Noncommutative geometry of the quantum clock

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Abstract

We introduce a model of noncommutative geometry that gives rise to the uncertainty relations recently derived from the discussion of a quantum clock. We investigate the dynamics of a free particle in this model from the point of view of doubly special relativity and discuss the geodesic motion in a Schwarzschild background.

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Several proposal exist for modifications of the Heisenberg uncertainty relations when effects of gravity are taken into account [1]. These are usually based on thought experiments and involve a dimensional parameter of the order of the Planck length (or mass) that sets the scale of the deformation. If one assumes that the Heisenberg algebra is deformed, such modifications can of course be derived formally by standard quantum mechanical arguments.

Deformed Heisenberg algebras have been considered in the literature mainly in relation with theories involving deformations of the Lorentz symmetry, like doubly special relativity (DSR) [2] or noncommutative geometries, especially of the κ-Poincaré class [3]. In fact, these theories are strongly related, although DSR investigates the deformations mainly from a classical (i.e. non-quantum) point of view. In these theories, the deformations are due to the introduction of a new fundamental scale, that cannot be invariant under the standard Lorentz transformations and whose appearance is justified as an effect of quantum gravity. For example, κ-Poincaré models are based on the deformed commutation relations of space and time coordinates $[x_0, x_i] = ix_i/\kappa$, with κ a constant proportional to the Planck mass[1], which imply a deformation of the full Heisenberg algebra. Note that such deformation is not unique and different models (usually called bases of the κ-Poincaré algebra) can be defined, leading to different modifications of the uncertainty relations.

While the modifications of the uncertainty relations considered in the literature usually concern the position-momentum relations, in a recent paper [4] a thought experiment has been discussed, which predicts an uncertainty relation connecting the measure of time and spatial intervals, given by

$$\Delta r \Delta t \geq \beta, \tag{1}$$

where $r = \sqrt{\mathbf{x}^2}$ is a radial coordinate and t is time. The constant β is given in terms of the Planck length L_P by $\beta = L_P^2/c$. Clearly, this uncertainty relation can be interpreted as due to noncommutativity of spatial and time coordinates, in analogy with the κ-Poincaré model cited above. The thought experiment is based on an ideal "quantum clock", namely a device that measures time by counting the decays of a sample of radioactive matter, that was first devised in ref. [5].

The quantum clock is defined as follows: given a set of N radioactive particles of mass m, with total mass $M = Nm$, the mean number of decays in a time interval Δt is $\Delta N = \lambda N \Delta t$, with variance $\sigma_N = \sqrt{\lambda N \Delta t}$. Therefore, it is possible to measure a time interval counting the number of decays. The relative error ϵ in the time measurement will be

$$\epsilon = \frac{\sigma_t}{\Delta t} = \frac{1}{\sqrt{\lambda N \Delta t}}, \tag{2}$$

where $\sigma_t = \sigma_N/\lambda N$. In order to measure short time intervals with small relative error it is therefore necessary to increase N. From eq. (2) it follows that

$$\Delta t = \frac{1}{\epsilon^2 \lambda N} = \frac{m}{\epsilon^2 \lambda M}, \tag{3}$$

or, in terms of the rest energy of the particles $E = mc^2$,

$$\Delta t = \frac{E}{\epsilon^2 \lambda M c^2}. \tag{4}$$

Now, from the Heisenberg uncertainty relation, one has for each particle

$$\delta E\, \delta t \geq \hbar/2, \tag{5}$$

where δE and δt are the uncertainties in the energy and time measurements. But $\delta E < E$, $\delta t < 1/\lambda$, and hence

$$\frac{E}{\lambda} \geq \hbar/2. \tag{6}$$

[1] We adopt the signature $(-1, 1, 1, 1)$ and denote spacetime coordinates as $(x_0, x_i) = (ct, \mathbf{x})$.

