilde{F}(x,t) = \int_0^t \tilde{f}(x,\tau)\, d\tau.$$

Since $u_\pm \in L^\infty(\Omega)$, $\tilde{f}$ satisfies $(\mathbf{H}_{7.1})$. Now set for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{J}(u) = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega \tilde{F}(x,u)\, dx.$$

By $(\mathbf{H}_{7.3})$ (i) (ii), reasoning as in the proof of Theorem 7.2.1 we see that $\tilde{J} \in C^1(W_0^{s,p}(\Omega))$ is coercive. As a consequence, $\tilde{J}$ satisfies (PS) (see [111, Proposition 2.1]). Whenever $u \in W_0^{s,p}(\Omega)$ is a critical point of $\tilde{J}$, then for all $v \in W_0^{s,p}(\Omega)$

$$\langle (-\Delta)_p^s\, u, v\rangle = \int_\Omega \tilde{f}(x,u) v\, dx. \tag{7.3.1}$$

By Lemmas 2.3.10, 2.3.11 we have $u \in C_s^0(\overline{\Omega})$. Besides, testing (7.3.1) with $(u-u_+)^+, -(u-u_-)^- \in W_0^{s,p}(\Omega)$ and arguing as in Lemma 7.1.3 we have $u_- \leqslant u \leqslant u_+$ in Ω, hence u solves (2.3.14) in Ω. Using the notation of Section 7.1, we can say that $u \in \mathcal{S}(u_-, u_+)$.

We introduce a further truncation setting for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{f}_+(x,t) = \tilde{f}(x,t^+), \quad \tilde{F}_+(x,t) = \int_0^t \tilde{f}_+(x,\tau)\, d\tau,$$

and for all $u \in W_0^{s,p}(\Omega)$

$$\tilde{J}_+(u) = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega \tilde{F}_+(x,u)\, dx.$$

Reasoning as above, we see that $\tilde{J}_+ \in C^1(W_0^{s,p}(\Omega))$ is coercive, and whenever $u \in W_0^{s,p}(\Omega)$ is a critical point of $\tilde{J}_+$ we have $u \in \mathcal{S}(0, u_+)$. By the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$, it is easily seen that $\tilde{J}_+$ is sequentially weakly lower semicontinuous, hence there exists $\tilde{u}_+ \in W_0^{s,p}(\Omega)$ such that

$$\tilde{J}_+(\tilde{u}_+) = \inf_{u \in W_0^{s,p}(\Omega)} \tilde{J}_+(u).$$

Arguing as in Theorem 7.2.1 we see that $\tilde{J}_+(\tilde{u}_+) < 0$, hence $\tilde{u}_+ \neq 0$. By $(\mathbf{H}_{7.3})$ *(iii)* and Lemma 2.3.13, we have $\tilde{u}_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. So, $\tilde{u}_+$ is a positive solution of (2.3.14), hence the minimality of u_+ implies $\tilde{u}_+ = u_+$. In particular, since $\tilde{J} = \tilde{J}_+$ in $C_s^0(\overline{\Omega})_+$, we see that $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$ is a local minimizer of $\tilde{J}$ in $C_s^0(\overline{\Omega})$. By Lemma 2.3.12, then u_+ is a local minimizer of $\tilde{J}$ in $W_0^{s,p}(\Omega)$ as well (recall that $\tilde{f}$ satisfies $(\mathbf{H}_{7.1})$).
Similarly we prove that $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$ is a local minimizer of $\tilde{J}$.
Now we argue by contradiction, assuming that there are no other critical points of $\tilde{J}$ than 0, u_+, and u_-, namely,

$$K_{\tilde{J}} = \{0, u_+, u_-\}. \tag{7.3.2}$$

In particular, both $u_\pm$ are strict local minimizers of $\tilde{J}$, which satisfies (PS). By the mountain pass Theorem ([150, Proposition 5.42], see also Theorem 1.1.3), there exists $\tilde{u} \in K_{\tilde{J}}^c$, where we have set

$$\Gamma = \big\{\gamma \in C([0,1], W_0^{s,p}(\Omega)) : \gamma(0) = u_+, \gamma(1) = u_-\big\},$$

and

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \tilde{J}(\gamma(t)) > \max\big\{\tilde{J}(u_+), \tilde{J}(u_-)\big\}.$$

In particular $\tilde{u} \neq u_\pm$, which by (7.3.2) implies $\tilde{u} = 0$ and hence $c = 0$. Set

$$\Sigma = \{u \in W_0^{s,p}(\Omega) \cap C_s^0(\overline{\Omega}) : \|u\|_p = 1\}.$$

By $(\mathbf{H}_{7.3})$ *(iii)* we can find $\mu > \lambda_2$, $\delta > 0$ such that for all $x \in \Omega$, $|t| \leqslant \delta$

$$F(x,t) \geqslant \frac{\mu}{p}|t|^p.$$

By definition of λ_2 (3.4.2) there is $\gamma_1 \in \Gamma_1$ such that

$$\max_{t\in[0,1]} \|\gamma_1(t)\|^p_{W_0^{s,p}(\Omega)} < \mu,$$

and by density we may assume $\gamma_1 \in C([0,1], \Sigma)$, continuous with respect to the $C_s^0(\overline{\Omega})$-norm (see [76] for details). Since $t \mapsto \|\gamma_1(t)\|_\infty$ is bounded in $[0,1]$, we can find $\varepsilon > 0$ such that $\|\varepsilon\gamma_1(t)\|_\infty \leqslant \delta$ for all $t \in [0,1]$.
Besides, taking $\varepsilon > 0$ even smaller if necessary, we have for all $t \in [0,1]$

$$u_+ - \varepsilon_t\gamma_1(t) \in \mathrm{int}(C_s^0(\overline{\Omega})_+), \quad u_- - \varepsilon_t\gamma_1(t) \in -\mathrm{int}(C_s^0(\overline{\Omega})_+),$$

in particular $u_- < \varepsilon\gamma_1(t) < u_+$ a.e. in Ω. So, for all $t \in [0,1]$ we get

$$\begin{aligned}
\tilde{J}(\varepsilon\gamma_1(t)) &= \frac{\varepsilon^p}{p}\|\gamma_1(t)\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega \tilde{F}(x, \varepsilon\gamma_1(t))\,dx \\
&\leqslant \frac{\varepsilon^p}{p}\|\gamma_1(t)\|^p_{W_0^{s,p}(\Omega)} - \frac{\mu\varepsilon^p}{p}\|\gamma_1(t)\|_p^p \\
&= \frac{\varepsilon^p}{p}(\|\gamma_1(t)\|^p_{W_0^{s,p}(\Omega)} - \mu) < 0.
\end{aligned}$$

Thus, $\varepsilon\gamma_1$ is a continuous path joining $\varepsilon\hat{u}_1$ to $-\varepsilon\hat{u}_1$, such that for all $t \in [0,1]$

$$\tilde{J}(\varepsilon\gamma_1(t)) < 0.$$

Besides, by (7.3.2) and Lemma 2.3.13 we have

$$K_{\tilde{J}_+} = \{0, u_+\},$$

Set $a = \tilde{J}_+(u_+)$, $b = \tilde{J}_+(\varepsilon\hat{u}_1)$, hence $a < b < 0$ and there is no critical level in $(a,b]$. Therefore, by the second deformation theorem [150, Theorem 5.34] there exists a continuous deformation $h : [0,1] \times \{\tilde{J}_+ \leqslant b\} \to \{\tilde{J}_+ \leqslant b\}$ such that for all $t \in [0,1]$, $\tilde{J}_+(u) \leqslant b$

$$h(0,u) = u, \quad h(1,u) = u_+, \quad \tilde{J}_+(h(t,u)) \leqslant \tilde{J}_+(u).$$

Set for all $t \in [0,1]$

$$\gamma_+(t) = h(t, \varepsilon\hat{u}_1)^+ \in W^{s,p}_0(\Omega)_+,$$

then $\gamma_+ \in C([0,1], W^{s,p}_0(\Omega))$ with $\gamma_+(0) = \varepsilon\hat{u}_1$, $\gamma(1) = u_+$, and for all $t \in [0,1]$

$$\tilde{J}(\gamma_+(t)) \leqslant b < 0.$$

Similarly we construct $\gamma_- \in C([0,1], W^{s,p}_0(\Omega))$ such that $\gamma_-(0) = -\varepsilon\hat{u}_1$, $\gamma(1) = u_-$, and for all $t \in [0,1]$

$$\tilde{J}(\gamma_-(t)) < 0.$$

Concatenating $\gamma_+, \varepsilon\gamma_1, \gamma_-$ we find a path $\gamma \in \Gamma$ such that for all $t \in [0,1]$

$$\tilde{J}(\gamma(t)) < 0,$$

hence $c < 0$, a contradiction. So, (7.3.2) is false, i.e., there exists $\tilde{u} \in K_{\tilde{J}} \setminus \{0, u_+, u_-\}$, so as ween above we have $\tilde{u} \in \mathcal{S}(u_-, u_+)$.

Finally, we prove that $\tilde{u}$ is nodal. Indeed, if $\tilde{u} \in W^{s,p}_0(\Omega)_+ \setminus \{0\}$, then by Lemma 2.3.13 we would have $\tilde{u} \in \mathrm{int}(C^0_s(\overline{\Omega})_+)$, along with $\tilde{u} \leqslant u_+$, which, by Theorem 7.2.1, would imply $\tilde{u} = u_+$, a contradiction. Similarly we see that $\tilde{u}$ cannot be negative.

Thus, $\tilde{u} \in C^0_s(\overline{\Omega}) \setminus \{0\}$ is a nodal solution of (2.3.14) such that $u_- \leqslant \tilde{u} \leqslant u_+$ a.e. in Ω. $\square$

Remark 7.3.2. The argument based on the characterization of λ_2 was already employed in [116, Theorem 4.1] and [76, Theorem 3.3] (for $p = 2$). The novelty of Theorem 7.3.1 above, with respect to such results (even for the linear case $p = 2$), lies in the detailed information about solutions, as we prove that $u_\pm$ are *extremal* constant sign solutions and $\tilde{u}$ is *nodal*. We also remark that the assumption $p \geqslant 2$ is essentially due to regularity theory (Lemma 2.3.11), but the arguments displayed in this chapter also work, with minor adjustments, for $p \in (1,2)$.

Chapter 8

Three nontrivial solutions for nonlocal anisotropic inclusions under resonance

In this chapter, we consider a Dirichlet problem for a pseudo-differential inclusion, driven by a nonlocal anisotropic integro-differential operator L_K defined as in (2.2.3), with the following form

$$\begin{cases} L_K u \in \partial F(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{8.0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$, $N > 2s$, $s \in (0,1)$ and $\partial F(x,\cdot)$ denotes the Clarke generalized subdifferential of a potential $F : \Omega \times \mathbb{R} \to \mathbb{R}$.

Problem (8.0.1) can be referred to as a pseudo-differential inclusion in Ω, coupled with a Dirichlet-type condition in Ω^c (due to the nonlocal nature of the operator L_K). The interest in extending the theory of nonlinear pseudo-differential equations to pseudo-differential inclusions can be motivated by the presence of discontinuous nonlinearities. In such case the corresponding energy functional may be nondifferentiable, but only locally Lipschitz continuous. Let us consider the equation $L_K u = f(x,u)$ in Ω and suppose that $f(x,u)$ is not a Carathéodory function, but it is only locally bounded in u. So a reasonable way to define solutions of such equations is to set $F(x,t) = \int_0^t f(x,\tau)\,d\tau$, then $F(x,\cdot)$ is locally Lipschitz and so it admits the Clarke generalized subdifferential. Under convenient growth assumption, we have $\partial F(x,t) \subseteq [f_-(x,t), f_+(x,t)]$, where

$$f_-(x,t) = \lim_{\delta\to 0^+} \operatorname{ess\,inf}_{|t-\tau|<\delta} f(x,\tau) \quad \text{and} \quad f_+(x,t) = \lim_{\delta\to 0^+} \operatorname{ess\,sup}_{|t-\tau|<\delta} f(x,\tau).$$

In particular, any smooth enough solution of the inclusion satisfies the equation at continuity points of f.

