A Nonlinear Perron-Frobenius Approach for Stability and Consensus of Discrete-Time Multi-Agent Systems ⋆

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Abstract

In this paper we propose a novel method to study stability and, in addition, convergence to a consensus state for a class of discrete-time Multi-Agent System (MAS) where agents evolve with nonlinear dynamics, possibly different for each agent (heterogeneous local interaction rules). In particular, we focus on a class of discrete-time MASs whose global dynamics can be represented by positive, sub-homogeneous and type-K order-preserving nonlinear maps. This paper generalizes results that apply to linear MASs to the nonlinear case by exploiting nonlinear Perron-Frobenius theory. We provide sufficient conditions on the structure of the nonlinear local interaction rules to guarantee stability of a MAS and an additional condition on the topology of the network ensuring the achievement of consensus as a particular case. Two examples are provided to corroborate the theoretical analysis. In the first one we consider a susceptible-infected-susceptible (SIS) model while in the second we consider a novel protocol to solve the max-consensus problem.

Key words: Nonlinear Perron-Frobenius theory; Multi-agent systems; Stability; Consensus; Positive systems; Heterogeneous agents.

1 Introduction

The study of complex systems where local interactions between individuals give rise to a global collective behavior has spurred much interest within the control community. Such complex systems consisting of multiple interacting agents, are often called Multi-Agent Systems (MASs). A topic that captured the attention of many researchers is the consensus problem [?], where the objective is to design local interaction rules among agents such that their state variables converge to the same value, the so called agreement or consensus state.

In the discrete-time linear case classical Perron-Frobenius theory is crucial in the convergence analysis of MASs. Indeed, in one of the most popular works in this topic [?], the authors established criteria for convergence to a consensus state for MASs whose global dynamics can be represented by linear time-varying systems with nonnegative row-stochastic state transition matrices, which are object of study of the classical Perron-Frobenius theory. The most notable aspect of this approach was a novel setting for proving convergence based on algebraic theory and graph theory instead of Lyapunov theory. This allows one to study systems for which finding a common Lyapunov function to establish convergence is difficult or even impossible: such is the case of switched systems as was later shown in [?], it allowed studying switched linear systems for which there does not exist a common quadratic Lyapunov function.

Along this line of thought, in this paper we aim to exploit nonlinear Perron-Frobenius theory [?], a generalization of nonnegative matrix theory, to study stability of MASs without Lyapunov based arguments. We consider MASs with state space $\mathcal{X} = \mathbb{R}^n$ and nonlinear dynamics given by map $f : \mathcal{X} \to \mathcal{X}$. Taking inspiration from nonlin-

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ear Perron-Frobenius theory, we identify a novel class of dynamics represented by *positive*, *sub-homogeneous*, *type-K order-preserving maps* and we prove special convergence properties in the positive orthant

$$\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R}^n : x \geq \mathbf{0}\},$$

of their iterative behavior. We also provide tools for stability and consensus analysis of MASs whose dynamics belongs to such a class by exploiting fixed-point theory and nonlinear Perron-Frobenius theory instead of Lyapunov theory. The **main contribution** of this paper is threefold.

(1) We provide sufficient conditions for stability of a novel class of nonlinear discrete-time systems represented by *positive*, *sub-homogeneous*, *type-K order-preserving maps* (see Theorem 13).
(2) We propose a sufficient condition on the local interaction rule of a generic agent (such a rule can be potentially different for each agent) which guarantees that the global model of the MAS falls into the considered class of systems (see Theorem 14).
(3) We propose a sufficient condition which links the topology of the network and the structure of the local interaction rules to guarantee the achievement of a consensus state, i.e., the network state in which all state variables have the same value (see Theorem 15).

The contributions of this paper go beyond the preliminary results presented in our previous work [**?**]. In particular, all three main theorems mentioned above introduce new original results: Theorem 13 is now supported by an accurate proof; Theorem 14 is new; Theorem 15 has been significantly generalized by considering general MASs which are not assumed a priori to have only consensus equilibrium points and consequently the proof method exploits different tools which are discussed in detail.

This paper is organized as follows. In Section 2 we present our notation and background material on multi-agent systems, positive, type-K order-preserving, subhomogenoeus maps and recall the concept of periodic fixed-points. In Section 3 we give some insights on the class of systems under study. In Section 4 we state our main results which formalized into three main theorems and theirs proofs; required technical lemmas can be found in the Appendix. In Section 5 we present two examples of application of our theoretical results. Analysis of a SIS model and a novel algorithm able to solve the max-consensus problem. In Section 6 it is explained how this work is related to the existing literature. Finally, in Section 7 we give our concluding remarks.

2 Background

In this work we propose novel tools to perform stability analysis (consensus as a special case) of MASs whose state update is represented by *positive*, *type-K order-preserving* and *sub-homogeneous* maps. In this section we define a model of autonomous nonlinear MASs in discrete-time, its associated graph and present the above mentioned properties which define the class of MAS under study.

2.1 Multi-agent systems

We consider a MAS composed by a set of agents $\mathcal{V} = \{1, \ldots, n\}$, which are modeled as autonomous discrete-time dynamical systems with scalar state in $\mathbb{R}$. Agents are interconnected and update their state as follows

$$\begin{array}{c} x_1(k+1) = f_1\left(x_1(k), \ldots, x_n(k)\right) \\ \vdots \\ x_n(k+1) = f_n\left(x_1(k), \ldots, x_n(k)\right) \end{array}, \tag{1}$$

where $k \in \{0, 1, 2, \ldots\}$ is a discrete-time index. Introducing the aggregate state $x = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, system (1) can be written as

$$x(k+1) = f(x(k)) \tag{2}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable. We now associate to the map f a graph $\mathcal{G}(f)$ which captures the pattern of interactions among agents and denote it as *inference graph* [**?**].

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a graph where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes representing the agents and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is a set of directed edges. A directed edge $(i, j) \in \mathcal{E}$ exists if node i sends information to node j. To each agent i is associated a set of nodes called neighbors of agent i defined as $\mathcal{N}_i = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$. A *directed path* between two nodes p and q in a graph is a finite sequence of m edges $e_k = (i_k, j_k) \in E$ that joins node p to node q, i.e., $i_1 = p$, $j_m = q$ and $j_k = i_{k+1}$ for $k = 1, \ldots, m-1$. A node j is said to be *reachable* from node i if there exists a directed path from node i to node j. A node is said to be *globally reachable* if it is reachable from all nodes $i \in \mathcal{V}$.

Definition 1 (Inference graph) *Given a map $f : \mathbb{R}^n \to \mathbb{R}^n$ its inference graph $\mathcal{G}(f) = (\mathcal{V}, \mathcal{E})$ is defined by a set of n nodes $\mathcal{V}$ and a set of directed edges $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$. An edge $(i, j) \in \mathcal{E}$ from node i to node j exists if*

$$\frac{\partial f_i(x)}{\partial x_j} \neq 0 \quad \forall x \in \mathbb{R}^n \setminus S,$$

where S is a set of measure zero in $\mathbb{R}^n$. ∎

2.2 Positive and Order-preserving maps

In this work we consider *positive* [?] and type-K order-preserving systems. Positivity is a term with different meanings in different contexts. In this work positivity is intended in the sense of cone invariance [?], with the invariant cone K being constant and equal to the positive orthant

$$\mathbb{R}^n_{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}.$$

Hence, a system (and the associated map) is positive if given an initial state $x(0) \in \mathbb{R}^n_{\geq 0}$ in the cone its trajectory remains confined within it.

