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On the whole spectrum of Timoshenko beams. Part II: further applications

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Abstract. The problem of free vibrations of the Timoshenko beam model has been addressed in the first part of this paper. A careful analysis of the governing equations has shown that the vibration spectrum consists of two parts, separated by a *transition frequency*, which, depending on the applied boundary conditions, might be itself part of the spectrum. Here, as an extension, the case of a doubly clamped beam is considered. For both parts of the spectrum the values of natural frequencies are computed and the expressions of eigenmodes are provided: this allows to acknowledge that the nature of vibration modes changes when moving across the transition frequency. This case is a meaningful example of more general ones, where the wave-numbers equation cannot be written in a factorized form and hence must be solved by general root-finding methods for non-linear transcendental equations. These theoretical results can be used as further benchmarks for assessing the correctness of the numerical values provided by several numerical techniques, *e.g.* finite element models.

Mathematics Subject Classification (2010). Primary 74K10, 74H45, 74H05; Secondary 70J10, 70J30, 34L10, 34L15, 35C05, 35L25.

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1. Introduction

A large number of papers on the same topic treated here have appeared since 1921, when Timoshenko published [1] a first paper on the dynamics of a beam model — which is now universally associated to his name — including the effects of both rotary inertia and shear strain, which he further extended in [2]. There are still, however, some issues which deserve some attention, in particular a complete and precise definition of the vibration spectrum of this beam model. Indeed, the most debated point about the Timoshenko beam theory is precisely the so-called *second spectrum*, which was first described by Traill-Nash and Collar [3]. Following that paper many contributions on this issue appeared; an updated, though inevitably incomplete list should mention at least the following contributions (in order of appearance):[4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18].

This paper is devoted to carry out the discussion, started in the companion paper [19], about the complete spectrum of the Timoshenko beam, *i.e.* the solution, in terms of natural frequencies and corresponding vibration modes, for this model in the most general case. Using only real-valued variables, general results have been specialized to some peculiar boundary conditions and therefore the numerical values of natural frequencies and eigenmodes have been constructed.

Among the ten basic configurations that a single-span Timoshenko beam can assume, in terms of end constraints, attention has been concentrated on two representative cases, namely the simply-supported beam and the doubly clamped one. The first case was presented and discussed in [19] whereas the second one is going to be presented here. The doubly clamped beam is indeed a prototype of the most general cases, where the wave-numbers equation (a non-linear transcendental one) cannot be written in a factorized form, and hence cannot be solved by direct methods. Thus, for evaluating natural frequencies it is necessary to apply general root-finding algorithms for non-linear equations. Eigenmodes, too, have now more complicated shapes than those of the simply-supported case, since they involve either circular and hyperbolic functions, in the first part of the spectrum, or circular functions depending on *two* different wave-numbers, in the second part of the spectrum. Moreover, the transition frequency does not belong, in general, to the vibration spectrum.

As it has been already done (in the first part [19]) for the simply-supported beam, in the case here analysed, the complete list of the first 50 natural frequencies is given, for suitably chosen geometric and material data, as well as some representative plots of the eigenmodes in different portions of the spectrum; moreover a comparison between the spectrum of the Euler-Bernoulli model and that of the Timoshenko one is presented for the same geometric and material data. Together, these theoretical results will be used, in a forthcoming paper [20], as reference solutions to validate, from a quantitative point of view, the accuracy of some finite element models.

The rest of the paper is structured in this way: in Section 2 the main tools to perform a modal analysis according to Timoshenko theory, along with the data used to build the spectrum are presented; in Section 3 a complete discussion on the solution for the case of a doubly clamped single-span beam is carried out; in Section 4 the specific complete spectrum is constructed. Finally, Section 5 contains an insight on the eigenmode related to the transition frequency. This anticipates the final remarks and future perspectives, which are reported in Section 6.

A complete list of symbols is here provided for the reader's convenience.

Symbol	Definition
$\mathbf{A}$	coefficient matrix for the homogenous system
$\mathbf{A}_r$	coefficient matrix for the reduced homogenous system
$\mathbf{X}$	unknown column matrix for the homogenous system
$\mathbf{X}_r$	unknown column matrix for the reduced homogenous system
$\mathbf{0}$	right-hand side column matrix for homogeneous system
$\mathbf{0}_r$	right-hand side column matrix for reduced homogeneous system
A	cross section area
A_1, A_2, A_3, A_4	integration constants for V, first part of the spectrum
A_{1n}, A_{2n}, A_{3n}, A_{4n}	integration constants for the n-th eigenmode
B	cross section depth (and width)
B_1, B_2, B_3, B_4,	integration constants for Φ, first part of the spectrum
C	constant factor (see Eq. (3.17))
C_1, C_3, C_4	integration constants for V, transition frequency
$\tilde{C}_1$, $\tilde{C}_3$, $\tilde{C}_4$, $\tilde{D}_1$	integration constants for the n-th eigenmode at transition frequency
D_1	integration constant for Φ, transition frequency
E	Young's modulus
E_1, E_2, E_3, E_4	integration constants for V, second part of the spectrum
E_{1n}, E_{2n}, E_{3n}, E_{4n}	integration constants for the n-th eigenmode
G	shear modulus
I	cross section mass moment of inertia
L	beam length
$\tilde{L}$	special value of beam length
V	vibration mode for transversal displacement
f_λ	space frequency associated to wave-number λ
f_{λ_n}	space frequency associated to the n-th vibration mode
k, k_1, k_2	integer values corresponding to wave-numbers of vibration modes
t	time variable
v	transversal displacement

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x	space variable (beam abscissa)
z	dummy space variable
$z^\star$	normalized dummy space variable
Φ	vibration mode for section rotation
$\hat{\alpha}_1$	coefficient of eigenmode for generalized wave-number
$\hat{\alpha}_{1n}$	value of $\hat{\alpha}_1$ for n-th vibration mode
α_1, α_2	eigenmode coefficients for first/second wave-number
α_{1n}, α_{2n}	values of α_1, α_2 for n-th vibration mode
$\tilde{\alpha}_2$	eigenmode coefficient for second wave-number at transition frequency
ε	predefined convergence tolerance
κ	shear correction factor
$\hat{\lambda}_1$	generalized wave-number (first part of the spectrum)
λ_1	first wave-number (second part of the spectrum)
λ_2	second wave-number (first and second part of the spectrum)
$\tilde{\lambda}_2$	second wave-number at transition frequency
λ_{1n}	first wave-number for n-th vibration mode
λ_{2n}	second wave-number for n-th vibration mode
$\lambda_1^{\star 2}$	first root (squared) of wave-numbers equations
$\lambda_2^{\star 2}$	second root (squared) of wave-numbers equations
ν	Poisson's ratio
ξ	dimensionless space variable (dimensionless beam abscissa)
ρ	beam density (mass per unit volume)
σ_n, $\hat{\chi}_n$	eigenmode coefficient for first part of spectrum for n-th vibration mode
τ_n, χ_n	eigenmode coefficient for second part of spectrum for n-th vibration mode
ϕ	section rotation
$\hat{\chi}$	eigenmode coefficient for first part of spectrum, doubly clamped beam
χ	eigenmode coefficients for second part of spectrum, doubly clamped beam
ω	angular frequency
$\tilde{\omega}$	angular frequency at the transition value (cut-off frequency)
$\omega^\star$	limiting value (upper/lower bound) for angular frequency
ω_n	angular frequency (theoretical value) for n-th vibration mode

2. Modal analysis of Timoshenko beams — a résumé

In this Section, the minimal tools to perform the modal analysis of a Timoshenko beam are briefly recalled, in order to devise its complete spectrum. Complete details can be found in the companion paper [19].

The coupled equations of motion, written in terms of kinematic variables $v = v(x,t)$ and $\phi = \phi(x,t)$ (*i.e.* the transversal displacement of the centroid and the cross-section rotation, which depend on both the abscissa, x, and time, t), for the Timoshenko beam model are:

$$G\kappa A\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial \phi}{\partial x}\right) - \rho A\frac{\partial^2 v}{\partial t^2} = 0, \tag{2.1}$$

$$EI\frac{\partial^2 \phi}{\partial x^2} - G\kappa A\left(\frac{\partial v}{\partial x} + \phi\right) - \rho I\frac{\partial^2 \phi}{\partial t^2} = 0. \tag{2.2}$$

In Eqs. (2.1)–(2.2) G and E are shear and Young's moduli, κ the shear-correction factor, ρ the density of the material constituting the beam, A and I the area and the area moment of inertia of the beam cross-section. The assumed positive conventions for v and ϕ are illustrated in Figure 1.

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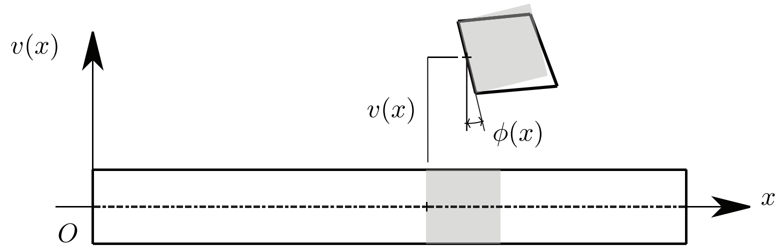

Figure 1. Timoshenko beam element showing the assumed conventions for generalized displacements (v, ϕ).

