THE OBSTACLE PROBLEM AT ZERO FOR THE FRACTIONAL p-LAPLACIAN

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Abstract. In this paper we establish a multiplicity result for a class of unilateral, nonlinear, nonlocal problems with nonsmooth potential (variational-hemivariational inequalities), using the degree map of multivalued perturbations of fractional nonlinear operators of monotone type, the fact that the degree at a local minimizer of the corresponding Euler functional is equal one, and controlling the degree at small balls and at big balls.

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1. Introduction

Over the past few years, nonlocal operators have taken increasing importance, due to the fact that they appear in a number of applications, in such fields as game theory, finance, image processing, and optimization, see [2, 7, 9, 41] and the references therein.

One of these operators is the fractional p-Laplacian, a nonlinear and nonlocal operator, that is defined for any sufficiently smooth function $u : \mathbb{R}^N \to \mathbb{R}$ and all $x \in \mathbb{R}^N$ by

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+ps}}\, dy,$$

where $s \in (0, 1)$, $p \in (1, \infty)$, in the linear case $p = 2$ gives to the fractional Laplacian up to a dimensional constant $C(N, p, s) > 0$ (see [8, 19]).

In [44] Teng studies hemivariational inequalities driven by nonlocal elliptic operator and he shows the existence of two nontrivial solutions, by applying critical point theory for nonsmooth functionals, while in [42] Servadei and Valdinoci prove Lewy-Stampacchia type estimates for variational inequalities driven by nonlocal operators. In [45] Xiang considers a variational inequality involving nonlocal elliptic operators, proving the existence of one solution, by exploiting variational methods combined with a penalization tecnique and Schauder's fixed point theorem. In [1] Aizicovici, Papageorgiou and Staicu study the degree theory for the operator $\partial\varphi + \partial\psi$, where $\partial\varphi$ is the Clarke generalized subdifferential of a nonsmooth and locally Lipschitz functional φ, and $\partial\psi$ the subdifferential of ψ, a proper, convex and lower semicontinuous functional, in the sense of convex analysis. They show a result regarding the degree of an isolated minimizer for Euler functionals of the form $\varphi + \psi$. Such extension allow to study nonlinear variational inequalities with a nonsmooth potential function (variational-hemivariational inequalities). Such variational-hemivariational inequalities are called in this way, because in them appear a maximal monotone term which is not in general everywhere defined (variational inequality), and a nonmonotone, but everywhere defined term (hemivariational inequality). In the last decade hemivariational inequalities have been actively studied through employing the techniques of nonlinear analysis (including degree theory and minimax methods), see [11, 31, 35, 36, 39] and the references therein. Furthermore hemivariational inequalities can be naturally applied in problems of mechanics and engineering, taking into account more realistic laws which involve multivalued (nonsmooth potential) and nonmonotone (nonconvex potential) operators, see [35].

A natural obstacle problem is given by an elastic membrane, with vertical movement u on a domain

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Ω, which is bound to its boundary ($u = 0$ along $\partial\Omega$) and it is forced to stay below some obstacle ($u \geq \gamma$). Afterwards, at the equilibrium, everytime the membrane does not come into contact with the obstacle, the elasticity provides a balance of the tension of the membrane, that, geometrically, reflects into a balance of the principal curvatures of the surface described by u. At the same time, whenever the membrane sticks to the obstacle, its principal curvatures are supposed to adapt to those of γ. In addition, when an external force w appears, the elastic tension of the membrane will balance up the force. These physical arguments are reflected in the following variational inequality in the case of Laplacian operator

$$\int_{\Omega} \nabla u(x)(\nabla v(x) - \nabla u(x))\, dx \geq \int_{\Omega} w(x)(v(x) - u(x))\, dx \tag{1.1}$$

for any test function v, with $v \geq \gamma$ and $v = 0$ along $\partial\Omega$ (see [42]). While in the case of p-Laplacian operator, looking at nonlinear elastic reactions of the membrane, the inequality becomes the following

$$\int_{\Omega} |\nabla u|^{p-2}\nabla u(x)(\nabla v(x) - \nabla u(x))\, dx \geq \int_{\Omega} w(x)(v(x) - u(x))\, dx$$

with $p \in (1, \infty)$ (see [1, 13, 40]). Likewise, one may take into account the long range interactions of particles, changing the local elastic reaction in (1.1) with a nonlocal one, for example substituting the Laplacian with the fractional Laplacian, hence (1.1) becomes the following nonlocal variational inequality

$$\int_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))(v(x) - v(y) - u(x) + u(y))}{|x - y|^{N+2s}}\, dxdy \geq \int_{\Omega} w(x)(v(x) - u(x))\, dx.$$

These type of obstacle problems have been intensively investigated in [10, 32, 43] and in [29, 30, 42] for other integrodifferential kernels.

Motivated by the above mentioned works, in this paper we show a multiplicity result for the obstacle problem at zero driven by the fractional p-Laplacian operator

$$\begin{cases} \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(v(x)-v(y)-u(x)+u(y))}{|x-y|^{N+ps}}\, dxdy \geq \int_{\Omega} w(x)(v(x) - u(x))\, dx \quad \text{for all } v \in W_0^{s,p}(\Omega)_+, \\ w(x) \in N(u) = \{\tilde{w} \in L^{p'}(\Omega) : \tilde{w}(x) \in \partial j(x, u(x)) \text{ for a.e. } x \in \Omega\}, \quad \left(\frac{1}{p} + \frac{1}{p'} = 1\right) \\ u \in W_0^{s,p}(\Omega)_+, \end{cases} \tag{1.2}$$

where $\Omega \subseteq \mathbb{R}^N$, $N > 1$, is a bounded domain with a C^2-boundary $\partial\Omega$, j is a nonsmooth potential that satisfies suitable assumptions, and $W_0^{s,p}(\Omega)$ is the fractional Sobolev space (that will be defined later). By a^{p-1} we mean $|a|^{p-2}a$ for all $a \in \mathbb{R}$ and denote by p' the conjugate exponent of p.

Using a combination of degree theory, based on the degree map for specific multivalued perturbations of $(S)_+$- nonlinear operators (see [1, 24]), and variational methods, we are able to prove that problem (1.2) admits at least two nontrivial solutions.

The paper has the following structure: in Section 2 we collect some basic notions from nonsmooth critical point theory, as well as some useful results on the degree theory, while in Section 3 we gather the results concerning the fractional weighted eigenvalue problem. In Section 4 we consider the obstacle problem at zero and we show our main result.