Our work stands at the conjunction of these two branches of research: nonsmooth problems studied in a variational perspective and nonlocal problems driven by fractional-type operators. Inspired by [119], we will extend to the anisotropic case their result about the existence of at least two constant sign solutions, by applying nonsmooth critical point theory. Moreover, we shall prove the existence of three nontrivial weak solutions for problem (8.0.1) (one positive, one negative and one with unknown sign) under the assumptions that the nonsmooth potential satisfies nonresonance conditions both at the origin and at infinity. In particular the existence of the third solution will require a nonsmooth version of the Sobolev vs. Hölder minimizers result.

Our existence result is, according to our knowledge, the first one for nonlocal problems involving anisotropic operators and set-valued reactions in higher dimension, while we should mention [197, 200] for the ordinary case (the first based on fixed point methods, the second on nonsmooth variational methods). We also recall an application of nonsmooth analysis to a single-valued nonlocal equation in [61].

The chapter has the following structure: in Section 8.1 we recall some useful results about regularity of solutions of inclusions, in particular we show the nonsmooth anisotropic principle of equivalence of minimizers and in Section 8.2 we prove our main result.

8.1 Preliminary results

In this section, we collect some results that will be used in our arguments.

8.1.1 A priori bound and regularity of weak solutions of inclusions

In this section we gather some useful results related to the nonlocal anisotropic operator L_K defined in (2.2.3). Now we consider integral functionals defined on L^2-spaces by means of locally Lipschitz continuous potentials.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a $C^{1,1}$ boundary, $N > 2s$, $s \in (0,1)$ and let F_0 be a potential satisfying the following:

($\mathbf{H}_{8.1}$) $F_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F_0(\cdot, 0) = 0$, $F_0(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $F_0(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover, there exists $a_0 > 0$ such that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial F_0(x,t)$

$$|\xi| \leqslant a_0 |t|.$$

We define for all $u \in L^2(\Omega)$ the functional

$$\hat{J}_0(u) = \int_\Omega F_0(x, u)\, dx, \tag{8.1.1}$$

and the set-valued Nemytzkij operator

$$N_0(u) = \{w \in L^2(\Omega) :\, w(x) \in \partial F_0(x, u(x)) \text{ for a.e. } x \in \Omega\}.$$

From [53, Theorem 2.7.5] we have the following lemma, that is a particular case of [119, Lemma 2.3].

Lemma 8.1.1. *If $F_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies ($\mathbf{H}_{8.1}$), then $\hat{J}_0 : L^2(\Omega) \to \mathbb{R}$, defined by (8.1.1), is Lipschitz continuous on any bounded subset of $L^2(\Omega)$. Moreover, for all $u \in L^2(\Omega)$, $w \in \partial \hat{J}_0(u)$ one has $w \in N_0(u)$.*

Now we consider the problem

$$\begin{cases} L_K u \in \partial F_0(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases} \tag{8.1.2}$$

where F_0 satisfies $(\mathbf{H}_{8.1})$.

Definition 8.1.2. *A function $u \in X_K(\Omega)$ is said to be a (weak) solution of (8.1.2) if there exists $w \in N_0(u)$ such that for all $v \in X_K(\Omega)$*

$$\langle L_K(u), v\rangle = \int_\Omega wv\,dx. \tag{8.1.3}$$

By the embedding of $X_K(\Omega)$ in $L^2(\Omega)$, we have that $L^2(\Omega)$ is embedded in $X_K(\Omega)^*$, so (8.1.3) can be rephrased by

$$L_K(u) = w \text{ in } X_K(\Omega)^*. \tag{8.1.4}$$

By means of (8.1.4), problem (8.0.1) may be seen as a pseudodifferential equation (with single-valued right hand side), to which we can apply most recent results from fractional calculus of variations. In [119, Lemma 2.5] the authors proved a uniform L^∞-bounds for the fractional p-Laplacian $(-\Delta)_p^s$, in particular this holds in the case $p = 2$, namely for the fractional Laplacian $(-\Delta)^s$. Using the previous fact and the embedding of $X_K(\Omega)$ in $H_0^s(\Omega)$, we obtain that

$$\|u\|_\infty \le C_0(1 + \|u\|_{H_0^s(\Omega)}) \le C(1 + \|u\|_{X_K(\Omega)}).$$

Hence, we have the following lemma.

Lemma 8.1.3. *If F_0 satisfies $(\mathbf{H}_{8.1})$, then there exists $C > 0$ such that for all solutions $u \in X_K(\Omega)$ of (8.1.2) one has $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \leqslant C(1 + \|u\|_{X_K(\Omega)}).$$

Lemma 8.1.4. *If F_0 satisfies $(\mathbf{H}_{8.1})$, then there exist $\alpha \in (0, s)$ and $C_1 > 0$ such that for all solutions $u \in X_K(\Omega)$ of (8.1.2) one has $u \in C_s^\alpha(\overline{\Omega})$ and*

$$\|u\|_{\alpha,s} \leqslant C_1\big(1 + \|u\|_{X_K(\Omega)}\big).$$

Proof. From Lemma 8.1.3, we obtain $u \in L^\infty(\Omega)$ such that $\|u\|_\infty \le C(1 + \|u\|_{X_K(\Omega)})$, with $C > 0$ independent of u. Let $w \in N_0(u)$ be as in Definition 8.1.2. Then by $(\mathbf{H}_{8.1})$, we have

$$\|w\|_\infty \le a_0\|u\|_\infty.$$

Now Proposition 2.3.4 and Theorem 2.3.6 imply $u \in C_s^\alpha(\overline{\Omega})$ and

$$\|u\|_{\alpha,s} \le (c_0 + c\|w\|_\infty) \le C_1\big(1 + \|u\|_{X_K(\Omega)}\big),$$

with $c_0, c, C_1 > 0$ independent of u. $\square$

The regularity C^s is the best result that we can obtain in the fractional framework, as was pointed out in [174] even for the fractional Laplacian. In particular, solutions do not, in general, admit an outward normal derivative at the points of $\partial\Omega$ and, for this reason, the Hopf property is stated in terms of a Hölder-type quotient (see [67] and Lemma 8.2.2 below).

8.1.2 Equivalence of minimizers in the two topologies

In the next theorem we prove an useful topological result, regarding the minimizers in the $X_K(\Omega)$-topology and in the $C_s^0(\overline{\Omega})$-topology, respectively. This is a nonsmooth anisotropic version of Theorem 2.3.8, for this reason here we give a sketch of the proof and we focus on the differences between the smooth and nonsmooth case.

Theorem 8.1.5. *(Hölder vs Sobolev minimizers) If F_0 satisfies $(\mathbf{H}_{8.1})$, then for all $u_0 \in X_K(\Omega)$ the following are equivalent:*

(i) there exists $\rho > 0$ such that $J(u_0 + v) \geq J(u_0)$ for all $v \in X_K(\Omega) \cap C_s^0(\overline{\Omega})$, $\|v\|_{0,s} \leq \rho$;

(ii) there exists $\epsilon > 0$ such that $J(u_0 + v) \geq J(u_0)$ for all $v \in X_K(\Omega)$, $\|v\|_{X_K(\Omega)} \leq \epsilon$.

Proof. Let J be the locally Lipschitz energy functional, defined as

$$J(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F_0(x, u(x))\, dx.$$

(i) $\Rightarrow$ (ii), Case $u_0 = 0$.
We point out that $J(0) = 0$, hence we can rewrite the hypothesis as

$$\inf_{u \in X_K(\Omega) \cap \overline{B}^s_\rho} J(u) = 0,$$

where $\overline{B}^s_\rho$ denotes the closed ball in $C_s^0(\overline{\Omega})$ centered at 0 with radius ρ.
We suppose by contradiction that (i) holds and that there exist a sequence $(\epsilon_n) \in (0, \infty)$ such that $\epsilon_n \to 0$ and for all $n \in N$ we have

$$\inf_{u \in \overline{B}^X_{\epsilon_n}} J(u) = m_n < 0,$$

where $\overline{B}^X_{\epsilon_n}$ denotes the closed ball in $X_K(\Omega)$ centered at 0 with radius ϵ_n.
Furthermore, the functional $u \mapsto \|u\|^2_{X_K(\Omega)}/2$ is convex, hence weakly lower semicontinuous in $X_K(\Omega)$, while $\hat{J}_0$ (defined in (8.1.1)) is continuous in $L^2(\Omega)$, which, by the compact embedding $X_K(\Omega) \hookrightarrow L^2(\Omega)$ and the Eberlein-Smulyan theorem, implies that $\hat{J}_0$ is sequentially weakly continuous in $X_K(\Omega)$. Hence, J is sequentially weakly lower semicontinuous in $X_K(\Omega)$. As a consequence, m_n is attained at some $u_n \in \overline{B}^X_{\epsilon_n}$ for all $n \in N$.
We state that, for all $n \in \mathbb{N}$, there exist $\mu_n \leq 0$, $w_n \in N(u_n)$ such that for all $v \in X_K(\Omega)$

$$\langle L_K(u_n), v\rangle - \int_\Omega w_n v\, dx = \mu_n \langle L_K(u_n), v\rangle\,. \tag{8.1.5}$$

Indeed, if $u_n \in B^X_{\epsilon_n}$, then u_n is a local minimizer of J in $X_K(\Omega)$, hence a critical point, so (8.1.5) holds with $\mu_n = 0$. If $u_n \in \partial B^X_{\epsilon_n}$, then u_n minimizes J restricted to the C^1- Banach manifold

$$\left\{ u \in X_K(\Omega) : \frac{\|u\|^2_{X_K(\Omega)}}{2} = \frac{\epsilon_n^2}{2} \right\},$$

so we can find a Lagrange multiplier $\mu_n \in \mathbb{R}$ such that (8.1.5) holds. More precisely, testing (8.1.5) with $-u_n$, we obtain

$$\langle B(u_n), -u_n\rangle := \langle L_K(u_n), -u_n\rangle - \int_\Omega w_n(-u_n)\, dx = -\mu_n \|u_n\|^2_{X_K(\Omega)},$$

where $B(u_n) \in X_K(\Omega)^*$, so recalling that $J(u) \geq J(u_n)$ for all $u \in B^X_{\epsilon_n}$, applying the definition of generalized subdifferential, the properties of the generalized directional derivative (see [95, Proposition 1.3.7]), and Lemma 1.3.1 *(vi)*, we get

$$\langle B(u_n), -u_n\rangle \geq J^0(u_n, -u_n) \geq 0,$$

hence $\mu_n \leq 0$.
Putting $C_n = (1-\mu_n)^{-1} \in (0,1]$, we obtain that for all $n \in \mathbb{N}$, $u_n \in X_K(\Omega)$ is a weak solution of the auxiliary boundary value problem

$$\begin{cases} L_K u_n = C_n w_n & \text{in } \Omega \\ u_n = 0 & \text{in } \Omega^c, \end{cases}$$

where $C_n w_n \in N(u_n)$ for all $n \in N$. By Lemma 8.1.3, $u_n \in L^\infty(\Omega)$, so by Lemma 8.1.4 we have $u_n \in C^\alpha_s(\overline{\Omega})$. Hence (u_n) is bounded in $C^\alpha_s(\overline{\Omega})$. By following exactly the proof of Theorem 2.3.8, we can conclude that for $n \in \mathbb{N}$ big enough $\|u_n\|_{0,s} \leq \rho$ together with $J(u_n) = m_n < 0$, a contradiction.