Definition 2 (Positiveness) *A map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be positive if $f(x) \in \mathbb{R}^n_{\geq 0}$ for all $x \in \mathbb{R}^n_{\geq 0}$, i.e., f maps nonnegative vectors into nonnegative vectors. Correspondingly, system* (2) *is said to be positive.* ■

The positive orthant $\mathbb{R}^n_{\geq 0}$ is a partially ordered set with respect to the natural order relation $\leq$. For $u, v \in \mathbb{R}^n_{\geq 0}$, we can write the partial ordering relations as follows

$$\begin{aligned} &u \leq v \Leftrightarrow u_i \leq v_i \quad \forall i \in \mathcal{V}, \\ &u \lneq v \Leftrightarrow u \leq v \text{ and } u \neq v, \\ &u < v \Leftrightarrow u_i < v_i \quad \forall i \in \mathcal{V}. \end{aligned}$$

The partial ordering $\leq$ yields an equivalence relation $\sim$ on $\mathbb{R}^n_{\geq 0}$, i.e., x is equivalent to y ($x \sim y$) if there exist $\alpha, \beta \geq 0$ such that $x \leq \alpha y$ and $y \leq \beta x$. The equivalence classes are called *parts* of the cone of nonnegative real vectors and the set of all parts is denoted by $\mathcal{P}$. It can be shown (see [?]) that the cone $\mathbb{R}^n_{\geq 0}$ has exactly 2^n parts, which are given by

$$P_I = \{x \in \mathbb{R}^n_{\geq 0}\, |x_i > 0,\ \forall i \in I \text{ and } x_i = 0 \text{ otherwise}\}\ ,$$

with $I \subseteq \{1, \ldots, n\}$. We define a partial ordering on the set of parts $\mathcal{P}$ given by $P_{I_1} \preceq P_{I_2}$ if $I_1 \subseteq I_2$. If between $u, v \in \mathbb{R}^n_{\geq 0}$ there exists an order relation and the map f is such that this relation is preserved for the points $f(u)$ and $f(v)$, then f is said to be *order-preserving.* Next, we provide a formal definition of three kinds of order-preserving maps present in the literature.

Definition 3 (Order-preservation) *A positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be*

- *order-preserving if $\forall x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \leq y \Rightarrow f(x) \leq f(y).$$

- *strictly order-preserving if $\forall x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \lneq y \Rightarrow f(x) \lneq f(y).$$

- *strongly order-preserving if $\forall x, y \in \mathbb{R}^n_{\geq 0}$ it holds*

$$x \lneq y \Rightarrow f(x) < f(y).$$

Correspondingly, system (2) *is said to be (strictly, strongly) order-preserving.* ■

In the next remark we clarify the context of the contribution of this paper.

Remark 4 *For linear maps, order-preservation and positivity are equivalent properties and correspond to mappings defined by nonnegative matrices, the object of study of classical Perron-Frobenius theory. This equivalence is not preserved for nonlinear maps, thus nonlinear Perron-Frobenius theory considers maps that are both positive and order-preserving. Since the aim of this work is to apply such a theory to MASs, we also consider this class of maps.* ■

In addition to these notions of order-preservation, we introduce in this paper a new notion that is in between strict and strong order-preservation. We denote this new notion as *type-K order-preservation*; it plays a pivotal role in the characterization of the class of nonlinear systems in which we are interested and which will be discussed at length in the proofs of our results.

Definition 5 (Type-K order-preservation) *A positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be type-K order-preserving if $\forall x, y \in \mathbb{R}^n_{\geq 0}$ and $x \lneq y$ it holds*

(i) $x_i = y_i \Rightarrow f_i(x) \leq f_i(y)$,
(ii) $x_i < y_i \Rightarrow f_i(x) < f_i(y)$,

for all $i = 1, \ldots, n$, where f_i is the i-th component of f. Correspondingly, system (2) *is said to be type-K order-preserving.* ■

As it will be shown later (see Remark 7), such a property is sufficient but not necessary for order-preservation. However, since it is easily identifiable from the sign structure of the Jacobian matrix, it allows to easily establish order-preservation of a given function. Furthermore, it constraints the behavior of the system, preventing the system from evolving with periodic trajectories and thus helping in proving convergence to a steady state.

2.3 Sub-homogeneous maps

Order-preserving dynamical systems and nonlinear Perron-Frobenius theory are closely related. In the theory of order-preserving dynamical systems, the emphasis is placed on strong order-preservation, which allows one to prove generic convergence to periodic trajectories under appropriate conditions [?]. An extensive overview of these results was given by Hirsch and Smith [?]. On the other hand, in nonlinear Perron-Frobenius theory dynamical systems are not assumed to be strongly order-preserving but are required to satisfy an additional concave assumption, which allows one to obtain similar results regarding periodic trajectories [?]. The concave assumption of interest in this paper is sub-homogeneity.

Definition 6 (Sub-homogeneity) *A positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be sub-homogeneous if*

$$\alpha f(x) \leq f(\alpha x)$$

for all $x \in \mathbb{R}^n_{\geq 0}$ and $\alpha \in [0, 1]$. ■

Order-preserving and sub-homogeneous maps arise in a variety of applications, including optimal control and game theory [?], mathematical biology [?], analysis of discrete event systems [?] and so on.

2.4 Periodic points

Concluding this section, we recall some basic concepts on periodic points which are instrumental to state our main results.

Consider the state trajectory of the system in eq. (2). A point $x \in \mathbb{R}^n$ is called a *periodic point* of map f if there exist an integer $p \geq 1$ such that $f^p(x) = x$. The minimal such $p \geq 1$ is called the *period* of x under f. A *fixed point* is a periodic point with period $p = 1$, i.e., it satisfies $f(x) = x$. Fixed points of a map are equilibrium points for a dynamical system. We denote $F_f = \{x \in X : f(x) = x\}$ the set of all fixed points of map f. The *trajectory* of the system in eq. (2) with initial state x is given by $\mathcal{T}(x, f) = \{f^k(x) : k \in \mathbb{Z}\}$. If f is clear from the context, we simply write $\mathcal{T}(x)$ to denote its trajectory, where x is the initial state. If x is a periodic point, we say that $\mathcal{T}(x)$ is a periodic trajectory. We denote the limit set of a point x of map f as $\omega(x, f)$ (or simply $\omega(x)$ if f is clear from the context), which is defined as

$$\omega(x) = \bigcap_{k\geq 0} cl\left(\{f^m(x) : m \geq k\}\right),$$

with $cl(\cdot)$ denoting the closure of a set, i.e., the set together with all of its limit points. If x is a fixed point it follows that the set $\omega(x)$ is a singleton, i.e., a set containing a single point.

3 Insights on positive, type-K order-preserving and sub-homogeneous systems

In this section we give some useful insights on the considered class of systems.

3.1 Kamke Condition

We begin by clarifying the relationships among the different kinds of order-preservation.

Remark 7 *Strong order-preservation $\Rightarrow$ Type-K order-preservation $\Rightarrow$ strict order-preservation $\Rightarrow$ order-preservation.* ■

Every converse relationship in Remark 7 does not hold. Given $x, y \in \mathbb{R}$, let $f : \mathbb{R}^2 \to \mathbb{R}^2$, we have the following counter-examples:

- $f(x, y) = [1, 1]^T$ is order-preserving but not strictly order-preserving;
- $f(x, y) = [y, x]^T$ is strictly order-preserving but not type-K order-preserving;
- $f(x, y) = [\sqrt{x} + y, y]^T$ is type-K order-preserving but not strongly order-preserving.

Usually, to verify order-preservation is not an easy task. For differentiable continuous-time dynamical systems $\dot{x} = f(x)$ a sufficient condition to ensure order-preservation is given by Kamke [?,?]. The *Kamke condition* usually exploited in the analysis of continuous time systems is shown next.