It is possible to reduce the system of two second-order Partial Differential Equations (PDEs), Eqs. (2.1)–(2.2), to a unique fourth-order PDE. Indeed, from Eq. (2.1), it follows immediately:

$$\frac{\partial \phi}{\partial x} = \frac{\rho}{G\kappa}\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2}, \tag{2.3}$$

and by proper use of multivariate differential calculus, ϕ can be eliminated in Eq. (2.1). The resulting fourth-order PDE in terms of v alone is:

$$EI\frac{\partial^4 v}{\partial x^4} - \rho I\left(1+\frac{E}{G\kappa}\right)\frac{\partial^4 v}{\partial t^2 \partial x^2} + \rho A\frac{\partial^2 v}{\partial t^2} + \frac{\rho^2 I}{G\kappa}\frac{\partial^4 v}{\partial t^4} = 0, \tag{2.4}$$

which is the equation first established by Timoshenko [1] in 1921 when developing a new beam theory able to deal with both shear strain and rotary inertia. Similarly, if v is eliminated in Eq. (2.2), a fully-decoupled fourth-order PDE in terms of ϕ alone is obtained:

$$EI\frac{\partial^4 \phi}{\partial x^4} - \rho I\left(1+\frac{E}{G\kappa}\right)\frac{\partial^4 \phi}{\partial t^2 \partial x^2} + \rho A\frac{\partial^2 \phi}{\partial t^2} + \frac{\rho^2 I}{G\kappa}\frac{\partial^4 \phi}{\partial t^4} = 0. \tag{2.5}$$

Solutions to Eqs. (2.4)–(2.5) are sought such that the independent variables, x and t, are separated. In particular an harmonic kind time-dependence is assumed, so that free vibrations are possible. Thus one states:

$$v(x,t) = V(x)\exp(\mathrm{i}\omega t), \qquad \phi(x,t) = \Phi(x)\exp(\mathrm{i}\omega t), \tag{2.6}$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit; then, if primes are used to denote derivatives with respect to x, it follows from Eq. (2.4) (a similar expressions follows for Eq. (2.5), too):

$$V'''' + \frac{\rho\,\omega^2}{E}\left(1+\frac{E}{G\kappa}\right)V'' + \frac{\rho\,\omega^2}{E}\left(\frac{\rho\,\omega^2}{G\kappa} - \frac{A}{I}\right)V = 0. \tag{2.7}$$

This is a fourth-order ODE with constant coefficients, whose solutions are to be found in the form of exponential functions $V(x) = \exp(\lambda^\star x)$, where, in general, $\lambda^\star \in \mathbb{C}$.

In particular, the *characteristic equation* associated to Eq. (2.7) is:

$$\lambda^{\star 4} + \frac{\rho\,\omega^2}{E}\left(1+\frac{E}{G\kappa}\right)\lambda^{\star 2} + \frac{\rho\,\omega^2}{E}\left(\frac{\rho\,\omega^2}{G\kappa} - \frac{A}{I}\right) = 0, \tag{2.8}$$

which is a biquadratic algebraic equation, where the independent variable is $\lambda^\star$. The squared roots of Eq. (2.8) are therefore:

$$\lambda_1^{\star 2} = -\frac{\rho\,\omega^2}{2E}\left(1+\frac{E}{G\kappa}\right) + \sqrt{\frac{\rho^2\omega^4}{4E^2}\left(1-\frac{E}{G\kappa}\right)^2 + \frac{\rho\,\omega^2 A}{EI}}, \tag{2.9}$$

$$\lambda_2^{\star 2} = -\frac{\rho\,\omega^2}{2E}\left(1+\frac{E}{G\kappa}\right) - \sqrt{\frac{\rho^2\omega^4}{4E^2}\left(1-\frac{E}{G\kappa}\right)^2 + \frac{\rho\,\omega^2 A}{EI}}. \tag{2.10}$$

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While, in Eq. (2.10), it is always $\lambda_2^{\star 2} < 0$, the sign of the other root, $\lambda_1^{\star 2}$, given by Eq. (2.9) depends on the value of ω^2; there is a special value of ω^2, which correspond to a *transition frequency*,

$$\tilde{\omega}^2 = \frac{G\kappa A}{\rho\, I}, \tag{2.11}$$

such that the value of $\lambda_1^{\star 2}$ changes from positive to negative. As a consequence, when solving Eq. (2.8), these three cases must be distinguished.

Case 1. $\omega^2 < \tilde{\omega}^2$**.** For this range of angular frequency, it results: $\lambda_1^{\star 2} > 0$ and $\lambda_2^{\star 2} < 0$. Hence, Eq. (2.8), has *two real roots*, namely $\pm\sqrt{\lambda_1^{\star 2}}$, and *two purely imaginary conjugate roots*, *viz.* $\pm \mathrm{i}\sqrt{-\lambda_2^{\star 2}}$.

Case 2. $\omega^2 = \tilde{\omega}^2$**.** In the present case (transition frequency) it follows: $\lambda_1^{\star 2} = 0$ and $\lambda_2^{\star 2} < 0$. In particular,

$$\lambda_2^{\star 2}|_{\omega^2=\tilde{\omega}^2} = -\tilde{\lambda}_2^2, \tag{2.12}$$

with

$$\tilde{\lambda}_2 = \sqrt{\frac{A}{I}\left(1 + \frac{G\kappa}{E}\right)} > 0. \tag{2.13}$$

Consequently there is *a null real root, whose multiplicity is two*, and *one couple of imaginary conjugate roots*, namely again $\pm \mathrm{i}\sqrt{-\lambda_2^{\star 2}}$.

Case 3. $\omega^2 > \tilde{\omega}^2$**.** This time it results $\lambda_1^{\star 2} < 0$ and $\lambda_2^{\star 2} < 0$. As a consequence, *all four roots* of Eq. (2.8) *are purely imaginary*. In particular, there are *two couples of conjugate roots*, *i.e.* $\pm \mathrm{i}\sqrt{-\lambda_1^{\star 2}}$ and $\pm \mathrm{i}\sqrt{-\lambda_2^{\star 2}}$.

2.1. The eigenmodes of Timoshenko beams

The complete solution to Eq. (2.7) — and to the corresponding equation which provides $\Phi(x)$ — can be computed in terms of real-valued quantities only; results will be presented separately for the three cases outlined above. Again, all relevant details are given in [19].

Case 1. $\omega^2 < \tilde{\omega}^2$**.** In the first part of the spectrum the eigenfunctions in terms of $V(x)$ and $\Phi(x)$ are:

$$V(x) = A_1 \cosh \hat{\lambda}_1 x + A_2 \sinh \hat{\lambda}_1 x + A_3 \cos \lambda_2 x + A_4 \sin \lambda_2 x, \tag{2.14}$$

$$\Phi(x) = -\frac{\hat{\alpha}_1}{\hat{\lambda}_1}(A_2 \cosh \hat{\lambda}_1 x + A_1 \sinh \hat{\lambda}_1 x) + \frac{\alpha_2}{\lambda_2}(A_4 \cos \lambda_2 x - A_3 \sin \lambda_2 x). \tag{2.15}$$

where A_1, A_2, A_3, A_4 are integration constants and the following *proper* (λ_2) and *generalized* ($\hat{\lambda}_1$) wave-numbers apply:

$$\hat{\lambda}_1 = +\sqrt{\lambda_1^{\star 2}} > 0, \qquad \lambda_2 = +\sqrt{-\lambda_2^{\star 2}} > 0. \tag{2.16}$$

In Eq. (2.15) the following short-hand notation has been adopted:

$$\hat{\alpha}_1 = \frac{\rho\,\omega^2}{G\kappa} + \hat{\lambda}_1^2, \qquad \alpha_2 = \frac{\rho\,\omega^2}{G\kappa} - \lambda_2^2. \tag{2.17}$$

Hence in the first part of the spectrum the eigenmodes are given, in general, by a linear combination of hyperbolic and trigonometric functions. It has to be remarked that only for particular choices of Boundary Conditions (BCs) it is possible to annihilate the contribution of hyperbolic functions: this happens, for instance, in the case of a simply-supported beam, see [19].

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Case 2. $\omega^2 = \tilde{\omega}^2$. At the transition frequency, $\tilde{\omega}$, the eigenfunctions have these expressions:

$$V(x) = C_1 + C_3 \cos \tilde{\lambda}_2 x + C_4 \sin \tilde{\lambda}_2 x, \tag{2.18}$$

$$\Phi(x) = D_1 - \frac{\rho \tilde{\omega}^2}{G\kappa} C_1 x - \frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}(C_3 \sin \tilde{\lambda}_2 x - C_4 \cos \tilde{\lambda}_2 x), \tag{2.19}$$

where C_1, C_3, C_4, D_1 are integration constants. In Eqs. (2.18)–(2.19), for the seek of a compact notation, the following definition has been adopted, see Eq. (2.13):

$$\tilde{\alpha}_2 = \frac{\rho \tilde{\omega}^2}{G\kappa} - \tilde{\lambda}_2^2 = -\frac{G\kappa A}{EI}. \tag{2.20}$$

Thus, from the above-written equations, it follows that at the *transition frequency* the V component of the vibration mode is a linear combination of trigonometric functions and of a constant, while the Φ component is obtained by combining a complete linear polynomial and the usual sine and cosine functions.