(i) $\Rightarrow$ (ii), Case $u_0 \neq 0$.
For all $v \in C^\infty_0(\Omega)$, we stress that in particular $v \in X_K(\Omega) \cap C^0_s(\overline{\Omega})$, so the minimality assures

$$\langle L_K(u_0), v\rangle = \int_\Omega w_0 v\, dx \text{ for some } w \in N_0(u), \text{ for all } v \in C^\infty_c(\Omega). \tag{8.1.6}$$

Since $C^\infty_0(\Omega)$ is dense in $X_K(\Omega)$ (see Proposition 2.1.7), and $L_K(u_0) \in X_K(\Omega)^*$, equality (8.1.6) holds for all $v \in X_K(\Omega)$, namely u_0 is a weak solution of (8.1.2). From Lemma 8.1.3, we get $u_0 \in L^\infty(\Omega)$, hence $w_0 \in L^\infty(\Omega)$. Applying Lemma 8.1.4, we have that $u_0 \in C^0_s(\overline{\Omega})$. We define for all $(x,t) \in \Omega \times \mathbb{R}$

$$\tilde{F}(x,t) = F(x, u_0(x)+t) - F(x, u_0(x)) - w_0(x)t,$$

and for all $v \in X_K(\Omega)$

$$\tilde{J}(v) = \frac{\|v\|^2_{X_K(\Omega)}}{2} - \int_\Omega \tilde{F}(x, v(x))\, dx,$$

where $\tilde{J}$ is locally Lipschitz, $\tilde{F}$ satisfies $(\mathbf{H}_{8.1})$ and $\tilde{w} \in \tilde{N}(v)$.
From this point forward, by reasoning exactly as in the proof of Theorem 2.3.8, we can find $\epsilon > 0$ such that for all $v \in X_K(\Omega)$, $\|v\|_{X_K(\Omega)} \leq \epsilon$, we get $\tilde{J}(v) \geq 0$, that is to say $J(u_0 + v) \geq J(u_0)$.

(ii) $\Rightarrow$ (i)
We argue by contradiction. We suppose that there exists a sequence (u_n) in $X_K(\Omega) \cap C_s^0(\overline{\Omega})$ such that $u_n \to u_0$ in $C_s^0(\overline{\Omega})$ and $J(u_n) < J(u_0)$.
We note that

$$\int_\Omega F(x, u_n)\, dx \to \int_\Omega F(x, u_0)\, dx \qquad \text{as } n \to \infty.$$

From this point on, the proof follows verbatim as in Theorem 2.3.8, providing that for $n \in \mathbb{N}$ big enough we have $\|u_n - u_0\|_{X_K(\Omega)} \leq \epsilon$, a contradiction. □

Using the hypothesis of nonresonance at infinity, we can show the coercivity of J, and this is fundamental to obtain the constant sign solutions of (8.0.1).

Lemma 8.1.6. *Let $\theta \in L^\infty(\Omega)_+$ be such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and $\varphi \in C^1(X_K(\Omega))$ be defined by*

$$\varphi(u) = \|u\|^2_{X_K(\Omega)} - \int_\Omega \theta(x)|u|^2\, dx.$$

Then there exists $\theta_0 \in (0, \infty)$ such that for all $u \in X_K(\Omega)$

$$\varphi(u) \geq \theta_0 \|u\|^2_{X_K(\Omega)}.$$

Proof. The claim follows from [119, Proposition 2.9] and recalling that $X_K(\Omega)$ is embedded in $H_0^s(\Omega)$. □

8.2 Application: a multiplicity result

In this section, we prove the existence of three nontrivial solutions of problem (8.0.1) (one positive, one negative and one of unknown sign), by means of the (nonsmooth) second deformation theorem and spectral theory. Precisely, on the nonsmooth potential F we will assume the following:

$(\mathbf{H}_{8.2})$ $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(\cdot, 0) = 0$, $j(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $F(x, \cdot)$ is locally Lipschitz continuous in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover,

- (i) for all $\rho > 0$ there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $|t| \leqslant \rho$, and all $\xi \in \partial F(x, t)$

$$|\xi| \leqslant a_\rho(x);$$

(*ii*) there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta \leqslant \lambda_1$, $\theta \not\equiv \lambda_1$, and uniformly for a.e. $x \in \Omega$

$$\limsup_{|t|\to\infty} \max_{\xi\in\partial F(x,t)} \frac{\xi}{t} \leqslant \theta(x);$$

(*iii*) there exist $\eta_1, \eta_2 \in L^\infty(\Omega)_+$, $\inf_\Omega \eta_1 > \lambda_2$ such that uniformly for a.e. $x \in \Omega$

$$\eta_1(x) \leqslant \liminf_{t\to 0} \min_{\xi\in\partial F(x,t)} \frac{\xi}{t} \leqslant \limsup_{t\to 0} \max_{\xi\in\partial F(x,t)} \frac{\xi}{t} \leqslant \eta_2(x);$$

(*iv*) for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial F(x,t)$

$$\xi t \geqslant 0.$$

Clearly, by hypothesis $(\mathbf{H}_{8.2})$, problem (8.0.1) always has the zero solution. The hypothesis $(\mathbf{H}_{8.2})$ (*ii*)-(*iii*) produce a nonresonance phenomenon both at infinity and at the origin, where we indicate with λ_1 and λ_2 the principal and the second eigenvalue of L_K with Dirichlet conditions in Ω (see Chapter 3).
Here we give an example of a potential satisfying $(\mathbf{H}_{8.2})$.

Example 8.2.1. Let $\theta, \eta \in L^\infty(\Omega)_+$ be such that $\theta < \lambda_1 < \lambda_2 < \eta$, and $F : \Omega \times \mathbb{R} \to \mathbb{R}$ be defined for all $(x,t) \in \Omega \times \mathbb{R}$ by

$$F(x,t) = \begin{cases} \dfrac{\eta(x)}{2}|t|^2 & \text{if } |t| \leqslant 1 \\[2ex] \dfrac{\theta(x)}{2}|t|^2 + \ln(|t|^2) + \dfrac{\eta(x)-\theta(x)}{2} & \text{if } |t| > 1. \end{cases}$$

As a first step we define two truncated, nonsmooth energy functionals, setting for all $u \in X_K(\Omega)$

$$J_\pm(u) = \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega F_\pm(x,u)\,dx,$$

where for all $(x,t) \in \Omega \times \mathbb{R}$

$$F_\pm(x,t) = F(x, \pm t^\pm), \text{ with } t^\pm = \max\{\pm t, 0\}.$$

Such functionals $J_\pm$ allow to find constant sign solutions of (8.0.1), as explained by the following lemma.

Lemma 8.2.2. *The functional $J_+ : X_K(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Moreover, if $u \in X_K(\Omega) \setminus \{0\}$ is a critical point of J_+, then $u \in C^\alpha_s(\overline{\Omega})$ is a solution of* (8.0.1) *such that*

(*i*) *$u(x) > 0$ for all $x \in \Omega$;*

(*ii*) *for all $y \in \partial\Omega$*

$$\liminf_{\substack{x\to y\\ x\in\Omega}} \frac{u(x)}{\mathrm{d}^s_\Omega(x)} > 0.$$

Analogously, the functional $J_- : X_K(\Omega) \to \mathbb{R}$ is locally Lipschitz continuous. Furthermore, if $u \in X_K(\Omega) \setminus \{0\}$ is a critical point of J_-, then $u \in C_s^\alpha(\overline{\Omega})$ is a solution of (8.0.1) such that

(i) $u(x) < 0$ for all $x \in \Omega$;

(ii) for all $y \in \partial\Omega$

$$\limsup_{\substack{x\to y\\ x\in\Omega}} \frac{u(x)}{\mathrm{d}_\Omega^s(x)} < 0.$$

Proof. By [119, Lemma 3.1, Lemma 3.2] this result holds in the case $p = 2$, namely for $(-\Delta)^s$. Exploiting the embedding of $X_K(\Omega)$ in $H_0^s(\Omega)$ and recalling the strong maximum principle (consequence of [173, Lemma 7.3]) and the Hopf lemma (see [173, Lemma 7.3]) for L_K we obtain the thesis. □

Now we can prove our main result, where Theorem 8.1.5 plays an essential part to relate critical points of $J_\pm$ with critical points of J.

Theorem 8.2.3. *If hypotheses ($\mathbf{H}_{8.2}$) hold, then problem (8.0.1) admits at least three nontrivial solutions $u_\pm \in \pm\mathrm{int}(C_s^0(\overline{\Omega})_+)$, and $\tilde{u} \in C_s^0(\overline{\Omega}) \setminus \{0\}$.*

Proof. We focus on the truncated functional J_+ and we show the existence of the positive solution, that will be a global minimizer of such functional. First of all, the generalized subdifferential $\partial F_+(x, \cdot)$ for all $t \in \mathbb{R}$ is given by

$$\begin{cases} \partial F_+(x,t) = \{0\} & \text{if } t < 0, \\ \partial F_+(x,t) \subseteq \{\mu\xi : \mu \in [0,1], \xi \in \partial F(x,0)\} & \text{if } t = 0, \\ \partial F_+(x,t) = \partial F(x,t) & \text{if } t > 0. \end{cases} \tag{8.2.1}$$

Exploiting ($\mathbf{H}_{8.2}$) *(ii)*, for any $\varepsilon > 0$ we can find $\rho > 0$ such that for a.e. $x \in \Omega$, all $t > \rho$ and all $\xi \in \partial F_+(x,t)$ we have

$$|\xi| \le (\theta(x) + \varepsilon)t$$

(we note that $\partial F_+(x,t) = \partial F(x,t)$ for $t > 0$). From ($\mathbf{H}_{8.2}$)*(i)* and using (8.2.1), there exists $a_\rho \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $t \le \rho$ and all $\xi \in \partial F_+(x,t)$

$$|\xi| \le a_\rho(x).$$

Hence, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial F_+(x,t)$ we obtain

$$|\xi| \le a_\rho(x) + (\theta(x) + \varepsilon)|t|. \tag{8.2.2}$$

From the Rademacher theorem and [53, Proposition 2.2.2], we know that for a.e. $x \in \Omega$ the mapping $F_+(x, \cdot)$ is differentiable for a.e. $t \in \mathbb{R}$ with

$$\frac{d}{dt} F_+(x,t) \in \partial F_+(x,t).$$

Hence, integrating and applying (8.2.2), we obtain for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$

$$F_+(x,t) \leq a_\rho(x)|t| + (\theta(x)+\varepsilon)\frac{|t|^2}{2}. \tag{8.2.3}$$

Applying (8.2.3), Proposition 3.1.2, Lemma 8.1.6, and the continuous embedding $X_K(\Omega) \hookrightarrow L^1(\Omega)$, for all $u \in X_K(\Omega)$ we have

$$\begin{aligned} J_+(u) &\geq \frac{\|u\|^2_{X_K(\Omega)}}{2} - \int_\Omega \Big(a_\rho(x)|u| + (\theta(x)+\varepsilon)\frac{|u|^2}{2}\Big)\,dx \\ &\geq \frac{1}{2}\Big(\|u\|^2_{X_K(\Omega)} - \int_\Omega \theta(x)|u|^2\,dx\Big) - \|a_\rho\|_\infty \|u\|_1 - \frac{\varepsilon}{2}\|u\|_2^2 \\ &\geq \frac{1}{2}\Big(\theta_0 - \frac{\varepsilon}{\lambda_1}\Big)\|u\|^2_{X_K(\Omega)} - c\|u\|_{X_K(\Omega)} \text{ for some } \ c>0. \end{aligned}$$

