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Blow-up phenomena in a parabolic–elliptic–elliptic attraction–repulsion chemotaxis system with superlinear logistic degradation

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This paper is concerned with the attraction–repulsion chemotaxis system with superlinear logistic degradation,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + \lambda u - \mu u^k, & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0, \end{cases}$$

under homogeneous Neumann boundary conditions, in a ball $\Omega\subset\mathbb{R}^n$ $(n\geq 3)$, with constant parameters $\lambda\in\mathbb{R}$, $k>1$, $\mu,\chi,\xi,\alpha,\beta,\gamma,\delta>0$. Blow-up phenomena in the system have been well investigated in the case $\lambda=\mu=0$, whereas the attraction–repulsion chemotaxis system with logistic degradation has been not studied. Under the condition that $k>1$ is close to 1, this paper ensures a solution which blows up in L^∞-norm and L^σ-norm with some $\sigma>1$ for some nonnegative initial data. Moreover, a lower bound of blow-up time is derived.

1. Introduction

Chemotaxis is a property of cells to move in response to the concentration gradient of a chemical substance produced by the cells. More precisely, it accounts for a process in which cells exhibit in response to chemoattractant and chemorepellent which are produced by themselves, that is, moving towards higher concentrations of an attractive signal and keeping away from a repulsive signal. A fully parabolic attraction–repulsion chemotaxis system was proposed by Painter and Hillen [20] to show the quorum effect in the

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chemotactic process and Luca et al. [12] to describe the aggregation of microglia observed in Alzheimer's disease, and can be approximated by a parabolic–elliptic–elliptic system.

In this paper we consider the parabolic–elliptic–elliptic attraction–repulsion chemotaxis system with superlinear logistic degradation,

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + \lambda u - \mu u^k, & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - \beta v, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - \delta w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial\nu} = \frac{\partial v}{\partial\nu} = \frac{\partial w}{\partial\nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega := B_R(0) \subset \mathbb{R}^n$ $(n \ge 3)$ is an open ball centered at the origin with radius $R > 0$; $\lambda \in \mathbb{R}$, $k > 1$ and $\mu, \chi, \xi, \alpha, \beta, \gamma, \delta$ are positive constants; $\frac{\partial}{\partial\nu}$ is the outward normal derivative on $\partial\Omega$. Moreover, the initial data u_0 is supposed to satisfy

$$u_0 \in C^0(\overline{\Omega}) \text{ is radially symmetric and nonnegative.} \tag{1.2}$$

The functions u, v and w represent the cell density, the concentration of attractive and repulsive chemical substances, respectively.

Blow-up phenomena correspond to the concentration of organisms on chemical substances. Hence it is important to investigate whether a solution of system (1.1) blows up or not. In this paper we show finite-time blow-up in L^∞-norm and L^σ-norm with some $\sigma > 1$, and derive a lower bound of blow-up time. Still more, not only blow-up phenomena but also global existence and boundedness have been studied in many literatures on chemotaxis systems (see [1,2,9]). Before presenting the main results, we give an overview of known results about some problems related to (1.1).

We first focus on the chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + g(u),\\ \tau v_t = \Delta v + \alpha u - \beta v \end{cases} \tag{1.3}$$

under homogeneous Neumann boundary conditions, where χ, α, β are positive constants and g is a function of logistic type, $\tau \in \{0, 1\}$. The system with $g(u) \equiv 0$ was proposed by Keller and Segel [10]. Since then, system (1.3) was extensively investigated as listed below.

- If $\tau = 1$, $g(u) \equiv 0$ and $\alpha = \beta = 1$, global existence and boundedness as well as finite-time blow-up were investigated as follows. In the one-dimensional setting, Osaki and Yagi [19] showed that all solutions are global in time and bounded. In the two-dimensional setting, Nagai et al. [17] established global existence and boundedness under the condition $\int_\Omega u_0(x)\,dx < \frac{4\pi}{\chi}$. On the other hand, Herrero and Velázquez [8] presented existence of radially symmetric solutions which blow up in finite time. Winkler in [28] with $\chi = 1$ and $n \ge 3$, derived that if $\|u_0\|_{L^{\frac{n}{2}+\varepsilon}(\Omega)}$ and $\|\nabla v_0\|_{L^{n+\varepsilon}(\Omega)}$ are small for sufficiently small $\varepsilon > 0$, then a solution is global and bounded. Also, Winkler in [29] proved finite-time blow-up under some conditions for initial data (u_0, v_0).
- If $\tau = 1$ and $g(u) = \lambda u - \mu u^k$ with $\lambda, \mu > 0$, global existence for any $k > 1$ and stabilization for $k \ge 2 - \frac{2}{n}$ were achieved in a generalized solution concept by Winkler [31]. Also, for certain choices of λ, μ, Yan and Fuest in [32], derived global existence of weak solutions under the condition $k > \min\{2 - \frac{2}{n},\ 2 - \frac{4}{n+4}\}$, $n \ge 2$ and $\alpha = \beta = 1$. In particular for $n = 2$, they showed that taking any $k > 1$ suffices to exclude the possibility of collapse into a persistent Dirac distribution.
- If $\tau = 0$, $g(u) \equiv 0$ and $\beta = 1$, Nagai in [15] proved global existence and boundedness when $n = 1$, or $n = 2$ and $\int_\Omega u_0(x)\,dx < \frac{4\pi}{\chi\alpha}$, and finite-time blow-up under some condition for the energy function

and the moment of u when $n \geq 2$. Also, in the two-dimensional setting, Nagai in [16] obtained global existence and boundedness under the condition $\int_\Omega u_0(x)\,dx < \frac{8\pi}{\chi\alpha}$, and finite-time blow-up under the conditions that $\alpha = 1$, $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi}$ and that

$$\int_\Omega u_0(x)|x - x_0|^2\,dx \text{ is sufficiently small for some } x_0 \in \Omega. \tag{1.4}$$

