Maxwellian Iteration of a Causal Relativistic Model of Polyatomic Gases and evaluation of Bulk, Shear viscosity and Heat conductivity

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Abstract

In the present paper, we give the parabolic limit of the field equations of a recent hyperbolic model of relativistic polyatomic gas in the framework of Rational Extended Thermodynamics (RET) theory. We obtain in this way the corresponding constitutive equations of the Thermodynamics of Irreversible Processes (TIP) obtained first in the context of relativity by Eckart. The limit is reached via the Maxwellian iteration procedure to make a connection between the phenomenological coefficients (shear, bulk viscosity, and heat conductivity) and the relaxation times. The classical and ultrarelativistic limit of these coefficients are also obtained finding that, in the case of classical limit, they coincide with the ones known in the literature. As a particular case, we study the monatomic gas and we can plot the dimensionless coefficients associated with bulk, shear, and heat conductivity. In contrast to a monatomic gas in which the bulk viscosity is very small and tends to the classical regime of the order of magnitude of $O(1/c^4)$, the bulk viscosity and the dynamical pressure for polyatomic gases, due to the rotation and vibration of internal modes, are of the order of unit (except in the ultrarelativistic limit), and this might open new perspectives in cosmology.

Keyword: Relativistic Fluids, Maxwellian Iteration, Bulk viscosity, Rational Extended Thermodynamics, Rarefied Polyatomic Gases

1 Introduction

In relativistic fluid dynamics, it is mandatory to adopt hyperbolic systems to satisfy the condition that perturbations propagate with finite velocities less than the light velocity. The Eckart theory of relativistic dissipative fluid mechanics [1] that is the relativistic counterpart of Navier-Stokes-Fourier theory is not appropriate due to its parabolic character. Moreover as proved by Hiscock and Lindblom [2] there exist generic short wavelength secular instabilities; and there is not a well-posed initial value problem for rotating fluids. Müller [3] and independently Israel [4] gave the first tentative of a causal relativistic phenomenological theory of gases. A successive modern approach being compatible, at the mesoscopic scale, with the kinetic theory was done by Liu, Müller, and Ruggeri (LMR) [5, 6]. A recent theory on this subject was given by Pennisi and Ruggeri that proposed a causal relativistic theory for a non-equilibrium rarefied gas with an internal structure that can take into account the energy exchange between translational modes and internal modes typical of polyatomic gas [7]. The theory includes the LMR as a singular limit [8] and in the classical limit converges to the model of Extended Thermodynamics for polyatomic gas [9]. This paper aims to complete the theory giving the so-called Maxwellian Iteration [10], a sort of Chapman-Enskog expansion [11], with respect to the relaxation times. In such a way we can obtain the parabolic limit of the theory that corresponds in relativity to the Eckart one. In this way, we can evaluate the expression of the phenomenological coefficients as shear, bulk viscosity, and heat conductivity in terms of the relaxations times, the degree of freedom of molecules D, and the relativistic parameter $\gamma = mc^2/k_BT$ (m is the atomic mass in the rest frame, c is the light velocity, k_B is the Boltzmann constant and T is the temperature). We evaluate the classical ($\gamma \to \infty$) and the ultrarelativistic limit ($\gamma \to 0$) and in the classical limit the coefficients coincide with the ones obtained in [9]. Then we consider the monatomic limit ($D \to 3$) obtaining the same coefficients of monatomic gas evaluated in the LMR paper [5] but with explicit expressions of the production terms coefficients. This permits to know in all ranges of γ the dimensionless bulk viscosity, shear viscosity, and heat conductivity. The bulk viscosity ν is of the order of unity in any range of γ including the classical case, except in the ultrarelativistic limit. This is in contrast with the monatomic gas in which in relativistic regime ν is small and in the classical limit tends to zero with the order of $O(1/c^4)$.

2 Summary of Casual Relativistic Theory of Polyatomic gas

In [7] the following system was considered to describe a relativistic polyatomic gas:

$$\partial_\alpha V^\alpha = 0 \quad , \quad \partial_\alpha T^{\alpha\beta} = 0 \quad , \quad \partial_\alpha A^{\alpha<\beta\gamma>} = I^{<\beta\gamma>} \, , \tag{1}$$

where the partial derivative ∂_α is calculated with respect to the space-time coordinates x^α with $\alpha = 0, 1, 2, 3$, and $< \cdots >$ denotes the traceless part of a tensor. V^α and $T^{\alpha\beta}$ can be written in terms of the usual physical variables:

$$V^\alpha = nmU^\alpha, \qquad T^{\alpha\beta} = t^{<\alpha\beta>} + (p+\pi)h^{\alpha\beta} + \frac{2}{c^2}\,U^{(\alpha}q^{\beta)} + \frac{e}{c^2}\,U^\alpha U^\beta,$$

so that it is clear that $(1)_{1,2}$ represent the conservation laws of particle number and energy momentum, where U^α is the four-velocity, n is the number density, p is the equilibrium pressure, π is the dynamical pressure, $h^{\alpha\beta} = -g^{\alpha\beta} + \frac{1}{c^2}U^\alpha U^\beta$ is the projector tensor, $g^{\alpha\beta} = g_{\alpha\beta} = diag(1\,,\,-1\,,\,-1\,,\,-1)$ is the metric tensor, $t^{<\alpha\beta>} = T^{\mu\nu}\left(h^\alpha_\mu h^\beta_\nu - \frac{1}{3}h^{\alpha\beta}h_{\mu\nu}\right)$ is the viscous deviatoric stress, e is the energy, $q^\alpha = -h^\alpha_\mu U_\nu T^{\mu\nu}$ is the heat flux and $\rho = nm$. The field variables are n, U^α, $t^{<\alpha\beta>}$, π, q^α and the absolute temperature T but only 14 of them are independent, because of the constraints $U_\alpha q^\alpha = 0$, $U_\alpha t^{<\alpha\beta>} = 0$, $g_{\alpha\beta}\,t^{<\alpha\beta>} = 0\,$, $U_\alpha U^\alpha = c^2$.
In [7] Pennisi and Ruggeri considered the generalized relativistic Boltzmann-Chernikov equation

$$p^\alpha \partial_\alpha f = Q\,, \tag{2}$$

where the distribution function $f \equiv f(x^\alpha, p^\alpha, \mathcal{I})$ depends not only on the space-time coordinates x^α and on the four-momentum p^β, ($p^\alpha p_\alpha = m^2c^2$), but also on a quantity $\mathcal{I}$ which takes into account the microscopic internal energy due to the rotation and vibrations of the molecules. In the present case the system (1) is the truncated system of the following moments associated with the Boltzmann-Chernikov equation (2):

$$\begin{aligned}
V^\alpha &= mc\int_{\mathbb{R}^3}\int_0^{+\infty} f p^\alpha \phi(\mathcal{I})\,d\mathcal{I}\,d\boldsymbol{P},\\
T^{\alpha\beta} &= \frac{1}{mc}\int_{\mathbb{R}^3}\int_0^{+\infty} f\left(mc^2+\mathcal{I}\right)p^\alpha p^\beta\,\phi(\mathcal{I})\,d\mathcal{I}\,d\boldsymbol{P}\,,\\
A^{\alpha\beta\gamma} &= \frac{1}{m^2c}\int_{\mathbb{R}^3}\int_0^{+\infty} f\left(mc^2+2\mathcal{I}\right)p^\alpha p^\beta p^\gamma \phi(\mathcal{I})\,d\mathcal{I}\,d\boldsymbol{P},\\
I^{\beta\gamma} &= \frac{c}{m}\int_{\mathbb{R}^3}\int_0^{+\infty} Q\,p^\beta p^\gamma\,\phi(\mathcal{I})\,d\mathcal{I}\,d\boldsymbol{P},
\end{aligned} \tag{3}$$