Case 3. $\omega^2 > \tilde{\omega}^2$. In this second part of the spectrum the eigenfunctions are:

$$V(x) = E_1 \cos \lambda_1 x + E_2 \sin \lambda_1 x + E_3 \cos \lambda_2 x + E_4 \sin \lambda_2 x, \tag{2.21}$$

$$\Phi(x) = \frac{\alpha_1}{\lambda_1}(E_2 \cos \lambda_1 x - E_1 \sin \lambda_1 x) + \frac{\alpha_2}{\lambda_2}(E_4 \cos \lambda_2 x - E_3 \sin \lambda_2 x), \tag{2.22}$$

where E_1, E_2, E_3, E_4 are integration constants (to be determined by boundary conditions) and the two independent, real-valued wave-numbers are given by:

$$\lambda_1 = +\sqrt{-\lambda_1^{\star 2}}, \qquad \lambda_2 = +\sqrt{-\lambda_2^{\star 2}}. \tag{2.23}$$

The following short-hand notation has been adopted in Eq. (2.22):

$$\alpha_1 = \frac{\rho \omega^2}{G\kappa} - \lambda_1^2, \tag{2.24}$$

while α_2 is still defined by Eq. $(2.17)_2$.

Therefore, in the second part of the spectrum mode shapes are given by a linear combination of trigonometric functions depending on *two* different wave-numbers, λ_1 and λ_2; in general, eigenmodes involve both λ_1 and λ_2, since wave-numbers are entwined (or even entangled); only for particular cases, *e.g.* the simply-supported beam, see [19], the contributions of wave-numbers become decoupled.

2.2. Comments on the construction of the spectrum

Here the spectrum will be explicitly computed only for the doubly clamped beam; together with the already analysed case of the simply-supported beam (see [19]), they are somehow representative of all general cases which can be encountered.

Indeed, differently from what happens in the case of the simply-supported beam, (where the wave-number transcendental equation can be written in a factorized form and, as a consequence, the frequency equation becomes a simple algebraic one, and allows for the evaluation of natural frequencies ω_n by a direct method), in the doubly clamped case — as it occurs, on the other hand for all the remaining cases — there is a complete coupling. Since the wave-number transcendental equation cannot be written as a product, the computation of natural frequencies ω_n must be performed by solving a complicated implicit transcendental equation.

Furthermore, in this case, as it occurs for any case but the simply-supported one, hyperbolic functions appear in the eigenmodes in the first part of the spectrum, while, each eigenmode depends simultaneously on *both* wave-numbers in the second part of the spectrum.

Finally, the transition frequency is *not* part of the spectrum *in general* (except for particular special values of the beam length), differently from what happen in the simply-supported beam, where it is *always* part of the spectrum.

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The expressions of the frequency equation and of the eigenmodes corresponding to other BCs can be found in [7], where only those which are valid for $\omega_n < \tilde{\omega}$ are reported, and in [21], where the complete expressions are given.

The geometric and material data used for building the spectrum are the *same* which have already adopted for the simply-supported case in [19]. They are the following: a straight uniform and homogeneous beam, whose length is $L = 2$ m, having a square cross-section with side length (either depth or width) $B = 0.1$ m. Consequently, the cross-section area and area moment of inertia are respectively $A = B^2 = 0.01$ m^2; $I = B^4/12 = 1/120,000$ m^4. The length-to-depth ratio (a rough measure of slenderness) is therefore: $L/B = 20$, so that shear strains are expected to be non-negligible.

Material density is assumed to be $\rho = 8000$ kg/m^3, Young's modulus $E = 260$ GPa, Poisson's ratio $\nu = 0.3$ so that, under the hypothesis of elastic isotropy, the shear modulus is $G = 100$ GPa. The shear correction factor has been chosen according to the standard value, first established by Goens [22] for a rectangular cross-section, $\kappa = 5/6$.

3. The doubly clamped beam

When both ends of the beam, whose length is L, are built-in, there are only kinematic type BCs:

$$@x = 0: \quad V = 0 \text{ and } \Phi = 0; \qquad @x = L: \quad V = 0 \text{ and } \Phi = 0. \tag{3.1}$$

Again, the two parts of the spectrum must be treated separately.

3.1. First part of the spectrum: $\omega^2 < \tilde{\omega}^2$

When BCs, Eqs. (3.1), are substituted into Eqs. (2.14) and (2.15), a homogeneous system of simultaneous linear algebraic equations:

$$\mathbf{A}\mathbf{X} = \mathbf{0}, \tag{3.2}$$

is obtained, where matrices $\mathbf{A}$ and $\mathbf{X}$ assume these expressions:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \dfrac{\hat{\alpha}_1}{\hat{\lambda}_1} & 0 & -\dfrac{\alpha_2}{\lambda_2} \\ \cosh \hat{\lambda}_1 L & \sinh \hat{\lambda}_1 L & \cos \lambda_2 L & \sin \lambda_2 L \\ \dfrac{\hat{\alpha}_1}{\hat{\lambda}_1} \sinh \hat{\lambda}_1 L & \dfrac{\hat{\alpha}_1}{\hat{\lambda}_1} \cosh \hat{\lambda}_1 L & \dfrac{\alpha_2}{\lambda_2} \sin \lambda_2 L & -\dfrac{\alpha_2}{\lambda_2} \cos \lambda_2 L \end{bmatrix}, \mathbf{X} = \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix}. \tag{3.3}$$

Since, as a simple check confirms, $\hat{\alpha}_1 \neq 0$ and $\hat{\lambda}_1 \neq 0$ it follows from the first two equations:

$$A_1 = -A_3, \quad A_2 = \hat{\chi} A_4, \quad \hat{\chi} = \frac{\alpha_2 \hat{\lambda}_1}{\hat{\alpha}_1 \lambda_2}, \tag{3.4}$$

and this *reduced* system of equations is arrived at:

$$\begin{bmatrix} (\cos \lambda_2 L - \cosh \hat{\lambda}_1 L) & \sin \lambda_2 L + \hat{\chi} \sinh \hat{\lambda}_1 L \\ \dfrac{\hat{\alpha}_1}{\hat{\lambda}_1} (\hat{\chi} \sin \lambda_2 L - \sinh \hat{\lambda}_1 L) & -\dfrac{\alpha_2}{\lambda_2} (\cos \lambda_2 L - \cosh \hat{\lambda}_1 L) \end{bmatrix} \begin{Bmatrix} A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.5}$$

After some cumbersome algebraic/trigonometric expansions and simplifications, it is found that, for the existence of non-trivial solutions, the following transcendental equation must be satisfied:

$$2(1 - \cosh \hat{\lambda}_1 L \cos \lambda_2 L) + \frac{\hat{\lambda}_1 \lambda_2}{\hat{\alpha}_1 \alpha_2} \left(\frac{\alpha_2^2}{\lambda_2^2} - \frac{\hat{\alpha}_1^2}{\hat{\lambda}_1^2} \right) \sinh \hat{\lambda}_1 L \sin \lambda_2 L = 0. \tag{3.6}$$

Hence, for the doubly clamped beam it is not possible to arrive at a closed form solution: natural frequencies ω_n have to be determined by solving Eq. (3.6) once the expressions of $\hat{\lambda}_1(\omega)$, $\lambda_2(\omega)$, $\hat{\alpha}_1(\omega)$, $\alpha_2(\omega)$ are plugged into it, producing a complicated implicit transcendental equation in ω.

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It has to be emphasized that computing these frequencies is generally an awkward task, since the presence of hyperbolic functions produces wide oscillations, such that automatic procedure can easily fail, and, on the other hand, it is difficult to produce *a priori* suitably bracketed intervals where the existence of just one solution is guaranteed.

It is necessary, of course, to restrict the search for solutions to the range $0 < \omega_n < \tilde{\omega}$, since only in this range, by the analysis performed in Section 2, Eq. (3.6) is guaranteed to assume real values. If the roots of Eq. (3.6) are then denoted by $\omega_n, (n = 1, \ldots, \tilde{n})$, with

$$\tilde{n} = \max\{n \in \mathbb{N} \,|\omega_n < \tilde{\omega}\}, \tag{3.7}$$

the corresponding values of $\hat{\lambda}_1$, λ_2, $\hat{\alpha}_1$, α_2 might be usefully denoted by:

$$\hat{\lambda}_{1n} = \hat{\lambda}_1(\omega_n), \quad \lambda_{2n} = \lambda_2(\omega_n), \tag{3.8}$$

$$\hat{\alpha}_{1n} = \hat{\alpha}_1(\omega_n), \quad \alpha_{2n} = \alpha_2(\omega_n). \tag{3.9}$$

Then, once a solution ω_n is found, the corresponding eigenfunctions can be determined by taking into account that the coefficient matrix of Eq. (3.5) becomes singular when $\omega = \omega_n$. Consequently one of the two equations, *e.g.* the former, can be used to compute A_{3n}, once A_{4n} is fixed or vice-versa, since the ratio between such coefficients is fixed. If, for instance, for normalizing purposes $A_{4n} = 1$ is assumed, then it follows:

$$A_{3n} = \sigma_n, \quad \sigma_n = \frac{\hat{\chi}_n \sinh \hat{\lambda}_{1n} L + \sin \lambda_{2n} L}{\cosh \hat{\lambda}_{1n} L - \cos \lambda_{2n} L}, \quad \hat{\chi}_n = \frac{\alpha_{2n} \hat{\lambda}_{1n}}{\hat{\alpha}_{1n} \lambda_{2n}}. \tag{3.10}$$

By substitution it is finally possible to evaluate the other coefficients:

$$A_{1n} = -\sigma_n, \quad A_{2n} = \hat{\chi}_n, \tag{3.11}$$

and completing by Eqs. (2.14) and (2.15) the construction of the eigenmodes. It is remarked here that, for a doubly clamped beam, *all* functions $\cosh \hat{\lambda}_{1n} x$, $\sinh \hat{\lambda}_{1n} x$, $\cos \lambda_{2n} x$, $\sin \lambda_{2n} x$ appear in the eigenmodes which are relevant to the first part of the spectrum.