- If $\tau = 0$, $g(u) \leq a - \mu u^2$ with $a > 0$, $\mu > 0$ $(n \leq 2)$, $\mu > \frac{n-2}{n}\chi$ $(n \geq 3)$ and $\alpha = \beta = 1$, Tello and Winkler in [26] showed global existence and boundedness.
- If $\tau = 0$ and $\chi = \alpha = \beta = 1$, when $g(u) = \lambda u - \mu u^k$ with $\lambda \in \mathbb{R}$, $\mu > 0$ and $k > 1$, Winkler in [30] established finite-time blow-up in L^∞-norm under suitable conditions on data; more precisely, the author asserted that if $\Omega = B_R(0) \subset \mathbb{R}^n$ with $n \geq 3$, $R > 0$ and $1 < k < \frac{7}{6}$ $(n \in \{3, 4\})$, $1 < k < 1 + \frac{1}{2(n-1)}$ $(n \geq 5)$, then system (1.3) admits a solution which blows up in L^∞-norm at finite time. In [14], Marras and Vernier derived finite-time blow-up in L^σ-norm with $\sigma > \frac{n}{2}$ and finally obtained a lower bound of blow-up time. Moreover, as to system (1.3) with nonlinear diffusion, finite-time blow-up in L^∞-norm was obtained by Black et al. in [3] (see also [21,22] for weak chemotactic sensitivity and [13] for finite-time blow-up in L^p-norm to more general chemotaxis system).

We now shift our attention to the attraction–repulsion chemotaxis system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + \xi\nabla\cdot(u\nabla w) + g(u), \\ \tau v_t = \Delta v + \alpha u - \beta v, \\ \tau w_t = \Delta w + \gamma u - \delta w \end{cases} \tag{1.5}$$

under homogeneous Neumann boundary conditions, where $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants and $\tau \in \{0, 1\}$. The system with $\tau = 0$ and $g(u) = \lambda u - \mu u^k$ coincides with (1.1), whereas the previous works on this system are collected as follows.

- If $\tau = 0$ and $g(u) \equiv 0$, existence of solutions which blow up in L^∞-norm at finite time was studied in [24] and [11]. More precisely, in the two-dimensional setting, Tao and Wang [24] derived finite-time blow-up under the conditions (1.4) and

  (i) $\chi\alpha - \xi\gamma > 0$, $\delta = \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha - \xi\gamma}$.

  Also, in the two-dimensional setting, Li and Li [11] extended the above (i) to the following two conditions:

  (ii) $\chi\alpha - \xi\gamma > 0$, $\delta \geq \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha - \xi\gamma}$;
  (iii) $\chi\alpha\delta - \xi\gamma\beta > 0$, $\delta < \beta$ and $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha\delta - \xi\beta\gamma}$.
- If $\tau = 0$ and $g(u) \equiv 0$, Yu et al. [33] replaced $\chi\alpha\delta - \xi\gamma\beta$ with $\chi\alpha - \xi\gamma$ in (iii) and filled the gap between the above (ii) and (iii). In [11,24,33], blow-up phenomena were analyzed by introducing the linear combination of the solution components v, w such that $z := \chi v - \xi w$ (as to the fully parabolic case $\tau = 1$, see [5]).
- If $\tau = 0$ and $g(u) \equiv 0$, explicit lower bound of blow-up time for system (1.5) was provided under the condition $\chi\alpha - \xi\gamma > 0$ in the two-dimensional setting (see [27]).

In summary, blow-up phenomena have been well studied in both a parabolic–elliptic Keller–Segel system and an attraction–repulsion one when logistic sources are missing. However, blow-up with effect of logistic degradation in a Keller–Segel system has been investigated, while for an attraction–repulsion system it is still an open problem.

The purpose of this paper is to solve the above open problem. Namely, we examine finite-time blow-up in the attraction–repulsion system (1.1) and we achieve a lower bound of the blow-up time. This paper shows that logistic degradation does not necessarily rule out blow-up in the system (1.1), while there are some related works studying whether signal consumption suppresses blow-up, see e.g., [23,25] for (1.3) and [6,7] for (1.5), in which both systems have the second equation $v_t = \Delta v - uv$. These literatures prove that signal consumption prevents blow-up in some special cases. However, whether this is true or not is still open in general.

We now state main theorems. The first one asserts finite-time blow-up in L^∞-norm. The statement reads as follows.

Theorem 1.1 (*Finite-time Blow-up in L^∞-norm*)**.** *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Assume that $k > 1$ satisfies*

$$k < \begin{cases} \frac{7}{6} & \text{if } n \in \{3,4\}, \\ 1 + \frac{1}{2(n-1)} & \text{if } n \geq 5, \end{cases} \tag{1.6}$$

and $\chi, \xi, \alpha, \gamma > 0$ fulfill $\chi\alpha - \xi\gamma > 0$. Then, for all $L > 0$, $m > 0$ and $m_0 \in (0, m)$ one can find $r_0 = r_0(R, \lambda, \mu, k, L, m, m_0) \in (0, R)$ with the property that whenever u_0 satisfies (1.2) and is such that

$$u_0(x) \leq L|x|^{-n(n-1)} \quad \text{for all } x \in \Omega \tag{1.7}$$

as well as

$$\int_\Omega u_0(x)\,dx \leq m \quad \text{but} \quad \int_{B_{r_0}(0)} u_0(x)\,dx \geq m_0,$$

there exist $T_{\max} \in (0,\infty)$ and a classical solution (u, v, w) of system (1.1), uniquely determined by

$$\begin{aligned} &u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \\ &v, w \in \bigcap_{\vartheta > n} C^0([0, T_{\max}); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})), \end{aligned}$$

which blows up at $t = T_{\max}$ in the sense that

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{1.8}$$

We next state a result which guarantees a solution blows up in L^σ-norm at the blow-up time in L^∞-norm. The theorem is the following.

Theorem 1.2 (*Finite-time Blow-up in L^σ-norm*)**.** *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, a classical solution (u, v, w) for $t \in (0, T_{\max})$, provided by Theorem 1.1, is such that for all $\sigma > \frac{n}{2}$,*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\sigma(\Omega)} = \infty. \tag{1.9}$$

Define for all $\sigma > 1$ the energy function

$$\Psi(t) := \frac{1}{\sigma}\|u(\cdot,t)\|^\sigma_{L^\sigma(\Omega)} \quad \text{with} \quad \Psi_0 := \Psi(0) = \frac{1}{\sigma}\|u_0\|^\sigma_{L^\sigma(\Omega)}.$$

The third theorem provides a lower bound of blow-up time. The result reads as follows.