where

$$d\boldsymbol{P} = \frac{dp^1\,dp^2\,dp^3}{p^0},$$

and $\phi(\mathcal{I})$ is the state density corresponding to $\mathcal{I}$, i.e., $\phi(\mathcal{I})d\mathcal{I}$ represents the number of internal state between $\mathcal{I}$ and $\mathcal{I}+d\mathcal{I}$. The measure $\phi(\mathcal{I})$ is necessary to recover in the classical theory the caloric equation of state of internal energy for polyatomic gases as observed first in classical framework in [12].
As usual for the truncated moment system there is the closure problem and we need to determine $A^{\alpha\beta\gamma}$ and $I^{\beta\gamma}$ in terms of the 14 independent field variables. To reach the closure in [7] the Principle of Maximum Entropy (MEP) was used. This principle was developed first by Jaynes in the context of the theory of information based on the Shannon entropy [13, 14]. It states that the probability distribution that represents the current state of knowledge in the best way is the one with the largest entropy. The MEP in nonequilibrium thermodynamics was originally applied by Kogan in [15, 16]. The MEP procedure was applied, by Müller

and Ruggeri in the first edition of 1993 of their book [6] to the general case of any number of moments, where it was proved for the first time that the closed system is symmetric hyperbolic if one chooses the Lagrange multipliers as field variables.
Following this variational procedure of MEP in [7] we obtained the generalization of the Jüttner equilibrium distribution function f_E:

$$f_E = \frac{n\gamma}{A(\gamma)K_2(\gamma)} \frac{1}{4\pi m^3 c^3} e^{-\frac{1}{k_B T}\left[\left(1+\frac{\mathcal{I}}{mc^2}\right)U_\beta p^\beta\right]}, \tag{4}$$

with $A(\gamma)$ given by

$$A(\gamma) = \frac{\gamma}{K_2(\gamma)} \int_0^{+\infty} \frac{K_2(\gamma *)}{\gamma *} \phi(\mathcal{I})\, d\mathcal{I},$$

and

$$\gamma^* = \gamma \left(1 + \frac{\mathcal{I}}{mc^2}\right).$$

Also $K_n(\gamma), (n = 0, 1, 2, \ldots)$ denote the modified second order Bessel functions. From (4), the following equilibrium constitutive equations were obtained [7]:

$$\begin{aligned} p &= \frac{nmc^2}{\gamma} = \frac{k_B}{m}\rho T\,, \\ e &= \frac{nmc^2}{A(\gamma)K_2(\gamma)} \int_0^{+\infty} \gamma^*\, J_{2,2}(\gamma^*)\phi(\mathcal{I})\, d\mathcal{I}, \end{aligned} \tag{5}$$

where

$$J_{m,n}(\gamma) = \int_0^{+\infty} e^{-\gamma \cosh s} (\sinh s)^m \,(\cosh s)^n\, ds\,. \tag{6}$$

We remark that the pressure has the same expression for a monatomic and for a polyatomic gas, while $(5)_2$ is the generalization of the Synge energy to the case of polyatomic gases, provided that the measure is

$$\phi(\mathcal{I}) = \mathcal{I}^a, \tag{7}$$

where the constant a is defined by

$$a = \frac{D-5}{2},$$

and $D = 3 + f^i$ is related to the degrees of freedom of a molecule given by the sum of the space dimension 3 for the translational motion and the contribution of the internal degrees of freedom $f^i \geq 0$ related to the rotation and vibration. The closure of the non-equilibrium 14 equations (1) using MEP in principle can be done in the fully nonlinear case but as usual in RET there are some difficulties to consider the nonlinear approach. The main problem is to obtain the nonlinear invertibility between the Lagrange multipliers and the physical variables. And in the classical case there is also a problem noticed first by Junk [17] that the domain of definition of the flux is not convex, the flux has a singularity, and the equilibrium state lies on the boarder of the domain of definition of the flux. To avoid these difficulties in the MEP approach, we consider, as usual, the processes near equilibrium. The consequence

is that the system is symmetric hyperbolic only in a neighborhood of the equilibrium of the state space.
This is, unfortunately, a limitation of RET and as was observed by one of the reviewers this can be an obstacle to apply the present theory for instance in heavy-ion collisions problems that maybe require the fully nonlinear equations. On the other hand, the theory is already very complex even under this approximation, and in the knowledge of the authors there exists no causal relativistic theory of gas that is fully nonlinear in the non-equilibrium variables. In the classical case instead, there are new tentative to enlarge the domain of validity with high expansion as was presented first by Brini and Ruggeri in [18]. The higher-order expansion has the advantage of having a larger hyperbolicity domain [19, 20].
In this linear approximation with respect the non-equilibrium variables in [7] the closure was obtained and we have the following expression for the triple tensor:

$$
\begin{aligned}
A^{\alpha\beta\gamma} = {} & A_1^0 U^\alpha U^\beta U^\gamma + 3A_{11}^0 h^{(\alpha\beta}U^{\gamma)} \\
& - \frac{3}{c^2}\frac{N_1^\pi}{D_1^\pi}\,\pi U^\alpha U^\beta U^\gamma - 3\frac{N_{11}^\pi}{D_1^\pi}\,\pi h^{(\alpha\beta}U^{\gamma)} + \frac{3}{c^2}\frac{N_3}{D_3}\,q^{(\alpha}U^\beta U^{\gamma)} \\
& + \frac{3}{5}\frac{N_{31}}{D_3} h^{(\alpha\beta}q^{\gamma)} + 3C_5 t^{(<\alpha\beta>}U^{\gamma)}\,,
\end{aligned}
\tag{8}
$$

where the scalar coefficients A_1^0, A_{11}^0, D_1^π, N_1^π, N_{11}^π, N_3, D_3, N_{31}, C_5 are expressed in terms of integrals involving $J_{m,n}^* = J_{m,n}(\gamma^*)$. The explicit expressions of the coefficients are reported in [7] (see Eqs. $(36)_1$, (49), (50), (57), (58), (59)).
The explicit expressions of the coefficients are reported in [7] (see Eqs. $(36)_1$, (49), (50), (57), (58), (59)). In this way, choosing the measure as in (7) the system was closed completely for what concerns the main part of the operator, i.e. the explicit expression of $A^{\alpha<\beta\gamma>}$ in terms of the physical variables.

The production term $I^{\beta\gamma}$ in $(1)_3$ in the neighborhood of equilibrium state can be represented as [5]:

$$
I^{\beta\gamma} = B_1^\pi \pi \left(g^{\beta\gamma} - \frac{4}{c^2} U^\beta U^\gamma \right) + B_3^t\, t^{<\beta\gamma>} + B_4^q \left(U^\beta q^\gamma + U^\gamma q^\beta \right). \tag{9}
$$

The explicit expression of the coefficients depends on the collisional term Q that is not easy to give in a relativistic context and in particular for polyatomic gas. Usual for monatomic gas a relativistic generalization of the BGK model is adopted first by Marle [21, 22] and successively by Anderson and Witting [23]. The Marle model is an extension of the classical BGK model in the Eckart frame [24, 1], and the Anderson-Witting model is such an extension using the Landau-Lifshitz frame [24, 25].