Lemma 8 (Kamke Condition) *[?,?] The map $f : \mathbb{R}^n \to \mathbb{R}^n$ of a continuous-time system*

$$\dot{x} = f(x)$$

is order-preserving if its Jacobian matrix is Metzler, i.e.,

$$\partial f_i/\partial x_j \geq 0 \text{ for } i \neq j\ .$$ ■

As a counterpart to Lemma 8, for discrete-time systems we propose a sufficient condition to ensure type-K order-preservation of a map f, which we denote *Kamke-like* condition.

Proposition 9 (Kamke-like condition) *The map $f : \mathbb{R}^n \to \mathbb{R}^n$ of a discrete-time system*

$$x(k + 1) = f(x(k))$$

is type-K order-preserving if its Jacobian matrix is Metzler with strictly positive diagonal elements, i.e., if

$$\partial f_i/\partial x_i > 0 \text{ and } \partial f_i/\partial x_j \geq 0 \text{ for } i \neq j\ . \tag{3}$$

Proof. Let $x \in \mathbb{R}^n$ and, without lack of generality, $y = x + \varepsilon e_j$ where $\varepsilon > 0$ and e_j denotes a canonical vector with all zero values but the j-th which is 1, thus $x \lneq y$. If (3) holds, then

a) If $i \neq j$ then $y_i = x_i + \varepsilon 0 = x_i$ and

$$\frac{\partial f_i(x)}{\partial x_j} = \lim_{\varepsilon \to 0} \frac{f_i(x + \varepsilon e_j) - f_i(x)}{\varepsilon} \geq 0 \,,$$

which implies that $f_i(x) \leq f_i(x + \varepsilon e_j) = f_i(y)$, i.e., condition (i) of Definition 5.

b) If $i = j$ then $y_i = x_i + \varepsilon 1 > x_i$ and

$$\frac{\partial f_i(x)}{\partial x_i} = \lim_{\varepsilon \to 0} \frac{f_i(x + \varepsilon e_i) - f_i(x)}{\varepsilon} > 0 \,,$$

which implies that $f_i(x) < f_i(x + \varepsilon e_i) = f_i(y)$, i.e., condition (ii) of Definition 5.

Since $a) \Rightarrow$ (3) and $b) \Rightarrow$ (3), the proof is complete. □

Having clarified how to verify the type-K order-preservation property for a discrete-time system, we move on in the next subsection to discuss a significant property of order-preserving maps which are also sub-homogeneous, i.e., non-expansivness with respect to the so-called Thompson's metric.

3.2 Non-expansive maps

In this section we show that dynamical systems defined by order-preserving and sub-homogeneous maps are *non-expansive* under the *Thompson's metric* [?].

Definition 10 (Thompson's metric) *For $x, y \in \mathbb{R}^n$ define*

$$M(x/y) = \inf\{\alpha \geq 0 : y \leq \alpha x\} \,,$$

with $M(x/y) = \infty$ if the set is empty. By means of function $M(y/x)$, Thompson's metric $d_T : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty]$ is defined for all $(x, y) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus (0, 0)$ as follows

$$d_T(x, y) = \log(\max\{M(x/y), M(y/x)\})$$

with $d_T(0, 0) = 0$. ■

Definition 11 (Non-expansiveness) *A positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ is called non-expansive with respect to a metric $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, if*

$$d(f(x), f(y)) \leq d(x, y)$$

for all $x, y \in \mathbb{R}^n$. ■

A main feature of the class of systems under study is to be non-expansive under the Thompson's metric. Non-expansiveness is not a common concept but it is closely related to the more common notion of contractivity. Contractivity differs from non-expansiveness in Definition 11 by the inequality relation holding strictly. The next result is taken from [?] but it is stated here for the special cone $K = \mathbb{R}^n_{\geq 0}$.

Lemma 12 *[?] If a positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ is an order-preserving map, then it is sub-homogeneous if and only if it is non-expansive with respect to Thompson's metric d_T.* ■

This is a key property of our theory. In fact, from this property follow a series of Lemmas given in the Appendix which are instrumental to prove our main results in the next section.

4 Main results

In this section we illustrate the main results of this paper, while the following sections are dedicated to their proof.

For positive maps which are also order-preserving and sub-homogeneous, existing results (see the discussion in Section 4.1) do not provide any condition to ensure convergence to a fixed point, but only to periodic points [?]. Furthermore, they assume that the initial state is strictly positive, i.e., $x \in \mathbb{R}^n_{>0}$, and, to the best of our knowledge, no result provides any information about trajectories whose initial state lies in the boundary of $\mathbb{R}^n_{\geq 0}$.

Our aim is thus to fill this void considering the previously defined class of order-preserving maps, called *type-K order-preserving*, for which we prove convergence to a fixed point for any initial state $x \in \mathbb{R}^n_{\geq 0}$ and not only for $x \in \mathbb{R}^n_{>0}$. This result is given in next theorem.

Theorem 13 (Convergence) *Let a positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ be type-K order-preserving and sub-homogeneous. If f has at least one positive fixed point in the interior of $\mathbb{R}^n_{\geq 0}$ then all periodic points are fixed points, i.e., the set $\omega(x)$ is a singleton and*

$$\lim_{k \to \infty} f^k(x) = \bar{x}, \quad \forall x \in \mathbb{R}^n_{\geq 0}$$

where $\bar{x}$ is a fixed point of f. ■

Using this result, another of our contributions consists in a sufficient condition on the local interaction rules under which a MAS is stable, i.e., its state converges to a fixed point [1]. This result is given in Theorem 14, whose statement is shown next.

Theorem 14 (Stability) *Consider a MAS as in* (2) *with at least one positive equilibrium point. If the set of differentiable local interaction rules $f_i : \mathbb{R}^n \to \mathbb{R}$, with $i = 1, \ldots, n$, satisfies the next conditions:*

[1] Fixed points are synonymous for equilibrium points. While in the literature the term fixed points is widely used in the context of iterated maps, the term equilibrium point is usually preferred in the context of discrete-time dynamical systems.

(*i*) $f_i(x) \in \mathbb{R}_{\geq 0}$ *for all* $x \in \mathbb{R}^n_{\geq 0}$*;*
(*ii*) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$*;*
(*iii*) $\alpha f_i(x) \leq f_i(\alpha x)$ *for all* $\alpha \in [0,1]$ *and* $x \in \mathbb{R}^n_{\geq 0}$*;*

then the MAS converges to one of its equilibrium points for any positive initial state $x(0) \in \mathbb{R}^n_{\geq 0}$*.* ■

As a special case, we also study the consensus problem for the considered class of MAS. We propose a sufficient condition based on the result in Theorem 14 so that, for any initial state in $\mathbb{R}^n_{\geq 0}$, the MAS asymptotically reaches the consensus state, i.e., all state variable converge to same value. The proposed sufficient condition is graph theoretical and based on the inference graph $\mathcal{G}(f)$. The condition is satisfied if there exists a globally reachable node in graph $\mathcal{G}(f)$ and the consensus state is a fixed point for the considered MAS. This result is given in the next theorem.