2. Preliminaries

In this section, we collect some basic definitions and results from nonsmooth and nonlinear analysis, as well as some useful results on the degree theory, which we will be required for our purposes (see [1, 3, 14, 16, 17, 21, 46]).

Let $(X, \|\cdot\|)$ be a reflexive Banach space and $(X^*, \|\cdot\|_*)$ its topological dual. We denote by $\langle\cdot,\cdot\rangle$ the duality pairing between X^* and X, and by $2^X \setminus \{\varnothing\}$ the family of all nonempty subsets of X.

By $\Gamma_0(X)$ we indicate the cone of all proper (not identically $+\infty$), convex and lower semicontinuous

functions $\psi : X \to \mathbb{R} \cup \{+\infty\}$.
Let C be a nonempty, closed convex subset of X, the *indicator function of* C is defined by

$$i_C : X \to \mathbb{R} \cup \{+\infty\} \quad i_C(u) = \begin{cases} 0 & \text{if } u \in C, \\ +\infty & \text{if } u \notin C. \end{cases}$$

If $C \neq \varnothing$, then $i_C \in \Gamma_0(X)$.
Given $\psi \in \Gamma_0(X)$, the *subdifferential of* ψ in the sense of convex analysis is given by the multifunction $\partial\psi : X \to 2^{X^*}$

$$\partial\psi(u) = \{u^* \in X^* : \langle u^*, v - u\rangle \leq \psi(v) - \psi(u) \text{ for all } v \in X\}.$$

Regarding the properties of the subdifferential of ψ in the sense of convex analysis, we refer the reader to [33] and the references therein. We stress that if $\psi \in \Gamma_0(X)$ is Gâteaux differentiable at $u \in X$, then $\partial\psi(u) = \{\psi'(u)\}$. Moreover we note that the subdifferential in the sense of convex analysis $\partial\psi : X \to 2^{X^*}$ of a function $\psi \in \Gamma_0(X)$ is a maximal monotone operator.
If ψ coincides with i_C, the indicator function of $C \subseteq X$, then we obtain a closed convex cone, called the *normal cone to* C *at* u, defined by

$$\partial i_C(u) = \left\{u^* \in X^* : \langle u^*, u\rangle = \sigma(u^*; C) = \sup_{v \in C} \langle u^*, v\rangle\right\}.$$

A function $\varphi : X \to \mathbb{R}$ is said to be locally Lipschitz, if for every $u \in X$ there exist a neighborhood U of u and $L > 0$ such that

$$|\varphi(v) - \varphi(w)| \leq L\|v - w\| \text{ for all } v, w \in U.$$

For such function φ, we define the *generalized directional derivative* of φ at u along $v \in X$ in the following way

$$\varphi^0(u; v) = \limsup_{u' \to u, \lambda \to 0^+} \frac{\varphi(u' + \lambda v) - \varphi(u')}{\lambda},$$

(see [21, Propositions 1.3.7]). The *Clarke generalized subdifferential* of φ at u is the set

$$\partial\varphi(u) = \{u^* \in X^* : \langle u^*, v\rangle \leq \varphi^\circ(u; v) \text{ for all } v \in X\}.$$

If φ is continuous and convex, then φ is locally Lipschitz and the generalized and convex subdifferentials coincide.
We say that u is a *critical point* of φ if $0 \in \partial\varphi(u)$. The following Lemma states some useful properties about $\partial\varphi$, see [21, Propositions 1.3.8-1.3.12].

Lemma 2.1. *If $\varphi, \psi : X \to \mathbb{R}$ are locally Lipschitz continuous, then*

(i) *$\partial\varphi(u)$ is convex, closed and weakly* compact for all $u \in X$;*
(ii) *the multifunction $\partial\varphi : X \to 2^{X^*}$ is upper semicontinuous with respect to the weak* topology on X^*;*
(iii) *if $\varphi \in C^1(X)$, then $\partial\varphi(u) = \{\varphi'(u)\}$ for all $u \in X$;*
(iv) *$\partial(\lambda\varphi)(u) = \lambda\partial\varphi(u)$ for all $\lambda \in \mathbb{R}$, $u \in X$;*
(v) *$\partial(\varphi + \psi)(u) \subseteq \partial\varphi(u) + \partial\psi(u)$ for all $u \in X$;*
(vi) *if u is a local minimizer (or maximizer) of φ, then $0 \in \partial\varphi(u)$.*

In the sequel we focus on the study of critical points of the functional $\varphi + \psi$, for this purpose we mention the following facts (see [28, 34]).

Definition 2.2. *Let $\varphi : X \to \mathbb{R}$ be a locally Lipschitz functional and $\psi : X \to \mathbb{R} \cup \{\infty\}$ be proper, convex and lower semicontinuous. We say that $u \in X$ is a critical point of $\varphi + \psi$ if*

$$\varphi^0(u; v - u) + \psi(v) - \psi(u) \geq 0 \quad \forall v \in X,$$

where $\varphi^0(u; z)$ is the generalized directional derivative of φ at the point $u \in X$ in the direction $z \in X$.

Proposition 2.3. *An element $u \in X$ is a critical point of $\varphi + \psi$ if and only if $0 \in \partial\varphi(u) + \partial\psi(u)$, where $\partial\varphi$ is the Clarke generalized subdifferential and $\partial\psi$ is the subdifferential in the sense of convex analysis.*

Throughout the paper, we denote by $\partial\varphi$ (or $\partial\widehat{\varphi}$) the Clarke generalized subdifferential of φ (or $\widehat{\varphi}$, which will be clear from the context) and by $\partial\psi$ the subdifferential of ψ in the sense of convex analysis.