These models have been widely employed for various relativistic problems, but several drawbacks have been also recognized in the literature. Starting from these considerations, Pennisi and Ruggeri proposed a variant of Anderson-Witting model in the Eckart frame both for monatomic and polyatomic gases, and proved that the conservation laws of particle number and energy-momentum are satisfied, the H-theorem holds and in the limit $\gamma \to \infty$ it reduces to the classical BGK equation [26] (see also [27]). The collisional term Q as the

BGK relativistic variant proposed in [26] is

$$Q = \frac{U_\alpha p^\alpha}{c^2 \tau}\left(f_E - f - f_E\, p^\mu q_\mu \frac{1+\frac{\mathcal{I}}{mc^2}}{bmc^2}\right). \tag{10}$$

with

$$b = \frac{\rho c^2}{3} \frac{\int_0^{+\infty} J^*_{4,1}\,\left(1+\frac{\mathcal{I}}{mc^2}\right)^2 \phi(\mathcal{I})\, d\,\mathcal{I}}{\int_0^{+\infty} J^*_{2,1}\,\phi(\mathcal{I})\, d\,\mathcal{I}},$$

and τ is the relaxation time. Recently in [28] was proved the unique existence and asymptotic behavior of classical solutions to the equation (2) with (10) when the initial data is sufficiently close to a global equilibrium.

Using the expression of Q given in (10) it is possible to evaluate all the coefficients that are present in (9) in terms of the field variables and of the relaxation time appearing in (10) [8]:

$$B^\pi_1 = -\frac{1}{4\,\tau}\frac{3N^\pi_1 + N^\pi_{11}}{D^\pi_1}, \quad B^t_3 = -\frac{1}{\tau}C_5 \quad B^q_4 = \frac{1}{c^2\tau}\left(\frac{B_2}{B_4} - \frac{N_3}{D_3}\right), \tag{11}$$

where the expressions of B_2 and B_4 are given in (A.6) and (A.7) of reference [7]. In this way, the system is fully closed and we have 14 equations with 14 unknowns. Even if the theory is not completely non-linear (linear in the non-equilibrium variables and non-linear with respect to the equilibrium ones) and therefore the validity is only in the region of hyperbolicity close to the equilibrium state, the Cauchy problem for the differential system is is well-posed at least locally in time with existence, uniqueness and stability [29, 6, 27]. Therefore it does not have the only linear stability as in the old Müller-Israel model [2]. This is due to the proof of the convexity of entropy and as a consequence of the symmetric form of the hyperbolic system [8]. Furthermore, it was proved (in the monatomic gas case) that the so-called K-condition is satisfied and the system has global smooth solutions for all time for sufficiently small initial data [30].

3 Maxwellian Iteration

To obtain the parabolic limit and to find the constitutive equations for π, $t_{<\mu\nu>}$ and q_μ we perform the first step of the Maxwellian iteration for equations (1), by considering their left-hand sides at equilibrium and their right-hand sides at order 1 with respect to equilibrium[1]. Taking into account that

$$\begin{aligned}
V^\alpha_E &= V^\alpha = n\, m\, U^\alpha, \\
T^{\alpha\beta}_E &= p\, h^{\alpha\beta} + \frac{e}{c^2}\, U^\alpha U^\beta, \\
A^{\alpha\beta\gamma}_E &= A^0_1 U^\alpha U^\beta U^\gamma + 3A^0_{11} h^{(\alpha\beta} U^{\gamma)}\,,
\end{aligned}$$

[1]For more details about the Maxwellian iteration, introduced for the first time in [10], in particular concerning RET see the books [6, 9].

eqs. (1) read

$$
\begin{aligned}
&\partial_\alpha(nmU^\alpha)=0\,,\\
&\partial_\alpha(ph^{\alpha\beta}+\frac{e}{c^2}\,U^\alpha U^\beta)=0\,,\\
&\partial_\alpha(A_1^0U^\alpha U^\delta U^\theta+3A_{11}^0h^{(\alpha\delta}U^{\theta)})(g_\delta^\beta g_\theta^\gamma-\frac{1}{4}g^{\beta\gamma}g_{\delta\theta})=I^{<\beta\gamma>}\,.
\end{aligned}
$$

We split the equations in the above system into their temporal and spatial components, and use eqs. (9) and (11) obtaining

$$
\begin{aligned}
&\partial_\alpha(nmU^\alpha)=0\,,\\
&U_\beta\partial_\alpha(ph^{\alpha\beta}+\frac{e}{c^2}\,U^\alpha U^\beta)=0\,,\\
&h_{\beta\gamma}\partial_\alpha(ph^{\alpha\beta}+\frac{e}{c^2}\,U^\alpha U^\beta)=0\,,\\
&U_\beta U_\gamma\partial_\alpha(A_1^0U^\alpha U^\delta U^\theta+3A_{11}^0h^{(\alpha\delta}U^{\theta)})(g_\delta^\beta g_\theta^\gamma-\frac{1}{4}g^{\beta\gamma}g_{\delta\theta})=-\,3\,c^2B_1^\pi\,\pi^{(1)}\,,\\
&U_\beta h_{\gamma\delta}\partial_\alpha(A_1^0U^\alpha U^\mu U^\nu+3A_{11}^0h^{(\alpha\mu}U^{\nu)})(g_\mu^\beta g_\nu^\gamma-\frac{1}{4}g^{\beta\gamma}g_{\mu\nu})=-\,c^2\,B_4^q\,q_\delta^{(1)}\,,\\
&\left(h_{\beta\mu}h_{\nu\gamma}-\frac{1}{3}h_{\mu\nu}h_{\beta\gamma}\right)\partial_\alpha(A_1^0U^\alpha U^\delta U^\theta+3A_{11}^0h^{(\alpha\delta}U^{\theta)})(g_\delta^\beta g_\theta^\gamma-\frac{1}{4}g^{\beta\gamma}g_{\delta\theta})=-\frac{C_5}{\tau}\,t_{<\mu\nu>}^{(1)}\,,
\end{aligned}
\tag{12}
$$

where $\pi^{(1)}$, $q_\delta^{(1)}$, $t_{<\mu\nu>}^{(1)}$ are the first iterates of π, q_δ, $t_{<\mu\nu>}$ respectively.
Now we use eqs. $(12)_{1-3}$ to obtain $U^\alpha\partial_\alpha n$, $U^\alpha\partial_\alpha\gamma$,$U^\alpha\partial_\alpha U^\beta$ and substitute them in $(12)_{4-6}$ to obtain the first iterates $\pi^{(1)}$, $q_\delta^{(1)}$, $t_{<\mu\nu>}^{(1)}$. In particular:

- from eq. $(12)_1$ it follows

$$\partial_\alpha\left(nU^\alpha\right)=0\quad\to\quad U^\alpha\partial_\alpha n=-n\partial_\alpha U^\alpha\,;\tag{13}$$

- from eqs. $(12)_2$, by reminding that the independent variables are n, γ (or T) and U_α, and that $U^\beta\partial_\alpha U_\beta=0$, we have

$$
\begin{aligned}
&U^\alpha\partial_\alpha\,e=-(p+e)\partial_\alpha U^\alpha\qquad\text{or}\\
&(\partial_n e)\,(U^\alpha\partial_\alpha n)+(\partial_\gamma e)\;U^\alpha\partial_\alpha\gamma=-(p+e)\partial_\alpha U^\alpha
\end{aligned}
\tag{14}
$$

  where we have put

$$\partial_n e=\left(\frac{\partial e}{\partial n}\right)_\gamma\,,\qquad \partial_\gamma e=\left(\frac{\partial e}{\partial\gamma}\right)_n\,.$$