Theorem 15 (Consensus) *Consider a MAS as in* (2)*. If the set of differentiable local interaction rules* $f_i : \mathbb{R}^n \to \mathbb{R}$*, with* $i = 1, \ldots, n$*, satisfies the next conditions:*

(*i*) $f_i(x) \in \mathbb{R}_{\geq 0}$ *for all* $x \in \mathbb{R}^n_{\geq 0}$*;*
(*ii*) $\partial f_i / \partial x_i > 0$ *and* $\partial f_i / \partial x_j \geq 0$ *for* $i \neq j$*;*
(*iii*) $\alpha f_i(x) \leq f_i(\alpha x)$ *for all* $\alpha \in [0,1]$ *and* $x \in \mathbb{R}^n_{\geq 0}$*;*
(*iv*) $f_i(x) = x_i$ *if* $x_i = x_j$ *for all* $j \in \mathcal{N}_i$*;*
(*v*) *The inference graph* $\mathcal{G}(f)$ *has a globally reachable node;*

then, the MAS converges asymptotically to a consensus state for any initial state $x(0) \in \mathbb{R}^n_{\geq 0}$*.* ■

In the remainder of the section, we discuss the proof of our main results in Theorem 13, 14 and 15.

4.1 Proofs of the main results

As pointed out in the previous section, for positive maps f which are also order-preserving and sub-homogeneous it is possible to establish the boundedness of any trajectory with initial states in the interior of the positive orthant, $x(0) \in int(\mathbb{R}^n_{\geq 0})$ and entirely enclosed in it. On the contrary, there are still no results for trajectories with initial state $x(0)$ in the boundary of $\mathbb{R}^n_{\geq 0}$.

Theorem 13 extends result about boundedness of trajectories starting at any point in $\mathbb{R}^n_{\geq 0}$ while excluding the existence of periodic trajectories and forcing them to converge to a fixed point of the map. The new assumption is that the map has to be type-K order-preservation, a notion that we have introduced in this paper and was firstly defined by [**?**]. We provide here the proof of Theorem 13 which makes use of some lemmas given in the Appendix for sake of readability.

Proof of Theorem 13. Trajectories of positive, sub-homogeneous and type-K order-preserving maps with a positive fixed point are bounded for all $x \in \mathbb{R}^n_{\geq 0}$ (see Lemma 18). Since all trajectories are bounded the system is stable, furthermore all periodic points are fixed points (see Theorem 2.3 in [**?**]), i.e., the set $\omega(x)$ is a singleton. Since f is continuous all trajectories are bounded and all periodic points are fixed points, by Lemma 3.1.3 in [**?**], it holds $\lim_{k\to\infty} f^k(x) = \bar{x}$ where $\bar{x}$ is a fixed point of f. □

By means of the technical result in Theorem 13 we prove our second main result, a sufficient condition on the structure of the local interaction rules of the MAS under consideration so that the global map (possibly unknown due to an unknown network topology) is positive, type-K order-preserving and sub-homogeneous map, thus falling within the class of systems considered in Theorem 13.

Proof of Theorem 14. We start the proof by establishing equivalence relationships between the properties $(i)-(iii)$ of the local interaction rules of the MAS listed in the statement of Theorem 14 and properties (a)-(c) shown next:

(*a*) f is positive;
(*b*) f is type-K order-preserving;
(*c*) f is sub-homogeneous;

We now prove all equivalences one by one.

- $[(i) \Leftrightarrow (a)]$ Condition (i) implies that f maps a point of $\mathbb{R}^n_{\geq 0}$ into $\mathbb{R}^n_{\geq 0}$ and is, therefore, a positive map (see Definition 2).
- $[(ii) \Rightarrow (b)]$ due to Proposition 9 (Kamke-like condition).
- $[(iii) \Leftrightarrow (c)]$ by Definition 6 of a sub-homogeneous map, sub-homogeneity can be verified element-wise for map f, thus the equivalence follows.

If conditions (i) to (iii) hold true for all local interaction rules f_i with $i = 1, \ldots, n$, since by assumption map f has at least one positive fixed point, we can exploit the result in Theorem 13 to establish that for all positive initial conditions, the state trajectories of the MAS converge to one of its positive equilibrium points. □

Finally, we detail a proof of our third and last main result, i.e., Theorem 15, which provides sufficient conditions for asymptotic convergence to the consensus state.

Proof of Theorem 15 We start the proof by establishing the relations between properties $(i)-(v)$ and the following:

(a) f is positive;
(b) f is type-K order-preserving;
(c) f is sub-homogeneous;
(d) $F_f = \{c\mathbf{1} : c \in \mathbb{R}_{\geq 0}\}$.

We go through all equivalences one by one.

(1) [$(i) \Leftrightarrow (a)$] See Proof of Theorem 14.
(2) [$(ii) \Leftrightarrow (b)$] See Proposition 9 (Kamke-like condition).
(3) [$(iii) \Leftrightarrow (c)$] See Proof of Theorem 14.
(4) [$(i-v) \Rightarrow (d)$] The proof of this implication is given below.

Condition (iv) implies that the consensus space $c\mathbf{1}$ is a subset of the set of fixed points F_f of map f, i.e.,

$$F_f \supseteq \{c\mathbf{1} : c \in \mathbb{R}_{\geq 0}\}.$$

By Lemma 19 the Jacobian matrix $J_f(c\mathbf{1})$ evaluated at a consensus point is row-stochastic, i.e., $J_f(c\mathbf{1})\mathbf{1} = \mathbf{1}$. By the definition of inference graph (see Definition 1), it holds that $\mathcal{G}(f) = \mathcal{G}(J_f(c\mathbf{1}))$. Thus $\mathcal{G}(J_f(c\mathbf{1}))$ has a globally reachable node by hypothesis and is aperiodic because condition (ii) ensures a self-loop at each node.

We are ready to prove by contradiction [$(i-v) \Rightarrow (d)$]. In particular, if there exists a fixed point $\bar{x} \neq c\mathbf{1}$, then by Lemma 20 the Jacobian matrix $J_f(c\mathbf{1})$ has a unitray eigenvalue with multiplicity strictly greater than one. On the other hand, by the widely known Theorem 5.1 in [?], if $\mathcal{G}(J_f(c\mathbf{1}))$ has a globally reachable node and is aperiodic then $J_f(c\mathbf{1})$ has a simple unitary eigenvalue with corresponding eigenvector equal to $\mathbf{1}$, unique up to a scaling factor c. This is a contradiction, therefore it does not exist a fixed point $\bar{x}$ such that $\bar{x} \neq c\mathbf{1}$ with $c > 0$. Thus, we conclude that the set of fixed points of map f satisfies

$$F_f = \{c\mathbf{1}, \quad c \in \mathbb{R}_{\geq 0}\}.$$

Finally, if conditions (a) to (c) are satisfied, then Theorem 14 the MAS converges to its set of fixed points F_f. If (d) is satisfied, the F_f contains only consensus points and thus the MAS in (2) converges to a consensus state for all $x \in \mathbb{R}^n_{\geq 0}$. □

5 Examples

As a **first example** we consider a susceptible-infected-susceptible (SIS) epidemic model [?] described by the following

$$x_i(k+1) = x_i(k) + h \cdot \varepsilon_i(1 - x_i(k)) - h \cdot x_i(k) \sum_{j \in \mathcal{N}_i} \beta_{ij}(1 - x_j(k)) \,. \tag{4}$$

Such model was originally derived to describe the propagation of an infectious diseases over a group of individuals. Each group is subdivided according to susceptible and infectious. Individuals can be cured and reinfected many times, there is not an immune group. Given n groups, let $x_i(k)$, $y_i(k)$ be the portion of, respectively, susceptibles and infectious of group i at time k, it is clear that $x_i(k), y_i(k) \geq 0$ and $x_i(k) + y_i(k) = 1$ for any k. Thus, it is sufficient to consider the dynamics of one of them to completely describe the system. In model (4) variables have the following meaning:

- $\beta_{ij} \geq 0$ are the infectious rates;
- $\varepsilon_i \geq 0$ is the healing rate;
- $h \geq 0$ is the sampling rate.