Now, we introduce the degree map that we will use in the sequel. For a fuller treatment we refer the reader to [1, 6, 24, 33] and the references therein.
Since X is a reflexive Banach space, by the Troyanski renorming theorem (see [21, Theorem A.3.9]), we can equivalently renorm X in such a way that both X and X^* are locally uniformly convex with Fréchet differentiable norms. Therefore, in the following, we suppose that both X and X^* are reflexive and locally uniformly convex.
From [33, Theorem 2.46, Proposition 2.70], the duality map $\mathcal{F} : X \to X^*$, defined by

$$\mathcal{F}(u) = \left\{ u^* \in X^* : \langle u^*, u \rangle = \|u\|^2 = \|u^*\|_*^2 \right\},$$

is single-valued, strictly monotone, a homeomorphism and a (S_+) operator.
An operator $A : X \to X^*$ satisfies the $(S)_+$*-property* if for every sequence $(u_n)_n \subseteq X$ such that

$$u_n \rightharpoonup u \text{ in } X \quad \text{ and } \quad \limsup_{n\to\infty} \langle A(u_n), u_n - u \rangle \le 0,$$

it follows that

$$u_n \to u \text{ in } X.$$

A multifunction $G : X \to 2^{X^*}$ *belongs to class* (P) if it is upper semicontinuous with closed, convex nonempty values and such that

$$G(A) = \bigcup_{u\in A} G(u)$$

is relatively compact in X^* for any bounded subset U of X.
Let U be a bounded open subset in X, $S : \overline{U} \to X^*$ a bounded, demicontinuous operator of type $(S)_+$ and $A : D(A) \subseteq X \to 2^{X^*} \setminus \{\varnothing\}$ a maximal monotone operator with $0 \in A(0)$, then for every $\lambda > 0$, the operator $S + A_\lambda$ is a bounded, demicontinuous operator of type $(S)_+$. For every $u^* \notin (S+A)(\partial U)$, $\deg_0(S + A, U, u^*)$ is defined by

$$deg_0(S + A, U, u^*) = deg_{(S)_+}(S + A_\lambda, U, u^*)$$

for all sufficiently small $\lambda > 0$, where $A_\lambda(u) = -\frac{1}{\lambda}\mathcal{F}(v - u)$ is everywhere defined, single valued, bounded and monotone.
In addition we have a multifunction G in the class (P), then for $u^* \notin (S + A + G)(\partial U)$, $deg(S + A + G, U, u^*)$ is defined by

$$deg(S + A + G, U, u^*) = deg_0(S + A + g_\varepsilon, U, u^*)$$

for $\varepsilon > 0$ small, where g_ε is a continuous $\varepsilon-$approximate selection of G (see [12, Cellina's approximate selection Theorem], [23, Theorem 4.41]).
Concerning the degree maps $\deg_{(S)_+}$ and $\deg_0$ we refer the reader to [6], while for the degree map deg we refer to [24]. The degree map preserves the usual properties: normalization, domain additivity, homotopy invariance, excision and solution property. One of such properties is the homotopy invariance with respect to a certain class of admissible homotopies. Now we introduce the admissible homotopies for the maps S, A and G (see [1]).

Definition 2.4. *The admissible homotopies for the maps S, A and G are defined in the next way.*

- *A one-parameter family $\{S_t\}_{t\in[0,1]}$ of maps from $\overline{U}$ into X^* is a **homotopy of class** $(S)_+$, if for any $(u_n)_n \subseteq \overline{U}$ such that $u_n \rightharpoonup u$ in X, and for any $(t_n)_n \subseteq [0, 1]$ with $t_n \to t$ for which*

$$\limsup_{n\to\infty} \langle S_{t_n}(u_n), u_n - u \rangle \le 0,$$

we have that $u_n \to u$ *in* X *and* $S_{t_n}(u_n) \rightharpoonup S_t(u)$ *in* X^*.

- *A family* $\{A^t\}_{t\in[0,1]}$ *of maximal monotone maps from* X *into* X^* *such that* $(0,0) \in \mathrm{Gr}A^t$ *(graph of* A^t*) for all* $t \in [0,1]$ *is a* ***pseudomonotone homotopy****, if it satisfies the following mutually equivalent conditions*
  - *if* $t_n \to t$ *in* $[0,1]$, $u_n \rightharpoonup u$ *in* X, $u_n^* \rightharpoonup u^*$ *in* X^*, $u_n^* \in A^{t_n}(u_n)$ *and*
  $$\limsup_{n\to\infty} \langle u_n^*, u_n - u\rangle \le 0,$$
  then $(u,u^*) \in \mathrm{Gr}A^t$ *and* $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$;
  - $(t,u^*) \mapsto \xi(t,u^*) = \left(A^t + \mathcal{F}\right)^{-1}(u^*)$ *is continuous from* $[0,1]\times X^*$ *into* X*, where both* X *and* X^* *are equipped with their respective norm topologies;*
  - *for every* $u^* \in X^*$, $t \mapsto \xi(t,u^*) = \left(A^t + \mathcal{F}\right)^{-1}(u^*)$ *is continuous from* $[0,1]$ *into* X *endowed with the norm topology;*
  - *if* $t_n \to t$ *in* $[0,1]$ *and* $u^* \in A^t(u)$*, then there exist sequences* $(u_n)_n$ *and* $(u_n^*)_n$ *such that* $u_n^* \in A^{t_n}(u_n)$, $u_n \to u$ *in* X *and* $u_n^* \to u^*$ *in* X^*.
- *A one-parameter family* $\{G_t\}_{t\in[0,1]}$ *of multifunctions* $G_t : \overline{U} \to 2^{X^*}\setminus\{\varnothing\}$ *is a* ***homotopy of class*** (P) *if* $(t,u) \mapsto G_t(u)$ *is usc from* $[0,1]\times\overline{U}$ *into* $2^{X^*}\setminus\{\varnothing\}$*, for every* $(t,u) \in [0,1]\times\overline{U}$, $G_t(u) \subseteq X^*$ *is closed and convex and*
$$\overline{\bigcup \left\{G_t(u) : t \in [0,1],\ u \in \overline{U}\right\}}$$
is compact in X^*.

Therefore the homotopy invariance of the degree map "deg", can be expressed in the following way. If $\{S_t\}_{t\in[0,1]}$ is a homotopy of class $(S)_+$ such that each S_t is bounded, $\{A^t\}_{t\in[0,1]}$ is a pseudomonotone homotopy of maximal monotone operators with $0 \in A^t(0)$ for all $t \in [0,1]$, $\{G_t\}_{t\in[0,1]}$ is a homotopy of class (P) and $u^* : [0,1] \to X^*$ is a continuous map such that
$$u_t^* \notin (S_t + A_t + G_t)(\partial U)$$
for all $t \in [0,1]$, then $\deg(S_t + A_t + G_t, U, u_t^*)$ is independent of $t \in [0,1]$. (This is the meaning of admissible homotopy for us in this paper.)
Now, we identify another class of pseudomonotone homotopies (see [1, Lemma 15]).