  Eq. (14), together with eq. (13) can be used to obtain the following expression for $U^\alpha\partial_\alpha\gamma$

$$U^\alpha\partial_\alpha\gamma=-\frac{(\partial_n e)\,(U^\alpha\partial_\alpha n)+(p+e)\partial_\alpha U^\alpha}{\partial_\gamma e}=-\,\frac{p}{\partial_\gamma e}\,\partial_\alpha U^\alpha\,;\tag{15}$$

$$U^\alpha\partial_\alpha\gamma=-\frac{(\partial_n e)\,(U^\alpha\partial_\alpha n)+(p+e)\partial_\alpha U^\alpha}{\partial_\gamma e}=-\,\frac{p}{\partial_\gamma e}\,\partial_\alpha U^\alpha\,;\tag{16}$$

- from eqs. $(12)_3$ we have

$$- (\partial_n p) \cdot h^{\alpha\psi} \partial_\alpha n - (\partial_\gamma p) \cdot h^{\alpha\psi} \partial_\alpha \gamma - \frac{p+e}{c^2} U^\alpha \partial_\alpha U^\psi = 0 \,, \tag{17}$$

$$\text{or,} \quad - (\partial_n p) \cdot h^{\alpha\psi} \left(\partial_\alpha n + \frac{e}{mc^2} \partial_\alpha \gamma \right) + 2 \frac{p+e}{\gamma c^2} h^{\psi(\alpha} U^{\mu)} \, \partial_\alpha (\gamma U_\mu) = 0 \,,$$

where eq. $(39)_4$ of the Appendix has been used. The above equation can be used to obtain the following expression

$$h^{\alpha\psi} \left(\partial_\alpha n + \frac{e}{mc^2} \, \partial_\alpha \gamma \right) = \frac{2 \frac{p+e}{\gamma c^2} \, h^{\psi(\alpha} U^{\mu)} \, \partial_\alpha (\gamma U_\mu)}{\partial_n p} \,. \tag{18}$$

Now we substitute these results in $(12)_{4-6}$.

- From eqs. $(12)_4$ we have

$$\frac{3}{4} c^2 \left(c^2 \partial_n A_1^0 + \partial_n A_{11}^0 \right) U^\alpha \partial_\alpha n + \frac{3}{4} c^2 \, \left(c^2 \partial_\gamma A_1^0 + \partial_\gamma A_{11}^0 \right) U^\alpha \partial_\alpha \gamma + \\ + \left(\frac{3}{4} \, c^4 A_1^0 + \frac{11}{4} \, c^2 A_{11}^0 \right) \partial_\alpha U^\alpha = - \, 3 \, c^2 \, B_1^\pi \, \pi^{(1)} \,,$$

that, thanks to eqs. (13), (16), becomes

$$\left[- \frac{3}{4} \frac{p \, c^2}{\partial_\gamma e} \left(c^2 \partial_\gamma A_1^0 + \partial_\gamma A_{11}^0 \right) + 2 c^2 A_{11}^0 \right] \partial^\alpha U_\alpha = - \, 3 \, c^2 \, B_1^\pi \, \pi^{(1)} \,.$$

It is possible to express the partial derivative with respect to γ as derivative with respect to T, obtaining that the above equation becomes

$$\left[- \frac{3}{4} \frac{p \, c^2}{\partial_T e} \left(c^2 \partial_T A_1^0 + \partial_T A_{11}^0 \right) + 2 c^2 A_{11}^0 \right] \partial_\alpha U^\alpha = - \, 3 \, c^2 \, B_1^\pi \, \pi^{(1)} \,. \tag{19}$$

- From eqs. $(12)_5$ we have

$$- c^2 \partial_n A_{11}^0 \cdot h_\psi^\alpha \partial_\alpha n - c^2 \partial_\gamma A_{11}^0 \cdot h_\psi^\alpha \partial_\alpha \gamma - (c^2 A_1^0 + 2 A_{11}^0) U^\alpha \partial_\alpha U_\psi = - \, c^2 \, B_4^q \, q_\psi^{(1)} \tag{20}$$

that, thanks to eq. (39) of the Appendix, can be written also as

$$- c^2 \partial_n A_{11}^0 \cdot h_\psi^\alpha \left(\partial_\alpha n + \frac{e}{mc^2} \partial_\alpha \, \gamma \right) + \frac{2}{\gamma} (c^2 A_1^0 + 2 A_{11}^0) \, h_\psi^{(\alpha} U^{\mu)} \partial_\alpha (\gamma U_\mu) = - \, c^2 \, B_4^q \, q_\psi^{(1)} \,.$$

This result can be expressed by using eq. (18) and becomes

$$\frac{2}{\gamma} \left(c^2 A_1^0 + 2 A_{11}^0 \; - \; \frac{e+p}{p} \right) \, h_\psi^{(\alpha} U^{\mu)} \partial_\alpha (\gamma U_\mu) = - \, c^2 \, B_4^q \, q_\psi^{(1)} \,.$$

Now we can proceed in two ways:

1. The first one makes easier the calculation of the monatomic limit and the comparison with the results of [5]. It consists in substituting here $h^\alpha_\psi \left(\partial_\alpha n + \frac{e}{mc^2}\partial_\alpha \gamma\right)$ from eq. (18) so that it becomes

$$- \left(c^2 A^0_1 + 2A^0_{11} - A^0_{11}\frac{e+p}{p}\right) c^2\, h^\alpha_\psi \left[\frac{1}{T}\partial_\alpha T - \frac{1}{c^2}U^\mu \partial_\mu U_\alpha\right] = -\, c^2\, B^q_4\, q^{(1)}_\psi \,. \tag{21}$$

2. The second one makes easier the calculation of the non relativistic limit. It consists in substituting $U^\alpha\, \partial_\alpha U^\psi$ from eq. $(17)_1$ into eq. (20) so that it becomes

$$\begin{aligned}
&-\, c^2\, B^q_4\, q^{(1)}_\psi = -\, c^2 \left(\frac{A^0_{11}}{n}\, h^\alpha_\psi \partial_\alpha\, n + \frac{\partial\, A^0_{11}}{\partial\, T} h^\alpha_\psi\, \partial_\alpha T\right) + \\
&+ \left(A^0_1 c^2 + 2\, A^0_{11}\right)\, \frac{p\, c^2}{e+p}\, \left(\frac{1}{n}\, h^\alpha_\psi \partial_\alpha\, n + \frac{1}{T} h^\alpha_\psi\, \partial_\alpha T\right) = \\
&= \left(\frac{1}{n}\, h^\alpha_\psi \partial_\alpha\, n\right) \left[-\, c^2\, \left(A^0_{11} - p\right)\, +\, \frac{p\, c^2}{e+p}\, \left(A^0_1 c^2 - e + 2\, A^0_{11} - p\right)\right] + \\
&+ \left(\frac{1}{T}\, h^\alpha_\psi \partial_\alpha\, T\right) \left[-\, c^2\, \left(T\, \frac{\partial\, A^0_{11}}{\partial\, T} - p\right)\, +\, \frac{p\, c^2}{e+p}\, \left(A^0_1 c^2 - e + 2\, A^0_{11} - p\right)\right] .
\end{aligned} \tag{22}$$

- From eqs. $(12)_6$ we have

$$-2A^0_{11}\, \partial_{<\delta} U_{\gamma>} = B^t_3\, t^{(1)}_{<\delta\gamma>} \,. \tag{23}$$