The first eigenmodes of a doubly clamped Timoshenko beam are shown in Figure 2; it is clear that hyperbolic functions *do contribute* to the eigenmodes: for instance, for $n > 3$ their effect produces different values for all positive (or negative) peaks.

Remark 1. It is useful noticing that the coefficients multiplying, in Eq. (3.6), the term $\sinh \hat{\lambda}_1 L \sin \lambda_2 L$ can be written also in these alternate ways:

$$\frac{\hat{\lambda}_1 \lambda_2}{\hat{\alpha}_1 \alpha_2} \left(\frac{\alpha_2^2}{\lambda_2^2} - \frac{\hat{\alpha}_1^2}{\hat{\lambda}_1^2} \right) = \left(\frac{\hat{\lambda}_1 \alpha_2}{\lambda_2 \hat{\alpha}_1} - \frac{\lambda_2 \hat{\alpha}_1}{\hat{\lambda}_1 \alpha_2} \right) = \frac{\hat{\lambda}_1^2 \alpha_2^2 - \lambda_2^2 \hat{\alpha}_1^2}{\hat{\alpha}_1 \alpha_2 \hat{\lambda}_1 \lambda_2}, \tag{3.12}$$

and this confirms that both expressions presented by Levinson and Cooke [7, p. 321, Eqs. (13),(15)], for the doubly clamped and for the completely free beam do coincide. The same expression can be found, as well, in Pilkey [23, p. 596–598].

3.2. Transition frequency: $\omega^2 = \tilde{\omega}^2$

When doubly clamped BCs are imposed, a new homogeneous system of simultaneous linear algebraic equations similar to Eq. (3.2) is obtained, where in this case:

$$\mathbf{A} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{\tilde{\alpha}_2}{\tilde{\lambda}_2} & 1 \\ 1 & \cos \tilde{\lambda}_2 L & \sin \tilde{\lambda}_2 L & 0 \\ -\dfrac{\rho \tilde{\omega}^2}{G\kappa} L & -\dfrac{\tilde{\alpha}_2}{\tilde{\lambda}_2} \sin \tilde{\lambda}_2 L & \dfrac{\tilde{\alpha}_2}{\tilde{\lambda}_2} \cos \tilde{\lambda}_2 L & 1 \end{bmatrix}, \quad \mathbf{X} = \begin{Bmatrix} C_1 \\ C_3 \\ C_4 \\ D_1 \end{Bmatrix}. \tag{3.13}$$

where $\tilde{\lambda}_2$ and $\tilde{\alpha}_2$ are given by Eqs. (2.13) and (2.20).

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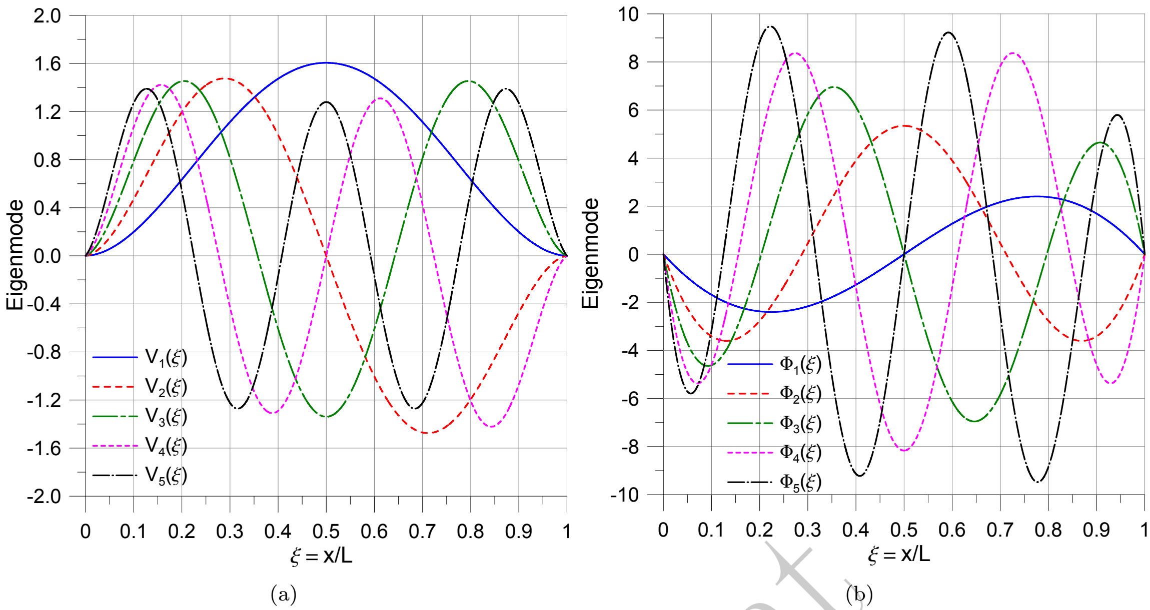

FIGURE 2. Vibration shapes corresponding to modes 1–5 for a doubly clamped Timoshenko beam, *first* part of the spectrum. Transversal displacement, V is shown in (a); section rotation, Φ in (b). Geometric and material data are given in Section 2

The first two equations allow eliminating two unknowns, namely:

$$C_1 = -C_3, \quad D_1 = -\frac{\tilde{\alpha}_2}{\tilde{\lambda}_2} C_4, \tag{3.14}$$

and thus this homogeneous reduced system of equations $\mathbf{A}_r \mathbf{X}_r = \mathbf{0}_r$ can be obtained:

$$\begin{bmatrix} -(1-\cos\tilde{\lambda}_2 L) & \sin\tilde{\lambda}_2 L \\ (\frac{\rho\,\tilde{\omega}^2}{G\kappa} L - \frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}\sin\tilde{\lambda}_2 L) & -\frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}(1-\cos\tilde{\lambda}_2 L) \end{bmatrix} \begin{Bmatrix} C_3 \\ C_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.15}$$

The determinant of the coefficient matrix $\mathbf{A}_r$ appearing in Eq. (3.15), since $\tilde{\alpha}_2 \neq 0$ and $\tilde{\lambda}_2 \neq 0$ might be written as:

$$\det(\mathbf{A}_r) = (1-\cos\tilde{\lambda}_2 L) - \frac{1}{2\tilde{\alpha}_2}\frac{\rho\,\tilde{\omega}^2}{G\kappa}\tilde{\lambda}_2 L \sin\tilde{\lambda}_2 L \tag{3.16}$$

For an assigned value of L, Eq. (3.16) is completely determined and, in general, it is $\det(\mathbf{A}_r) \neq 0$: this implies that the coefficient matrix is non-singular, so that the only solution to Eq. (3.15) is the trivial one, namely $C_3 = 0$, $C_4 = 0$: as a consequence both $V(x) = 0$ and $\Phi(x) = 0$, and it comes out that (in general) *the transition frequency is not part of the spectrum for the doubly clamped beam*, since at such frequency there are no vibrations.

This situation corresponds to values of the beam length such that it is impossible to place a suitable number of sine/cosine waves along the beam span which can, at the same time, satisfy the BCs at both ends of the beam. The only possibility for having frequency $\omega = \tilde{\omega}$ in the spectrum is that the beam length L has been chosen in such a way, $L = \tilde{L}$, that Eq. (3.16), as a function of $\tilde{L}$, vanishes. Then, if the following notation is adopted:

$$z = \tilde{\lambda}_2 \tilde{L}, \quad C = \frac{1}{2\tilde{\alpha}_2}\frac{\rho\,\tilde{\omega}^2}{G\kappa}, \tag{3.17}$$

it follows from Eq. (3.16):

$$(1-\cos z) - C z \sin z = 0. \tag{3.18}$$

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In such conditions, the coefficient matrix $\mathbf{A}_r$ appearing in Eq. (3.15) becomes singular (*i.e.* it has $\mathrm{rank}(\mathbf{A}_r) < 2$) and a non-trivial solution is obtained. In this particular situation, beam length is such that it becomes possible to place a suitable number of sine/cosine waves along the beam span which can, at the same time, satisfy the BCs at both ends of the beam. This occurrence is presented and discussed in details in Section 5.