Using (4), one finally obtains

$$\Delta t \geq \frac{\hbar}{2\epsilon^2 c^2 M}, \tag{7}$$

which gives a lower limit for the mass of a clock capable of measuring time intervals with accuracy Δt.

However, it is not possible to arbitrarily increase the mass of the clock holding it in a small volume, since the radial size R of the clock must be such that a black hole cannot form, and therefore greater than its Schwarzschild radius,

$$R > \frac{2GM}{c^2}. \tag{8}$$

Setting $\Delta r = R$, with $r = \sqrt{\mathbf{x}^2}$, from (7) it follows that

$$\Delta r \Delta t \geq \frac{G\hbar}{c^4}, \tag{9}$$

which is the relation (1). In a quantum theory, this uncertainty relation can be derived assuming (up to numerical factors) the commutation relation $[t, r] = i\beta$. In fact, by the usual quantum mechanical argument, in the case of vanishing expectation values of r and t:

$$\Delta r \Delta t \geq \ \frac{1}{2} \left| \langle\, [r, t]\, \rangle \right| = \frac{\beta}{2}. \tag{10}$$

It is therefore natural to assume that the uncertainty relation (1) can be obtained starting from a deformation of the Heisenberg algebra of the kind investigated in noncommutative geometry or in DSR theories. In particular, a deformed commutation relation leading to (1) is, in relativistic notation[2],

$$[x_0, x_i] = i\beta \frac{x_i}{r}, \tag{11}$$

which clearly implies a noncommutative geometry, and in particular recalls the κ-Poincaré commutation relations $[x_0, x_i] = ix_i/\kappa$. It is therefore likely that it can be obtained from a similar construction.

Actually, assuming (11), one can construct several deformations of the Heisenberg algebra obeying the Jacobi identities. We consider here the simplest deformation compatible with (11), and investigate its classical limit, with commutators replaced by Poisson brackets, and its DSR implementation. Investigation of the quantum theory may result difficult, since the commutation relations (11) are nonlinear in the coordinates x_i, contrary to the models usually investigated in the context of noncommutative geometry.

We define the deformed algebra through the Poisson brackets

$$\begin{aligned} &\{x_i, x_j\} = 0, \qquad \{x_0, x_i\} = \beta \frac{x_i}{r}, \qquad \{p_\mu, p_\nu\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}, \\ &\{x_i, p_0\} = 0, \qquad \{x_0, p_0\} = -1, \qquad \{x_0, p_i\} = -\frac{\beta}{r} \left(p_i - \frac{\mathbf{x}\cdot\mathbf{p}}{r^2} x_i \right). \end{aligned} \tag{12}$$

It is easy to check that this algebra implies $\{x_0, r\} = \beta$, as required, and that the Poisson brackets are covariant under spatial rotations. This algebra can be realized in terms of canonical coordinates $\tilde{x}_\mu$, p_μ by the simple rule

$$x_0 = \tilde{x}_0 - \beta \frac{\tilde{x}_i p_i}{r}, \qquad x_i = \tilde{x}_i, \tag{13}$$

while the momenta maintain their canonical form.

The Poisson brackets (12) cannot however be covariant under boosts, since x_0 and r are not. Defining the Lorentz generators as $J_{\mu\nu} \equiv \tilde{x}_\mu p_\nu - \tilde{x}_\nu p_\mu$, so that the Lorentz algebra is not deformed, the infinitesimal action on the spacetime coordinates of a boost in the i direction is given by $\delta_L x_\mu = \{J_{0i}, x_\mu\}$. The covariance under boosts is obtained if their action is deformed so that

$$\delta_L x_j = \delta_{ij} \left(x_0 + \frac{\mathbf{x}\cdot\mathbf{p}}{r} \right), \qquad \delta_L x_0 = -x_i - \frac{\beta}{r} \left(p_0 + \frac{\mathbf{x}\cdot\mathbf{p}}{r^2} \right) x_i + \frac{\beta}{r} \left(x_0 + \beta \frac{\mathbf{x}\cdot\mathbf{p}}{r^2} \right) p_i. \tag{14}$$

[2] In the following we use natural unities, $\hbar = c = G = 1$.