At the end, with eqs. (19), (21) and (23) we have found the following constitutive equations (we omit the index (1))

$$\begin{aligned}
\pi &= -\frac{1}{B^\pi_1}\left[-\frac{1}{4}\frac{p}{\partial_T e}\left(c^2 \partial_T A^0_1 + \partial_T A^0_{11}\right) + \frac{2}{3} A^0_{11}\right] \partial_\alpha U^\alpha \\
q_\beta &= \frac{1}{B^q_4}\left(c^2 A^0_1 + 2A^0_{11} - A^0_{11}\frac{e+p}{p}\right) h^\alpha_\beta \left[\frac{1}{T}\partial_\alpha T - \frac{1}{c^2}U^\mu \partial_\mu U^\alpha\right] \\
t_{<\beta\delta>} &= -\frac{2}{B^t_3} A^0_{11}\, h^\alpha_\beta\, h^\mu_\delta \partial_{<\alpha} U_{\mu>},
\end{aligned} \tag{24}$$

that can be rewritten in the Eckart form:

$$\begin{aligned}
\pi &= -\nu\, \partial_\alpha U^\alpha \\
q_\beta &= -\chi\, h^\alpha_\beta \left[\partial_\alpha T - \frac{T}{c^2} U^\mu \partial_\mu U^\alpha\right], \\
t_{<\beta\delta>} &= 2\mu\, h^\alpha_\beta\, h^\mu_\delta \partial_{<\alpha} U_{\mu>}
\end{aligned} \tag{25}$$

where ν, χ and μ are the bulk viscosity, the heat conductivity and the shear viscosity, respectively.

By comparing (24) and (25) we obtain the precise explicit expression for these phenomenological coefficients for a generic polyatomic gas in all the range of γ:

$$\begin{aligned}
\nu &= \frac{1}{B_1^\pi}\left[-\frac{1}{4}\frac{p}{\partial_T e}\partial_T\left(c^2 A_1^0 + A_{11}^0\right) + \frac{2}{3}A_{11}^0\right] \\
\chi &= -\frac{1}{B_4^q T}\left(c^2 A_1^0 + A_{11}^0 - A_{11}^0\frac{e}{p}\right) \\
\mu &= -\frac{A_{11}^0}{B_3^t}\,,
\end{aligned} \tag{26}$$

with B_1^π, B_4^q, B_3^t explicitly given by (11) except for the relaxation time τ.

Remark 1: As we observed in Section 2, the causal relativistic theory is valid only in the vicinity of the equilibrium states even if the local equilibrium hypothesis is abandoned. As is known, the parabolic theories such as Eckart's theory in relativity or the Navier-Stokes-Fourier theory in the classical case take the hypothesis of local equilibrium as a starting point and therefore are valid only near equilibrium. Consequently, the evaluation of the phenomenological coefficients is not affected by the approximation made for the causal theory.

Remark 2: We observe that the Maxwellian iteration procedure starting from the truncated moment equations has always the same result as the Chapman-Enskog procedure starting directly from the kinetic equation. Even if we have not verified it directly with the BGK equation we used, it should be part of the general procedure treated for example in the book by Cercignani and Kremer [24] for what concerns the relativistic case.

4 Non relativistic limit

We perform now the non relativistic limit of eqs. (24) with $(24)_1$ expressed as in eq. (22). To this end we need the non relativistic limits of the coefficients

$$\begin{aligned}
&-\frac{1}{4}\frac{p}{\partial_T e}\left(c^2\partial_T A_1^0 + \partial_T A_{11}^0\right) + \frac{2}{3}A_{11}^0 \quad , \quad A_{11}^0\,, \text{ and} \\
&\left(\frac{1}{n}\,h_\psi^\alpha \partial_\alpha\, n\right)\left[-c^2\,\left(A_{11}^0\, - p\right)\, + \,\frac{p\,c^2}{e+p}\,\left(A_1^0 c^2 - e + 2\,A_{11}^0 - p\right)\right] + \\
&+\left(\frac{1}{T}\,h_\psi^\alpha \partial_\alpha\, T\right)\left[-c^2\,\left(T\,\frac{\partial\, A_{11}^0}{\partial\, T}\, - p\right)\, + \,\frac{p\,c^2}{e+p}\,\left(A_1^0 c^2 - e + 2\,A_{11}^0 - p\right)\right].
\end{aligned} \tag{27}$$

For the first one of these we note that

$$\begin{aligned}
&e\, - \rho\, c^2 = p\,\frac{D}{2} + O\left(1/c^2\right), \quad A_1^0 c^2\, - \rho\, c^2 = pD + O\left(1/c^2\right), \quad A_{11}^0 = p\, + O\left(1/c^2\right) \\
&\partial_T e = \frac{p}{T}\,\frac{D}{2} + O\left(1/c^2\right), \quad \partial_T A_1^0 c^2 = \frac{p}{T}D\, + O\left(1/c^2\right), \quad \partial_T A_{11}^0 = \frac{p}{T} + O\left(1/c^2\right).
\end{aligned}$$

It follows that

$$\lim_{c\to+\infty}\left[-\frac{1}{4}\frac{p}{\partial_T e}\left(c^2\partial_T A_1^0 + \partial_T A_{11}^0\right) + \frac{2}{3}A_{11}^0\right] = \frac{p}{3}\,\frac{a+1}{2a+5}\,.$$

The second element in eq. (27) has clearly p as limit. For the third element, we have that

$$\begin{aligned} & A^0_1 c^2 - e + 2\, A^0_{11} - p = \left(A^0_1 - \, m\, n\right)\, c^2 \, - \, (e - m\, n\, c^2) + 2\, A^0_{11} - p = \\ & = \left(\frac{A^0_1}{m\, n} - 1\right)\, p\, \gamma \, - \, (e - m\, n\, c^2) + 2\, A^0_{11} - p = \\ & = p\,(2a+5) + \, \frac{1}{c^2}(\cdots) \, - \, p\, \frac{2a+5}{2} \, + \, \frac{1}{c^2}(\cdots) + \, p \, + \, \frac{1}{c^2}(\cdots)\, . \end{aligned}$$

so that

$$\lim_{c\to+\infty} A^0_1 c^2 - e + 2\, A^0_{11} - p = p\, \frac{2a+7}{2}\, .$$

Moreover, we have

$$\lim_{c\to+\infty} \left(A^0_{11} - p\right)\, c^2 = \frac{p^2}{m\, n}\, \frac{2a+7}{2}\, , \quad \lim_{c\to+\infty} \frac{p\, c^2}{e+p} = \frac{p}{m\, n}\, . \tag{28}$$

It follows that the coefficient of $h^\alpha_\psi \partial_\alpha\, n$ in eq. $(27)_3$ has limit zero, while the coefficient of $\frac{1}{T}\, h^\alpha_\psi \partial_\alpha\, T$ has limit

$$-\frac{p^2}{m\, n}\, \frac{2a+7}{2}\, , \quad \text{because} \quad c^2 \left(T\, \frac{\partial\, A^0_{11}}{\partial\, T} - p\right) = T\, \frac{\partial\, \left(A^0_{11} - \, p\right)\, c^2}{\partial\, T} \;\to\; \frac{p^2}{m\, n}\,(2a+7)\, ,$$

where $(28)_1$ was used. So we have found that

$$\begin{aligned} \lim_{c\to+\infty} \pi^{(1)} &= \frac{-1}{\lim_{c\to+\infty}\, B^\pi_1}\, \frac{p}{3}\, \frac{a+1}{2a+5}\, \partial_i v^i\, , \\ \lim_{c\to+\infty} q^{(1)}_i &= \frac{1}{\lim_{c\to+\infty}\, (c^2\, B^q_4)}\, \frac{p^2}{m\, n\, T}\, \frac{2a+7}{2} \partial_i\, T\, , \\ \lim_{c\to+\infty} t^{(1)}_{<ij>} &= \frac{2\, p}{\lim_{c\to+\infty}\, B^t_3}\, \partial_{<i} v_{j>}\, . \end{aligned} \tag{29}$$