3.3. Second part of the spectrum: $\omega^2 > \tilde{\omega}^2$

As in the already considered cases, substitution of the BCs into Eqs. (2.21) and (2.22), gives a homogeneous system of simultaneous linear algebraic equations analogous to Eq. (3.2), where in the present case:

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & \dfrac{\alpha_1}{\lambda_1} & 0 & \dfrac{\alpha_2}{\lambda_2} \\ \cos\lambda_1 L & \sin\lambda_1 L & \cos\lambda_2 L & \sin\lambda_2 L \\ -\dfrac{\alpha_1}{\lambda_1}\sin\lambda_1 L & \dfrac{\alpha_1}{\lambda_1}\cos\lambda_1 L & -\dfrac{\alpha_2}{\lambda_2}\sin\lambda_2 L & \dfrac{\alpha_2}{\lambda_2}\cos\lambda_2 L \end{bmatrix}, \mathbf{X} = \begin{Bmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \end{Bmatrix}, \tag{3.19}$$

By taking into account Eq. (2.24), it follows that:

$$E_1 = -E_3, \quad E_2 = \chi E_4, \quad \chi = -\frac{\alpha_2\lambda_1}{\alpha_1\lambda_2}, \tag{3.20}$$

so that Eqs. (3.2) can be reduced to the following:

$$\begin{bmatrix} \cos\lambda_2 L - \cos\lambda_1 L & \chi\sin\lambda_1 L + \sin\lambda_2 L \\ \dfrac{\alpha_1}{\lambda_1}(\sin\lambda_1 L + \chi\sin\lambda_2 L) & \dfrac{\alpha_2}{\lambda_2}(\cos\lambda_2 L - \cos\lambda_1 L) \end{bmatrix} \begin{Bmatrix} E_3 \\ E_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{3.21}$$

Non-trivial solutions might be shown to exist, provided that the following transcendental equation is satisfied:

$$2(1 - \cos\lambda_1 L\cos\lambda_2 L) - \frac{\lambda_1\lambda_2}{\alpha_1\alpha_2}\left(\frac{\alpha_1^2}{\lambda_1^2} + \frac{\alpha_2^2}{\lambda_2^2}\right)\sin\lambda_1 L\sin\lambda_2 L = 0. \tag{3.22}$$

Again, for the doubly clamped beam it is not possible to arrive at a closed form solution: natural frequencies ω_n have to be determined by solving Eq. (3.22) once the expressions of $\lambda_1(\omega)$, $\lambda_2(\omega)$, $\alpha_1(\omega)$, $\alpha_2(\omega)$ are plugged into it, producing a complicated implicit transcendental equation in ω.

It has to be emphasized that also computing these frequencies is generally a difficult task — even though it is less awkward than in the case of solving Eq. (3.6), since the presence of trigonometric functions depending on two wave-numbers, whose ratio is not (in general) a rational number produces non periodic oscillations, such that automatic procedure can still fail, while, on the other hand, it is difficult to identify *a priori* suitably bracketed intervals where the existence of just one solution is guaranteed.

It is necessary, of course, to restrict the search for solutions to the range $\omega_n > \tilde{\omega}$, since only in this range, by the analysis performed in Section 2, Eq. (3.22) is guaranteed to assume real values. If the roots of Eq. (3.6) are denoted by $\omega_n, (n = \tilde{n} + 1, \ldots, \infty)$, with $\tilde{n}$ defined by Eq. (3.7), the corresponding values of λ_1, λ_2, α_1, α_2 might be usefully denoted by:

$$\lambda_{1n} = \lambda_1(\omega_n), \quad \lambda_{2n} = \lambda_2(\omega_n), \tag{3.23}$$

$$\alpha_{1n} = \alpha_1(\omega_n), \quad \alpha_{2n} = \alpha_2(\omega_n). \tag{3.24}$$

Once a solution ω_n is found, the corresponding eigenfunctions can be determined by taking into account that the coefficient matrix of Eq. (3.21) becomes singular when $\omega = \omega_n$; consequently one of the two equations, *e.g.* the former, can be used to compute E_{3n}, once E_{4n} is chosen or vice-versa, since the ratio between such coefficients is fixed. If, for instance, for normalizing purposes $E_{4n} = 1$ is assumed, then it follows:

$$E_{3n} = \tau_n, \quad \tau_n = \frac{\chi_n\sin\lambda_{1n}L + \sin\lambda_{2n}L}{\cos\lambda_{1n}L - \cos\lambda_{2n}L}, \quad \chi_n = -\frac{\alpha_{2n}\lambda_{1n}}{\alpha_{1n}\lambda_{2n}}. \tag{3.25}$$

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By substitution, see Eq. (3.20), it is finally possible to evaluate the other coefficients:

$$E_{1n} = -\tau_n, \quad E_{2n} = \chi_n, \tag{3.26}$$

and completing by Eqs. (2.21) and (2.22) the construction of the eigenmodes. For a doubly clamped beam all four functions $\cos \lambda_{1n} x$, $\sin \lambda_{1n} x$, $\cos \lambda_{2n} x$, $\sin \lambda_{2n} x$ appear in the eigenmodes which are relevant to the second part of the spectrum. The first eigenmodes belonging to this part of the spectrum are portrayed in Figure 3.

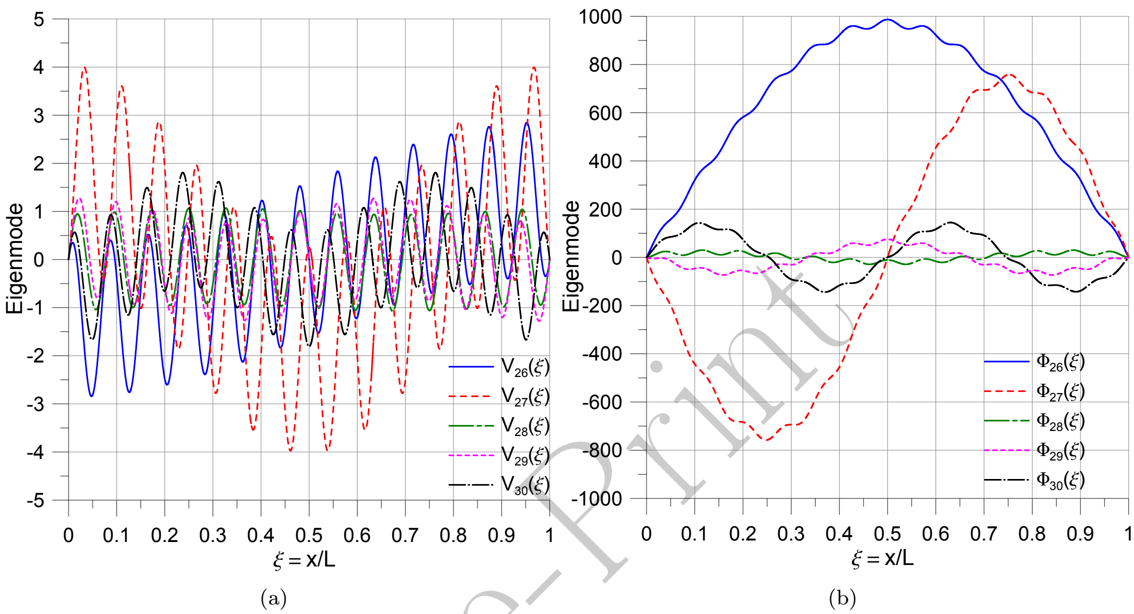

FIGURE 3. Vibration shapes corresponding to modes 26–30 for a doubly clamped Timoshenko beam, *second* part of the spectrum. Transversal displacement, V is shown in (a); section rotation, Φ in (b). Geometric and material data are given in Section 2.

4. Construction of the spectrum for the doubly clamped beam

For the same data set which has been explained in Section 2 the spectrum for the doubly clamped beam has been constructed. Of course, as already mentioned, in the present case it is not possible to identify natural frequencies by a closed-form procedure. It is indeed necessary to numerically find the roots of the transcendental equations (3.6), for the first part of the spectrum, and (3.22), for the second one.

Solutions were obtained with a Computer Algebra System (CAS), namely Mathematica™(version 6.0). The roots of the above mentioned transcendental equations (3.6) or (3.22) have been computed by using the native function `FindRoot`, see [24], which implements a variant of the secant method. Bracketing intervals to isolate roots were defined by properly magnified plots of the corresponding functions.

In the present paper, all roots have been computed by assigning variables with 250 digit precision. Moreover, any root, once computed, has been back-substituted into the equation and the error obtained, ε, has been checked against a predefined small tolerance: it has been verified that all provided roots satisfy the corresponding transcendental equation to within $|\varepsilon| \leq 1 \cdot 10^{-200}$.