The deformation of the action of boosts is a well-known consequence of the modification of the Heisenberg algebra.

Contrary to standard DSR models, the transformation rules of the momenta are instead not modified, and hence the Poincaré algebra is preserved. It follows in particular that the Casimir invariant of the Poincaré algebra is p^2, as in special relativity. One can therefore take as Hamiltonian for a free particle

$$H = \frac{p^2}{2m}. \tag{15}$$

Starting from this Hamiltonian, and taking into account the deformed Poisson brackets (12), the Hamilton equations for a free particle are then

$$m\dot{x}_i = \{x_i, H\} = p_i, \qquad m\dot{x}_0 = \{x_0, H\} = -p_0 - \frac{\beta}{r}\left(\mathbf{p}^2 - \frac{(\mathbf{x}\cdot\mathbf{p})^2}{r^2}\right), \qquad \dot{p}_\mu = \{p_\mu, H\} = 0, \tag{16}$$

where a dot denotes a derivative with respect to the evolution parameter.

The equations of motion can also be obtained varying the action

$$S = -\int ds\left(x^\mu \dot{p}_\mu + \beta\frac{\mathbf{x}\cdot\mathbf{p}}{r}\dot{p}_0 + H\right). \tag{17}$$

In fact, varying with respect to x^μ and p_μ, one gets

$$\dot{x}_0 = -\frac{p_0}{m} - \beta\left(\frac{\dot{\mathbf{x}}\cdot\mathbf{p} + \mathbf{x}\cdot\dot{\mathbf{p}}}{r} - \frac{\mathbf{x}\cdot\mathbf{p}\ \mathbf{x}\cdot\dot{\mathbf{x}}}{r^2}\right), \qquad m\dot{x}_i = \frac{p_i}{m} + \beta\frac{x_i\dot{p}_0}{r},$$

$$\dot{p}_0 = 0, \qquad \dot{p}_i + \beta\frac{p_i}{r}\dot{p}_0 = 0, \tag{18}$$

which are equivalent to (16).

As usual in theories containing deformations of the Lorentz symmetry, some problems arise in the definition of the velocity [6]. In fact, in relativistic theories the 3-velocity of a particle can be defined either as $v_i^H = \frac{\partial p^0}{\partial p^i}$ or as $v_i^K = \frac{\dot{x}_i}{\dot{x}_0}$, and the two definitions can yield different results. In our model, the first definition gives the standard relativistic relation $v_i^H = p_i/p_0$, while the second definition leads to

$$v_i^K = \frac{p_i}{p_0 + \frac{\beta}{r}\left(\mathbf{p}^2 - \frac{(\mathbf{x}\cdot\mathbf{p})^2}{r^2}\right)} = \frac{p_i}{p_0 + \beta\frac{\mathbf{L}^2}{r^3}}, \tag{19}$$

where $\mathbf{L} = \mathbf{x}\times\mathbf{p}$ is the angular momentum. The velocity v_i^K is always less than 1 for positive β.

Notice that this result is in contrast with most DSR models where v_i^K has the standard relativistic form, while v_i^H is deformed. This is related to the fact that in our case the Poincaré algebra, and hence the dispersion relation for particles, is not deformed, but its action on coordinates is.

In this respect, our model differs from the standard DSR models, that assume a deformation of the action of the Lorentz group on momentum space, rather than spacetime. However, like in that case, the deformation extends to the full phase space, since the position-momentum commutation relations are deformed (cf. (12)). This implies also a deformation of the standard relativistic position-momentum uncertainty relations.

The phenomenological implications of our model are not easy to disclose. The problem is that we have chosen a minimal deformation compatible with (11), affecting only the time variable, and hence its effects can appear only in fully relativistic situations.