Note: Taking into account the decomposition $U^\alpha \equiv (\Gamma\, c\, ,\, v^i)$, where Γ is the Lorentz factor, we have $\partial_\alpha U^\alpha = \frac{1}{c}\, \partial_t\,(\Gamma\, c) + \, \partial_k\, \left(\Gamma\, v^k\right)$ whose limit is $\partial_i v^i$ because $\partial_t\, \Gamma = -\, \Gamma^3 \frac{v_i}{c^2}\, \partial_t v^i$ which has zero limit. Similarly for $\partial_k\, \Gamma$. We have changed sign in the right hand side of eq. $(29)_3$ because v^j was defined with the above index and, with our choice of the metric, we have $v_j = -\, v^j$. Moreover, $\lim_{c\to+\infty} h^{ij} = \delta^{ij}$ and

$$\begin{aligned} \frac{1}{c^2}\, U^\mu \partial_\mu U^0 &= \frac{1}{c^2}\, \Gamma\, c\, \frac{1}{c}\, \partial_t\, \left(\Gamma\, c\right) \, + \, \frac{1}{c^2}\, \Gamma\, v^k\, \partial_k\, \left(\Gamma\, c\right) \quad \text{whose limit is}\, 0\, , \\ \frac{1}{c^2}\, U^\mu \partial_\mu U^i &= \frac{1}{c^2}\, \Gamma\, c\, \frac{1}{c}\, \partial_t\, \left(\Gamma\, v^i\right) \, + \, \frac{1}{c^2}\, \Gamma\, v^k\, \partial_k\, \left(\Gamma\, v^i\right) \quad \text{whose limit is}\, 0\, . \end{aligned}$$

We can also take into account the expressions (11), and the fact that

$$\lim_{c\to+\infty} \frac{N^\pi_1}{D^\pi_1} = 0\, , \; \lim_{c\to+\infty} \frac{N^\pi_{11}}{D^\pi_1} = -1\, , \; \lim_{c\to+\infty} C_5 = 1\, , \; \lim_{c\to+\infty} \frac{N_3}{D_3} = 2\, , \; \lim_{c\to+\infty} \frac{B_2}{B_4} = 1\, .$$

The last one of these comes immediately from the expressions of B_2 and B_4 in [7], while the others can be found in the new Ruggeri-Sugiyama book [27].
As consequence, the limit of (11) is

$$\lim_{c\to+\infty} B_1^\pi = \frac{1}{4\,\tau}\,, \quad \lim_{c\to+\infty} B_3^t = -\frac{1}{\tau}\,, \quad \lim_{c\to+\infty} \left(c^2\,B_4^q\right) = -\frac{1}{\tau}\,.$$

In this case, (29) become

$$\begin{aligned} &\lim_{c\to+\infty} \pi^{(1)} = -\frac{4}{3}\,\tau\,p\,\frac{a+1}{2a+5}\,\partial_i v^i\,, \\ &\lim_{c\to+\infty} q_i^{(1)} = -\tau\,\frac{p^2}{m\,n\,T}\frac{2a+7}{2}\partial_i\,T\,, \\ &\lim_{c\to+\infty} t_{<ij>}^{(1)} = -2\,\tau\,p\,\partial_{<i}v_{j>}\,. \end{aligned}$$

By comparison with the classical Navier-Stokes-Fourier equations:

$$\pi = \nu\,\partial_i v^i, \qquad \sigma_{<ij>} = -t_{<ij>} = 2\mu\,\partial_{<i}v_{j>}, \qquad q_i = -\chi\,\partial_i\,T,$$

($\sigma_{<ij>} = -t_{<ij>}$ is the traceless viscosity tensor), we obtain as limit the classical shear, viscosity, bulk viscosity and heat conductivity:

$$\mu = p\tau, \quad \nu = \frac{2(D-3)}{3D}p\tau, \quad \chi = \frac{D+2}{2}\frac{p^2}{\rho T}\tau, \tag{30}$$

that coincide with the one obtained in [9] when the relaxation times are all equal to τ due to the fact that we have used here a BGK approximation. In the phenomenological theory there are three different relaxations time $\tau_\pi, \tau_\sigma, \tau_q$ that can be very different each other. In particular τ_π is very large in several polyatomic gas [9]. We notice that the bulk viscosity is of the order of unit provided that $D > 3$, while it vanishes for $D = 3$ corresponding to the monatomic gas.

5 Ultrarelativistic limit

Now we want to evaluate the phenomenological coefficients (26) in the ultrarelativistic limit, i.e. when $\gamma \to 0$. According to the results in [31], in the following limit the triple tensor (8) assumes a special form:

$$\begin{aligned} A^{\alpha\beta\gamma} = &\left(f_0(a,\gamma)\rho + f_1(a,\gamma)\frac{\pi}{c^2}\right)\left(U^\alpha U^\beta U^\gamma + c^2 h^{(\alpha\beta}U^{\gamma)}\right) + \\ &+ f_2(a,\gamma)\left(\frac{1}{c^2}q^{(\alpha}U^\beta U^{\gamma)} + \frac{1}{5}h^{(\alpha\beta}q^{\gamma)}\right) + f_3(a,\gamma)t^{(<\alpha\beta>}U^{\gamma)}, \end{aligned} \tag{31}$$

where f_0, f_1, f_2, f_3 are explicit functions of a and γ that change expressions depending on the range of a and can be found in [31].

Here we limit the analysis to the case in which $-1 < a < 2$. Comparing (31) with (8) and taking into account (11), (26) and that $e = 3p$ (see [32]) we have:

$$\bar{\nu} = \frac{\gamma}{3f_1}\left(2f_0 + \gamma\partial_\gamma f_0\right), \qquad \bar{\mu} = \gamma\frac{f_0}{f_3}, \qquad \bar{\chi} = -\gamma^2\frac{f_0}{f_4}$$

where we introduce the dimensionless variables:

$$\bar{\nu} = \frac{\nu}{p\tau}, \qquad \bar{\mu} = \frac{\mu}{p\tau}, \qquad \bar{\chi} = \chi\frac{\rho T}{p^2\tau}, \tag{32}$$

and we indicate with $f_4(a,\gamma)$ the quantity $\frac{B_2}{B_4} - \frac{N_3}{D_3}$ evaluated in the ultrarelativistic regime.

Now in [31] it was proved that in this range of a, when $\gamma \to 0$, we have

$$f_0 = \frac{g_0(a)}{\gamma^2}, \qquad f_3 = \frac{g_3(a)}{\gamma},$$

and

$$f_1 = O\left(\frac{1}{\gamma^2}\right) \quad \text{if} \quad -1 < a < 1, \qquad f_1 = O\left(\frac{1}{\gamma^{(3-a)}}\right) \quad \text{if} \quad 1 < a < 2.$$

In [31] it is possible to find the explicit expression of $g_0(a)$ and $g_3(a)$, while here we find with the same methodology that:

$$f_4 = \frac{g_4(a)}{\gamma}. \tag{33}$$

Therefore we have

$$\lim_{\gamma\to 0}\bar{\nu} \;=\; \lim_{\gamma\to 0}\bar{\chi} = 0, \qquad \lim_{\gamma\to 0}\bar{\mu} = \frac{g_0(a)}{g_3(a)} = \frac{5(a-2)(a+5)^2}{3(a-3)(a+4)(a+11)}. \tag{34}$$

In conclusion, we can summarize the results with the following

Statement: Using the Maxwellian iteration we have evaluated the bulk viscosity, shear viscosity, and heat conductivity for a general relativistic polyatomic gas in terms of ρ, γ, a, τ (see (26) and (11)). In the classical limit, we obtain the classical results (30), while in the ultrarelativistic limit in the range $-1 < a < 2$, the bulk viscosity and the heat conductivity tend to zero, while the shear viscosity tends to a finite value (see (34)).