We now evaluate conditions $(i) - (iv)$ of Theorem 14 to establish the convergence of the associated MAS to a positive fixed point. Due to space limitations, we omit all steps and give directly conditions under which the theorem holds.

- First we notice that $x_i(k)$ belongs to $[0, 1]$ for all k. It is guaranteed that for all $x_i(k) \in [0, 1]$ also $x_i(k + 1)+ \in [0, 1]$ if and only if (5) and (6) hold,

$$h\varepsilon_i \leq 1 \,, \quad h \sum_{j \neq i} \beta_{ij} \leq 1 \,, \tag{5}$$

$$h\beta_{ii} \leq \left[\sqrt{1 - h \sum_{j \neq i} \beta_{ij}} + \sqrt{h\varepsilon_i} \right]^2 . \tag{6}$$

  We conclude that for any $x \in [0, 1]^n \subset \mathbb{R}^n_{\geq 0}$ then $f_i(x) \in [0, 1] \subset \mathbb{R}_{\geq 0}$, thus condition (i) holds.
- Condition (ii) holds if and only if

$$h\beta_{ii} < 1 - h\varepsilon_i - h \sum_{j \neq i} \beta_{ij}. \tag{7}$$

- Condition (iii) holds if and only if

$$\varepsilon_i \geq \sum_{j \neq i} \beta_{ij}. \tag{8}$$

- Condition (iv) is satisfied since $\bar{x} = \mathbf{1} \in \mathbb{R}^n_{\geq 0}$ is a positive fixed point.

One can prove that (5), (6), (7), (8) are equivalent to (9), given $\beta = \sum_{j \neq i} \beta_{ij}$,

$$h\varepsilon_i + h\beta < 1 - h\beta_{ii} \,, \quad h\beta \leq 0.5 \,. \tag{9}$$

If (9) holds, then conditions of Theorem 14 are satisfied, and we conclude that the MAS converges to a fixed point for all $x \in [0, 1]^n$. For a MAS described by graph $\mathcal{G}_1$ in Figure 1a, a numerical simulation is given in Figure 2a.

As a **second example** we propose a novel algorithm to solve the max-consensus problem, i.e., we design local interaction rules steering all agents to the maximum value among all initial agents' states. Consider a MAS described by graph $\mathcal{G}_2$ in Figure 1b and nonlinear local interaction rule

$$x_i(k+1) = x_i(k) + \varepsilon_i \sum_{j \in \mathcal{N}_i} \Big(d_{ji}(k) + |d_{ji}(k)| \Big), \quad (10)$$

with $d_{ji} = x_j(k) - x_i(k)$. We first evaluate conditions $(i) - (v)$ of Theorem 15 to establish the convergence of the associated MAS to a consensus state.

- Condition (i) can be evaluated in the worst case scenario, i.e., when $x_j > x_i$ for each neighbor $j \in \mathcal{N}_i$. In this case

$$f_i(x) = (1 - 2|\mathcal{N}_i|\Delta_i)x_i(k) + 2\Delta_i \sum_{j \in \mathcal{N}_i} x_j(k).$$

This is true if and only if

$$\Delta_i \leq \frac{1}{2|\mathcal{N}_i|}.$$

- Condition (ii) is analyzed as follows

$$\begin{aligned} \frac{\partial f_i}{\partial x_i} &= 1 + \Delta_i \left[-|\mathcal{N}_i| - \sum_{j \in \mathcal{N}_i} \mathrm{sign}(x_j(k) - x_i(k)) \right] \\ &- 2|\mathcal{N}_i|\Delta_i \qquad \text{if } x_j(k) > x_i(k), \forall j \in \mathcal{N}_i \\ &> 0 \qquad\qquad\ \ \text{if } \Delta_i < \frac{1}{2|\mathcal{N}_i|} \end{aligned}$$

and

$$\begin{aligned} \frac{\partial f_i}{\partial x_j} &= 0 + \Delta_i \left[1 + \mathrm{sign}(x_j(k) - x_i(k)) \right] \\ &\geq \Delta_i \left[1 - 1 \right] \qquad \text{if } x_j(k) < x_i(k) \forall j \in \mathcal{N}_i \\ &\geq 0\,. \end{aligned}$$

Thus, condition (ii) holds true if and only if

$$\Delta_i < \frac{1}{2|\mathcal{N}_i|} \quad \forall i = 1, \ldots, 6. \quad (11)$$

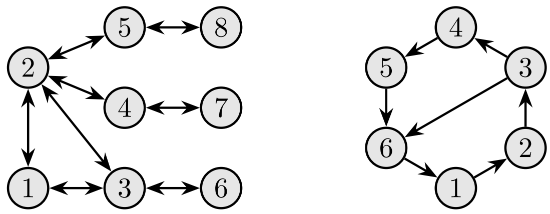
- Condition (iii) holds true for any $x \in \mathbb{R}^n_{\geq 0}$. This follows from the fact that the only nonlinear term of f_i is $|\cdot|$ for which it holds

$$|\alpha x| = \alpha |x|.$$

Thus, we can conclude that

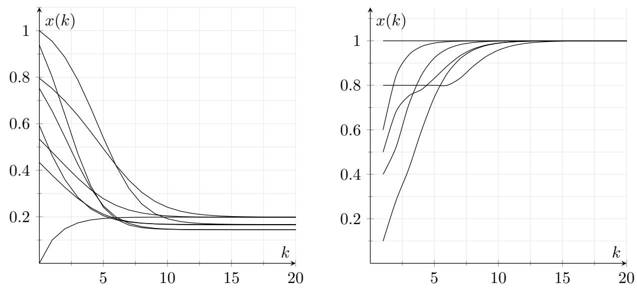
$$\alpha f_i(x) = f_i(\alpha x),$$

which implies sub-homogeneity.
- Condition (iv) is satisfied since $\bar{x} = c\mathbf{1}$ for any $c > 0$ is a positive fixed point.
- Condition (v) is satisfied since graph $\mathcal{G}_2$ in Figure 1b contains a globally reachable node.



(a) Graph $\mathcal{G}_1$. (b) Graph $\mathcal{G}_2$.

Fig. 1. Graphs of Examples 1 and 2.



(a) Parameters: $h = 0.5$, $\beta_{ij} = 0.5$, $\varepsilon_i = 0.5$ for all i and $j \in \mathcal{N}_i$. (b) Parameters: $\varepsilon = 0.1$.

Fig. 2. Evolutions of Examples 1 and 2.

When (11) holds, then all conditions of Theorem 15 are satisfied, and we conclude that the MAS in (10) converges to a consensus state.

It is straightforward to show that the consensus state reached by a MAS in (10) is exactly the maximum among the initial states of all agents. In fact, without loss of generality, suppose that node 1 has the maximum value, i.e., $x_1 \geq x_i$ for all $i = 1, \ldots, 6$. In this case, $x_1(k+1) = x_1(k)$, since the second term in (10) is 0. Since agent i keeps its value for all $k \geq 0$, it is clear that the MAS in (10) reaches consensus on the maximum value among all initial states. A numerical simulation is given in Figure 2b.

6 Discussion

The literature on nonlinear consensus problems is vast and it is mostly composed by particular nonlinear consensus protocols which offer advantages such as finite-time convergence [**?**,**?**], resilience to non-uniform time-delays [**?**] and many more. These protocols are usually proved to converge to the consensus state via ad hoc Lyapunov functions.