The simplest effect in DSR phenomenology is the time delay in the detection of photons of different energies coming from a distant source. In [7] it has been shown that, taking into account the effect of the nontrivial action of translations, the time delay predictions calculated using either the phase or the group velocity are identical at least for some models of DSR. In our case no time delay is present. In fact, for particles moving on a straight line from the source to the observer, both phase and group velocity are equal and coincide with the relativistic ones, since in that case $\mathbf{L} = 0$. Some effects could however be present if the particle travels on a curved trajectory.

As an illustration, we therefore consider the orbital motion in Schwarzschild spacetime, although it is not likely that the associated corrections be observable in practice. We shall show that deviations from the predictions of relativity only occur in the time of travel, while the trajectories are unaltered.

In fact, due to the conservation of angular momentum, the problem can as usual be reduced to 1+2 dimensions. Going to spherical coordinates

$$t = x_0 = -x^0, \qquad r = \sqrt{(x^1)^2 + (x^2)^2}, \qquad \theta = \arctan \frac{x^2}{x^1}, \tag{20}$$

with momentum components

$$p_t = p_0, \qquad p_r = \frac{x^1 p_1 + x^2 p_2}{\sqrt{(x^1)^2 + (x^2)^2}} = \frac{\mathbf{x} \cdot \mathbf{p}}{r}, \qquad p_\theta \equiv J_{12} = x_1 p_2 - x_2 p_1, \tag{21}$$

it is easy to check that the only nontrivial brackets are

$$\{t, r\} = \beta, \qquad \{t, p_t\} = -1, \qquad \{r, p_r\} = 1, \qquad \{\theta, p_\theta\} = 1. \tag{22}$$

The Hamiltonian for the motion of a free particle of mass m in Schwarzschild spacetime is[3]

$$H = \frac{1}{2m}\left[-\frac{p_t^2}{A} + A p_r^2 + \frac{p_\theta^2}{r^2}\right], \tag{23}$$

with

$$A(r) = 1 - \frac{2M}{r}. \tag{24}$$

The Hamilton equations read

$$m\dot{t} = \frac{p_t}{A} + \frac{\beta M}{r^2}\left(p_r^2 + \frac{p_t^2}{A^2}\right) - \beta\frac{p_\theta^2}{r^3}, \qquad m\dot{r} = A p_r, \qquad m\dot{\theta} = \frac{p_\theta}{r^2}, \tag{25}$$

$$\dot{p}_t = \dot{p}_\theta = 0, \qquad m\dot{p}_r = -\frac{M}{r^2}\left(p_r^2 + \frac{p_t^2}{A^2}\right) + \frac{p_\theta^2}{r^3}. \tag{26}$$

Two conserved quantities are present,

$$p_t \equiv mE = mA\dot{t} - \frac{\beta M A}{r^2}\left(p_r^2 + \frac{p_t^2}{A^2}\right) + \beta\frac{p_\theta^2 A}{r^3}, \qquad p_\theta \equiv ml = mr^2\dot{\theta}, \tag{27}$$

where we have introduced the normalized momenta E and l. Moreover, p_r can be obtained in terms of the other momenta from the constraint $H = -m/2$, as

$$p_r^2 = \frac{p_t^2}{A^2} - \frac{p_\theta^2}{Ar^2} - \frac{m^2}{A}. \tag{28}$$

From (25) it follows that the equation of the orbits is independent of β and has the standard relativistic form

$$\frac{dr}{d\theta} = \frac{r^2 A p_r}{p_\theta}. \tag{29}$$

[3] We use this unusual normalization in order to keep track of possible breakdowns of the equivalence principle.