If we choose $\varepsilon \in (0, \theta_0\lambda_1)$ in the last term of the inequality, then $J_+(u)$ tends to $+\infty$ as $\|u\|_{X_K(\Omega)} \to \infty$, hence J_+ is coercive in $X_K(\Omega)$.
Furthermore, the functional $u \mapsto \|u\|^2_{X_K(\Omega)}/2$ is convex, so weakly lower semicontinuous in $X_K(\Omega)$, while $\hat{J}_+$ is continuous in $L^2(\Omega)$, which, by the compact embedding $X_K(\Omega) \hookrightarrow L^2(\Omega)$ and the Eberlein-Smulyan theorem, implies that $\hat{J}_+$ is sequentially weakly continuous in $X_K(\Omega)$. Hence, J_+ is sequentially weakly lower semicontinuous in $X_K(\Omega)$. Consequently, there exists $u_+ \in X_K(\Omega)$ such that

$$J_+(u_+) = \inf_{u \in X_K(\Omega)} J_+(u) =: m_+. \tag{8.2.4}$$

From Lemma 1.3.1 *(vi)*, u_+ is a critical point of J_+. We state that

$$m_+ < 0. \tag{8.2.5}$$

Indeed, by $(\mathbf{H}_{8.2})$ *(iii)*, for any $\varepsilon > 0$, we can find $\delta > 0$ such that for a.e. $x \in \Omega$, all $t \in [0,\delta)$, and all $\xi \in \partial F_+(x,t)$

$$\xi \geq (\eta_1(x) - \varepsilon)t.$$

Arguing as before, integrating we have

$$F_+(x,t) \geq \frac{\eta_1(x)-\varepsilon}{2}t^2. \tag{8.2.6}$$

Let $u_1 \in X_K(\Omega) \cap C^\alpha_s(\overline{\Omega})$ be the first eigenfunction. We can find $\mu > 0$ such that $0 < \mu u_1(x) \leq \delta$ for all $x \in \Omega$. Then, applying (8.2.6) and Proposition 3.1.2, we obtain

$$\begin{aligned} J_+(\mu u_1) &\leq \frac{\mu^2}{2}\|u_1\|^2_{X_K(\Omega)} - \frac{\mu^2}{2}\int_\Omega (\eta_1(x)-\varepsilon)u_1^2\,dx \\ &= \frac{\mu^2}{2}\Big(\int_\Omega (\lambda_1 - \eta_1(x))u_1^2\,dx + \varepsilon\Big). \end{aligned}$$

Using the fact that $\inf_\Omega \eta_1 > \lambda_2$ with $\lambda_2 > \lambda_1$, and that $u_1(x) > 0$ for all $x \in \Omega$, we get that

$$\int_\Omega (\lambda_1 - \eta_1(x))u_1^2\,dx < 0.$$

Hence, for $\varepsilon > 0$ small enough, the estimates above imply $J_+(\mu u_1) < 0$. Therefore, (8.2.5) is true.

Moreover, from (8.2.4) we obtain $u_+ \neq 0$. From Lemma 8.2.2 we have that $u_+ \in C_s^\alpha(\overline{\Omega})$, $u_+(x) > 0$ for all $x \in \Omega$, and

$$\liminf_{\substack{x \to y \\ x \in \Omega}} \frac{u_+(x)}{\mathrm{d}_\Omega^s(x)} > 0$$

for all $y \in \partial\Omega$, so we deduce $u_+ \in \mathrm{int}(C_s^0(\overline{\Omega})_+)$. Noting that $J \equiv J_+$ on $C_s^0(\overline{\Omega})_+$, we see that u_+ is a Hölder local minimizer of J, hence by Theorem 8.1.5, u_+ is as well a Sobolev local minimizer of J. In particular, $u_+ \in K_J$ is a positive solution of (8.0.1).

Working on J_- and recalling Lemma 8.2.2, we can find another solution $u_- \in C_s^\alpha(\overline{\Omega})$ such that $u_-(x) < 0$ for all $x \in \Omega$, and

$$\limsup_{\substack{x \to y \\ x \in \Omega}} \frac{u_-(x)}{\mathrm{d}_\Omega^s(x)} < 0$$

for all $y \in \partial\Omega$. Therefore $u_- \in -\mathrm{int}(C_s^0(\overline{\Omega})_+)$ and similarly u_- is a local minimizer of J.

We want to show the existence of another nontrivial solution, and in order to do it, first we observe that J is coercive. Now we show that J and $J_\pm$ satisfy the Palais-Smale condition. Let (u_n) be a bounded sequence in $X_K(\Omega)$ such that $(J(u_n))$ is bounded and $m_J(u_n) \to 0$. By Lemma 1.3.1 *(i)*, the definition of $m_J(u_n)$, and recalling that $\partial J(u_n) \subset L_K(u_n) - N(u_n)$ for all $n \in \mathbb{N}$, there exists $w_n \in N(u_n)$ such that $m_J(u_n) = ||L_K(u_n) - w_n||_*$. Due to reflexivity of $X_K(\Omega)$ and the compact embedding $X_K(\Omega) \to L^2(\Omega)$, passing if necessary to a subsequence, we have $u_n \rightharpoonup u$ in $X_K(\Omega)$ and $u_n \to u$ in $L^2(\Omega)$ for some $u \in X_K(\Omega)$. Besides, by $(\mathbf{H}_{8.1})$ we see that (w_n) is bounded in $L^2(\Omega)$. By what was stated above, we have

$$\begin{aligned}
\|u_n - u\|_{X_K(\Omega)}^2 &= \langle u_n, u_n - u\rangle - \langle u, u_n - u\rangle \\
&= \langle L_K(u_n) - w_n, u_n - u\rangle + \int_\Omega w_n(u_n - u)\,dx - \langle L_K(u), u_n - u\rangle \\
&\leq m_J(u_n)\|u_n - u\|_{X_K(\Omega)} + \|w_n\|_2\|u_n - u\|_2 - \langle L_K(u), u_n - u\rangle
\end{aligned}$$

for all $n \in \mathbb{N}$ and the latter tends to 0 as $n \to \infty$. Thus, $u_n \to u$ in $X_K(\Omega)$.

From $(\mathbf{H}_{8.2})$, we have $0 \in K_J$, while from the first part of the proof we already know that $u_\pm \in K_J \setminus \{0\}$. By contradiction, we suppose there is no more critical point $\tilde{u} \in X_K(\Omega)$, which means

$$K_J = \{0, u_+, u_-\}. \tag{8.2.7}$$

Without loss of generality, we assume that $J(u_+) \geq J(u_-)$ and that u_+ is a strict local minimizer of J, so we can find $r \in (0, \|u_+ - u_-\|_{X_K(\Omega)})$ such that $J(u) > J(u_+)$ for all

$u \in X_K(\Omega)$ and $0 < \|u - u_+\|_{X_K(\Omega)} \leq r$. Furthermore, we have

$$\eta_r = \inf_{\|u-u_+\|_{X_K(\Omega)}=r} J(u) > J(u_+). \tag{8.2.8}$$

We could also find a sequence (u_n) in $X_K(\Omega)$ such that $\|u_n - u_+\|_{X_K(\Omega)} = r$ for all $n \in \mathbb{N}$, $J(u_n) \to J(u_+)$ and $m_J(u_n) \to 0$. Then by Palais - Smale condition, we would have $u_n \to \overline{u}$ in $X_K(\Omega)$ for some $\overline{u} \in X_K(\Omega)$ and $\|\overline{u} - u_+\|_{X_K(\Omega)} = r$, hence in turn $J(\overline{u}) = J(u_+)$, which is a contradiction.
Now we introduce

$$\Gamma = \{\gamma \in C([0,1], X_K(\Omega)) : \gamma(0) = u_+, \gamma(1) = u_-\} \text{ and } c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t)).$$

From Theorem 1.3.2, we have $c \geq \eta_r$ and there exists $\tilde{u} \in K_J^c$. By (8.2.8), $\tilde{u} \neq u_\pm$. Hence, from (8.2.7) we deduce that $\tilde{u} = 0$, so $c = 0$. In order to achieve a contradiction, we will construct a path $\gamma \in \Gamma$ such that

$$\max_{t\in[0,1]} J(\gamma(t)) < 0, \tag{8.2.9}$$

so that $c < 0$.
Let $0 < \eta_1' < \eta_1(x)$ and $\tau > 0$ be such that

$$\eta_1' > \lambda_2 + \tau. \tag{8.2.10}$$

By $(\mathbf{H}_{8.2})$ *(iii)*, there exists $\sigma > 0$ such that $F(x,t) > \eta_1' \frac{|t|^2}{2}$ for a.e. in Ω and all $|t| \leq \sigma$. Moreover, by definition of λ_2 (3.1.5), there exists $\gamma_1 \in \Gamma_1$ such that

$$\max_{t\in[0,1]} \|\gamma_1(t)\|^2_{X_K(\Omega)} < \lambda_2 + \tau. \tag{8.2.11}$$

Since $C_0^\infty(\Omega)$ is dense in $X_K(\Omega)$ (see Proposition 2.1.7), we can picking out $\gamma_1(t) \in L^\infty(\Omega)$ for all $t \in [0,1]$ and γ_1 continuous with respect to the L^∞-topology. Hence, by choosing $\tilde{\mu} > 0$ small enough, we have $||\tilde{\mu}\gamma_1(t)||_\infty \leq \sigma$ for all $t \in [0,1]$. We define $\tilde{\gamma}(t) = \tilde{\mu}\gamma_1(t)$. Therefore, by (8.2.11) and recalling that $||\gamma_1(t)||_2 = 1$ (by definition of λ_2 (3.1.5)), we get for all $t \in [0,1]$ that

$$J(\tilde{\gamma}(t)) \leq \frac{\tilde{\mu}^2}{2}\|\gamma_1(t)\|^2_{X_K(\Omega)} - \int_\Omega \eta_1' \frac{\tilde{\mu}^2}{2}|\gamma_1(t)|^2\,dx \leq \frac{\tilde{\mu}^2}{2}(\lambda_2 + \tau - \eta_1') < 0,$$

and the latter is negative by (8.2.10). Then $\tilde{\gamma}$ is a continuous path joining $\tilde{\mu}u_1$ and $-\tilde{\mu}u_1$ such that

$$\max_{t\in[0,1]} J(\tilde{\gamma}(t)) < 0. \tag{8.2.12}$$

By $(\mathbf{H}_{8.2})$ *(iv)* and Lemma 8.2.2, we see that $K_{J_+} \subset K_J$, actually, by (8.2.7), we get $K_{J_+} = \{0, u_+\}$. We fix $a = J_+(u_+)$ and $b = 0$, in this way $J_+^a = \{u_+\}$ and J_+ fulfill all the hypothesis of Theorem 1.3.3, so there exists a continuous deformation $h_+ : [0,1] \times (J_+^0 \setminus \{0\}) \to (J_+^0 \setminus \{0\})$ such that

$$\begin{cases} h_+(0,u) = u, \quad h_+(1,u) = u_+ & \text{for all } u \in (J_+^0 \setminus \{0\}), \\ h_+(t,u_+) = u_+ & \text{for all } t \in [0,1], \\ t \mapsto J_+(h_+(t,u)) & \text{is decreasing for all } u \in (J_+^0 \setminus \{0\}). \end{cases}$$

Moreover, the set $J_+^0 \setminus \{0\}$ is contractible. We define

$$\gamma_+(t) = h_+(t, \tilde{\mu} u_1)$$

for all $t \in [0,1]$. Then $\gamma_+ \in C([0,1], X_K(\Omega))$ is a path joining $\tilde{\mu} u_1$ and u_+, such that $J_+(\gamma_+(t)) < 0$ for all $t \in [0,1]$. Observing that $J(u) \leq J_+(u)$ for all $u \in X_K(\Omega)$, we get

$$J_+(u) - J(u) = \int_\Omega (F(x,u) - F_+(x,u))\, dx = \int_{\{u<0\}} F(x,u)\, dx,$$

and the latter is non-negative by $(\mathbf{H}_{8.2})$ *(iv)*. Hence we obtain

$$\max_{t \in [0,1]} J(\gamma_+(t)) < 0. \tag{8.2.13}$$

In the same way, we construct a path $\gamma_- \in C([0,1], X_K(\Omega))$ joining $-\tilde{\mu} u_1$ and u_-, such that

$$\max_{t \in [0,1]} J(\gamma_-(t)) < 0. \tag{8.2.14}$$

Concatenating $\gamma_+, \tilde{\gamma}$ and γ_- (with a convenient changes of parameter) and using (8.2.12)-(8.2.14), we construct a path $\gamma \in \Gamma$ satisfying (8.2.9), against (8.2.7) and the definition of the mountain pass level c.
Hence, we deduce that there exists a fourth critical point $\tilde{u} \in K_J \setminus \{0, u_+, u_-\}$, that is a nontrivial solution of (8.0.1). □

Remark 8.2.4. A similar result can be proved for a fractional p-Laplacian inclusion

$$\begin{cases} (-\Delta)_p^s u \in \partial F(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \Omega^c, \end{cases}$$

studied in [119], by extending the result of [116] to locally Lipschitz functionals and applying the characterization of λ_2 from [29]. Hence, besides the existence of a positive solution and a negative solution, the problem above should admit a third nontrivial solution.