The result is not so easy to interpret from the physical point of view. The authors expectation was that in the ultrarelativistic limit also the shear viscosity would be zero as was expected that in this regime every gas loses the dissipativeness and becomes substantially an Euler gas. Instead we have that shear viscosity still remains in this limit and is zero only for $a \to 2$.

6 Monatomic limit

For $D \to 3$ i.e. $a \to -1$ we have the singular limit of monatomic relativistic gas. The Synge energy converges to its monatomic expression:

$$e = p\gamma\frac{J_{2,2}(\gamma)}{J_{2,1}(\gamma)} = p\left(\gamma\frac{K_3(\gamma)}{K_2(\gamma)} - 1\right) = p\left(\gamma G - 1\right),$$

where

$$G = \frac{K_3(\gamma)}{K_2(\gamma)}$$

First, we recall the limit values of the coefficients that were evaluated in [33][2] :

$$A_1^0 = \rho \frac{J_{2,3}(\gamma)}{J_{2,1}(\gamma)} \qquad A_{11}^0 = \frac{\rho c^2}{3} \frac{J_{4,1}(\gamma)}{J_{2,1}(\gamma)}$$

$$D_1^\pi = \begin{vmatrix} J_{2,0}(\gamma) & J_{2,1}(\gamma) & J_{2,2}(\gamma) \\ J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) \end{vmatrix}, \quad N_1^\pi = N_{11}^\pi = \begin{vmatrix} J_{2,1}(\gamma) & J_{2,2}(\gamma) & J_{2,3}(\gamma) \\ J_{2,2}(\gamma) & J_{2,3}(\gamma) & J_{2,4}(\gamma) \\ J_{2,3}(\gamma) & J_{2,4}(\gamma) & J_{2,5}(\gamma) \end{vmatrix},$$

$$D_3 = \begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,1}(\gamma) & J_{4,2}(\gamma) \end{vmatrix}, \quad N_3 = N_{31} = \begin{vmatrix} J_{4,0}(\gamma) & J_{4,1}(\gamma) \\ J_{4,2}(\gamma) & J_{4,3}(\gamma) \end{vmatrix}, \tag{35}$$

$$C_5 = \frac{J_{6,1}(\gamma)}{J_{6,0}(\gamma)}, \qquad \frac{B_2}{B_4} = \frac{J_{4,2}(\gamma)}{J_{4,1}(\gamma)}.$$

Taking into account that all the $J_{p,q}$ given in (6) can be expressed in terms of the Bessel functions K_j and using the properties of K_j they can express in terms only of two independent Bessel functions for example K_2 and K_3, then after cumbersome calculations we obtain the following expressions for the bulk viscosity, heat conductivity and shear viscosity:

$$\begin{aligned} \nu &= \frac{p}{B_1^\pi}\left[\frac{\gamma + G(6-\gamma G)}{\gamma^2\left(G^2-1\right) - 5\gamma G + 1} + \frac{2G}{3}\right], \\ \chi &= \frac{m\rho k_B}{B_4^q}\left(\gamma + 5G - \gamma G^2\right), \\ \mu &= -\frac{p}{B_3^t} G, \end{aligned} \tag{36}$$

that are coincident with the one deduced in [5]. But differently from [5], we have now the explicit expression for B_1^π, B_3^t, B_4^1 and therefore we can give the explicit expression for ν, χ and μ as function of ρ, γ and the relaxation time τ. Taking into account eqs. (11) and (35) we obtain:

[2]We use the same symbols to avoid heavy notation omitting the symbol of limit.

$$
\begin{aligned}
B_1^\pi &= -\frac{1}{\tau}\frac{-2\gamma^3\left(G^2-1\right)+\gamma^2 G\left(19-9G^2\right)+5\gamma\left(9G^2-1\right)-30G}{\gamma\left\{3\gamma+G\left[2\gamma^2\left(G^2-1\right)-13\gamma G+20\right]\right\}} \\
B_4^q &= \frac{1}{c^2\tau}\left(-\frac{1}{\gamma}+\frac{G^2-1}{\gamma\left(G^2-1\right)-5G}+\frac{1}{G}\right) \\
B_3^t &= -\frac{1}{\tau}\left(\frac{6}{\gamma}+\frac{1}{G}\right).
\end{aligned}
\tag{37}
$$

Substituting (37) in (36) we have that the dimensionless variables (32) becomes now

$$
\begin{aligned}
&\bar{\nu} = \frac{\gamma\left\{3\gamma+G\left[2\gamma^2\left(G^2-1\right)-13\gamma G+20\right]\right\}^2}{3\left[\gamma^2\left(G^2-1\right)-5\gamma G+1\right]\left[2\gamma^3\left(G^2-1\right)+\gamma^2 G\left(9G^2-19\right)+5\gamma\left(1-9G^2\right)+30G\right]}, \\
&\bar{\mu} = \frac{\gamma\, G^2}{\gamma+6G}, \qquad \bar{\chi} = \frac{\gamma^2 G\left[\gamma+G(5-\gamma G)\right]^2}{\left(\gamma^2+5\right)G^2-\gamma(\gamma+5G)}.
\end{aligned}
\tag{38}
$$

Taking into account (38), the dimensionless variables are explicitly function only on γ and therefore they can be plotted. In Figure **??** we have the plot of $\bar{\nu}$ versus γ in the whole range of γ from $\gamma \to 0$ (ultrarelativistic case) to $\gamma \to \infty$ (classical limit) and according with (30) for large γ go to zero as $O(1/\gamma^2)$, i.e. of order of $O(1/c^4)$. In agreement with (34) is interesting that also in the ultrarelativistic regime for small γ the bulk viscosity tends to 0 as order of $O(\gamma^4)$. In the remaining range $\bar{\nu}$ is in any way very small with maximum value of $\bar{\nu}_{\max} \simeq 0.00302$ for $\gamma \simeq 4.57$. In Figure **??**a) we plot the normalized heat conductivity that according with (30) tend in the classical case to $5/2$ and tend to zero when $\gamma \to 0$. In Figure **??**b) we plot the normalized shear viscosity that according with (30) tend in the classical case to 1 and tend to $2/3$ in the ultrarelativistic limit.

Unfortunately, it is very difficult to plot the figures of $\bar{\nu}, \bar{\mu}$ and $\bar{\chi}$ in the polyatomic gas due to the improper integrals appearing in the coefficients but we expect similar behavior of monatomic gas taking into account the classical (30) and ultrarelativistic (32) limits. We aim to study in another paper the case of diatomic gas in which, at least for the energy, we have that the integrals have an analytic expression like in the Synge case as was recently proved in [34]. It would be extremely interesting as suggested by one of the reviewers to compare the trend of the bulk and shear viscosity with the recent paper of Bernhard, Moreland and Bass [35] concerning the Bayesian estimation of the specific shear and bulk viscosity of quark-gluon plasma. According to the reviewer's suggestion, it may be that the peak of the bulk viscosity due to the appearance of other degrees of freedom, resembles what is expected to occur in QCD and also in the transition from quarks and gluons to hadrons.