Lemma 2.5. *Let* $A : X \to X^*$ *be a bounded demicontinuous operator of type* $(S)_+$ *and* $\psi \in \Gamma_0(X)$*. Then*
$$(t,u) \mapsto h(t,u) = A(u) + t\partial\psi(u), \quad (t,u) \in [0,1]\times X$$
is a pseudomonotone homotopy.

3. Fractional weighted eigenvalue problems

Let $\Omega \subseteq \mathbb{R}^N (N > 1)$, be a bounded domain with a C^2-boundary $\partial\Omega$, $p > 1$ and $s \in (0,1)$ are real numbers such that $N > ps$. In this section we focus on the study of the following weighted fractional eigenvalue problem (see [15, 22])
$$\begin{cases} (-\Delta)_p^s\, u = \lambda m(x)|u|^{p-2}u & \text{in } \Omega,\\ u = 0 & \text{on } \Omega^c, \end{cases} \tag{3.1}$$
where $m \in L^\infty(\Omega)_+$, $m \ne 0$ is a weight function, λ a real parameter and $\Omega^c = \mathbb{R}^N \setminus \Omega$. The Dirichlet boundary condition is given in Ω^c and not simply on $\partial\Omega$, accordingly with the nonlocal character of the operator $(-\Delta)_p^s$. For all $1 \le q \le \infty$, $\|\cdot\|_q$ denotes the standard norm of $L^q(\Omega)$ (or $L^q(\mathbb{R}^N)$), which will be clear from the context.
As a first step, we fix a functional-analytical framework. For all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ we set
$$[u]_{s,p}^p = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x-y|^{N+ps}}\, dxdy.$$

Then we define the fractional Sobolev spaces (see [19]) as follows

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : [u]_{s,p} < \infty\},$$

$$W_0^{s,p}(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \Omega^c\},$$

this last one is a separable, uniformly convex (hence, reflexive) Banach space, endowed with the norm $||u|| = [u]_{s,p}$. We denote by $(W^{-s,p'}(\Omega), ||\cdot||_*)$ the topological dual of $(W_0^{s,p}(\Omega), ||\cdot||)$ and by $\langle\cdot,\cdot\rangle$ the duality pairing between $W^{-s,p'}(\Omega)$ and $W_0^{s,p}(\Omega)$. The critical exponent is defined as $p_s^* = \frac{Np}{N-ps}$, and the embedding $W_0^{s,p}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for all $q \in [1, p_s^*]$ and compact for all $q \in [1, p_s^*)$ (in particular, we will use $q = p$), see [19, Corollary 7.2]. Furthermore, we introduce the positive order cone

$$W_0^{s,p}(\Omega)_+ = \{u \in W_0^{s,p}(\Omega) : u(x) \geq 0 \text{ for a.e. } x \in \Omega\},$$

which has an empty interior with respect to the $W_0^{s,p}(\Omega)-$ topology.

Remark 3.1. Since $W_0^{s,p}(\Omega)$ is a reflexive Banach space, applying the Troyanski's renorming theorem, such space can be equivalently renormed so that both $W_0^{s,p}(\Omega)$ and $W^{-s,p'}(\Omega)$ are locally uniformly convex (and thus also strictly convex) and with Fréchet differentiable norms.

The operator $(-\Delta)_p^s$ can be represented by the nonlinear operator $A : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ defined for all $u, v \in W_0^{s,p}(\Omega)$ by

$$\langle A(u), v\rangle = \int_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^{p-1}(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy. \tag{3.2}$$

Moreover, we define the operators $\tilde{J}_\lambda, K_m : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ by

$$\langle K_m(u), v\rangle = \int_\Omega m(x)|u(x)|^{p-2}u(x)v(x)\, dx, \text{ with } m \in L^\infty(\Omega)_+, m \neq 0, \tag{3.3}$$

$$\langle \tilde{J}_\lambda(u), v\rangle = \langle A(u) - \lambda K_m(u), v\rangle, \text{ with } \lambda \in \mathbb{R}, \tag{3.4}$$

for any $v \in W_0^{s,p}(\Omega)$. In the sequel we will change the function m in (3.3) with a suitable function, but the definition of the operator $K_{(\cdot)}$ remains the same. In the following lemma some important features of such operators are stated.

Lemma 3.2. *The operators $A, K_m, \tilde{J}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$, defined above, satisfy the following properties:*

(i) $A : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ *is a maximal monotone, bounded and continuous operator of type* $(S)_+$,
(ii) $K_m : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ *is a bounded, continuous and compact operator,*
(iii) $\tilde{J}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ *is a bounded, continuous operator that satisfies the condition* $(S)_+$.

Proof. We start proving the first assertion. The operator A is odd, $(p-1)-$homogeneous, and satisfies for all $u \in W_0^{s,p}(\Omega)$

$$\langle A(u), u\rangle = ||u||^p, \quad ||A(u)||_* \leq ||u||^{p-1}.$$

Hence, A is bounded (see [25]). Since $W_0^{s,p}(\Omega)$ is uniformly convex, by [37, Proposition 1.3], A is an operator of type $(S)_+$. Now we show that A is a continuous operator. In order to do this, we define a support mapping $f(u) = \frac{A(u)}{||u||^{p-2}}$ for every $u \in \partial B_1(0) \subset W_0^{s,p}(\Omega)$ (for definition and properties we refer the reader to [18]). Recalling Remark 3.1, we obtain that the norm of $W^{-s,p'}(\Omega)$ is Fréchet differentiable and, applying [18, Theorem 1], we obtain that $f : \partial B_1(0) \subset W_0^{s,p}(\Omega) \to \partial B_1(0) \subset$