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TABLE 1. Computed natural frequencies, wave-numbers and vibration amplitudes of a doubly clamped Timoshenko beam for the first $N = 50$ vibration modes, *first* part of the spectrum. Circular frequency, ω_n, is expressed in rad/s, wave-numbers $\hat{\lambda}_{1n}$ (a *generalized* one) and λ_{2n} in rad/m; all other parameters are dimensionless.

n	ω_n	$\hat{\lambda}_{1n}$	λ_{2n}	A_{4n}	A_{2n}	A_{3n}
1	904.9409611	2.333875334	2.356010971	1.	−0.9810279387	−0.9996330292
2	2441.571820	3.802561816	3.900664347	1.	−0.9496312793	−0.9486861187
3	4657.856049	5.190071344	5.448594130	1.	−0.9060985311	−0.9061547886
4	7455.126708	6.466375170	6.989971677	1.	−0.8539603412	−0.8539562122
5	10742.87361	7.620475330	8.526546585	1.	−0.7964371880	−0.7964375710
6	14436.14841	8.647183238	10.05946258	1.	−0.7363144266	−0.7363143812
7	18460.58387	9.545829233	11.58986376	1.	−0.6757860398	−0.6757860467
8	22753.49430	10.31852142	13.11880486	1.	−0.6164595057	−0.6164595044
9	27263.30693	10.96882488	14.64722347	1.	−0.5594329277	−0.5594329280
10	31948.16369	11.50082121	16.17591918	1.	−0.5053970445	−0.5053970444
11	36774.29503	11.91847562	17.70554362	1.	−0.4547343471	−0.4547343471
12	41714.47640	12.22521743	19.23659970	1.	−0.4076033781	−0.4076033780
13	46746.69904	12.42364924	20.76944902	1.	−0.3640052529	−0.3640052529
14	51853.08522	12.51531507	22.30432521	1.	−0.3238336691	−0.3238336691
15	57019.02924	12.50046891	23.84135072	1.	−0.2869111247	−0.2869111247
16	62232.52671	12.37778669	25.38055471	1.	−0.2530141407	−0.2530141407
17	67483.65139	12.14395276	26.92188965	1.	−0.2218897524	−0.2218897525
18	72764.14249	11.79301522	28.46524490	1.	−0.1932647190	−0.1932647190
19	78067.06975	11.31532041	30.01045514	1.	−0.1668478188	−0.1668478188
20	83386.54582	10.69564009	31.55730091	1.	−0.1423238333	−0.1423238331
21	88717.44914	9.909609302	33.10549525	1.	−0.1193340557	−0.1193340563
22	94055.08973	8.916157340	34.65464041	1.	−0.0974277442	−0.0974277407
23	99394.62680	7.638505501	36.20410188	1.	−0.0759317595	−0.0759317947
24	104729.3945	5.900796868	37.75255648	1.	−0.0534963800	−0.0534955784
25	110040.4377	3.009915176	39.29497761	1.	−0.0249531228	−0.0250746945

The frequencies corresponding to the first 50 eigenmodes are reported in Tables 1 and 2. In order to allow the interested reader to reproduce the whole set of results provided here (up to $N = 100$), the relevant data for vibration modes 51–100 are reported in Appendix.

For comparison purpose the full spectrum relevant to the first 100 vibration modes for a doubly clamped beam according to both Euler-Bernoulli and Timoshenko models is shown in Figure 4. For the Euler-Bernoulli beam the natural frequencies are given, in the doubly clamped case, by the roots of this transcendental equation:

$$\cos(\lambda_{EB}L)\cosh(\lambda_{EB}L) - 1 = 0, \quad \text{with} \quad \lambda_{EB} = \sqrt[4]{\frac{\rho A\omega^2}{EI}} \tag{4.1}$$

and, as it is well-known, see *e.g.* [25], for sufficiently large values of k, $k \geq 5$, this asymptotic estimate of the natural frequencies holds:

$$\omega_{k,EB} = (2k+1)^2 \sqrt{\frac{EI}{\rho A}} \left(\frac{\pi}{2L}\right)^2. \tag{4.2}$$

Also in this case, it is apparent that for the Timoshenko beam model vibration frequencies are much less separated than for the Euler-Bernoulli one.

5. Eigenmodes corresponding to the transition frequency for a doubly clamped beam

In Section 3 the existence of eigenmodes corresponding to the transition frequency $\tilde{\omega}$ for a doubly clamped beam has been shown to be possible only for particular values of the beam length, $L = \tilde{L}$.

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TABLE 2. Computed natural frequencies, wave-numbers and vibration amplitudes of a doubly clamped Timoshenko beam for the first $N = 50$ vibration modes, *second* part of the spectrum. Circular frequency, ω_n, is expressed in rad/s, wave-numbers λ_{1n} and λ_{2n} in rad/m; all other parameters are dimensionless.

n	ω_n	λ_{1n}	λ_{2n}	E_{4n}	E_{2n}	$E_{3n} = -E_{1n}$
26	112269.3096	1.560860889	39.94262807	1.	0.01247129284	1.255192040
27	113685.9451	3.154610786	40.35438770	1.	0.02462586984	−1.891552346
28	115148.1517	4.228220178	40.77950049	1.	0.03222873189	0.061280546
29	116187.3003	4.859519139	41.08168542	1.	0.03642139904	−0.245756623
30	118829.9614	6.212298595	41.85044613	1.	0.04462137566	0.628419760
31	120909.4747	7.125155636	42.45567519	1.	0.04951036164	−0.044203124
32	122643.0746	7.821648728	42.96043358	1.	0.05288126459	1.634954669
33	126211.5471	9.129479716	44.00005262	1.	0.05837541997	−0.017757338
34	127024.2937	9.409313719	44.23695352	1.	0.05941225256	3.841605639
35	131561.6327	10.88276457	45.56035559	1.	0.06410273088	−0.007262243
36	131903.3768	10.98865818	45.66009148	1.	0.06439191971	9.310275297
37	136934.5768	12.48511319	47.12941462	1.	0.06784249797	−0.005524872
38	137198.2900	12.56074267	47.20648235	1.	0.06798682147	12.08008627
39	142323.5083	13.98660738	48.70533795	1.	0.07021545310	−0.010652219
40	142838.2723	14.12571436	48.85599156	1.	0.07038556925	6.145557559
41	147724.4485	15.41546836	50.28704089	1.	0.07160241714	−0.021561772
42	148764.8159	15.68366221	50.59198395	1.	0.07177899462	2.953157858
43	153137.7707	16.79001879	51.87468473	1.	0.07225813281	−0.038424837
44	154923.9204	17.23315452	52.39904651	1.	0.07234599571	1.585258689
45	158572.9376	18.12466188	53.47108258	1.	0.07236128056	−0.064095022
46	161247.2816	18.76743475	54.25744496	1.	0.07225032165	0.877822730
47	164062.9884	19.43550401	55.08598730	1.	0.07203773291	−0.108539094
48	167595.3748	20.26224859	56.12630413	1.	0.07165147028	0.449410558
49	169718.3241	20.75356403	56.75200209	1.	0.07136378463	−0.206183737
50	173670.4499	21.65813465	57.91775093	1.	0.07073370830	0.204494218

Now, the occurrence of such eigenmodes is further investigated. The frequency equation which needs to be satisfied is given by Eq. (3.18), here reproduced for the reader's convenience:

$$(1 - \cos z) - Cz \sin z = 0,$$

where $z = \tilde{\lambda}_2 \tilde{L}$, and C is a constant factor, whose definition is given by Eq. (3.17).

It is not difficult to acknowledge that Eq. (3.18) admits two kind of solutions:

1. *periodic solutions*, of the form:

$$z = 2\pi j, \quad (j = 1, 2, \ldots, \infty), \tag{5.1}$$

corresponding to even multiples of π. Consequently, it follows that the beam length must have these precise values:

$$\tilde{L}_j = \frac{z}{\tilde{\lambda}_2} = \frac{2\pi j}{\tilde{\lambda}_2}, \quad (j = 1, 2, \ldots, \infty). \tag{5.2}$$

For such values of z, both $\sin z$ and $(\cos z - 1)$ do vanish: as a consequence, the first Eq. (3.15) becomes an identity, while the second one ensures the existence of non-trivial solutions only for $C_3 = \tilde{C}_3 = 0$, $C_4 = \tilde{C}_4 \neq 0$.

Then, if for normalization purposes $\tilde{C}_4 = 1$ is assumed, the remaining coefficients assume these values:

$$\tilde{C}_1 = 0, \quad \tilde{D}_1 = -\frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}. \tag{5.3}$$