Theorem 1.3 (*Lower Bound of Blow-up Time*)**.** *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$. Then, for all $\sigma > \frac{n}{2}$ there exist $B_1 \geq 0$, $B_2, B_3 > 0$, depending on λ, μ, σ, n, such that for all u_0 fulfilling the same conditions as in Theorem 1.1, the blow-up time $T_{\max}$ in (1.9) satisfies the estimate*

$$T_{\max} \geq \int_{\Psi_0}^\infty \frac{d\eta}{B_1\eta + B_2\eta^{\gamma_1} + B_3\eta^{\gamma_2}}, \tag{1.10}$$

with $\gamma_1 := \frac{\sigma+1}{\sigma}$, $\gamma_2 := \frac{2(\sigma+1)-n}{2\sigma-n}$.

Theorems 1.2 and 1.3 provide additional information about blow-up in the system (1.1), which cannot be found for the attraction–repulsion chemotaxis system with/without logistic source.

One of the difficulties in the proofs of the above theorems is that the transformation $z := \chi v - \xi w$ does not work to reduce (1.1) to the Keller–Segel system in the case $\beta \neq \delta$, in contrast to the case $\beta = \delta$ which ensures the simplification of (1.1) as

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla z) + \lambda u - \mu u^k, & x \in \Omega,\ t > 0, \\ 0 = \Delta z + (\chi\alpha - \xi\gamma)u - \beta z, & x \in \Omega,\ t > 0, \end{cases}$$

which has already been studied in [14,30]. To overcome the difficulty, we carry out the arguments in the literatures without using the above transformation z.

This paper is organized as follows. In Section 2 we give preliminary results on local existence of classical solutions to (1.1) and some basic and useful facts. In Sections 3 and 4 we prove finite-time blow-up in L^∞-norm and L^σ-norm (Theorems 1.1 and 1.2), respectively. Section 5 is devoted to the derivation of a lower bound of blow-up time (Theorem 1.3).

2. Preliminaries

We start with the following lemma on local existence of classical solutions to (1.1). This lemma can be proved by a standard fixed point argument (see e.g., [26]).

Lemma 2.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^n$, $n \geq 3$ and $R > 0$, and let $\lambda \in \mathbb{R}$, $\mu > 0$, $k > 1$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Then for all nonnegative $u_0 \in C^0(\overline{\Omega})$ there exists $T_{\max} \in (0, \infty]$ such that (1.1) possesses a unique classical solution (u, v, w) such that*

$$u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$
$$v, w \in \bigcap_{\vartheta > n} C^0([0, T_{\max}); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

and

$$u \geq 0, \quad v \geq 0, \quad w \geq 0 \quad \text{for all } t \in (0, T_{\max}).$$

Moreover,

$$\text{if } T_{\max} < \infty, \quad \text{then } \limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Remark 2.1. We can use $\lim_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)}$ instead of $\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)}$ in the blow-up criterion (2.1), because we can construct a classical solution on $[0, T]$ with some positive time T depending only on $\|u_0\|_{L^\infty(\Omega)}$ and discuss the extension of the classical solution in a neighborhood of its maximal existence time $T_{\max}$, if $T_{\max} < \infty$.

We next give some properties of the Neumann heat semigroup which will be used later. For the proof, see [4, Lemma 2.1] and [28, Lemma 1.3].

Lemma 2.2. *Suppose $(e^{t\Delta})_{t \geq 0}$ is the Neumann heat semigroup in Ω, and let $\mu_1 > 0$ denote the first non zero eigenvalue of $-\Delta$ in Ω under Neumann boundary conditions. Then there exist $k_1, k_2 > 0$ which only depend on Ω and have the following properties:*

(i) if $1 \leq q \leq p \leq \infty$, then

$$\|e^{t\Delta} z\|_{L^p(\Omega)} \leq k_1 t^{-\frac{n}{2}(\frac{1}{q} - \frac{1}{p})} \|z\|_{L^q(\Omega)}, \quad \forall\, t > 0 \tag{2.2}$$

holds for all $z \in L^q(\Omega)$.

(ii) If $1 < q \le p \le \infty$, then

$$\|e^{t\Delta}\nabla\cdot \mathbf{z}\|_{L^p(\Omega)} \le k_2\big(1+t^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{q}-\frac{1}{p})}\big)e^{-\mu_1 t}\|\mathbf{z}\|_{L^q(\Omega)}, \quad \forall\, t>0 \tag{2.3}$$

is valid for any $\mathbf{z} \in (L^q(\Omega))^n$, where $e^{t\Delta}\nabla\cdot$ is the extension of the operator $e^{t\Delta}\nabla\cdot$ on $(C_0^\infty(\Omega))^n$ to $(L^q(\Omega))^n$.

In Section 5 we will use the following lemma which is obtained by a minor adjustment of the power of the Gagliardo–Nirenberg inequality.

Lemma 2.3. *Let Ω be a bounded and smooth domain of $\mathbb{R}^n$ with $n \ge 1$. Let $\mathsf{r} \ge 1$, $0 < \mathsf{q} \le \mathsf{p} \le \infty$, $\mathsf{s} > 0$. Then there exists a constant $C_{\rm GN} > 0$ such that*

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \le C_{\rm GN}\Big(\|\nabla f\|^{\mathsf{p}a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{\mathsf{p}(1-a)}_{L^{\mathsf{q}}(\Omega)} + \|f\|^{\mathsf{p}}_{L^{\mathsf{s}}(\Omega)}\Big) \tag{2.4}$$

for all $f \in L^{\mathsf{q}}(\Omega)$ with $\nabla f \in (L^{\mathsf{r}}(\Omega))^n$, and $a := \frac{\frac{1}{\mathsf{q}}-\frac{1}{\mathsf{p}}}{\frac{1}{\mathsf{q}}+\frac{1}{n}-\frac{1}{\mathsf{r}}} \in [0,1]$.