Chapter 9

The obstacle problem at zero for the fractional p-Laplacian

In this chapter we present a multiplicity result for the obstacle problem at zero driven by the fractional p-Laplacian operator

$$(9.0.1)\qquad \begin{cases} \langle(-\Delta)_p^s u, v-u\rangle \geq \int_\Omega w(x)(v(x)-u(x))\,dx \quad \text{for all } v \in W_0^{s,p}(\Omega)_+,\\ w(x) \in N(u) = \{\tilde{w} \in L^{p'}(\Omega) : \tilde{w}(x) \in \partial F(x,u(x)) \text{ for a.e. } x \in \Omega\},\\ u \in W_0^{s,p}(\Omega)_+, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $s \in (0,1)$, $p \in (1,\infty)$ such that $N > ps$, is a bounded domain with a $C^{1,1}$ boundary $\partial\Omega$ and F is a nonsmooth potential that satisfies suitable assumptions.
Using a combination of degree theory, based on the degree map for specific multivalued perturbations of $(S)_+$- nonlinear operators (see [1, 107]), and variational methods, we are able to prove that problem (9.0.1) admits at least two nontrivial solutions.
A natural obstacle problem is given by an elastic membrane, with vertical movement u on a domain Ω, which is bound to its boundary ($u = 0$ along $\partial\Omega$) and it is forced to stay below some obstacle ($u \geq \gamma$). Afterwards, at the equilibrium, everytime the membrane does not come into contact with the obstacle, the elasticity provides a balance of the tension of the membrane, that, geometrically, reflects into a balance of the principal curvatures of the surface described by u. At the same time, whenever the membrane sticks to the obstacle, its principal curvatures are supposed to adapt to those of γ. In addition, when an external force w appears, the elastic tension of the membrane will balance up the force. These physical arguments are reflected in the following variational inequality in the case of Laplacian operator

$$(9.0.2)\qquad \int_\Omega \nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geq \int_\Omega w(x)(v(x)-u(x))\,dx$$

for any test function v, with $v \geq \gamma$ and $v = 0$ along $\partial\Omega$ (see [184]). While in the case of p-Laplacian operator, looking at nonlinear elastic reactions of the membrane, the inequality becomes the following

$$\int_\Omega |\nabla u|^{p-2}\nabla u(x)(\nabla v(x) - \nabla u(x))\,dx \geq \int_\Omega w(x)(v(x)-u(x))\,dx$$

with $p \in (1, \infty)$ (see [1, 49, 171]). Likewise, one may take into account the long range interactions of particles, changing the local elastic reaction in (9.0.2) with a nonlocal one, for example substituting the Laplacian with the fractional Laplacian, hence (9.0.2) becomes the following nonlocal variational inequality

$$\iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(u(x) - u(y))(v(x) - v(y) - u(x) + u(y))}{|x - y|^{N+2s}} \, dxdy \geq \int_\Omega w(x)(v(x) - u(x)) \, dx.$$

These type of obstacle problems have been intensively investigated in [42, 144, 188] and in [124, 140, 141, 184] for other integrodifferential kernels.

This chapter has the following structure: in Section 9.1 we introduce the obstacle problem at zero and we collect preliminary results, in Section 9.2 we compute the degree of a suitable operator in an isolated minimizer, and in small and big balls, finally in Section 9.3 we show our multiplicity result.

9.1 Preliminary results

In this section, we collect preliminary results, which are useful to prove our main result. Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a $C^{1,1}-$boundary $\partial\Omega$, $p > 1$ and $s \in (0, 1)$ are real numbers such that $N > ps$, we consider the following obstacle problem at 0

$$\begin{cases} \langle (-\Delta)_p^s u, v - u \rangle \geq \int_\Omega w(x)(v(x) - u(x)) \, dx \quad \text{for all } v \in W_0^{s,p}(\Omega)_+, \\ w(x) \in N(u) = \{\tilde{w} \in L^{p'}(\Omega) : \, \tilde{w}(x) \in \partial F(x, u(x)) \text{ for a.e. } x \in \Omega\}, \\ u \in W_0^{s,p}(\Omega)_+. \end{cases}$$

We assume the following hypotheses on the nonsmooth potential.

($\mathbf{H}_{9.1}$) $F : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $F(\cdot, 0) = 0$ a.e. on Ω, $F(\cdot, t)$ is measurable in Ω for all $t \in \mathbb{R}$, $F(x, \cdot)$ is locally Lipschitz in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover

(*i*) $|\xi| \leq a(x) + c|t|^{p-1}$ with $a \in L^\infty(\Omega)_+$, $c > 0$, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial F(x, t)$,

(*ii*) there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and

$$0 \leqslant \liminf_{t \to +\infty} \frac{\xi}{t^{p-1}} \leqslant \limsup_{t \to +\infty} \frac{\xi}{t^{p-1}} \leqslant \theta(x)$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \partial F(x, t)$;

(*iii*) there exist $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that $\lambda_1 \leq \eta$, $\eta \not\equiv \lambda_1$, and

$$\eta(x) \leqslant \liminf_{t \to 0^+} \frac{\xi}{t^{p-1}} \leqslant \limsup_{t \to 0^+} \frac{\xi}{t^{p-1}} \leqslant \hat{\eta}(x)$$

uniformly for a.e. $x \in \Omega$ and all $\xi \in \partial F(x, t)$.

Remark 9.1.1. We denote by λ_1 the first eigenvalue of $(-\Delta)_p^s$ with Dirichlet conditions in Ω (see Chapter 3), hence $(\mathbf{H}_{9.1})$ (ii)-(iii) invoke nonuniform nonresonance conditions at $+\infty$ and at 0^+. The condition at $+\infty$ is from below λ_1 and the condition at 0^+ is from above with respect to λ_1.

Example 9.1.2. A nonsmooth locally Lipschitz potential satisfying hypotheses $(\mathbf{H}_{9.1})$ is defined as follows, which for simplicity we dropped the $x-$dependence:

$$F(t) = \begin{cases} \frac{\eta}{p}|t|^p - \frac{1}{p}\cos|t|^p & \text{if } |t| \leq 1, \\ \frac{\theta}{p}|t|^p + \frac{\eta-\theta}{p} - \frac{1}{p}\cos 1 & \text{if } |t| > 1, \end{cases}$$

with $\theta < \lambda_1 < \eta$.

Now we define the integral functional $\widehat{J} : L^p(\Omega) \to \mathbb{R}$ by

$$\widehat{J}(u) = \int_\Omega F(x, u(x))\, dx \text{ for all } u \in L^p(\Omega). \tag{9.1.1}$$

From $(\mathbf{H}_{9.1})$ (i) such functional $\widehat{J}$ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see [95, Theorem 1.3.10]).
Let $N : L^p(\Omega) \to 2^{L^{p'}(\Omega)}$ be defined by

$$N(u) = \{w \in L^{p'}(\Omega) : w(x) \in \partial F(x, u(x)) \text{ a.e. on } \Omega\},\ u \in L^p(\Omega).$$

Let us mention an important result about N, for the proof of the following proposition we refer to [1, Proposition 3, Corollary 4].

Proposition 9.1.3. *Let* $(\mathbf{H}_{9.1})$ (i) *hold. Therefore*

(i) *N has nonempty, weakly compact and convex values in $L^{p'}(\Omega)$ and it is upper semi-continuous from $L^p(\Omega)$ with the norm topology into $L^{p'}(\Omega)$ with the weak topology.*

(ii) *Moreover, $N : W_0^{s,p}(\Omega) \to 2^{W^{-s,p'}(\Omega)} \setminus \{\varnothing\}$ is a multifunction of class (P).*

For the second point we take into account that $W_0^{s,p}(\Omega)$ is embedded compactly and densely in $L^p(\Omega)$, and $L^{p'}(\Omega)$ is embedded compactly and densely in $W^{-s,p'}(\Omega)$.

Now we can introduce the Euler functional associated to problem (9.0.1), which is given for $u \in W_0^{s,p}(\Omega)$ by

$$J : W_0^{s,p}(\Omega) \to \mathbb{R} \cup \{+\infty\} \qquad J(u) = \tilde{J}(u) + I(u)$$

where

$$\tilde{J}(u) = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega F(x, u(x))\, dx$$

and

$$I(u) = i_{W_0^{s,p}(\Omega)_+}(u) = \begin{cases} 0 & \text{if } u \in W_0^{s,p}(\Omega)_+ \\ +\infty & \text{if } u \notin W_0^{s,p}(\Omega)_+. \end{cases}$$

From $(\mathbf{H}_{9.1})$ *(i)*, $\tilde{J}$ is locally Lipschitz (see [95, Theorem 1.3.10]). Furthermore, $W_0^{s,p}(\Omega)_+ \subseteq W_0^{s,p}(\Omega)$ is closed, convex, hence $I \in \Gamma_0(W_0^{s,p}(\Omega))$.
Throughout this chapter, we denote by $\partial \tilde{J}$ the Clarke generalized subdifferential of $\tilde{J}$ and by ∂I the subdifferential of I in the sense of convex analysis.

The next Lemma emphasizes the importance of the hypothesis $(\mathbf{H}_{9.1})$ *(ii)* (for the proof we refer to [119, Proposition 2.9]).

Lemma 9.1.4. *Let $\theta \in L^\infty(\Omega)_+$ be such that $\theta \leqslant \lambda_1$, $\theta \not\equiv \lambda_1$, and $\tau \in C^1(W_0^{s,p}(\Omega))$ be defined by*

$$\tau(u) = \|u\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega \theta(x)|u|^p \, dx.$$

Then there exists $\theta_0 \in (0, \infty)$ such that for all $u \in W_0^{s,p}(\Omega)$

$$\tau(u) \geqslant \theta_0 \|u\|^p_{W_0^{s,p}(\Omega)}.$$

The next proposition shows the existence of a minimizer, which belongs to $W_0^{s,p}(\Omega)_+$.