$W^{-s,p'}(\Omega)$ is continuous. Hence, by definition of f, A is continuous in $W_0^{s,p}(\Omega)\setminus\{0\}$. Indeed, we suppose that $v_n=\frac{u_n}{||u_n||}$ strongly converges to $v=\frac{u}{||u||}$ in $W_0^{s,p}(\Omega)$. Hence,

$$\begin{aligned}A(u_n)=A(||u_n||v_n)&=||u_n||^{p-1}A(v_n)=||u_n||^{p-1}f(v_n)\to||u||^{p-1}f(v)\\&=||u||^{p-1}f\left(\frac{u}{||u||}\right)=||u||^{p-2}f(u)=A(u)\end{aligned}$$

as n goes to infinity. The continuity in the origin is trivial, then A is continuous in the whole space $W_0^{s,p}(\Omega)$. By [38, Lemma 3.3], A is strictly monotone and by [21, Corollary 1.4.2] it is maximal monotone.
Now we show the second point. By Schwarz and Hölder inequalities, we get $|\langle K_m(u),v\rangle|\le||m||_\infty||u||^{p-1}||v||$, hence $||K_m(u)||_*\le c||u||^{p-1}$. Therefore, K_m is bounded. Let $(u_n)_n\subset W_0^{s,p}(\Omega)$ be bounded, we may assume, passing to a subsequence, $u_n\rightharpoonup u$ in $W_0^{s,p}(\Omega)$, $u_n\to u$ in $L^p(\Omega)$, hence, by [5, Theorem 4.9], up to a subsequence, $u_n(x)\to u(x)$ a.e. on Ω and $|u_n(x)|\le h(x)$ a.e. on Ω, with $h\in L^p(\Omega)$. Now, applying the dominated convergence Theorem, we obtain that

$$\langle K_m(u_n),v\rangle\to\langle K_m(u),v\rangle\quad\text{ as } n\to\infty.$$

Hence, K_m is compact. Similarly, we see that K_m is also continuous.
Using the previous fact, we get the third assertion. From (i) and (ii) we obtain that $\tilde{J}_\lambda$ is a bounded, continuous operator. Moreover, by [20, Lemma 1.2] and using again (i)-(ii) we get that $\tilde{J}_\lambda$ is an operator of type $(S)_+$. □

Definition 3.3. *A function $u\in W_0^{s,p}(\Omega)$ is called a (weak) solution of (3.1) if for all $v\in W_0^{s,p}(\Omega)$, we have*

$$\langle A(u),v\rangle=\lambda\langle K_m(u),v\rangle.$$

In an equivalent way, $u\in W_0^{s,p}(\Omega)$ solves (3.1) if $\tilde{J}_\lambda(u)=0$ in $W^{-s,p'}(\Omega)$.

We say that λ is an *eigenvalue* of $(-\Delta)_p^s$ related to the weight m if (3.1) has a nontrivial solution $u\in W_0^{s,p}(\Omega)\setminus\{0\}$ and such solution u is called an *eigenfunction* corresponding to the eigenvalue λ. In the following proposition we focus on the properties of the first eigenpair of (3.1), that will be required in the sequel to prove our main result (see [15, 22]).

Proposition 3.4. *Let $m\in L^\infty(\Omega)_+$, $m\neq 0$. The first eigenvalue is given by*

$$\lambda_1(m)=\inf_{u\in W_0^{s,p}(\Omega)\setminus\{0\}}\frac{||u||^p}{\int_\Omega m(x)|u|^p\,dx}.$$

Then,

- *(i) $\lambda_1(m)$ is positive, simple, isolated and it is attained by some positive a.e. eigenfunction $u_1\in W_0^{s,p}(\Omega)$ such that $\int_\Omega m(x)|u|^p\,dx=1$;*
- *(ii) if u is an eigenfunction of (3.1) associated with an eigenvalue $\lambda>\lambda_1(m)$, then u must be nodal (sign-changing);*
- *(iii) the first eigenfunction satisfies the so-called* unique continuation property *(u.c.p.) and hence, we have the strict monotonicity of the map $m\mapsto\lambda_1(m)$.*

Proof. We refer to [15, 22] for the proof of i) and ii).
We show the third point. Let $m_1,m_2\in L^\infty(\Omega)_+$ be such that $m_1,m_2\neq 0$, $m_1(x)\le m_2(x)$ for a.e. $x\in\Omega$, $m_1\not\equiv m_2$. Let u_1 and u_2 the first eigenfunctions corresponding to the weights m_1 and m_2, respectively. By i) such eigenfunctions are positive a.e., hence u_1 and u_2 clearly satisfy the u.c.p. From the definition of λ_1, we obtain

$$\lambda_1(m_1)=\frac{||u_1||^p}{\int_\Omega m_1(x)u_1^p\,dx}>\frac{||u_1||^p}{\int_\Omega m_2(x)u_1^p\,dx}\ge\lambda_1(m_2),$$

so $\lambda_1(m_1)>\lambda_1(m_2)$. □

When $m \equiv 1$ we will just write $\lambda_1(1) = \lambda_1$.
The following result about the degree of the operator $\tilde{J}_\lambda$ is fundamental for the sequel, whose proof closely follows that of [20, Theorem 3.7]. Moreover we point out that $\tilde{J}_\lambda$ is a monotone map, so we can apply the properties of the degree for generalized monotone maps (see [20]).

Proposition 3.5. *Let $A, K_m, \tilde{J}_\lambda : W_0^{s,p}(\Omega) \to W^{-s,p'}(\Omega)$ be defined by (3.2), (3.3), (3.4) and $\delta > 0$. Then*

$$\deg(\tilde{J}_\lambda, B_r(0), 0) = 1 \text{ for } \lambda \in (0, \lambda_1(m)),$$

and

$$\deg(\tilde{J}_\lambda, B_r(0), 0) = -1 \text{ for } \lambda \in (\lambda_1(m), \lambda_1(m) + \delta).$$

Proof. From the variational characterization of $\lambda_1(m)$ we have that

$$\langle \tilde{J}_\lambda(u), u \rangle > 0,$$

for $\lambda \in (0, \lambda_1(m))$ and any $u \in W_0^{s,p}(\Omega)$ with $||u|| \neq 0$. Hence, by [20, Theorem 1.5], the degree $\deg(\tilde{J}_\lambda, B_r(0), 0)$ is well defined for any $\lambda \in (0, \lambda_1(m))$ and any ball $B_r(0) \subset W_0^{s,p}(\Omega)$, moreover, applying [20, Theorem 1.6], we obtain

$$\deg(\tilde{J}_\lambda, B_r(0), 0) = 1 \text{ for } \lambda \in (0, \lambda_1(m)). \tag{3.5}$$