Proof. Following from the Gagliardo–Nirenberg inequality (see [18] for more details):

$$\|f\|^{\mathsf{p}}_{L^{\mathsf{p}}(\Omega)} \le \Big[c_{\rm GN}\Big(\|\nabla f\|^{a}_{L^{\mathsf{r}}(\Omega)}\|f\|^{1-a}_{L^{\mathsf{q}}(\Omega)} + \|f\|_{L^{\mathsf{s}}(\Omega)}\Big)\Big]^{\mathsf{p}},$$

with some $c_{\rm GN} > 0$, and then from the inequality

$$(\mathsf{a}+\mathsf{b})^\alpha \le 2^\alpha(\mathsf{a}^\alpha+\mathsf{b}^\alpha) \quad \text{for any } \mathsf{a},\mathsf{b} \ge 0,\ \alpha > 0,$$

we arrive to (2.4) with $C_{\rm GN} = 2^{\mathsf{p}}c^{\mathsf{p}}_{\rm GN}$. □

3. Finite-time blow-up in L^∞-norm

Throughout the sequel, we suppose that $\Omega = B_R(0) \subset \mathbb{R}^n$ ($n \ge 3$) with $R > 0$ and u_0 satisfies condition (1.2) as well as $\lambda \in \mathbb{R}$, $\mu > 0$, $k > 1$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$. Then we denote by $(u,v,w) = (u(r,t), v(r,t), w(r,t))$ the local classical solution of (1.1) given in Lemma 2.1 and by $T_{\max} \in (0,\infty)$ its maximal existence time.

The goal of this section is to prove finite-time blow-up in L^∞-norm. To this end, noting that u_0 is radially symmetric and so are u, v, w, we first define the functions

$$\begin{aligned}
U(s,t) &:= \int_0^{s^{\frac{1}{n}}} \rho^{n-1}u(\rho,t)\,d\rho, \quad s \in [0,R^n],\ t \in [0,T_{\max}),\\
V(s,t) &:= \int_0^{s^{\frac{1}{n}}} \rho^{n-1}v(\rho,t)\,d\rho, \quad s \in [0,R^n],\ t \in [0,T_{\max}),\\
W(s,t) &:= \int_0^{s^{\frac{1}{n}}} \rho^{n-1}w(\rho,t)\,d\rho, \quad s \in [0,R^n],\ t \in [0,T_{\max}).
\end{aligned}$$

Then we prove the following lemma.

Lemma 3.1. *Under the above notation, we have*

$$\begin{aligned}
U_t(s,t) &= n^2 s^{2-\frac{2}{n}}U_{ss}(s,t) + n\chi\alpha U(s,t)U_s(s,t) - n\chi\beta V(s,t)U_s(s,t)\\
&\quad - n\xi\gamma U(s,t)U_s(s,t) + n\xi\delta W(s,t)U_s(s,t)\\
&\quad + \lambda U(s,t) - n^{k-1}\mu\int_0^s U_s^k(\sigma,t)\,d\sigma
\end{aligned} \tag{3.1}$$

for all $s \in (0,R^n)$, $t \in (0,T_{\max})$.

Proof. By the definitions of U, V, W, we obtain

$$U_s(s,t) = \frac{1}{n} u(s^{\frac{1}{n}}, t), \quad U_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} u_r(s^{\frac{1}{n}}, t),$$
$$V_s(s,t) = \frac{1}{n} v(s^{\frac{1}{n}}, t), \quad V_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} v_r(s^{\frac{1}{n}}, t),$$
$$W_s(s,t) = \frac{1}{n} w(s^{\frac{1}{n}}, t), \quad W_{ss}(s,t) = \frac{1}{n^2} s^{\frac{1}{n}-1} w_r(s^{\frac{1}{n}}, t),$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max})$. Since u, v, w are radially symmetric functions, we see from the second and third equations in (1.1) that

$$\frac{1}{r^{n-1}} (r^{n-1} v_r(r,t))_r = -\alpha u(r,t) + \beta v(r,t),$$
$$\frac{1}{r^{n-1}} (r^{n-1} w_r(r,t))_r = -\gamma u(r,t) + \delta w(r,t),$$

from which we obtain

$$r^{n-1} v_r(r,t) = -\alpha U(r^n, t) + \beta V(r^n, t), \tag{3.2}$$
$$r^{n-1} w_r(r,t) = -\gamma U(r^n, t) + \delta W(r^n, t) \tag{3.3}$$

for all $r \in (0, R)$, $t \in (0, T_{\max})$. Moreover, rewriting the first equation in (1.1) in the radial coordinates as

$$\begin{aligned} u_t(r,t) = {} & \frac{1}{r^{n-1}} (r^{n-1} u_r(r,t))_r - \chi \frac{1}{r^{n-1}} (u(r,t) r^{n-1} v_r(r,t))_r \\ & + \xi \frac{1}{r^{n-1}} (u(r,t) r^{n-1} w_r(r,t))_r \\ & + \lambda u(r,t) - \mu u^k(r,t) \end{aligned} \tag{3.4}$$

and integrating it with respect to r over $[0, s^{\frac{1}{n}}]$, we have

$$\begin{aligned} U_t(s,t) = {} & n^2 s^{2-\frac{2}{n}} U_{ss}(s,t) - n\chi U_s(s,t) s^{1-\frac{1}{n}} v_r(s^{\frac{1}{n}}, t) \\ & + n\xi U_s(s,t) s^{1-\frac{1}{n}} w_r(s^{\frac{1}{n}}, t) \\ & + \lambda U(s,t) - n^{k-1} \mu \int_0^s U_s^k(\sigma, t)\, d\sigma \end{aligned}$$

for all $s \in (0, R^n)$, $t \in (0, T_{\max})$. Thanks to (3.2) and (3.3), we arrive at (3.1). □

Given $p \in (0,1)$, $s_0 \in (0, R^n)$, we next derive a differential inequality for the moment-type function Φ defined as

$$\Phi(t) := \int_0^{s_0} s^{-p} (s_0 - s) U(s,t)\, ds, \quad t \in [0, T_{\max}).$$