6.1 About Moreau's convexity condition

Most approaches that aim to establish convergence to consensus for some class of nonlinear MAS fall in the general convexity theory of [?], i.e., each agent's next state is strictly inside the convex hull spanned by its own state and the state value of its neighbors. The class of systems studied in our paper is not limited to Moreau's theory, as our second example in Section 5 shows. In fact, the considered dynamical system is a distributed algorithm to estimate the maximum among all initial agents' state, therefore not satisfying a strict convex condition. The study of dynamical equations not satisfying a strict convexity assumption is beyond the scope of [?].

6.2 About differential positivity

The differential positivity framework, developed by Forni et al. in [?], addresses a problem setup similar to the one in this paper. The main common point between our work and the work in [?] is the fact that we both consider *positive* systems. Positivity is intended in the sense of cone invariance; positivity is said to be *strict* if the boundary of the cone is eventually mapped to the interior of the cone. Our work is restricted to systems with a state space $\mathcal{X} = \mathbb{R}$ and a constant invariant cone $K = \mathbb{R}^n_{\geq 0}$. For such systems, by Theorem 1 in [?], it follows

$$\begin{aligned} \text{differential positivity} &\Leftrightarrow \text{order-preservation}, \\ \text{strict differential positivity} &\Leftrightarrow \text{strong order-preservation}. \end{aligned}$$

The results in [?] are limited to strictly differentially positive systems: no convergence results are provided for differentially positive systems. One can easily derive that the class of maps addressed in our work is differentially positive but not strict. A simple example is given by

$$x(k+1) = Ax(k), \quad A = \begin{bmatrix} 1 & 0 \\ \gamma & 1-\gamma \end{bmatrix}, \quad \gamma \in (0,1).$$

Map Ax is not strictly differentially positive because it does not map the boundary of the positive orthant to its interior; this is true for any $x = [0, \alpha]^T$, with $\alpha \in \mathbb{R}$. However, map A is type-K order-preserving since it is Metzler with strictly positive diagonal (Proposition 9).

6.3 About continuous-time models

While the literature on nonlinear discrete-time MASs is limited, a huge literature exists for continuous-time MASs. We start mentioning the work in [?], which is the continuous-time counterpart to the result of Moreau in [?], where the authors identify a class of nonlinear interactions denoted as *sub-tangent* and establish necessary and sufficient conditions on the network topology for convergence to consensus. It is reasonable to think that a generalization of our theory to the continuous-time case would lead to a generalization of this work.

A widely studied class of continuous-time systems is the one of cooperative systems [?,?,?]. Cooperative systems are closely related to type-K order-preserving discrete-time systems since the solution mappings for cooperative ordinary differential equations are always type-K order-preserving (see [?]). However, the converse relation does not hold, i.e., not all trajectories generated by type-K order-preserving mappings are solution of a cooperative ordinary differential equation. Since several results are obtained for continuous-time systems which are cooperative, we believe that studying discrete-time dynamical systems which are type-K order-preserving can lead to a fertile research field both because we enlarged the class of discrete-time MASs for which there exist convergence results and because results as in [?,?] assume stronger assumptions than ours, such as homogeneity (which implies sub-homogeneity) and irreducibility of the map, two properties not required by our theory.

There is also an ISS condition for nonlinear consensus in continuous-time in [?] for systems of single-integrator agents $\dot{x} = f(\sum_{j \in \mathcal{N}_i} a_{ij}(x_j - x_i))$ where a_{ij} are the components of the Laplacian of the associated graph. Since they require f to be an odd and increasing function, then the system is cooperative and their solutions are type-K order-preserving. This is the only common point between the two theories. In fact, there is a gap between them. For example, the work in [?] can handle systems which are not sub-homogeneous while our work can handle systems where the nonlinearity is applied first, and then summation is applied, i.e., $x(k+1) = x(k) + \sum_{j \in \mathcal{N}_i} f(a_{ij}(x_j(k) - x_i(k)))$.

7 Conclusions and future work

In this paper we presented three main results related to a class of nonlinear discrete-time multi-agent systems represented by a state transition map which is positive, sub-homogeneous and type-K order-preserving.

The first result establishes that a general discrete-time dynamical system converges to one of its equilibrium points asymptotically if its corresponding state transition map is positive, sub-homogeneous and type-K order-preserving. The second result provides sufficient conditions for a set of nonlinear, discrete-time local interaction rules (possibly different for each agent) which define the MAS to establish stability of the MAS, independently of its graph topology (which is considered unknown) by exploiting our first main result. Finally, the third result provides sufficient conditions for a set of nonlinear, discrete-time local interaction rules (possibly different for each agent) which define the MAS to establish asymptotic convergence to a consensus state if the inference graph of the MAS has a globally reachable node.

This paper generalizes results for discrete-time linear MAS whose state transition matrix is stochastic to the nonlinear case thanks to nonlinear Perron-Frobenius theory. Examples are provided to show the effectiveness of the stability analysis of a MAS based on our method.

Future work will consider MAS represented by a time-varying set of local interaction rules and an extension to the continuous-time case.

A Appendix

Lemma 16 *Let a positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ be type-K order-preserving. For all $x \in \mathbb{R}^n_{\geq 0}$ it holds that $f_i^k(x) > 0$ for all i such that $x_i > 0$ and $k \geq 1$.*

Proof. For any $x \in \mathbb{R}^n_{\geq 0}$ let $I(x) \subset \{1, \dots, n\}$ be such that $x_i > 0$ for $i \in I(x)$ and $x_i = 0$ otherwise. Since $\mathbf{0} \leq x$, by type-K order-preservation of f follows $f(\mathbf{0}) \leq f(x)$. More precisely it holds $f_i(x) > f_i(\mathbf{0}) \geq 0$ for $i \in I(x)$ and $f_i(x) \geq f_i(\mathbf{0}) \geq 0$ otherwise, implying $I(x) \subseteq I(f(x))$. By induction, $I(x) \subseteq I(f^k(x))$, i.e., $f_i^k(x) > 0$ for all $i \in I(x)$, completing the proof. □

Lemma 17 *Let a positive map f be sub-homogeneous and type-K order-preserving. For all $x \in \mathbb{R}^n_{\geq 0}$ there exists a part $P \in \mathcal{P}(\mathbb{R}^n_{\geq 0})$ and an integer $k_0 \in \mathbb{Z}$ such that $f^k(x) \in P$ for all $k \geq k_0$.*

Proof. Since f is order-preserving and sub-homogeneous, then f is non-expansive under Thompson's metric (see Definition 10) by Lemma 12. Thus, $x \sim y$ implies $f(x) \sim f(y)$. This can be easily proved by noticing that $d_T(f(x), f(y)) \leq d_T(x, y) < \infty$ since $x \sim y$. This means that f maps parts into parts, i.e., for all $x \in \mathbb{R}^n_{\geq 0}$ and $x' \in [x] = P_{I_0}$ it holds $f(x') \in [f(x)] = P_{I_1}$. By Lemma 16 it follows $P_{I_0} \preceq P_{I_1}$ and therefore $[x] \preceq [f(x)]$. Generalizing, we say that $f^k(x) \in P_{I_k}$ with $k \in \mathbb{Z}$ and $I_k \subseteq I_{k+1} \subseteq \{1, \dots, n\}$. There exists $k_0 \in \mathbb{Z}$ such that $I_k = I_{k_0}$ for all $k > k_0$ and thus $P_k = P_{k_0}$. This completes the proof. □

Lemma 18 *Let a positive map $f : \mathbb{R}^n \to \mathbb{R}^n$ be sub-homogeneous and type-K order-preserving. If f has a positive fixed point $\bar{x} \in \mathbb{R}^n_{\geq 0}$, then for all $x \in \mathbb{R}^n_{\geq 0}$ the trajectory $\mathcal{T}(x)$ is bounded.*

Proof. Function $g = \log \circ f \circ \exp$ is a sup-norm non-expansive map that has the same dynamical properties as f for all $x \in \mathbb{R}^n_{\geq 0}$ (see [?]). By Lemma 4.2.1 in [?] we know that one of the two cases can occur:

(*i*) all trajectories $\mathcal{T}(\log(x), g)$ are unbounded;
(*ii*) all trajectories $\mathcal{T}(\log(x), g)$ are bounded.