Now we show that $\deg(\tilde{J}_\lambda, B_r(0), 0) = -1$ for $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$. On account of Proposition 3.4 there exists $\delta > 0$ such that the interval $(\lambda_1(m), \lambda_1(m) + \delta)$ does not include any eigenvalue for the problem (3.1). Therefore the degree $\deg(\tilde{J}_\lambda, B_r(0), 0)$ is well defined also for $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$.
Let us compute $\operatorname{Ind}(\tilde{J}_\lambda, 0)$ for $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$. We introduce a function $\phi : \mathbb{R} \to \mathbb{R}$ by

$$\phi(t) = \begin{cases} 0 & \text{if } t \leq k \\ \frac{2\delta}{\lambda_1(m)}(t - 2k) & \text{if } t \geq 3k, \end{cases}$$

for a fixed number $k > 0$. We note that $\phi(t)$ is continuously differentiable, positive and strictly convex in $(k, 3k)$.
Now we can introduce the functional

$$\Phi_\lambda(u) = \frac{1}{p}\langle A(u), u \rangle - \frac{\lambda}{p}\langle K_m(u), u \rangle + \phi\left(\frac{1}{p}\langle A(u), u \rangle\right),$$

that is Fréchet differentiable and its critical point $u_0 \in W_0^{s,p}(\Omega)$ coincides to a solution of the equation

$$A(u_0) - \frac{\lambda}{1 + \phi'\left(\frac{1}{p}\langle A(u_0), u_0 \rangle\right)} K_m(u_0) = 0.$$

Nevertheless, since $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$, the only nontrivial critical points of Φ_λ turn up if

$$\phi'\left(\frac{1}{p}\langle A(u_0), u_0 \rangle\right) = \frac{\lambda}{\lambda_1(m)} - 1. \tag{3.6}$$

Owing to the definition of ϕ it follows that $\frac{1}{p}\langle A(u_0), u_0 \rangle \in (k, 3k)$ and because of (3.6) and the simplicity of $\lambda_1(m)$, it deduces that either $u_0 = -u_1$ or $u_0 = u_1$, where $u_1 > 0$ is the first eigenfunction (which is not necessarily normed by 1). Therefore, we may conclude that for $\lambda \in (\lambda_1(m), \lambda_1(m) + \delta)$ the derivative Φ'_λ has precisely three isolated critical points $\{-u_1, 0, u_1\}$ (in the sense of [20, Definition 1.2]).
We now show that Φ_λ is weakly lower semicontinuous. Indeed, suppose that $u_n \rightharpoonup \tilde{u}_0$ in $W_0^{s,p}(\Omega)$. Owing to the compactness of K_m, we get

$$\langle K_m(u_n), u_n \rangle \to \langle K_m(\tilde{u}_0), \tilde{u}_0 \rangle, \tag{3.7}$$

and recalling that $\liminf_{n\to\infty} ||u_n|| \geq ||\tilde{u}_0||$, (3.7) holds, and ϕ is nondecreasing, we obtain

$$\liminf_{n\to\infty} \left[\frac{1}{p}\langle A(u_n), u_n \rangle - \frac{\lambda}{p}\langle K_m(u_n), u_n \rangle + \phi\left(\frac{1}{p}\langle A(u_n), u_n \rangle\right)\right] \geq \Phi_\lambda(\tilde{u}_0).$$

Furthermore, Φ_λ is coercive, i.e. $\lim_{||u||\to\infty}\Phi_\lambda(u)=\infty$. Indeed, we get

$$\Phi_\lambda(u)=\frac{1}{p}\langle A(u),u\rangle-\frac{\lambda_1(m)}{p}\langle K_m(u),u\rangle+\frac{\lambda_1(m)-\lambda}{p}\langle K_m(u),u\rangle+\phi\left(\frac{1}{p}\langle A(u),u\rangle\right)$$

and, by the variational characterization of $\lambda_1(m)$,

$$\langle A(u),u\rangle-\lambda_1(m)\langle K_m(u),u\rangle\geq 0 \tag{3.8}$$

for any $u\in W_0^{s,p}(\Omega)$. From (3.8) we have that

$$\begin{aligned}\frac{\lambda_1(m)-\lambda}{p}\langle K_m(u),u\rangle+\phi\left(\frac{1}{p}\langle A(u),u\rangle\right)&\geq\frac{\lambda_1(m)-\lambda}{p\lambda_1(m)}\langle A(u),u\rangle+\phi\left(\frac{1}{p}\langle A(u),u\rangle\right)\\&\geq-\frac{\delta}{p\lambda_1(m)}\langle A(u),u\rangle+\frac{2\delta}{\lambda_1(m)}\left(\frac{1}{p}\langle A(u),u\rangle-2k\right)\to\infty\end{aligned}$$

for $||u||\to\infty$ because of the definition of ϕ. Therefore we obtain the coercivity. We observe that Φ_λ is even, the minimum of Φ_λ is achieved exactly in two points $-u_1,u_1$, while the origin is an isolated critical point, but it is not a minimum. Indeed, by definition of Φ_λ and ϕ, we get that

$$\begin{aligned}\Phi_\lambda(tu_1)&=\left(\frac{1}{p}\langle A(u_1),u_1\rangle-\frac{\lambda}{p}\langle K_m(u_1),u_1\rangle\right)t^p+\phi\left(\frac{t^p}{p}\langle A(u_1),u_1\rangle\right)\\&=\frac{t^p}{p}(\lambda_1(m)-\lambda)\langle K_m(u_1),u_1\rangle<0\quad\forall t\in(0,t_0).\end{aligned}$$

In accordance with [20, Theorem 1.8] we get

$$\mathrm{Ind}(\Phi'_\lambda,-u_1)=\mathrm{Ind}(\Phi'_\lambda,u_1)=1.$$

At the same time, we have $\langle\Phi'_\lambda(u),u\rangle>0$ for any $u\in W_0^{s,p}(\Omega)$, $||u||=\kappa$, with $\kappa>0$ large enough. Indeed

$$\begin{aligned}\langle\Phi'_\lambda(u),u\rangle&=\langle A(u),u\rangle-\lambda\langle K_m(u),u\rangle+\phi'\left(\frac{1}{p}\langle A(u),u\rangle\right)\langle A(u),u\rangle\\&=\langle A(u),u\rangle-\lambda_1(m)\langle K_m(u),u\rangle+\phi'\left(\frac{1}{p}\langle A(u),u\rangle\right)\\&\qquad\left(\langle A(u),u\rangle-\frac{\lambda-\lambda_1(m)}{\phi'\left(\frac{1}{p}\langle A(u),u\rangle\right)}\langle K_m(u),u\rangle\right)\\&\geq\frac{2\delta}{\lambda_1(m)}\left(\langle A(u),u\rangle-\frac{\lambda_1(m)}{p}\langle K_m(u),u\rangle\right)\to\infty\text{ as }||u||\to\infty.\end{aligned}$$