The solution of (29) can be obtained in the usual way by an expansion in the parameter $\eta = \frac{M^2}{l^2}$, as [8]

$$u \equiv \frac{1}{r} \sim \frac{M}{l^2}(1 + e\cos\theta'), \tag{30}$$

where e is the eccentricity of the orbit, related to the energy E by

$$E^2 \sim 1 - \eta(1 - e^2), \tag{31}$$

and [8]

$$\theta' \sim \theta - \eta(3\theta + e\sin\theta), \tag{32}$$

from which the standard perihelion shift $\Delta\theta = 6\pi\eta$ follows.

However, the time dependence of the orbit is modified, and so its period. In fact, from (25) and (28),

$$\frac{dt}{d\theta} = \frac{E}{lAu^2} + \beta m \left[\frac{M}{l} \left(\frac{2E^2}{A^2} - \frac{l^2u^2 + 1}{A} \right) - lu \right], \tag{33}$$

and after substituting (30), (31) and (32) one obtains, up to order η,

$$\frac{dt}{d\theta'} \sim \frac{l^3}{M^2} \frac{1 + 3\eta(1 + e\cos\theta')}{(1 + e\cos\theta')^2} + \frac{\beta mM}{l} \left[-e\cos\theta' + \eta(3 + 2e^2 + e\cos\theta' - e^2\cos^2\theta') \right]. \tag{34}$$

To compute the period T of the orbit (defined as the time between two successive passages through the perihelion) we integrate (34) in θ' between 0 and 2π. We get

$$T = \frac{2\pi l^3[1 + 3\eta(1 - e^2)]}{M^2(1 - e^2)^{3/2}} + \frac{2\pi\beta mM^3(3 + e^2)}{l^3}. \tag{35}$$

The first term is the classical one [8], while the second comes from the deformation of the symplectic structure. At leading order, the relative correction to the orbital period due to the second term is of order $\frac{\beta M^5}{l^6}\, m$, and depends linearly on the mass of the planet. Thus this correction breaks the equivalence principle, which is a common feature of DSR models [9].

The relative correction for the orbital period of the Earth would be 10^{-25}, i.e. of the order of 10^{-20} seconds, and similarly for other planets. The corrections are therefore extremely tiny and there is no chance to detect them. A different system that might give rise to stronger effects is a particle orbitating in a cyclotron. Also in this case the frequency should depend on the energy of the particle. The problem is presently being investigated.

It would also be interesting to consider the effects of our deformation in the quantum domain. First of all, we notice that, although the position-momentum Poisson brackets in (11) look odd, they take a perfectly standard form if spherical spatial coordinates r, θ, φ are used, namely

$$\{t, p_t\} = -1, \qquad \{r, p_r\} = \{\theta, p_\theta\} = \{\varphi, p_\varphi\} = 1. \tag{36}$$

Thus, under quantization in these coordinates, only the t-r commutation relations are deformed and hence the relativistic Heisenberg uncertainty relations stay unchanged, except (1).

One may ask if any effect might nevertheless occur in the relativistic hydrogen atom. To study this, one should find a Hilbert space realization of the commutation relations corresponding to the Poisson brackets (22). It is easy to see that the only difference from the standard realization is in the time operator, which now reads $t \to t - i\partial/\partial r$, cfr. (13). However in the relativistic Schrödinger (or Dirac) equation for the hydrogen atom only the operators p_μ and r appear and therefore no corrections arise in the energy spectrum. The only possible effects could occur for time-dependent observables.

To conclude, it seems that our model predicts only extremely tiny observable effects, related to the measure of time. However, we again remark that different models compatible with (11) could be constructed, presenting a more involved algebra than (12), that includes deformations also in the momentum sector, and is therefore more similar to standard DSR theories. The aim of our investigation has been anyway to investigate the deformations of the Heisenberg algebra compatible with the quantum clock uncertainty relations presenting a minimal departure from general relativity, rather than phenomenologically relevant ones.

It would also be interesting to study our model from the point of view of noncommutative geometry, to obtain for example the correct momentum addition law. This problem is not trivial because, as mentioned above, the commutation relations (11) are nonlinear in the coordinates and this may give rise to considerable technical problems.

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