Proposition 9.1.5. *Let $(\mathbf{H}_{9.1})$ (i)-(ii) hold, then there exists $u_0 \in W_0^{s,p}(\Omega)_+$ such that*

$$J(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} J(u).$$

Proof. By $(\mathbf{H}_{9.1})$ *(ii)*, given $\epsilon > 0$, there exists $M_\epsilon > 0$ such that for a.e. $x \in \Omega$, all $t \geq M_\epsilon$ and all $\xi \in \partial F(x, t)$, we obtain

$$\xi \leq (\theta(x) + \epsilon) t^{p-1}. \tag{9.1.2}$$

Moreover, by $(\mathbf{H}_{9.1})$ *(i)*, we can find $\beta_\epsilon \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $t \in [0, M_\epsilon]$ and all $\xi \in \partial F(x, t)$, we get

$$|\xi| \leq \beta_\epsilon(x). \tag{9.1.3}$$

By Rademacher's theorem for a.e. $x \in \Omega$, $F(x, \cdot)$ is differentiable almost everywhere and

$$\frac{d}{dr} F(x, r) \in \partial F(x, r).$$

Therefore, for a.e. $x \in \Omega$ and for all $t \geq 0$, we have

$$\begin{aligned} F(x,t) &= \int_0^t \frac{d}{dr} F(x,r) dr \\ &\leq \int_0^t [(\theta(x) + \epsilon) r^{p-1} + \beta_\epsilon(x)] \, dr \quad \text{(by (9.1.2), (9.1.3))} \\ &= \frac{1}{p} (\theta(x) + \epsilon) t^p + \beta_\epsilon(x) t. \end{aligned} \tag{9.1.4}$$

We stress that J coincides with $\tilde{J}$ for all $u \in W_0^{s,p}(\Omega)_+$, since $I(u) = 0$. Moreover, by (9.1.4) we have for every $u \in W_0^{s,p}(\Omega)_+$

$$\begin{aligned}
J(u) &= \frac{1}{p}\|u\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega F(x, u(x))\, dx \\
&\geqslant \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \int_\Omega \Big(\beta_\epsilon(x)u + (\theta(x) + \epsilon)\frac{|u|^p}{p}\Big)\, dx \\
&\geqslant \frac{1}{p}\Big(\|u\|^p_{W_0^{s,p}(\Omega)} - \int_\Omega \theta(x)|u|^p\, dx\Big) - \|\beta_\epsilon\|_\infty \|u\|_1 - \frac{\epsilon}{p}\|u\|_p^p \\
&\geqslant \frac{1}{p}\Big(\theta_0 - \frac{\epsilon}{\lambda_1}\Big)\|u\|^p_{W_0^{s,p}(\Omega)} - c\|u\|_{W_0^{s,p}(\Omega)} \ (\theta_0, c > 0),
\end{aligned}$$

where in the final passage we have used Lemma 9.1.4, and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$. If we choose $\epsilon \in (0, \theta_0\lambda_1)$, the latter tends to $+\infty$ as $\|u\|_{W_0^{s,p}(\Omega)} \to \infty$, hence J is coercive in $W_0^{s,p}(\Omega)$.
Moreover, recalling the definition of J, the functional $u \mapsto \|u\|^p_{W_0^{s,p}(\Omega)}/p$ is convex, hence weakly lower semicontinuous in $W_0^{s,p}(\Omega)$, while $\widehat{J}$ is continuous in $L^p(\Omega)$, which, by the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the Eberlein-Smulyan theorem, implies that $\widehat{J}$ is sequentially weakly continuous in $W_0^{s,p}(\Omega)$. Hence, J is sequentially weakly lower semicontinuous on $W_0^{s,p}(\Omega)$. Therefore, by the Weierstrass theorem, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that $J(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} J(u)$. $\square$

Remark 9.1.6. By Proposition 9.1.5 we observe that u_0 is a minimizer of $\tilde{J}$, hence, by Lemma 1.3.1 *(vi)* $0 \in \partial\tilde{J}(u_0)$, i.e. there exists $w \in N(u_0)$ such that $(-\Delta)_p^s u_0 = w$ in $W^{-s,p'}(\Omega)$. By [119, Definition 2.4] u_0 is a weak solution of $(-\Delta)_p^s u \in \partial F(x, u)$ in Ω, $u = 0$ in Ω^c. Moreover, exploiting $(\mathbf{H}_{9.1})$ *(i)* and *(iii)*, and arguing as in the proof of Proposition 9.2.2, we deduce that

$$|\xi| \leq c_1|t|^{p-1} \text{ for some } c_1 > 0,$$

for a.a. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial F(x, t)$. Therefore, from [119, Lemma 2.5], we obtain that $u_0 \in L^\infty(\Omega)$, hence, $w \in L^\infty(\Omega)$. By [119, Lemma 2.7] there exist $\alpha \in (0, s]$, $C > 0$ such that $u_0 \in C^\alpha(\overline{\Omega})$ with $\|u_0\|_{C^\alpha(\overline{\Omega})} \leq C(1 + \|u_0\|_{W_0^{s,p}(\Omega)})$. In particular, by [115, Theorem 1.1], if $p \geq 2$ then $u_0 \in C_s^\alpha(\overline{\Omega})$ and it holds the following estimate $\|u_0\|_{\alpha,s} \leq C(1 + \|u_0\|_{W_0^{s,p}(\Omega)})$.

9.2 Degree in a point of minimum, and in small and big balls

In order to show our multiplicity result, we need to prove some facts about the degree theory, extending the results proved in the nonlinear local case in [1].
A fundamental result for the sequel is a generalization of Amann's theorem to operators which are the sum of a Clarke generalized subdifferential and a subdifferential in the sense of convex analysis, that allow us to know the degree in an isolated local minimum (see [1, Theorem 8]). In order to do this, it is better clarifying some important facts. First

of all, we observe that $(-\Delta)_p^s$ is the Fréchet derivative of $u \mapsto \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p}$, viewed as a functional on $W_0^{s,p}(\Omega)$, moreover we know by Lemma 3.4.2 that $(-\Delta)_p^s$ is a bounded, $(S)_+$ operator. We set $\bar{J} = \widehat{J}\,|_{W_0^{s,p}(\Omega)}$ and $\tilde{J} = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \bar{J}$, then it makes sense to talk about the degree of $\partial\tilde{J} = (-\Delta)_p^s - N$ with

$$N = \partial\bar{J} = \partial\widehat{J}$$

(see [95, Proposition 1.3.17], for the last equality). Now we can state the extension of Amann's theorem for our problem.

Proposition 9.2.1. *Let $\tilde{J} : W_0^{s,p}(\Omega) \to \mathbb{R}$ $\tilde{J}(u) = \frac{\|u\|^p_{W_0^{s,p}(\Omega)}}{p} - \bar{J}(u)$ be locally Lipschitz and $I \in \Gamma_0(W_0^{s,p}(\Omega))$, $I \geq 0$. If $u_0 \in W_0^{s,p}(\Omega)$ is an isolated minimizer of $\tilde{J} + I$, then there exists $r > 0$ such that*

$$\deg(\partial\tilde{J} + \partial I, B_r(u_0), 0) = 1.$$

Now, exploiting the hypothesis $(\mathbf{H}_{9.1})$ *(iii)*, we prove that for small balls the degree map of $\partial\tilde{J} + \partial I$ is equal to -1.

Proposition 9.2.2. *Let $(\mathbf{H}_{9.1})$ hold. Then there exists $\rho_0 > 0$ such that for all $0 < \rho \leq \rho_0$, we obtain*

$$\deg(\partial\tilde{J} + \partial I, B_\rho(0), 0) = -1.$$

Proof. Let $m \in L^\infty(\Omega)_+$ be such that $\eta(x) \leq m(x) \leq \widehat{\eta}(x)$ a.e. on Ω. Let look at the homotopy $h : [0,1] \times W_0^{s,p}(\Omega) \to 2^{W^{-s,p'}(\Omega)} \setminus \{\varnothing\}$ defined by

$$h(t,u) = (-\Delta)_p^s u - tN(u) - (1-t)K_m(u) + t\partial I(u).$$

From Proposition 9.1.3 and Lemma 3.4.2 (i)-(ii), we obtain that $h_1(t,u) = (-\Delta)_p^s u - (1-t)K_m(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a $(S)_+-$ homotopy, $h_2(t,u) = -tN(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a $(P)-$ homotopy and $h_3(t,u) = t\partial I(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a pseudomonotone homotopy (see [1]), hence h is an admissible homotopy (see Chapter 1).

Claim: There exists $\rho_0 > 0$ such that for all $t \in [0,1]$, all $0 < \rho \leq \rho_0$ and all $u \in \partial B_\rho(0) \subseteq W_0^{s,p}(\Omega)$ we get

$$0 \notin h(t,u).$$

By contradiction, we can find $(t_n) \subseteq [0,1]$ and $u_n \in W_0^{s,p}(\Omega)_+$, $n \geq 1$, such that

$$t_n \to t \text{ in } [0,1], \quad \|u_n\|_{W_0^{s,p}(\Omega)} \to 0$$

and

$$0 \in (-\Delta)_p^s u_n - t_n N(u_n) - (1-t_n)K_m(u_n) + t_n\partial I(u_n), \quad n \geq 1. \tag{9.2.1}$$

We set

$$v_n = \frac{u_n}{\|u_n\|_{W_0^{s,p}(\Omega)}}, \quad n \geq 1,$$

hence, passing to a suitable subsequence, we can deduce that

$$v_n \rightharpoonup v \text{ in } W_0^{s,p}(\Omega),\ v_n \to v \text{ in } L^p(\Omega) \text{ and } v_n(x) \to v(x) \text{ a.e. in } \Omega,$$

hence $v \geq 0$ a.e. in Ω.
From (9.2.1), we have that there exists $w_n \in N(u_n)$ such that

$$-(-\Delta)_p^s u_n + t_n w_n + (1-t_n)K_m(u_n) \in t_n \partial I(u_n),$$

therefore,

$$\left\langle (-\Delta)_p^s u_n, \bar{v} - u_n \right\rangle - t_n \int_\Omega w_n(\bar{v} - u_n)\, dx - (1-t_n) \int_\Omega m|u_n|^{p-2} u_n(\bar{v} - u_n)\, dx \geq 0.$$

for all $\bar{v} \in W_0^{s,p}(\Omega)_+$. Dividing the last inequality with $\|u_n\|^p_{W_0^{s,p}(\Omega)}$, we have
(9.2.2)

$$\left\langle (-\Delta)_p^s v_n, \widehat{v} - v_n \right\rangle - t_n \int_\Omega \frac{w_n}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}}(\widehat{v} - v_n)\, dx - (1-t_n) \int_\Omega m|v_n|^{p-2} v_n(\widehat{v} - v_n)\, dx \geq 0,$$

for all $\widehat{v} \in W_0^{s,p}(\Omega)_+$.
By $(\mathbf{H}_{9.1})$ *(iii)*, there exists $\delta > 0$ such that for a.e. $x \in \Omega$, all t with $|t| < \delta$ and all $\xi \in \partial F(x,t)$, we obtain

$$|\xi| \leq (\widehat{\eta}(x) + 1)|t|^{p-1}. \tag{9.2.3}$$

While, from $(\mathbf{H}_{9.1})$ *(i)*, for a.e. $x \in \Omega$, and all $t \in \mathbb{R}$ with $|t| \geq \delta$ and all $\xi \in \partial F(x,t)$ we get

$$|\xi| \leq a(x) + c|t|^{p-1} \leq \left(\frac{a(x)}{\delta^{p-1}} + c \right) |t|^{p-1}. \tag{9.2.4}$$

The expressions (9.2.3) and (9.2.4) imply that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial F(x,t)$, we obtain

$$|\xi| \leq c_1 |t|^{p-1} \text{ for some } c_1 > 0. \tag{9.2.5}$$

Therefore, from (9.2.5), we deduce that $\left(\frac{w_n}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}} \right) \subseteq L^{p'}(\Omega)$ is bounded and, passing to a subsequence, we can state that

$$\frac{w_n}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}} \rightharpoonup f_0 \text{ in } L^{p'}(\Omega).$$

For every $\epsilon > 0$ and $n \geq 1$, we define the set

$$C^+_{\epsilon,n} = \left\{ x \in \Omega : u_n(x) > 0, \eta(x) - \epsilon \leq \frac{w_n(x)}{(u_n(x))^{p-1}} \leq \widehat{\eta}(x) + \epsilon \right\}.$$

Since $\|u_n\|_{W_0^{s,p}(\Omega)} \to 0$, we may suppose (at least for a subsequence) that

$$u_n(x) \to 0 \text{ a.e. on } \Omega \text{ as } n \to \infty.$$

Hence, by (**H**) *(iii)*, we get

$$\chi_{C^+_{\epsilon,n}}(x) \to 1 \text{ a.e. on } \{v > 0\}.$$

We observe that

$$\left\| \left(1 - \chi_{C^+_{\epsilon,n}}\right) \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} \right\|_{L^{p'}(\{v>0\})} \to 0,$$

then

$$\chi_{C^+_{\epsilon,n}} \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} \rightharpoonup f_0 \text{ in } L^{p'}(\{v > 0\}).$$

Recalling the definition of the set $C^+_{\epsilon,n}$, we obtain

$$\chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} = \chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{(u_n(x))^{p-1}} (v_n(x))^{p-1},$$

therefore

$$\chi_{C^+_{\epsilon,n}}(x)(\eta(x)-\epsilon)(v_n(x))^{p-1} \le \chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} \le \chi_{C^+_{\epsilon,n}}(x)(\widehat{\eta}(x)+\epsilon)(v_n(x))^{p-1} \text{ a.e. on } \Omega.$$