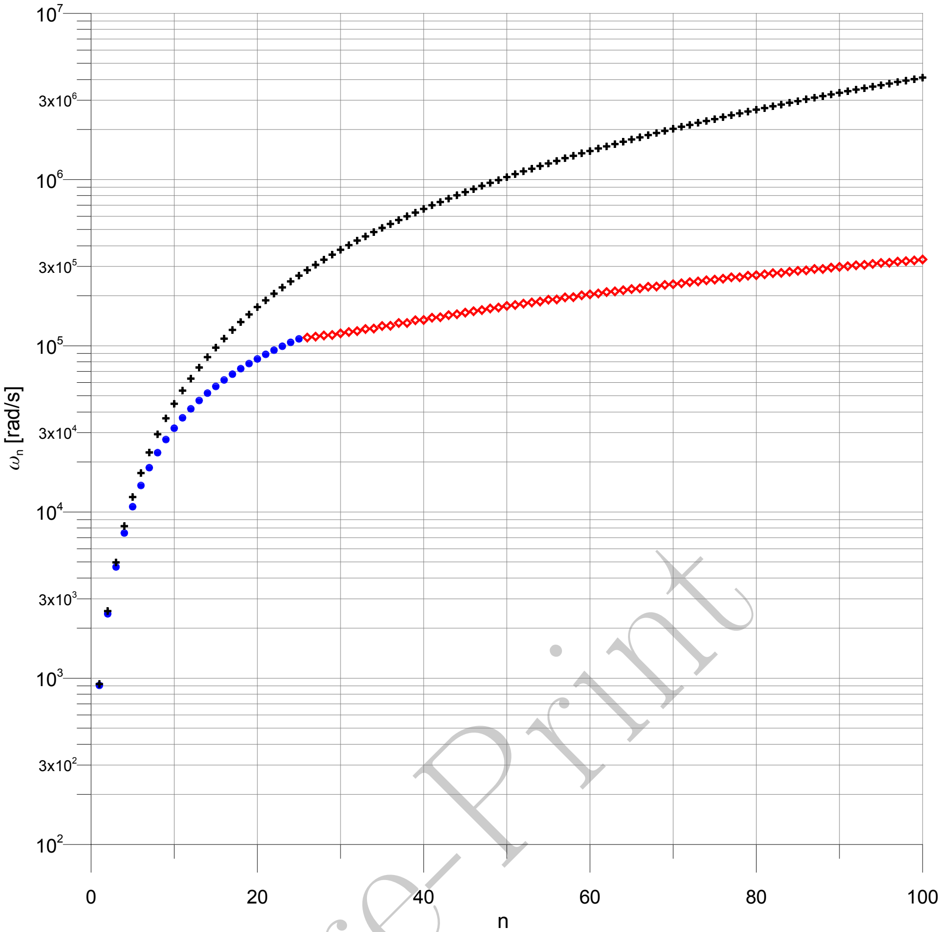

FIGURE 4. Full frequency spectrum, *i.e.* ω_n *vs.* n plot (for N=100 modes) for the doubly clamped Euler-Bernoulli beam model (denoted by *crosses*) and for the Timoshenko one. For the latter, modes corresponding to the *first* part of the spectrum are marked by solid dots, modes corresponding to the *second* part are denoted by hollow diamonds.

So, by Eqs. (2.18)–(2.19) the eigenmodes are the following:

$$\tilde{V}(x) = \sin \tilde{\lambda}_2 x, \quad \tilde{\Phi}(x) = \frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}(\cos \tilde{\lambda}_2 x - 1). \tag{5.4}$$

2. *non-periodic solutions*, such that:

$$1 - \cos z = C z \sin z, \quad z \neq 2\pi j, \; j \in \mathbb{N}, \tag{5.5}$$

such that both $\sin z \neq 0$ and $(1 - \cos z) \neq 0$. It is possible to show, see Figure 5 that there is *one and only one* such solution for each interval $2\pi j < z < 2\pi(j+1)$, with $j \in \mathbb{N}$.

In this case, the coefficient matrix $\mathbf{A}_r$ of Eq. (3.15) becomes singular, and any of the two equations allows identifying the ratio between C_3 and C_4; if the first equation is used, under the normalization assumption $\tilde{C}_4 = 1$, it results:

$$\tilde{C}_3 = \frac{\sin \tilde{\lambda}_2 \tilde{L}}{1 - \cos \tilde{\lambda}_2 \tilde{L}}, \quad \tilde{C}_1 = \frac{-\sin \tilde{\lambda}_2 \tilde{L}}{1 - \cos \tilde{\lambda}_2 \tilde{L}}, \quad \tilde{D}_1 = -\frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}. \tag{5.6}$$

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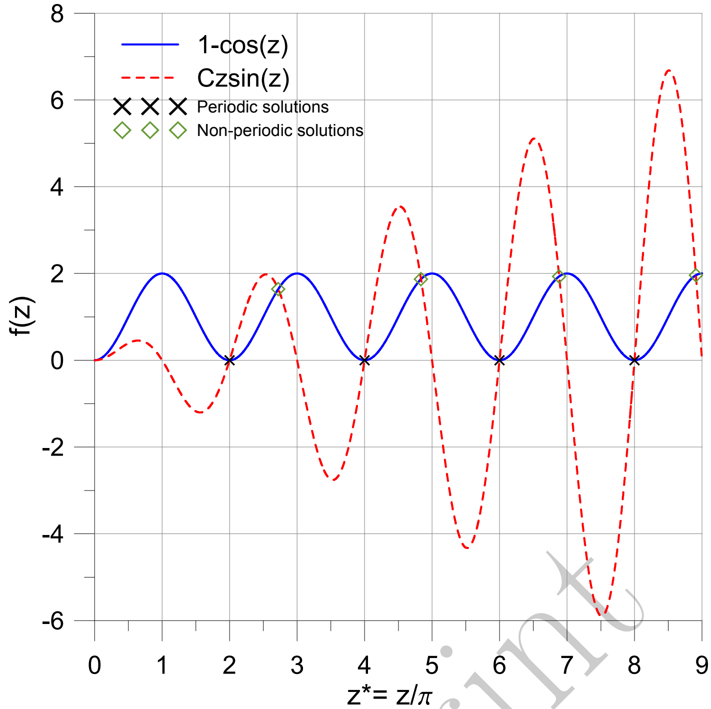

FIGURE 5. Illustration of the solutions to equation $(1-\cos z)-Cz\sin z=0$. Periodic solutions are marked by crosses, non-periodic ones by hollow diamonds. In the present case, $C=1/4$. For better readability a dimensionless coordinate $z^\star=z/\pi$ has been adopted.

So, for the non-periodic case, by Eqs. (2.18)–(2.19) the eigenmodes become:

$$\tilde{V}(x)=\frac{\sin\tilde{\lambda}_2\tilde{L}}{1-\cos\tilde{\lambda}_2\tilde{L}}(\cos\tilde{\lambda}_2 x-1)+\sin\tilde{\lambda}_2 x, \tag{5.7}$$

$$\tilde{\Phi}(x)=\frac{\tilde{\alpha}_2}{\tilde{\lambda}_2}\left[(\cos\tilde{\lambda}_2 x-1)+\frac{\sin\tilde{\lambda}_2\tilde{L}}{1-\cos\tilde{\lambda}_2\tilde{L}}\left(\frac{1}{\tilde{\alpha}_2}\frac{\rho\tilde{\omega}^2}{G\kappa}\tilde{\lambda}_2 x-\sin\tilde{\lambda}_2 x\right)\right]. \tag{5.8}$$

Then, by recognizing that, according to Eq. (3.17),

$$\frac{1}{\tilde{\alpha}_2}\frac{\rho\tilde{\omega}^2}{G\kappa}=2C$$

it is not difficult verifying that $\tilde{\Phi}(x)$ complies with BCs at both ends of the beam.
Finally, it is useful to remark that *periodic* solutions are always characterized by a space-frequency value which is an integer value, *i.e.*

$$f_{\tilde{\lambda}_2}=\frac{\tilde{\lambda}_2}{2\pi}\in\mathbb{N}^+,$$

while for the *non-periodic* ones this does not happen, and consequently, it results $f_{\tilde{\lambda}_2}\in\mathbb{R}^+$.

6. Final remarks and perspectives

The complete analysis of free vibrations for the Timoshenko beam model has been presented and carefully discussed in order to highlight the nature of the vibration spectrum, which has often been overlooked in the past. The analysis reveals indeed that there is a transition frequency (or *cut-off*

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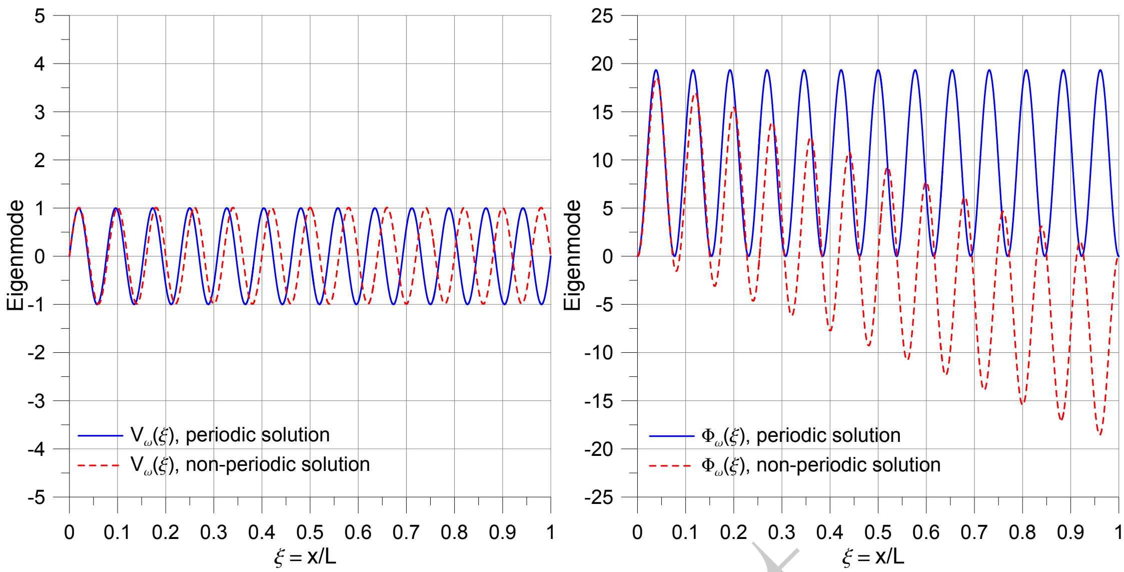

Figure 6. Vibration shapes corresponding to the *transition frequency* for a doubly clamped Timoshenko beam. Transversal displacement, V is shown in (a); section rotation, Φ in (b). The case of a periodic solution (corresponding to a length $\tilde{L} = 2.0519240731$ m) is marked by solid lines; that of a non-periodic solution (which is relevant to a length $\tilde{L} = 1.9734138880$ m) is denoted by dashed lines. Geometric and material data are given in Section 2.

frequency, in the language of wave propagation analysis, see [26]) which subdivide the spectrum corresponding to natural frequencies in two parts; each one of them exhibits a rather different shape. The transition frequency itself might be part of the spectrum, and has a characteristic vibration mode.