We again used the variational characterization of $\lambda_1(m)$ and the definition of ϕ. Then, [20, Theorem 1.6] and $\langle\Phi'_\lambda(u),u\rangle>0$ imply

$$\deg(\Phi'_\lambda,B_\kappa(0),0)=1.$$

We pick $\kappa>0$ so large that $\pm u_1\in B_\kappa(0)$. By [20, Theorem 1.7] and $\mathrm{Ind}(\Phi'_\lambda,-u_1)=\mathrm{Ind}(\Phi'_\lambda,u_1)=1$, and $\deg(\Phi'_\lambda,B_\kappa(0),0)=1$, we have

$$\mathrm{Ind}(\Phi'_\lambda,0)=-1. \tag{3.9}$$

Furthermore, by the definition of ϕ, we have

$$\deg(\tilde{J}_\lambda,B_r(0),0)=\mathrm{Ind}(\Phi'_\lambda,0) \tag{3.10}$$

for $r>0$ small enough. Then we deduce from (3.9), (3.10), that

$$\mathrm{Ind}(\tilde{J}_\lambda,0)=-1\text{ for }\lambda\in(\lambda_1(m),\lambda_1(m)+\delta).$$

It follows from the previous relations that

$$\deg(\tilde{J}_\lambda,B_r(0),0)=-1.$$

□

4. The obstacle problem at zero

In this section, we study the obstacle problem at 0 and we show that such problem admits at least two nontrivial solutions. In order to do this, we need to prove some facts about the degree theory, extending the results proved in the nonlinear local case in [1]. For this purpose, let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$, $p > 1$ and $s \in (0,1)$ are real numbers such that $N > ps$, and recalling (3.2), we can rewrite the obstacle problem (1.2) at 0 in the following way

$$\begin{cases} \langle A(u), v-u\rangle \geq \int_\Omega w(x)(v(x)-u(x))\,dx \quad \text{for all } v \in W_0^{s,p}(\Omega)_+, \\ w(x) \in N(u) = \{\tilde{w} \in L^{p'}(\Omega) : \tilde{w}(x) \in \partial j(x,u(x)) \text{ for a.e. } x \in \Omega\}, \\ u \in W_0^{s,p}(\Omega)_+. \end{cases} \tag{4.1}$$

We assume the following hypotheses on the nonsmooth potential

(H): $j : \Omega \times \mathbb{R} \to \mathbb{R}$ is a function such that $j(\cdot,0) = 0$ a.e. on Ω, $j(\cdot,t)$ is measurable in Ω for all $t \in \mathbb{R}$, $j(x,\cdot)$ is locally Lipschitz in $\mathbb{R}$ for a.e. $x \in \Omega$. Moreover

- (i) $|\xi| \leq a(x) + c|t|^{p-1}$ with $a \in L^\infty(\Omega)_+$, $c > 0$, for a.e. $x \in \Omega$, all $t \in \mathbb{R}$, and all $\xi \in \partial j(x,t)$,
- (ii) there exists $\theta \in L^\infty(\Omega)_+$ such that $\theta \leq \lambda_1$, $\theta \not\equiv \lambda_1$, and
$$0 \leqslant \liminf_{t\to+\infty} \frac{\xi}{t^{p-1}} \leqslant \limsup_{t\to+\infty} \frac{\xi}{t^{p-1}} \leqslant \theta(x)$$
uniformly for a.e. $x \in \Omega$ and all $\xi \in \partial j(x,t)$;
- (iii) there exist $\eta, \hat{\eta} \in L^\infty(\Omega)_+$ such that $\lambda_1 \leq \eta$, $\eta \not\equiv \lambda_1$, and
$$\eta(x) \leqslant \liminf_{t\to0^+} \frac{\xi}{t^{p-1}} \leqslant \limsup_{t\to0^+} \frac{\xi}{t^{p-1}} \leqslant \hat{\eta}(x)$$
uniformly for a.e. $x \in \Omega$ and all $\xi \in \partial j(x,t)$.

Remark 4.1. We denote by λ_1 the first eigenvalue of $(-\Delta)_p^s$ with Dirichlet conditions in Ω (see Section 3), hence **(H)** (ii)-(iii) invoke nonuniform nonresonance conditions at $+\infty$ and at 0^+. The condition at $+\infty$ is from below λ_1 and the condition at 0^+ is from above with respect to λ_1.

Example 4.2. A nonsmooth locally Lipschitz potential satisfying hypotheses **(H)** is defined as follows, which for simplicity we dropped the $x-$dependence:

$$j(t) = \begin{cases} \frac{\eta}{p}|t|^p - \frac{1}{p}\cos|t|^p & \text{if } |t| \leq 1, \\ \frac{\theta}{p}|t|^p + \frac{\eta-\theta}{p} - \frac{1}{p}\cos 1 & \text{if } |t| > 1, \end{cases}$$

with $\theta < \lambda_1 < \eta$.

Now we define the integral functional $\widehat{J} : L^p(\Omega) \to \mathbb{R}$ by

$$\widehat{J}(u) = \int_\Omega j(x,u(x))\,dx \text{ for all } u \in L^p(\Omega). \tag{4.2}$$

From **(H)** (i) such functional $\widehat{J}$ is Lipschitz continuous on bounded sets, hence it is locally Lipschitz (see [21, Theorem 1.3.10]).

Let $N : L^p(\Omega) \to 2^{L^{p'}(\Omega)}$ be defined by

$$N(u) = \{w \in L^{p'}(\Omega) : w(x) \in \partial j(x,u(x)) \text{ a.e. on } \Omega\},\ u \in L^p(\Omega).$$

Let us mention an important result about N, for the proof of the following proposition we refer to [1, Proposition 3, Corollary 4].

Proposition 4.3. *Let **(H)** (i) hold. Therefore*

- *N has nonempty, weakly compact and convex values in $L^{p'}(\Omega)$ and it is upper semicontinuous from $L^p(\Omega)$ with the norm topology into $L^{p'}(\Omega)$ with the weak topology.*
- *Moreover, $N : W_0^{s,p}(\Omega) \to 2^{W^{-s,p'}(\Omega)} \setminus \{\varnothing\}$ is a multifunction of class (P).*

For the second point we take into account that $W_0^{s,p}(\Omega)$ is embedded compactly and densely in $L^p(\Omega)$, and $L^{p'}(\Omega)$ is embedded compactly and densely in $W^{-s,p'}(\Omega)$.