Passing to weak limits in $L^{p'}(\{v > 0\})$ and applying Mazur's lemma, we have

$$(\eta(x) - \epsilon)(v(x))^{p-1} \le f_0(x) \le (\widehat{\eta}(x) + \epsilon)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \to 0$ and get

$$\eta(x)(v(x))^{p-1} \le f_0(x) \le \widehat{\eta}(x)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}. \tag{9.2.6}$$

Further, from (9.2.5), we get that

$$f_0(x) = 0 \text{ a.e. on } \{v = 0\}. \tag{9.2.7}$$

Hence, the conditions (9.2.6) and (9.2.7) imply that

$$f_0(x) = g_0(x)|v(x)|^{p-2}v(x) \text{ a.e. on } \Omega,$$

with $g_0 \in L^\infty(\Omega)_+$ such that $\eta(x) \le g_0(x) \le \widehat{\eta}(x)$ a.e. on Ω. In addition, if we set $\widehat{v} = v$ in (9.2.2), then since

$$\int_\Omega \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} (v_n(x) - v(x))\, dx \to 0$$

and

$$\int_\Omega m(x)|v_n(x)|^{p-2}v_n(x)(v(x) - v_n(x))\, dx \to 0,$$

from (9.2.2) we deduce

$$\limsup_{n\to\infty} \left\langle (-\Delta)^s_p v_n, v_n - v \right\rangle \le 0,$$

then

$$v_n \to v \text{ in } W^{s,p}_0(\Omega)$$

(we are using the fact that $(-\Delta)_p^s$ is a $(S)_+$-map). Hence, if n goes to ∞ in (9.2.2), we have

$$\left\langle(-\Delta)_p^s v, \widehat{v}-v\right\rangle - t\int_\Omega g_0|v|^{p-2}v(\widehat{v}-v)\,dx - (1-t)\int_\Omega m|v|^{p-2}v(\widehat{v}-v)\,dx \geq 0$$

for all $\widehat{v}\in W_0^{s,p}(\Omega)_+$. We set

$$\widehat{g}_t = tg_0 + (1-t)m,$$

hence we can rephrase the last inequality as

$$\left\langle(-\Delta)_p^s v, \widehat{v}-v\right\rangle - \int_\Omega \widehat{g}_t(x)(v(x))^{p-1}(\widehat{v}(x)-v(x))\,dx \geq 0 \text{ for all } \widehat{v}\in W_0^{s,p}(\Omega)_+. \tag{9.2.8}$$

Let $w\in W_0^{s,p}(\Omega)_+$ and set $\widehat{v}=v+w$, then we can rewrite (9.2.8) as

$$\left\langle(-\Delta)_p^s v, w\right\rangle \geq \int_\Omega \widehat{g}_t(x)(v(x))^{p-1}w(x)\,dx \text{ for all } w\in W_0^{s,p}(\Omega)_+.$$

Hence, applying the strong maximum principle [111, Proposition 2.2], we obtain that $v>0$ a.e. in Ω.
Let $z\in W_0^{s,p}(\Omega)$, $\epsilon>0$ and consider $(v+\epsilon z)^+ = v+\epsilon z+(v+\epsilon z)^-$. We take $\widehat{v}=(v+\epsilon z)^+\in W_0^{s,p}(\Omega)_+$ in (9.2.8) and we get

$$\left\langle(-\Delta)_p^s v - K_{\widehat{g}_t}(v), (v+\epsilon z)^+ - v\right\rangle \geq 0,$$

hence

$$\left\langle(-\Delta)_p^s v - K_{\widehat{g}_t}(v), \epsilon z\right\rangle \geq -\left\langle(-\Delta)_p^s v - K_{\widehat{g}_t}(v), (v+\epsilon z)^-\right\rangle. \tag{9.2.9}$$

We observe that

$$-\left\langle(-\Delta)_p^s v - K_{\widehat{g}_t}(v), (v+\epsilon z)^-\right\rangle = -\left\langle(-\Delta)_p^s v, (v+\epsilon z)^-\right\rangle + \int_\Omega \widehat{g}_t(x)v(x)(v(x)+\epsilon z(x))^-\,dx$$

and, since $\widehat{g}_t$, $v\geq 0$, we have that

$$\int_\Omega \widehat{g}_t v(v+\epsilon z)^-\,dx \geq 0.$$

Now, we want to study the sign of $-\left\langle(-\Delta)_p^s v, (v+\epsilon z)^-\right\rangle$. In order to do this, we introduce the sets

$$\Omega_\epsilon^- = \{v+\epsilon z<0\}$$

and

$$Q_\epsilon = \{(x,y)\in\Omega\times\mathbb{R}^\epsilon : v(x)+\epsilon z(x) < 0 \leq v(y)+\epsilon z(y), v(x)>v(y)\}.$$

By applying definition of $(-\Delta)^s_p$ (2.2.6), we have that

$$
\begin{aligned}
&- \left\langle (-\Delta)^s_p v, (v+\epsilon z)^- \right\rangle \\
&= - \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{(v(x)-v(y))^{p-1}((v+\epsilon z)^-(x) - (v+\epsilon z)^-(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x) - v(y) - \epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \iint_{\Omega_\epsilon^- \times (\Omega \setminus \Omega_\epsilon^-)} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&- \iint_{(\Omega \setminus \Omega_\epsilon^-) \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \iint_{\Omega_\epsilon^- \times \Omega^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&- \iint_{\Omega^c \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&+ \epsilon \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \iint_{\Omega_\epsilon^- \times (\Omega_\epsilon^-)^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&- \iint_{(\Omega_\epsilon^-)^c \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy \\
&+ \epsilon \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ 2 \iint_{\Omega_\epsilon^- \times (\Omega_\epsilon^-)^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&\geq \epsilon \iint_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ 2 \iint_{Q_\epsilon} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&= o(1)\epsilon \text{ as } \epsilon \to 0^+.
\end{aligned}
$$

In the last passage we use the fact that $|\Omega_\epsilon^-| \to 0$ as $\epsilon \to 0^+$ for the first integral, while for the second integral we note that for every $(x, y) \in Q_\epsilon$

$$
0 < v(x) - v(y) < \epsilon(z(y) - z(x))
$$

and

$$
\begin{aligned}
0 > v(x) + \epsilon z(x) &\geq v(x) + \epsilon z(x) - (v(y) + \epsilon z(y)) \\
&= (v(x) - v(y)) + \epsilon(z(x) - z(y)) \\
&> \epsilon(z(x) - z(y)).
\end{aligned}
$$

Then,

$$|(v(x) - v(y))^{p-1}(v(x) + \epsilon z(x))| \leq \epsilon^p |z(x) - z(y)|^p,$$

integrating,

$$\iint_{Q_\epsilon} \frac{|(v(x) - v(y))^{p-1}(v(x) + \epsilon z(x))|}{|x - y|^{N+ps}} \, dxdy \leq \epsilon^p \iint_{\mathbb{R}^N \times \mathbb{R}^N} \frac{|(z(x) - z(y)|^p}{|x - y|^{N+ps}} \, dxdy = o(\epsilon).$$

Going back to (9.2.9), we have that

$$\epsilon \left\langle (-\Delta)_p^s v - K_{\widehat{g}_t}(v), z \right\rangle \geq o(1)\epsilon,$$

hence, taking the limit when $\epsilon \to 0$, we get

$$\left\langle (-\Delta)_p^s v - K_{\widehat{g}_t}(v), z \right\rangle \geq 0.$$

Since $z \in W_0^{s,p}(\Omega)$ is arbitrary, it follows that $(-\Delta)_p^s v - K_{\widehat{g}_t}(v) = 0$, hence

$$(-\Delta)_p^s v = K_{\widehat{g}_t}(v),$$

therefore

$$
\begin{cases}
(-\Delta)_p^s v(x) = \widehat{g}_t(x)|v(x)|^{p-2}v(x) & \text{in } \Omega \\
v(x) = 0 & \text{on } \Omega^c.
\end{cases}
\tag{9.2.10}
$$

Since $\|v\|_{W_0^{s,p}(\Omega)} = 1$, we deduce that $v \neq 0$ and hence v is an eigenfunction of the weighted eigenvalue problem (9.2.10), with weight $\widehat{g}_t \in L^\infty(\Omega)_+$. Since

$$\widehat{g}_t(x) \geq \eta(x) \text{ a.e. on } \Omega,$$

by exploiting Lemma 3.4.1 *(iii)*, we have that

$$\lambda_1(\widehat{g}_t) \leq \lambda_1(\eta) < \lambda_1(\lambda_1) = 1,$$

so we discover that v cannot be the principal eigenfunction of the weighted eigenvalue problem with weight $\widehat{g}_t \in L^\infty(\Omega)_+$, hence, v must be nodal, but $v \in W_0^{s,p}(\Omega)_+$, a contradiction. Therefore, the claim is true.

Applying the homotopy invariance property of the degree map, we deduce that

$$\deg((-\Delta)_p^s - N + \partial I, B_\rho(0), 0) = \deg_{S_+}((-\Delta)_p^s - K_m, B_\rho(0), 0)$$

for all $0 < \rho \leq \rho_0$.

But from Proposition 3.4.3, we know that

$$\deg_{(S)_+}((-\Delta)_p^s - K_m, B_\rho(0), 0) = -1.$$

Therefore, we get

$$\deg(\partial\tilde{J}+\partial I, B_\rho(0), 0) = -1$$

for all $0 < \rho \leq \rho_0$. $\square$

Analogously, we show a corresponding result for big balls.

Proposition 9.2.3. *Let* $(\mathbf{H}_{9.1})$ *hold. Therefore there exists* $R_0 > 0$ *such that for all* $R \geq R_0$*, we obtain*

$$\deg(\partial\tilde{J}+\partial I, B_R(0), 0) = 1.$$

Proof. We take into account the homotopy

$$h(t,u) = (-\Delta)^s_p u - tN(u) + t\partial I(u) \text{ for } (t,u) \in [0,1] \times W_0^{s,p}(\Omega).$$

From Proposition 9.1.3 and Lemma 1.4.2, we have that $\widehat{h}(t,u) = -tN(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a $(P)-$homotopy and $\tilde{h}(t,u) = (-\Delta)^s_p u + t\partial I(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a pseudomonotone homotopy, hence $h(t,u)$ is an admissible homotopy.
Claim: There exists $R_0 \geq 0$ such that for all $t \in [0,1]$, all $R \geq R_0$ and all $u \in \partial B_R(0)$, we have

$$0 \notin h(t,u).$$

By contradiction, we can find $(t_n) \subseteq [0,1]$ and $u_n \in W_0^{s,p}(\Omega)_+$, $n \geq 1$, such that

$$t_n \to t \text{ in } [0,1], \|u_n\|_{W_0^{s,p}(\Omega)} \to \infty \text{ and } 0 \in h(t_n,u_n), \quad n \geq 1.$$

Hence, there exists $w_n \in N(u_n)$ such that

$$-(-\Delta)^s_p u_n + t_n w_n \in t_n \partial I(u_n), \quad \forall n \geq 1,$$

then

$$\left\langle (-\Delta)^s_p u_n, \bar{v} - u_n \right\rangle - t_n \int_\Omega w_n(x)(\bar{v}(x) - u_n(x))\, dx \geq 0 \text{ for all } \bar{v} \in W_0^{s,p}(\Omega)_+. \tag{9.2.11}$$