Lemma 3.2. *Let $\lambda \in \mathbb{R}$, $\mu > 0$, $\chi, \xi, \alpha, \beta, \gamma, \delta > 0$ and let $\chi\alpha - \xi\gamma > 0$. Assume that $k > 1$ satisfies (1.6). Then there is $p \in (1 - \frac{2}{n}, 1)$ with the following property: For all $m > 0$ and $L > 0$ there exist $s_* \in (0, R^n)$ and $C_1 > 0$ such that whenever u_0 fulfills (1.2), (1.7) and $\int_\Omega u_0(x)\, dx \le m$, for any $s_0 \in (0, s_*)$ the function Φ satisfies*

$$\Phi'(t) \ge \frac{1}{C_1} s_0^{p-3} \Phi^2(t) - C_1 s_0^{\frac{2}{n}+1-p} \tag{3.5}$$

for all $t \in (0, \widehat{T}_{\max})$, where $\widehat{T}_{\max} := \min\{1, T_{\max}\}$. Moreover, for all $m_0 \in (0, m)$ one can find $s_0 \in (0, s_*)$ and $r_0 = r_0(R, \lambda, \mu, k, L, m, m_0) \in (0, R)$ such that if $\int_{B_{r_0}(0)} u_0(x)\, dx \ge m_0$ and $\widehat{T}_{\max} > \frac{1}{2}$, then for all $t \in (0, \frac{1}{2})$,

$$\Phi'(t) \ge C_2 s_0^{p-3} \Phi^2(t), \tag{3.6}$$

where C_2 is a positive constant.

Proof. By the definition of Φ and Eq. (3.1), we have

$$\begin{aligned}\Phi'(t) &= \int_0^{s_0} s^{-p}(s_0-s)U_t(s,t)\,ds \\ &= n^2\int_0^{s_0} s^{2-\frac{2}{n}-p}(s_0-s)U_{ss}(s,t)\,ds \\ &\quad + n(\chi\alpha-\xi\gamma)\int_0^{s_0} s^{-p}(s_0-s)U(s,t)U_s(s,t)\,ds \\ &\quad - n\chi\beta\int_0^{s_0} s^{-p}(s_0-s)V(s,t)U_s(s,t)\,ds \\ &\quad + n\xi\delta\int_0^{s_0} s^{-p}(s_0-s)W(s,t)U_s(s,t)\,ds \\ &\quad + \lambda\int_0^{s_0} s^{-p}(s_0-s)U(s,t)\,ds - n^{k-1}\mu\int_0^{s_0} s^{-p}(s_0-s)\Big[\int_0^s U_s^k(\sigma,t)\,d\sigma\Big]\,ds. \end{aligned} \tag{3.7}$$

Since $U_s(s,t)=\frac{1}{n}u(s^{\frac{1}{n}},t)\geq 0$ and hence the fourth term on the right-hand side of (3.7) is nonnegative, we obtain

$$\begin{aligned}\Phi'(t) &\geq n^2\int_0^{s_0} s^{2-\frac{2}{n}-p}(s_0-s)U_{ss}(s,t)\,ds \\ &\quad + n(\chi\alpha-\xi\gamma)\int_0^{s_0} s^{-p}(s_0-s)U(s,t)U_s(s,t)\,ds \\ &\quad - n\chi\beta\int_0^{s_0} s^{-p}(s_0-s)V(s,t)U_s(s,t)\,ds \\ &\quad + \lambda\int_0^{s_0} s^{-p}(s_0-s)U(s,t)\,ds - n^{k-1}\mu\int_0^{s_0} s^{-p}(s_0-s)\Big[\int_0^s U_s^k(\sigma,t)\,d\sigma\Big]\,ds \end{aligned}$$

for all $t\in(0,T_{\max})$. Since $\chi\alpha-\xi\gamma>0$ by assumption, following the steps in [30, (4.3)], we can derive the differential inequalities (3.5) and (3.6); note that, in the assumption $\widehat{T}_{\max}>\frac{1}{2}$ for (3.6) the value $\frac{1}{2}$ can be replaced with other positive values less than 1. □

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. Thanks to Lemma 3.2, in particular, from (3.6), we can see that $T_{\max}<\infty$. Therefore, from blow-up criterion (2.1), we conclude that the finite-time blow-up in L^∞-norm occurs. Namely, (1.8) is proved. □

4. Finite-time blow-up in L^σ-norm

In these next sections we will assume the conditions contained in Theorem 1.1. In order to prove Theorem 1.2, first we state the following lemmas.

Lemma 4.1. *Let $\Omega\subset\mathbb{R}^n$, $n\geq 3$ be a bounded and smooth domain, and $\lambda\in\mathbb{R}$, $\mu>0$, $k>1$. Then for a classical solution (u,v,w) of (1.1) we have*

$$\int_\Omega u\,dx \leq m_* := \max\Big\{\int_\Omega u_0\,dx,\ \Big(\frac{\lambda_+}{\mu}|\Omega|^{k-1}\Big)^{\frac{1}{k-1}}\Big\} \quad \textit{for all } t\in(0,T_{\max}), \tag{4.1}$$

where $\lambda_+ := \max\{0,\lambda\}$.

Proof. Integrating the first equation in (1.1) and applying the divergence theorem and boundary conditions of (1.1), we obtain

$$\frac{d}{dt}\int_\Omega u\,dx = \lambda\int_\Omega u\,dx - \mu\int_\Omega u^k\,dx \le \lambda_+\int_\Omega u\,dx - \mu|\Omega|^{1-k}\Big(\int_\Omega u\,dx\Big)^k, \tag{4.2}$$

where in the last term we used Hölder's inequality: $\int_\Omega u \le |\Omega|^{\frac{k-1}{k}}(\int_\Omega u^k)^{\frac{1}{k}}$. From (4.2) we deduce that $y := \int_\Omega u\,dx$ fulfills

$$\begin{cases} y'(t) \le \lambda_+ y(t) - \overline{\mu} y^k(t), & \overline{\mu} := \mu|\Omega|^{1-k} \quad \text{for all } t \in (0, T_{\max}),\\ y(0) = y_0, & y_0 := \int_\Omega u_0\,dx. \end{cases}$$

Upon an ODE comparison argument this implies that $y(t) \le m_*$ for all $t \in (0, T_{\max})$. The lemma is proved. □

We next prove the following lemma which plays an important role in the proof of Theorem 1.2.