Since f has a fixed point $x_f \in \mathbb{R}^n_{\geq 0}$, such that $f(x_f) = x_f$, then $x_g = \log(x_f)$ is a fixed point of g, i.e., $g(x_g) = x_g$. The trajectory $\mathcal{T}(\log(x_f), g)$ is obviously bounded and therefore case (*ii*) holds.

By Lemma 17, we can partition $\mathbb{R}^n_{\geq 0}$ in two disjoint sets S_1, S_2 such that if for x there exists $k_0 \in \mathbb{Z}$ such that $f^{k_0}(x) \in \mathbb{R}^n_{\geq 0}$, then $x \in S_1$, otherwise $x \in S_2$. We analyze these two cases.

1) For all $x \in S_1$, by Lemma 17, it holds that $f^k(x) \in \mathbb{R}^n_{\geq 0}$ for all $k \geq k_0$. Let $x_0 = f^{k_0}(x)$. Since case (*ii*) holds $\mathcal{T}(\log(x_0), g)$ is bounded, because of the isometry also $\mathcal{T}(x_0, f)$ is bounded, and therefore also $\mathcal{T}(x, f)$. We conclude that for all $x \in S_1$ trajectories $\mathcal{T}(x, f)$ are bounded.

2) For all $x \in S_2$, by Lemma 17, there exists $k_0 \in \mathbb{Z}$ such that $f^k(x) \in P_I$ with $I(x) \subset N = \{1, \dots, n\}$ for all $k \geq k_0$. Without loss of generality, here we assume $I = \{1, \dots, m\}$, where $m < n$. Let $x = [z_1^T, z_2^T]^T$ with $z_1 \in \mathbb{R}^m_{\geq 0}$ and consider the following m-dimensional map $f^* : \mathbb{R}^m_{\geq 0} \to \mathbb{R}^m_{\geq 0}$ defined by

$$f_i^*(z_1) = f_i(z_1, z_2), \quad z_2 = \mathbf{0}\,,$$

with $i \in I(x)$. It is not difficult to check that f^* is still sub-homogeneous and type-K order-preserving. Accordingly, $g^* = \log \circ f^* \circ \exp$ is a sup-norm non-expansive map that has the same dynamical properties as f^* for all $x \in \mathbb{R}^m_{\geq 0}$. The main point now is to prove that if (*ii*) occurs then all trajectories $\mathcal{T}(\log(z_1), g^*)$ are also bounded. To this aim, we first need to show that for all $i \in I(x)$ it holds

$$g_i^*(z_1) \leq g_i(z_1, z_2). \tag{A.1}$$

Since both the exponential and the logarithmic functions are strictly increasing, (A.1) is equivalent to

$$f_i^*(z_1) \leq f_i(z_1, z_2). \tag{A.2}$$

By definition, (A.2) holds if $z_2 = 0$. If $z_2 \neq 0$, for any $x = [z_1^T, z_2^T]^T$ consider $\bar{x} = [z_1^T, \bar{z}_2^T]^T$ such that $\bar{z}_2 = \mathbf{0}$. Since f is order-preserving, for all $i \in I$ it holds that $f_i(\bar{x}) \leq f_i(x)$, which is equivalent to write $f_i(z_1, \bar{z}_2) \leq f_i(z_1, z_2)$. By definition, $f_i^*(z_1) = f_i(z_1, \bar{z}_2)$. Therefore, $f_i^*(z_1) \leq f_i(z_1, z_2)$ for all $z_2 \neq 0$, i.e., (A.2) and (A.1) hold. Suppose that (*ii*) occurs and there exist $\hat{z}_1 \in \mathbb{R}^m_{\geq 0}$ such that $\mathcal{T}(\log(\hat{z}_1), g^*)$ is unbounded. By (A.1) it is clear that given $\hat{x} = [\hat{z}_1^T, z_2^T]^T$ the trajectory $\mathcal{T}(\log(\hat{x}), g)$ is also unbounded, contradicting (*ii*). Let $x_0 = f^{k_0}(x)$. Since all trajectories $\mathcal{T}(\log(x_0), g)$ are bounded, because of the isometry also $\mathcal{T}(x_0, f)$ is bounded, and therefore also $\mathcal{T}(x,f)$. We conclude that for all $x \in S_2$ trajectories $\mathcal{T}(x,f)$ are bounded. □

Lemma 19 *Let a map $f : \mathbb{R}^n \to \mathbb{R}^n$ be positive and differentiable. If the set of fixed points F_f of map f satisfies $F_f \supseteq \{c\mathbf{1}, \quad c \in \mathbb{R}_{\geq 0}\}$, i.e., the set of fixed points contains at least all positive consensus states, then the Jacobian matrix J_f of map f computed at a consensus point $c\mathbf{1}$ is stochastic, i.e.,*

$$J_f(c\mathbf{1})\mathbf{1} = \mathbf{1} \quad \forall c \in \mathbb{R}_{\geq 0} .$$

Proof. Since f is differentiable, we can apply directly the definition of directional derivative in a point $x \in \mathbb{R}^n_{\geq 0}$ along a vector $v \in \mathbb{R}^n$ obtaining

$$J_f(x)v = \lim_{h \to 0} \frac{f(x+hv) - f(x)}{h} .$$

Evaluating this expression in a consensus point $x = c\mathbf{1} \in F_f$ and along the direction $v = \mathbf{1}$ (which is an invariant direction of f) we obtain

$$\begin{aligned} J_f(c\mathbf{1})\mathbf{1} &= \lim_{h \to 0} \frac{f(c\mathbf{1} + h\mathbf{1}) - f(c\mathbf{1})}{h} , \\ &= \lim_{h \to 0} \frac{\cancel{c\mathbf{1}} + h\mathbf{1} - \cancel{c\mathbf{1}}}{h} = \mathbf{1} , \end{aligned}$$

thus proving the statement. □

Lemma 20 *Let f be positive, sub-homogeneous, type-K order-preserving and have a set of fixed points F_f such that*

$$F_f \supseteq \{c\mathbf{1}, \quad c \in \mathbb{R}_{\geq 0}\} .$$

If there exists a fixed point $\bar{x} \in \mathbb{R}^n_{\geq 0}$ such that

$$\bar{x} \neq c\mathbf{1} , \quad \forall c \in \mathbb{R}_{\geq 0}$$

then there exists $\bar{c}(\bar{x}) > 0$ such that the Jacobian matrix $J_f(\bar{c}(\bar{x})\mathbf{1})$ of map f computed at $\bar{c}(\bar{x})\mathbf{1}$ has a unitary eigenvalue with multiplicity strictly greater than one.