Set $v_n = \frac{u_n}{\|u_n\|_{W_0^{s,p}(\Omega)}}$, $n \geq 1$ and, passing to a subsequence, we can suppose that

$$v_n \rightharpoonup v \text{ in } W_0^{s,p}(\Omega),\ v_n \to v \text{ in } L^p(\Omega) \text{ and } v_n(x) \to v(x) \text{ a.e. in } \Omega,$$

hence $v \geq 0$ a.e. in Ω. Dividing (9.2.11) by $\|u_n\|^p_{W_0^{s,p}(\Omega)}$, we have

$$\left\langle (-\Delta)^s_p v_n, \widehat{v} - v_n \right\rangle - t_n \int_\Omega \frac{w_n(x)}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}}(\widehat{v}(x) - v_n(x))\, dx \geq 0 \tag{9.2.12}$$

for all $\widehat{v} \in W_0^{s,p}(\Omega)_+$. Using (9.2.5), we obtain that $\left(\frac{w_n}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}}\right) \subseteq L^{p'}(\Omega)$ is bounded, hence, we can suppose that

$$\frac{w_n}{\|u_n\|^{p-1}_{W_0^{s,p}(\Omega)}} \rightharpoonup f_\infty \text{ in } L^{p'}(\Omega), \text{ as } n \to \infty.$$

For every $\epsilon > 0$ and $n \geq 1$, we define the set

$$D^+_{\epsilon,n} = \left\{ x \in \Omega : u_n(x) > 0, -\epsilon \leq \frac{w_n(x)}{(u_n(x))^{p-1}} \leq \theta(x) + \epsilon \right\}.$$

From $(\mathbf{H}_{9.1})$ (ii), we get

$$\chi_{D^+_{\epsilon,n}}(x) \to 1 \text{ a.e. on } \{v > 0\}.$$

We observe that

$$\left\| \left(1 - \chi_{D^+_{\epsilon,n}}(x)\right) \frac{w_n}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} \right\|_{L^{p'}(\{v>0\})} \to 0,$$

therefore,

$$\chi_{D^+_{\epsilon,n}}(x) \frac{w_n}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} \rightharpoonup f_\infty \text{ in } L^{p'}(\{v > 0\}).$$

By the definition of $D^+_{\epsilon,n}$, we get that

$$\begin{aligned}
\chi_{D^+_{\epsilon,n}}(x)(-\epsilon)(v_n(x))^{p-1} \leq \chi_{D^+_{\epsilon,n}}(x) \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} &= \chi_{D^+_{\epsilon,n}}(x) \frac{w_n(x)}{(u_n(x))^{p-1}} (v_n(x))^{p-1} \\
&\leq \chi_{D^+_{\epsilon,n}}(x)(\theta(x) + \epsilon)(v_n(x))^{p-1} \text{ a.e. on } \Omega.
\end{aligned}$$

Passing to weak limits in $L^{p'}(\{v > 0\})$ and applying Mazur's lemma, we have

$$-\epsilon(v(x))^{p-1} \leq f_\infty(x) \leq (\theta(x) + \epsilon)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

Let $\epsilon \to 0$, we obtain

$$0 \leq f_\infty(x) \leq \theta(x)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

While, by (9.2.5), we obtain that

$$f_\infty(x) = 0 \text{ a.e. on } \{v = 0\}.$$

Since $\Omega = \{v > 0\} \cup \{v = 0\}$ (recalling that $v \in W^{s,p}_0(\Omega)_+$), we get

$$0 \leq f_\infty(x) \leq \theta(x)(v(x))^{p-1} \text{ a.e. on } \Omega,$$

hence

$$f_\infty = g_\infty v^{p-1} \text{ with } g_\infty \in L^\infty(\Omega)_+,\ g_\infty(x) \leq \theta(x) \text{ a.e. on } \Omega.$$

Since $v \in W^{s,p}_0(\Omega)_+$, then in (9.2.12) we can set $\widehat{v} = v$ to obtain

$$\left\langle (-\Delta)^s_p v_n, v_n - v \right\rangle \leq t_n \int_\Omega \frac{w_n(x)}{\|u_n\|^{p-1}_{W^{s,p}_0(\Omega)}} (v_n(x) - v(x))\, dx,$$

therefore

$$\limsup_{n\to\infty} \left\langle (-\Delta)^s_p v_n, v_n - v \right\rangle \leq 0,$$

and since $(-\Delta)^s_p$ is of type $(S)_+$,

$$v_n \to v \text{ in } W^{s,p}_0(\Omega).$$

If n goes to ∞ in (9.2.12), we get

$$\langle (-\Delta)_p^s v, \widehat{v} - v \rangle \geq t \int_\Omega g_\infty(x)(v(x))^{p-1}(\widehat{v}(x) - v(x))\, dx, \quad \forall \widehat{v} \in W_0^{s,p}(\Omega)_+.$$

Set $\widehat{g}_t = tg_\infty(x)$. Using the test function $\widehat{v} = (v + \epsilon z)^+$ for any $z \in W_0^{s,p}(\Omega)$ and $\epsilon > 0$, then, as in the proof of Theorem 9.2.2, we have

$$\langle (-\Delta)_p^s v - K_{\widehat{g}_t}(v), z \rangle \geq 0,$$

by the arbitrariety of z, it follows that

$$(-\Delta)_p^s v = K_{\widehat{g}_t}(v),$$

therefore

$$\begin{cases} (-\Delta)_p^s v = tg_\infty(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \Omega^c. \end{cases} \tag{9.2.13}$$

Since $\|v\|_{W_0^{s,p}(\Omega)} = 1$, we deduce that $v \neq 0$ and hence v is an eigenfunction of the weighted eigenvalue problem (9.2.13), with weight $tg_\infty \in L^\infty(\Omega)_+$. Since

$$0 \leq tg_\infty \leq g_\infty \leq \theta,$$

by Lemma 3.4.1 *(iii)*, we obtain

$$\lambda_1(tg_\infty) \geq \lambda_1(g_\infty) \geq \lambda_1(\theta) > \lambda_1(\lambda_1) = 1.$$

Then from (9.2.13) we deduce that $v = 0$, a contradiction.
Therefore, from the homotopy invariance of the degree map, we obtain that

$$\deg((-\Delta)_p^s - N + \partial I, B_R(0), 0) = \deg_{S_+}((-\Delta)_p^s, B_R(0), 0) \text{for all } R \geq R_0. \tag{9.2.14}$$

We take the $(S)_+-$homotopy (see [150, Proposition 4.41])

$$h_1(t, u) = t(-\Delta)_p^s u + (1 - t)\mathcal{F}(u) \text{ for all } (t, u) \in [0, 1] \times W_0^{s,p}(\Omega).$$

We have that $\langle h_1(t, u), u \rangle \neq 0$ for all $u \neq 0$ and hence, by the homotopy invariance of $\deg_{(S)_+}$, we have

$$\deg_{(S)_+}((-\Delta)_p^s, B_R(0), 0) = \deg_{(S)_+}(\mathcal{F}, B_R(0), 0) = 1. \tag{9.2.15}$$

(The last passage follows from the normalization property). From (9.2.14) and (9.2.15), we can state that

$$\deg(\partial \tilde{J} + \partial I, B_R(0), 0) = 1$$

for all $R \geq R_0$. □

9.3 The obstacle problem at zero

In this section we show that the obstacle problem at zero (9.0.1) admits at least two nontrivial solutions.

Theorem 9.3.1. *Let* ($\mathbf{H}_{9.1}$) *hold. Therefore the problem* (9.0.1) *admits at least two nontrivial solutions* $u_0, \widehat{u} \in W_0^{s,p}(\Omega)$.

Proof. By Proposition 9.1.5, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that

$$J(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} J(u). \tag{9.3.1}$$

Since u_0 is a minimizer, by applying Proposition 9.2.1, there exists $r > 0$ such that

$$\deg(\partial \tilde{J} + \partial I, B_r(u_0), 0) = 1. \tag{9.3.2}$$

Therefore, (9.3.2) and Proposition 9.2.2 imply $u_0 \neq 0$. We choose $\rho_0 > 0$ small such that

$$B_r(u_0) \cap B_{\rho_0}(0) = \varnothing$$

and $R_0 > 0$ large such that

$$B_{\rho_0}(0), B_r(u_0) \subseteq B_{R_0}(0).$$

Exploiting the additivity of the domain property of the degree map and applying Proposition 9.2.1, Proposition 9.2.2 and Proposition 9.2.3, we get

$$\begin{aligned}
&\deg(\partial \tilde{J} + \partial I, B_{R_0}(0), 0) \\
&= \deg(\partial \tilde{J} + \partial I, B_r(u_0), 0) + \deg(\partial \tilde{J} + \partial I, B_{\rho_0}(0), 0) \\
&+ \deg(\partial \tilde{J} + \partial I, B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0)), 0),
\end{aligned}$$

therefore

$$1 = \deg(\partial \tilde{J} + \partial I, B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0)), 0).$$

Hence, by the existence property of the degree map we deduce that there exists

$$\widehat{u} \in B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0))$$

hence $\widehat{u} \neq u_0$, $\widehat{u} \neq 0$, such that

$$0 \in \partial \tilde{J}(\widehat{u}) + \partial I(\widehat{u}) = (-\Delta)_p^s \widehat{u} - N(\widehat{u}) + \partial I(\widehat{u}),$$

namely, there exists $w \in N(\widehat{u})$ such that

$$-(-\Delta)_p^s \widehat{u} + w \in \partial I(\widehat{u}).$$

From the latter we deduce

$$\left\langle (-\Delta)_p^s \widehat{u}, v - \widehat{u} \right\rangle - \int_\Omega w(x)(v(x) - \widehat{u}(x))\, dx \geq 0 \text{ for all } v \in W_0^{s,p}(\Omega)_+,$$

hence $\widehat{u} \in W_0^{s,p}(\Omega)$ is a nontrivial solution of (9.0.1).
Now we have to show that u_0 is a critical point of J and it is a second nontrivial solution of (9.0.1). By (9.3.1), for all $\lambda > 0$ and all $v \in W_0^{s,p}(\Omega)$ one has

$$0 \leq J(u_0 + \lambda v) - J(u_0) = \tilde{J}(u_0 + \lambda v) - \tilde{J}(u_0) + I(u_0 + \lambda v) - I(u_0)$$

hence

$$\begin{aligned} 0 &\leq \frac{1}{\lambda}(\tilde{J}(u_0 + \lambda v) - \tilde{J}(u_0)) + \frac{1}{\lambda}(I(u_0 + \lambda v) - I(u_0)) \\ &\leq \frac{1}{\lambda}[\tilde{J}(u_0 + \lambda v) - \tilde{J}(u_0)] + (I(u_0 + v) - I(u_0)) \end{aligned}$$

(since I is convex). When λ goes to 0, we get

$$0 \leq \tilde{J}^0(u_0; v) + I(u_0 + v) - I(u_0). \tag{9.3.3}$$

Let $z \in W_0^{s,p}(\Omega)$, we set $v = z - u_0$ in (9.3.3) and we obtain

$$0 \leq \tilde{J}^0(u_0; z - u_0) + I(z) - I(u_0).$$

Therefore, by Definition 1.3.4, $u_0 \in W_0^{s,p}(\Omega)$ is a critical point of $J = \tilde{J} + I$, hence, by Proposition 1.3.5

$$0 \in \partial\tilde{J}(u_0) + \partial I(u_0).$$

Therefore we can deduce that there exists $w \in N(u_0)$ such that

$$-(-\Delta)_p^s u_0 + w \in \partial I(u_0),$$

hence

$$\left\langle (-\Delta)_p^s u_0, v - u_0 \right\rangle - \int_\Omega w(x)(v(x) - u_0(x))\, dz \geq 0 \text{ for all } v \in W_0^{s,p}(\Omega)_+.$$

Consequently $u_0 \in W_0^{s,p}(\Omega)$ is a second nontrivial solution of (9.0.1). □

Remark 9.3.2. In the linear case ($p = 2$), a solution $\widehat{u}$ of problem (9.0.1) belongs to $C(\overline{\Omega})$, under the additional assumptions that Ω satisfies the exterior ball condition and $w \in N(\widehat{u})$ such that $w \in L^2(\Omega)$ with $N < 4s$ (see [27, Proposition 2.12]). Regularity results of solutions of (9.0.1) can be obtained by strengthening the assumptions of w, moreover, in the case of a general obstacle it is necessary that such obstacle has some regularity properties (see [27, Proposition 2.12] and the references therein).

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