Lemma 4.2. *Let $\Omega \subset \mathbb{R}^n$, $n \ge 3$ be a bounded and smooth domain. Let (u, v, w) be a classical solution of system (1.1). If for some $\sigma_0 > \frac{n}{2}$ there exists $C > 0$ such that*

$$\|u(\cdot,t)\|_{L^{\sigma_0}(\Omega)} \le C \quad \text{for all } t \in (0, T_{\max}),$$

then, for some $\widehat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \widehat{C} \quad \text{for all } t \in (0, T_{\max}). \tag{4.3}$$

Proof. For any $x \in \Omega$, $t \in (0, T_{\max})$, we set $t_0 := \max\{0, t-1\}$ and we consider the representation formula for u:

$$\begin{aligned} u(\cdot,t) &= e^{(t-t_0)\Delta}u(\cdot,t_0) - \chi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla v(\cdot,s))\,ds \\ &\quad + \xi\int_{t_0}^t e^{(t-s)\Delta}\nabla\cdot(u(\cdot,s)\nabla w(\cdot,s))\,ds + \int_{t_0}^t e^{(t-s)\Delta}\big[\lambda u(\cdot,s) - \mu u^k(\cdot,s)\big]\,ds \\ &=: u_1(\cdot,t) + u_2(\cdot,t) + u_3(\cdot,t) + u_4(\cdot,t) \end{aligned}$$

and

$$0 \le u(\cdot,t) \le \|u_1(\cdot,t)\|_{L^\infty(\Omega)} + \|u_2(\cdot,t)\|_{L^\infty(\Omega)} + \|u_3(\cdot,t)\|_{L^\infty(\Omega)} + u_4(\cdot,t). \tag{4.4}$$

We have

$$\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le \max\{\|u_0\|_{L^\infty(\Omega)}, m_* k_1\} =: C_5, \tag{4.5}$$

with $k_1 > 0$ and m_* defined in (4.1). In fact, if $t \le 1$, then $t_0 = 0$ and hence the maximum principle yields $u_1(\cdot,t) \le \|u_0\|_{L^\infty(\Omega)}$. If $t > 1$, then $t - t_0 = 1$ and from (2.2) with $p = \infty$ and $q = 1$, we deduce from (4.1) that $\|u_1(\cdot,t)\|_{L^\infty(\Omega)} \le k_1(t-t_0)^{-\frac{n}{2}}\|u(\cdot,t_0)\|_{L^1(\Omega)} \le m_* k_1$. We next use (2.3) with $p = \infty$, which leads to

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le k_2\chi\int_{t_0}^t (1 + (t-s)^{-\frac{1}{2}-\frac{n}{2q}})e^{-\mu_1(t-s)}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q(\Omega)}\,ds. \tag{4.6}$$

Here, we may assume that $\frac{n}{2} < \sigma_0 < n$, and then we can fix $q > n$ such that $1 - \frac{(n-\sigma_0)q}{n\sigma_0} > 0$, which enables us to pick $\theta \in (1, \infty)$ fulfilling $\frac{1}{\theta} < 1 - \frac{(n-\sigma_0)q}{n\sigma_0}$, that is, $\frac{q\theta}{\theta-1} < \frac{n\sigma_0}{n-\sigma_0}$. Then by Hölder's inequality, we can estimate

$$\begin{aligned}\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q(\Omega)} &\le \|u(\cdot,s)\|_{L^{q\theta}(\Omega)}\|\nabla v(\cdot,s)\|_{L^{\frac{q\theta}{\theta-1}}(\Omega)}\\ &\le C_6\|u(\cdot,s)\|_{L^{q\theta}(\Omega)}\|\nabla v(\cdot,s)\|_{L^{\frac{n\sigma_0}{n-\sigma_0}}(\Omega)} \quad \text{for all } s\in(0,T_{\max}),\end{aligned}$$

with some $C_6 > 0$. The Sobolev embedding theorem and elliptic regularity theory applied to the second equation in (1.1) tell us that $\|v(\cdot,s)\|_{W^{1,\frac{n\sigma_0}{n-\sigma_0}}(\Omega)} \le C_7\|v(\cdot,s)\|_{W^{2,\sigma_0}(\Omega)} \le C_8$ with some $C_7, C_8 > 0$. Thus again by Hölder's inequality and (4.1), we obtain

$$\|u(\cdot,s)\nabla v(\cdot,s)\|_{L^q(\Omega)} \le C_9\|u(\cdot,s)\|_{L^{q\theta}(\Omega)} \le C_{10}\|u(\cdot,s)\|^{\overline{\theta}}_{L^\infty(\Omega)} \quad \text{for all } s\in(0,T_{\max}),$$

with some $\overline{\theta}\in(0,1)$, $C_9 := C_6C_8$ and $C_{10} := C_9 m_*^{1-\overline{\theta}}$. Hence, combining this estimate and (4.6), we infer

$$\|u_2(\cdot,t)\|_{L^\infty(\Omega)} \le C_{10}k_2\chi\int_{t_0}^t (1+(t-s)^{-\frac12-\frac{n}{2q}})e^{-\mu_1(t-s)}\|u(\cdot,s)\|^{\overline{\theta}}_{L^\infty(\Omega)}\,ds.$$

Now fix any $T\in(0,T_{\max})$. Then, since $t-t_0\le 1$, we have

$$\begin{aligned}\|u_2(\cdot,t)\|_{L^\infty(\Omega)} &\le C_{10}k_2\chi\int_{t_0}^t (1+(t-s)^{-\frac12-\frac{n}{2q}}e^{-\mu_1(t-s)})\,ds\cdot\sup_{t\in[0,T]}\|u(\cdot,t)\|^{\overline{\theta}}_{L^\infty(\Omega)}\\ &\le C_{11}\chi\sup_{t\in[0,T]}\|u(\cdot,t)\|^{\overline{\theta}}_{L^\infty(\Omega)},\end{aligned} \tag{4.7}$$

where $C_{11} := C_{10}k_2(1+\mu_1^{\frac{n}{2q}-\frac12}\int_0^\infty r^{-\frac12-\frac{n}{2q}}e^{-r}\,dr) > 0$ is finite, because $-\frac12-\frac{n}{2q} > -1$. Similarly, we conclude

$$\|u_3(\cdot,t)\|_{L^\infty(\Omega)} \le C_{11}\xi\sup_{t\in[0,T]}\|u(\cdot,t)\|^{\overline{\theta}}_{L^\infty(\Omega)}. \tag{4.8}$$