Proof. Let $\bar{x} = [\bar{x}_1, \ldots, \bar{x}_n]^T \in \mathbb{R}^n_{\geq 0}$ be a fixed point of map f and let $c_1, c_2 \in \mathbb{R}_{\geq 0}$ be such that

$$\begin{aligned} c_1 &= \min_{i=1,\ldots,n} \bar{x}_i , \\ c_2 &= \max_{i=1,\ldots,n} \bar{x}_i . \end{aligned}$$

We define three sets

$$\begin{aligned} I_{min}(\bar{x}) &= \{i : \bar{x}_i = c_1\}, \\ I_{max}(\bar{x}) &= \{i : \bar{x}_i = c_2\}, \\ I(\bar{x}) &= \{i : \bar{x}_i \neq c_1, c_2\}. \end{aligned}$$

Consider a point y such that the i-th component is defined by

$$y_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{x}) \\ c_3 & \text{otherwise} \end{cases} \tag{A.3}$$

and such that

$$c_1\mathbf{1} \lneq y \lneq \bar{x} \lneq c_2\mathbf{1} . \tag{A.4}$$

By (A.3) and (A.4) it follows that

$$y \leq c_3\mathbf{1}. \tag{A.5}$$

- Since map f is type-K order-preserving, from (A.4) it follows $c_1 \leq f_i(y) \leq \bar{x}_i$ and from (A.5) $f_i(y) \leq c_3$ for $i = 1, \ldots, n$. For $i \in I_{min}(\bar{x})$, by definition $\bar{x}_i = c_1$ and thus $f_i(y) = c_1$, otherwise for $i \in I(\bar{x}) \cup I_{max}(\bar{x})$ by (A.4) $\bar{x}_i \geq y_i = c_3$ and it follows $c_1 \leq f_i(y) \leq c_3$. Therefore, it holds

$$f(y) \leq y. \tag{A.6}$$

- Since f is order-preserving and sub-homogeneous, then f is non-expansive under the Thompson's metric (see Definition 10) by Lemma 12. Now, by exploiting the definition of non-expansive map, we compute an upper bound to $d_T(\bar{x}, f(y))$. It holds

$$\begin{aligned} d_T(\bar{x}, f(y)) &\leq d_T(\bar{x}, y) \\ &= \log\left(\max\{M(\bar{x}/y), M(y/\bar{x})\}\right) \end{aligned}$$

where

$$\begin{aligned} M(\bar{x}/y) &= \inf\{\alpha \geq 0 : y \leq \alpha\bar{x}\} = \max_i \frac{y_i}{x_i} = 1, \\ M(y/\bar{x}) &= \inf\{\alpha \geq 0 : \bar{x} \leq \alpha y\} = \max_i \frac{x_i}{y_i} \leq \frac{c_2}{c_3}. \end{aligned}$$

Since $c_2 \geq c_3$, then

$$d_T(\bar{x}, f(y)) \leq \log\left(\frac{c_2}{c_3}\right). \tag{A.7}$$

On the other hand

$$d_T(\bar{x}, f(y)) = \log\left(\max\{M(\bar{x}/f(y)), M(f(y)/\bar{x})\}\right)$$

where

$$\begin{aligned} M(\bar{x}/f(y)) &= \inf\{\alpha \geq 0 : f(y) \leq \alpha\bar{x}\} = \max_i \frac{f_i(y)}{x_i} = 1, \\ M(f(y)/\bar{x}) &= \inf\{\alpha \geq 0 : \bar{x} \leq \alpha f(y)\} = \max_i \frac{x_i}{f_i(y)} \geq \frac{c_2}{c_3}. \end{aligned}$$

Since $\max\{M(\bar{x}/f(y)), M(f(y)/\bar{x})\} \geq \frac{c_2}{c_3}$, then

$$d_T(\bar{x}, f(y)) \geq \log\left(\frac{c_2}{c_3}\right). \tag{A.8}$$

By the upperbound (A.7) and the lowerbound (A.8) it follows

$$d_T(\bar{x}, f(y)) = \log\left(\max_i \frac{x_i}{f_i(y)}\right) = \log\left(\frac{c_2}{c_3}\right).$$

Therefore, it holds

$$\frac{c_3}{c_2}\bar{x}_i \leq f_i(y) \leq c_3, \quad i = 1, \ldots, n\,. \tag{A.9}$$

Here we are interested in understanding the iterative behavior of f over y. We proceed by considering all components f_i by knowing a priori that due to Theorem 13 it holds $\lim_{k\to\infty} f_i^k(y) = \bar{y}_i$. Three cases may occur:

(1) If $i \in I_{min}(\bar{x})$ then $\bar{x}_i = c_1$ and by (A.4) it follows $f_i(y) = c_1$.
(2) If $i \in I_{max}(\bar{x})$ then $\bar{x}_i = c_2$ and by (A.9) it follows $f_i(y) = c_3$.
(3) If $i \in I(\bar{x})$, by (A.6) two cases may occur:
   (a) There exists $k^* > 0$ such that $f_i^{k^*}(y) < y_i$. In this case, by type-K order-preservation it holds that $f_i^k(y) < f_i^{k-1}(y) \quad \forall k \geq k^* + 1$ and therefore
   $$\lim_{k\to\infty} f_i^k(y) = c_1.$$
   (b) Otherwise $f_i^k(y) = f^{k-1}(y) \quad \forall k > 0$ and therefore
   $$\lim_{k\to\infty} f_i^k(y) = y_i = c_3.$$

These consideration can be summarized as follows.

$$\bar{y}_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{x}), \\ c_3 & \text{if } i \in I_{max}(\bar{x}), \\ c_1 & \text{if } i \in I(\bar{x}) \text{ and} \\ & \exists k^* : f_i^{k^*}(y) < f_i^{k^*-1}(y), \\ c_3 & \text{otherwise.} \end{cases} \tag{A.10}$$

So far we proved in (A.10) that for any fixed point $\bar{x}$ different from a consensus point $c\mathbf{1}$ there exists a fixed point $\bar{y}$ with elements corresponding to either c_1 or c_3 and such that $I(\bar{y}) = \emptyset$. Last step in the proof is to consider a point z such that its i-th component is defined as follows

$$z_i = \begin{cases} c_1 & \text{if } i \in I_{min}(\bar{y}) \\ c_4 & \text{if } i \in I_{max}(\bar{y}) \end{cases} \tag{A.11}$$

with $c_4 \in [c_1, c_3]$. By (A.10) and (A.11), we can conclude that z is fixed point, i.e., $f(z) = z$, for all values of c_4 in the interval $c_4 \in [c_1, c_3]$. Now, let $v(\bar{x})$ be a vector such that

$$v_i(\bar{x}) = \begin{cases} 0 & \text{if } i \in I_{min}(\bar{x}) \\ 1 & \text{if } i \in I_{max}(\bar{x}) \\ 0 \textit{ or } 1 & \text{if } i \in I(\bar{x}) \end{cases}, \tag{A.12}$$

Thus, by (A.12) the point $c_1\mathbf{1} + hv(\bar{x})$ is a fixed point of map f for all $h \in [0, c_3 - c_1]$. Thus, it follows that

$$f(c_1\mathbf{1} + hv(\bar{x})) = c_1\mathbf{1} + hv\,, \quad h \in [0, c_3 - c_1]\,.$$

Since $v(\bar{x}) \neq \mathbf{1}$, it holds (by reasoning along the lines of Lemma 19) that the Jacobian of map f computed at $c_1\mathbf{1}$ has a right eigenvector equal to $v(\bar{x})$, i.e., $J_f(c_1\mathbf{1})v(\bar{x}) = v(\bar{x})$. By Lemma 19 it holds that the Jacobian of f satisfies $J_f(c\mathbf{1})\mathbf{1} = \mathbf{1}$ for all $c > 0$. Thus, if there exists $\bar{x} \neq c\mathbf{1}$ then there exists $\bar{c}(\bar{x}) = \min_{i=1,\ldots,n} \bar{x}_i = c_1$ such that matrix $J_f(\bar{c}(\bar{x})\mathbf{1})$ has a untiray eigenvalue with multiplicity strictly greater than one, thus proving the statement of this lemma. □