As a consequence, for a Timoshenko beam the vibration spectrum is obviously unique, but it has to be considered as formed by two parts, none of which can be, in principle, disregarded. Particular attention has been devoted to two special cases of boundary conditions: the simply supported beam and the doubly clamped one. They provide a rather simple — but exhaustive enough — representative view of the 10 independent combinations which can be formed with the four elementary end constraints (*e.g.* clamped, free, guided, supported) in a single-span beam.

For simply supported boundary conditions, the transcendental equation which provides the wave-numbers corresponding to natural frequencies has been shown to be factorized. Hence, this property produces vibration modes which have in both part of the spectrum a simple shape, consisting of an integer number of sine/cosine half-waves, while both components of the eigenmode corresponding to the transition frequency are constant functions.

Conversely, for doubly clamped boundary conditions the transcendental equation does not factorize, and this produces much more complicated vibration modes. In particular, in the first part of the spectrum, both circular and hyperbolic sine/cosine functions are combined in each eigenmode, while in the second part of it, there appears a combination of sine/cosine functions depending, however, on *two* different wave-numbers. And the transition frequency is not, in general, part of the spectrum: this is a common feature shared by the doubly clamped beam and by all other combinations of end constraints, with the only exception of the simply supported case.

For both considered cases, the simply supported, which has been analysed in [19] and the doubly clamped, which has been considered here, and for the *same* mechanical and geometric data (which have been chosen in such a way that they are representative of a beam model where shear strain effects are expected to be non-negligible), a complete list of the first 50 natural frequencies has been provided,

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along with the parameters which are necessary to completely identify the corresponding vibration modes, for both kinematic variables, transversal displacement, V, and cross-section rotation, Φ. The plots of some representative modes have been given, too, to better illustrate the Timoshenko beam response in terms of free vibrations.

The results presented in this work could be used for an in-depth analysis of some current and more complicated problems. For instance, the case of curved Timoshenko beams would be interesting and useful for technical applications. Particularly interesting is the extension to the computational framework, for example by applying the isogeometric approach: see for 1D problem these recently appeared contributions [27, 28, 29, 30, 31, 32, 33]. Also the use of highly-efficient discretisation techniques, such as those reported in [34, 35, 36] is interesting: indeed they provide more refined stress description and might therefore improve the accuracy of numerical results. Geometric nonlinearities have to be considered, as well, *viz.* by using the suggestions presented in [37, 38, 39, 40, 41, 42, 43], while a complete dynamic approach for the generalized beam theory has been addressed in [44, 45], and in the references cited therein, and in [46, 47] for wave propagation problem in second gradient continua and micromorphic materials. Also the effects of piecewise–smooth non linearities due to impact, see [48, 49, 50], will have the role of modifying the frequency spectrum

As it is well-known, the Timoshenko beam model is a particularly simple micro-mechanical model and can therefore be thought of as a simple prototype for providing fruitful clues for the development of new and refined mathematical models of continua. The interested readers will find many insight looking at the current research trend on generalized continua and their applications, for example in [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64], taking also into consideration the hints reported in [65, 66, 67].

Finally, it has to be pointed out that an accurate evaluation of the spectrum is fundamental in problems which consider damage detection, see for example [68, 69, 70, 71] and references provided therein, or try to optimize the structural response of smart structures such as the one described in [72, 73].

Appendix

To complete the solution of the doubly clamped beam case for the first $N = 100$ vibration modes, data for vibration modes 51–100 are presented in Table 3.

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Table 3. Computed natural frequencies, wave-numbers and vibration amplitudes of a doubly clamped Timoshenko beam for vibration modes from $n = 51$ to $n = 100$, *second* part of the spectrum. Circular frequency, ω_n, is expressed in rad/s, wave-numbers λ_{1n} and λ_{2n} in rad/m; all other parameters are dimensionless.

n	ω_n	λ_{1n}	λ_{2n}	E_{4n}	E_{2n}	$E_{3n} = -E_{1n}$
51	175811.7676	22.14318932	58.54987532	1.	0.07034766756	−0.459118843
52	179346.3332	22.93673625	59.59406036	1.	0.06965236462	0.096497915
53	182447.1401	23.62615083	60.51088478	1.	0.06899139578	−1.073056114
54	184827.8872	24.15150587	61.21529678	1.	0.06845651881	0.045791933
55	189397.2140	25.15090897	62.56843836	1.	0.06737486928	−3.708081370
56	190239.8805	25.33401394	62.81814915	1.	0.06716863480	0.013704778
57	195629.3486	26.49692725	64.41644240	1.	0.06581006167	−0.013793874
58	196507.2437	26.68508240	64.67698585	1.	0.06558325856	3.553251405
59	201029.2903	27.64903037	66.01990300	1.	0.06439639396	−0.046490722
60	203656.2278	28.20518365	66.80067643	1.	0.06369551989	0.919649317
61	206513.8295	28.80718278	67.65053975	1.	0.06292628699	−0.106699874
62	210476.5167	29.63712649	68.82996305	1.	0.06185162743	0.293057401
63	212427.0269	30.04367433	69.41087680	1.	0.06132060651	−0.304782294
64	216378.2871	30.86348974	70.58841974	1.	0.06024340662	0.097724284
65	219341.5607	31.47518116	71.47217618	1.	0.05943609639	−1.001888398
66	221843.2034	31.98959594	72.21868357	1.	0.05875607346	0.037965691
67	226756.4264	32.99490284	73.68593591	1.	0.05742748407	−7.020338225
68	227210.5725	33.08750564	73.82163241	1.	0.05730527632	0.005795019
69	232559.3969	34.17428743	75.42074693	1.	0.05587553482	−0.022527055
70	234289.3350	34.52430729	75.93829394	1.	0.05541738156	1.667986824
71	237946.5884	35.26203611	77.03299622	1.	0.05445661209	−0.064063636
72	241635.1978	36.00313044	78.13783836	1.	0.05349928147	0.425126730
73	243607.8676	36.39830005	78.72901330	1.	0.05299245796	−0.191487223
74	247859.9811	37.24744650	80.00400579	1.	0.05191294574	0.107012586
75	250452.9197	37.76354957	80.78196205	1.	0.05126371034	−0.794452877
76	253325.4219	38.33385082	81.64419920	1.	0.05055278308	0.037226979
77	258100.0914	39.27858356	83.07832744	1.	0.04939098762	−5.693086331
78	258659.1277	39.38894229	83.24631432	1.	0.04925661775	0.005891068
79	263972.1627	40.43523696	84.84360170	1.	0.04799725469	−0.020602970
80	265874.0444	40.80868893	85.41570118	1.	0.04755430234	1.484841673
81	269330.4257	41.48597742	86.45583536	1.	0.04676001807	−0.062117094
82	273327.3000	42.26699962	87.65929911	1.	0.04585885540	0.315147702
83	275081.9347	42.60915760	88.18784698	1.	0.04546913093	−0.227037252
84	279312.7263	43.43245109	89.46283964	1.	0.04454420287	0.072678546
85	282346.9033	44.02143957	90.37769229	1.	0.04389371871	−1.120812042
86	284684.9669	44.47450264	91.08291914	1.	0.04339976231	0.023272013
87	289974.9416	45.49711286	92.67935404	1.	0.04230543863	−0.002370762
88	290246.5133	45.54952090	92.76134072	1.	0.04225012542	11.826186890
89	295263.4370	46.51617022	94.27645773	1.	0.04124336139	−0.028685991
90	298156.7416	47.07237102	95.15068206	1.	0.04067564611	0.788831809
91	300656.8796	47.55226579	95.90636456	1.	0.04019258572	−0.088014847
92	305122.6989	48.40784962	97.25676278	1.	0.03934684125	0.133392313
93	307044.7656	48.77546848	97.83819064	1.	0.03898951107	−0.481598553
94	310630.8363	49.46038258	98.92333332	1.	0.03833341307	0.036851941
95	314917.5299	50.27751118	100.22107170	1.	0.03756695275	−3.122952859
96	315915.5746	50.46751520	100.52330632	1.	0.03739124924	0.007658744
97	321172.8013	51.46690015	102.11588447	1.	0.03648255848	−0.014124168
98	322997.6762	51.81324202	102.66890827	1.	0.03617366093	1.569979987
99	326467.9150	52.47108277	103.72085052	1.	0.03559535901	−0.048330526
100	330680.6049	53.26834291	104.99836301	1.	0.03490909911	0.250012985

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