We next prove that there exists a constant $C_{12}\ge 0$ such that $u_4(\cdot,t)\le C_{12}$. To this end, we observe that

$$h(u) := \lambda u - \mu u^k \le h(u_*) =: C_{12},$$

with $u_* := (\frac{\lambda_+}{\mu k})^{\frac{1}{k-1}}$. We have

$$u_4(\cdot,t) = \int_{t_0}^t e^{(t-s)\Delta}\big[\lambda u(\cdot,s)-\mu u^k(\cdot,s)\big]\,ds \le C_{12}\int_{t_0}^t ds \le C_{12}. \tag{4.9}$$

Plugging (4.5), (4.7), (4.8) and (4.9) into (4.4), we see that

$$0\le u(x,t)\le C_5 + C_{12} + C_{11}(\chi+\xi)\sup_{t\in[0,T]}\|u(\cdot,t)\|^{\overline{\theta}}_{L^\infty(\Omega)},$$

which implies

$$\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C_{13} + C_{14}\Big(\sup_{t\in[0,T]}\|u(\cdot,t)\|_{L^\infty(\Omega)}\Big)^{\overline{\theta}} \quad \text{for all } T\in(0,T_{\max}),$$

with $C_{13} := C_5 + C_{12}$ and $C_{14} := C_{11}(\chi+\xi)$. From this inequality with $\overline{\theta}\in(0,1)$, we arrive at (4.3). □

Proof of Theorem 1.2. Since Theorem 1.1 holds, the unique local classical solution of (1.1) blows up at $t = T_{\max}$ in the sense $\limsup_{t\nearrow T_{\max}}\|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$ (i.e., (1.8)). By contradiction, we prove that it blows up also in L^σ-norm. In fact, if there exist $\sigma_0 > \frac{n}{2}$ and $C > 0$ such that

$$\|u(\cdot,t)\|_{L^{\sigma_0}(\Omega)} \le C \quad \text{for all } t\in(0,T_{\max}),$$

then, from Lemma 4.2, there exists $\widehat{C} > 0$ such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \widehat{C} \quad \text{for all } t\in(0,T_{\max}),$$

which contradicts (1.8), so that, if u blows up in L^∞-norm, then u blows up in L^σ-norm for all $\sigma > \frac{n}{2}$. □

5. A lower bound for $T_{\max}$, the proof of Theorem 1.3

Let us consider $\Psi(t) = \frac{1}{\sigma}\int_\Omega u^\sigma(x,t)\,dx$, $u(x,t)$ the first component of solutions to (1.1) and we prove that Ψ satisfies a first order differential inequality.

In the proof of Theorem 1.3 we need an estimate for $\int_\Omega u^{\sigma+1}\,dx$. To this end, we use the Gagliardo–Nirenberg inequality (2.4) with $f = u^{\frac{\sigma}{2}}$, $\mathsf{p} = \frac{2(\sigma+1)}{\sigma}$, $\mathsf{r} = 2$, $\mathsf{q} = 2$, $\mathsf{s} = 2$. Since $\sigma > \frac{n}{2}$, we have

$$
\begin{aligned}
\int_\Omega u^{\sigma+1}\,dx &= \|u^{\frac{\sigma}{2}}\|^{\frac{2(\sigma+1)}{\sigma}}_{L^{\frac{2(\sigma+1)}{\sigma}}(\Omega)} \\
&\le C_{\mathrm{GN}}\|\nabla u^{\frac{\sigma}{2}}\|^{\frac{2(\sigma+1)}{\sigma}\theta_0}_{L^2(\Omega)}\|u^{\frac{\sigma}{2}}\|^{\frac{2(\sigma+1)}{\sigma}(1-\theta_0)}_{L^2(\Omega)} + C_{\mathrm{GN}}\|u^{\frac{\sigma}{2}}\|^{\frac{2(\sigma+1)}{\sigma}}_{L^2(\Omega)} \\
&= C_{\mathrm{GN}}\Big(\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx\Big)^{\frac{\sigma+1}{\sigma}\theta_0}\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+1}{\sigma}(1-\theta_0)} + C_{\mathrm{GN}}\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+1}{\sigma}} \\
&\le C_{\mathrm{GN}}\varepsilon_1\beta_0\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx + C_{\mathrm{GN}}\varepsilon_1^{-\frac{\beta_0}{1-\beta_0}}(1-\beta_0)\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{(\sigma+1)(1-\theta_0)}{\sigma(1-\beta_0)}} \\
&\quad + C_{\mathrm{GN}}\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+1}{\sigma}} \\
&= c_1(\varepsilon_1)\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx + c_2(\varepsilon_1)\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{2(\sigma+1)-n}{2\sigma-n}} + c_3\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+1}{\sigma}},
\end{aligned}
\tag{5.1}
$$

with $\varepsilon_1 > 0$, $\theta_0 := \frac{n}{2(\sigma+1)} \in (0,1)$ and $\beta_0 := \frac{\sigma+1}{\sigma}\theta_0 = \frac{n}{2\sigma} \in (0,1)$. Now, we derive a differential inequality of the first order for $\Psi(t)$.

$$
\begin{aligned}
\Psi'(t) &= \int_\Omega u^{\sigma-1}\Delta u\,dx - \chi\int_\Omega u^{\sigma-1}\nabla\cdot(u\nabla v)\,dx + \xi\int_\Omega u^{\sigma-1}\nabla\cdot(u\nabla w)\,dx \\
&\quad + \lambda\int_\Omega u^\sigma\,dx - \mu\int_\Omega u^{\sigma+k-1}\,dx \\
&=: \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4 + \mathcal{I}_5.
\end{aligned}
\tag{5.2}
$$