7 Conclusions

In this paper, by using the Maxwellian iteration and the production terms that are compatible with a BGK model for the collisional term of Boltzmann-Chernikov equation, we deduce the parabolic limit of the causal hyperbolic theory of polyatomic relativistic gas, and this permits us to give an explicit relation between the bulk viscosity, shear viscosity, and heat conductivity in terms of the relaxation time. We have also evaluated the ultrarelativistic limit and the classical limit. In particular, in the classical limit, the bulk viscosity is not zero as for monatomic gas, and therefore we expect a finite bulk viscosity in all ranges of γ except for $\gamma \to 0$. As we said before, at the end of Section 4, we used a BGK model and therefore we have only one relaxation time, but as usual in RET we can assume by comparison with the phenomenological approach that instead of only one τ, we have 3 different relaxation times. Therefore if the corresponding relaxation time τ_π is large, with respect to τ_σ and τ_q as it is for several gases at least in the classical regime, we can have large bulk viscosity that can be very useful to explain some cosmological problems of universe expansion. We recall in fact that the bulk viscosity plays a fundamental role to calculate the rate of damping of protogalactic fluctuations in the period immediately before the recombination of hydrogen as was firstly observed by Weinberg [36] that was able to evaluate the cosmological entropy production associated with a non-vanishing mean free time of photons, neutrinos, or gravitons. More recently the importance of bulk viscosity was the object of several studies. We quote in particular, the paper of Li and Barrow [37] that investigate the possibility that a single imperfect fluid with bulk viscosity can replace the need for separate dark matter and dark energy in cosmological models. Now we believe that the model of relativistic fluid that takes into account the internal structure of gas and, as a consequence, with a relevant bulk viscosity can be more natural to improve the previous models. This will be the object of a future paper.

A Some useful properties

Let's remind that n, γ or T, and U_α are independent variables so we expect to express ∂_α as a composite derivative with respect to the independent variables. It will be useful to consider the following derivatives

$$\bullet\, \partial_n e = \frac{e}{n}\,, \qquad \partial_n A_1^0 = \frac{A_1^0}{n}\,, \qquad \partial_n A_{11}^0 = \frac{A_{11}^0}{n}\,, \quad \frac{e}{mc^2}\,\partial_n p = \partial_\gamma p + \frac{p+e}{\gamma}\,,$$

where the last equation is simply a consequence of the definition $(5)_1$ and the others are consequences of the fact that e, A_1^0, A_{11}^0 are linear in the variable n.

$$\bullet\, \gamma \partial_\gamma A_{11}^0 = -c^2 A_1^0 - 2A_{11}^0 + \gamma A_{11}^0 \frac{e}{nmc^2}\,, \tag{40}$$

or

$$\frac{e}{mc^2}\partial_n A_{11}^0 = \partial_\gamma A_{11}^0 + \frac{1}{\gamma}(c^2 A_1^0 + 2A_{11}^0)\,.$$

To prove this property, we start from eq. $(7.6)_{2,3}$ of [5]

$$\begin{aligned} \gamma J_{m+2,n}(\gamma) &= -nJ_{m,n-1}(\gamma) + (n+m+1)J_{m,n+1}(\gamma) \\ \partial_\gamma J_{m,n}(\gamma) &= -J_{m,n+1}(\gamma) \end{aligned}$$

from which it follows

$$\gamma\left(1+\frac{\mathcal{I}}{mc^2}\right)J_{m+2,n}(\gamma^*) = -nJ_{m,n-1}(\gamma^*) + (n+m+1)J_{m,n+1}(\gamma^*)\,,$$
$$\partial_\gamma J_{m,n}(\gamma^*) = -\left(1+\frac{\mathcal{I}}{mc^2}\right)J_{m,n+1}(\gamma^*)\,. \tag{41}$$

By using this result and (49), (50), (42) of [7] we have

$$\gamma\partial_\gamma A^0_{11} = \frac{1}{3}\gamma nmc^2\frac{\int_0^{+\infty} -J_{4,2}(\gamma*)\left(1+\frac{2\mathcal{I}}{mc^2}\right)\left(1+\frac{\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}} +$$
$$+\frac{1}{3}\gamma nmc^2\frac{\left[\int_0^{+\infty} J_{4,1}(\gamma*)\left(1+\frac{2\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}\right]\left[\int_0^{+\infty} J_{2,3}(\gamma*)\left(1+\frac{\mathcal{I}}{mc^2}\right)\left(1+\frac{\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}\right]}{\left[\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}\right]^2}\,.$$

By using eq. $(41)_1$ this expression becomes

$$\gamma\partial_\gamma A^0_{11} = \frac{1}{3}nmc^2\frac{\int_0^{+\infty}\left[2J_{2,1}(\gamma*) - 5J_{2,3}(\gamma*)\right]\left(1+\frac{2\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}} + \gamma A^0_{11}\frac{e}{nmc^2}\,.$$

By using $(7.6)_1$ from [5] it becomes

$$\gamma\partial_\gamma A^0_{11} = \frac{1}{3}nmc^2\frac{\int_0^{+\infty}\left[-3J_{2,3}(\gamma*) - 2J_{4,1}(\gamma*)\right]\left(1+\frac{2\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}} + A^0_{11}\frac{\gamma e}{nmc^2} =$$
$$= -A^0_1c^2 - 2A^0_{11} + A^0_{11}\frac{\gamma e}{nmc^2}\,,$$

which concludes the proof of eq. (40).

- Moreover we have

$$c^2\partial_\gamma A^0_1 + \partial_\gamma A^0_{11} = \frac{e}{nmc^2}(A^0_1c^2 + A^0_{11}) - mc^2B_3 - \frac{1}{3}mB_2\,. \tag{42}$$

In fact:

$$c^2\partial_\gamma A^0_1 + \partial_\gamma A^0_{11} = c^2\partial_\gamma\frac{mn\int_0^{+\infty} J_{2,3}(\gamma*)\left(1+\frac{2\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}} +$$
$$\frac{1}{3}\partial_\gamma\frac{nmc^2\int_0^{+\infty} J_{4,1}(\gamma*)\left(1+\frac{2\mathcal{I}}{mc^2}\right)\phi(\mathcal{I})d\mathcal{I}}{\int_0^{+\infty} J_{2,1}(\gamma*)\phi(\mathcal{I})d\mathcal{I}}\,.$$

This expression, by using eq. (41) becomes

$$
\begin{aligned}
&= nmc^2 \left[\int_0^{+\infty} J_{2,1}(\gamma*) \phi(\mathcal{I}) d\mathcal{I} \right]^{-2} \cdot \\
&\left\{ \left[- \int_0^{+\infty} J_{2,4}(\gamma*) \left(1 + \frac{\mathcal{I}}{mc^2}\right) \left(1 + \frac{2\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \left[\int_0^{+\infty} J_{2,1}(\gamma*) \phi(\mathcal{I}) d\mathcal{I} \right] \right. \\
&+ \left[\int_0^{+\infty} J_{2,3}(\gamma*) \left(1 + \frac{2\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \left[\int_0^{+\infty} J_{2,2}(\gamma*) \left(1 + \frac{\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \\
&- \frac{1}{3} \left[\int_0^{+\infty} J_{4,2}(\gamma*) \left(1 + \frac{\mathcal{I}}{mc^2}\right) \left(1 + \frac{2\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \left[\int_0^{+\infty} J_{2,1}(\gamma*) \phi(\mathcal{I}) d\mathcal{I} \right] \\
&\left. + \frac{1}{3} \left[\int_0^{+\infty} J_{4,1}(\gamma*) \left(1 + \frac{2\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \left[\int_0^{+\infty} J_{2,2}(\gamma*) \left(1 + \frac{\mathcal{I}}{mc^2}\right) \phi(\mathcal{I}) d\mathcal{I} \right] \right\} .
\end{aligned}
$$

This result, jointly with (42), (49), (50) and $(A.6)_{1,2}$ of [7] concludes the proof of eq. (42).

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