We have:

$$
\begin{aligned}
\mathcal{I}_1 &= \int_\Omega u^{\sigma-1}\Delta u\,dx = -(\sigma-1)\int_\Omega u^{\sigma-2}|\nabla u|^2\,dx \\
&= -\frac{4(\sigma-1)}{\sigma^2}\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx,
\end{aligned}
\tag{5.3}
$$

and

$$
\begin{aligned}
\mathcal{I}_2 &= -\chi\int_\Omega u^{\sigma-1}\nabla\cdot(u\nabla v)\,dx = \chi\frac{\sigma-1}{\sigma}\int_\Omega \nabla u^\sigma\cdot\nabla v\,dx \\
&= -\chi\frac{\sigma-1}{\sigma}\int_\Omega u^\sigma\Delta v\,dx \\
&= -\chi\beta\frac{\sigma-1}{\sigma}\int_\Omega u^\sigma v\,dx + \chi\alpha\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx \\
&\le \chi\alpha\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx
\end{aligned}
\tag{5.4}
$$

as well as

$$\begin{aligned}
\mathcal{I}_3 &= \xi \int_\Omega u^{\sigma-1}\nabla\cdot(u\nabla w)\,dx \\
&= \xi\delta\frac{\sigma-1}{\sigma}\int_\Omega u^\sigma w\,dx - \xi\gamma\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx \\
&\le \xi\delta\frac{\sigma-1}{\sigma}\Big(\int_\Omega u^{\sigma+1}\,dx\Big)^{\frac{\sigma}{\sigma+1}}\Big(\int_\Omega w^{\sigma+1}\,dx\Big)^{\frac{1}{\sigma+1}} - \xi\gamma\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx \\
&\le \xi\gamma\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx - \xi\gamma\frac{\sigma-1}{\sigma}\int_\Omega u^{\sigma+1}\,dx \\
&= 0, \qquad (5.5)
\end{aligned}$$

where the last inequality holds from $(\int_\Omega w^{\sigma+1})^{\frac{1}{\sigma+1}} \le \frac{\gamma}{\delta}(\int_\Omega u^{\sigma+1})^{\frac{1}{\sigma+1}}$ established by standard testing procedures in the equation for w. We now use (5.1) in (5.4) to obtain

$$\mathcal{I}_2 \le \widetilde{c}_1(\varepsilon_1)\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx + \widetilde{c}_2(\varepsilon_1)\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{2(\sigma+1)-n}{2\sigma-n}} + \widetilde{c}_3\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+1}{\sigma}}, \qquad (5.6)$$

with $\widetilde{c}_1(\varepsilon_1) := \chi\alpha\frac{\sigma-1}{\sigma}c_1(\varepsilon_1)$, $\widetilde{c}_2(\varepsilon_1) := \chi\alpha\frac{\sigma-1}{\sigma}c_2(\varepsilon_1)$, $\widetilde{c}_3 := \chi\alpha\frac{\sigma-1}{\sigma}c_3$. Also, using Hölder's inequality, we see that

$$\begin{aligned}
\mathcal{I}_4 + \mathcal{I}_5 &= \lambda\int_\Omega u^\sigma\,dx - \mu\int_\Omega u^{\sigma+k-1}\,dx \\
&\le \lambda_+\int_\Omega u^\sigma\,dx - \mu|\Omega|^{\frac{1-k}{\sigma}}\Big(\int_\Omega u^\sigma\,dx\Big)^{\frac{\sigma+k-1}{\sigma}}. \qquad (5.7)
\end{aligned}$$

Substituting (5.3), (5.5), (5.6) and (5.7) in (5.2) we get

$$\begin{aligned}
\Psi' \le{}& B_1\Psi + B_2\Psi^{\frac{\sigma+1}{\sigma}} + B_3\Psi^{\frac{2(\sigma+1)-n}{2\sigma-n}} - B_4\Psi^{\frac{\sigma+k-1}{\sigma}} \\
&+ \Big(\widetilde{c}_1(\varepsilon_1) - \frac{4(\sigma-1)}{\sigma^2}\Big)\int_\Omega |\nabla u^{\frac{\sigma}{2}}|^2\,dx, \qquad (5.8)
\end{aligned}$$

with $B_1 := \lambda_+\sigma$, $B_2 := \widetilde{c}_3\sigma^{\frac{\sigma+1}{\sigma}}$, $B_3 := \widetilde{c}_2(\varepsilon_1)\sigma^{\frac{2(\sigma+1)-n}{2\sigma-n}}$, $B_4 := \mu|\Omega|^{\frac{1-k}{\sigma}}\sigma^{\frac{\sigma+k-1}{\sigma}}$. In (5.8) we choose ε_1 such that $\widetilde{c}_1(\varepsilon_1) - \frac{4(\sigma-1)}{\sigma^2} \le 0$ and neglecting the negative terms, we obtain

$$\Psi' \le B_1\Psi + B_2\Psi^{\frac{\sigma+1}{\sigma}} + B_3\Psi^{\frac{2(\sigma+1)-n}{2\sigma-n}}. \qquad (5.9)$$

Integrating (5.9) from 0 to $T_{\max}$, we arrive to (1.10). □

Remark 5.1. Since u blows up in $L^\sigma(\Omega)$-norm at finite time $T_{\max}$, then there exists a time $t_1 \in [0, T_{\max})$, where $\Psi(t_1) = \Psi_0$. As a consequence, $\Psi(t) \ge \Psi_0$, $t \in [t_1, T_{\max})$ so that $\Psi^\rho \le \Psi^{\gamma_2}\Psi_0^{\rho-\gamma_2}$ for some $\rho \le \gamma_2$. Moreover, taking into account that $1 < \frac{\sigma+1}{\sigma} \le \frac{2(\sigma+1)-n}{2\sigma-n} = \gamma_2$, it follows that

$$\Psi' \le A\Psi^{\gamma_2} \quad \text{in } (t_1, T_{\max}), \qquad (5.10)$$

with $A := B_1\Psi_0^{-\frac{2}{2\sigma-n}} + B_2\Psi_0^{-\frac{n}{\sigma(2\sigma-n)}} + B_3$. Integrating (5.10) from t_1 to $T_{\max}$, we derive the following explicit lower bound of the blow-up time $T_{\max}$:

$$T_{\max} \ge \frac{1}{A(\gamma_2-1)\Psi_0^{\gamma_2-1}}.$$

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