Now we can introduce the Euler functional associated to problem (4.1), which is given for $u \in W_0^{s,p}(\Omega)$ by

$$\varphi : W_0^{s,p}(\Omega) \to \mathbb{R} \cup \{+\infty\} \qquad \varphi(u) = \widehat{\varphi}(u) + \psi(u)$$

where

$$\widehat{\varphi}(u) = \frac{||u||^p}{p} - \int_\Omega j(x, u(x))\, dx \quad \text{and} \quad \psi(u) = i_{W_0^{s,p}(\Omega)_+}(u) = \begin{cases} 0 & \text{if } u \in W_0^{s,p}(\Omega)_+ \\ +\infty & \text{if } u \notin W_0^{s,p}(\Omega)_+. \end{cases}$$

From **(H)** *(i)*, $\widehat{\varphi}$ is locally Lipschitz (see [21, Theorem 1.3.10]). Furthermore, $W_0^{s,p}(\Omega)_+ \subseteq W_0^{s,p}(\Omega)$ is closed, convex, hence $\psi \in \Gamma_0(W_0^{s,p}(\Omega))$.
The next Lemma emphasizes the importance of the hypothesis **(H)** *(ii)* (for the proof we refer to [27, Proposition 2.9]).

Lemma 4.4. *Let $\theta \in L^\infty(\Omega)_+$ be such that $\theta \leqslant \lambda_1$, $\theta \not\equiv \lambda_1$, and $\psi \in C^1(W_0^{s,p}(\Omega))$ be defined by*

$$\tau(u) = \|u\|^p - \int_\Omega \theta(x)|u|^p\, dx.$$

Then there exists $\theta_0 \in (0, \infty)$ such that for all $u \in W_0^{s,p}(\Omega)$

$$\tau(u) \geqslant \theta_0 \|u\|^p.$$

The next proposition shows the existence of a minimizer, which belongs to $W_0^{s,p}(\Omega)_+$.

Proposition 4.5. *Let **(H)** (i)-(ii) hold, then there exists $u_0 \in W_0^{s,p}(\Omega)_+$ such that*

$$\varphi(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} \varphi(u).$$

Proof. By **(H)** *(ii)*, given $\epsilon > 0$, there exists $M_\epsilon > 0$ such that for a.e. $x \in \Omega$, all $t \geq M_\epsilon$ and all $\xi \in \partial j(x, t)$, we obtain

$$\xi \leq (\theta(x) + \epsilon) t^{p-1}. \tag{4.3}$$

Moreover, by **(H)** *(i)*, we can find $\beta_\epsilon \in L^\infty(\Omega)_+$ such that for a.e. $x \in \Omega$, all $t \in [0, M_\epsilon]$ and all $\xi \in \partial j(x, t)$, we get

$$|\xi| \leq \beta_\epsilon(x). \tag{4.4}$$

By Rademacher's theorem for a.e. $x \in \Omega$, $j(x, \cdot)$ is differentiable almost everywhere and

$$\frac{d}{dr} j(x, r) \in \partial j(x, r).$$

Therefore, for a.e. $x \in \Omega$ and for all $t \geq 0$, we have

$$\begin{aligned} j(x,t) &= \int_0^t \frac{d}{dr} j(x,r) dr \\ &\leq \int_0^t [(\theta(x) + \epsilon) r^{p-1} + \beta_\epsilon(x)]\, dr \quad \text{(by (4.3), (4.4))} \\ &= \frac{1}{p}(\theta(x) + \epsilon) t^p + \beta_\epsilon(x) t. \end{aligned} \tag{4.5}$$

We stress that φ coincides with $\widehat{\varphi}$ for all $u \in W_0^{s,p}(\Omega)_+$, since $\psi(u) = 0$. Moreover, by (4.5) we have for every $u \in W_0^{s,p}(\Omega)_+$

$$\begin{aligned}
\varphi(u) &= \frac{1}{p}||u||^p - \int_\Omega j(x, u(x))\, dx \\
&\geqslant \frac{\|u\|^p}{p} - \int_\Omega \Big(\beta_\epsilon(x)u + (\theta(x) + \epsilon)\frac{|u|^p}{p}\Big)\, dx \\
&\geqslant \frac{1}{p}\Big(\|u\|^p - \int_\Omega \theta(x)|u|^p\, dx\Big) - \|\beta_\epsilon\|_\infty \|u\|_1 - \frac{\epsilon}{p}\|u\|_p^p \\
&\geqslant \frac{1}{p}\Big(\theta_0 - \frac{\epsilon}{\lambda_1}\Big)\|u\|^p - c\|u\| \ (\theta_0, c > 0),
\end{aligned}$$

where in the final passage we have used Lemma 4.4, and the continuous embedding $W_0^{s,p}(\Omega) \hookrightarrow L^1(\Omega)$. If we choose $\epsilon \in (0, \theta_0\lambda_1)$, the latter tends to $+\infty$ as $\|u\| \to \infty$, hence φ is coercive in $W_0^{s,p}(\Omega)$. Moreover, recalling the definition of φ, the functional $u \mapsto \|u\|^p/p$ is convex, hence weakly lower semicontinuous in $W_0^{s,p}(\Omega)$, while $\widehat{J}$ is continuous in $L^p(\Omega)$, which, by the compact embedding $W_0^{s,p}(\Omega) \hookrightarrow L^p(\Omega)$ and the Eberlein-Smulyan theorem, implies that $\widehat{J}$ is sequentially weakly continuous in $W_0^{s,p}(\Omega)$. Hence, φ is sequentially weakly lower semicontinuous on $W_0^{s,p}(\Omega)$. Therefore, by the Weierstrass theorem, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that $\varphi(u_0) = \inf_{u\in W_0^{s,p}(\Omega)} \varphi(u)$. □

Remark 4.6. By Proposition 4.5 we observe that u_0 is a minimizer of $\widehat{\varphi}$, hence, by Lemma 2.1 *(vi)* $0 \in \partial\widehat{\varphi}(u_0)$, i.e. there exists $w \in N(u_0)$ such that $A(u_0) = w$ in $W^{-s,p'}(\Omega)$. By [27, Definition 2.4] u_0 is a weak solution of $(-\Delta)_p^s\, u \in \partial j(x, u)$ in Ω, $u = 0$ in Ω^c. Moreover, exploiting **(H)** *(i)* and *(iii)*, and arguing as in the proof of Proposition 4.8, we deduce that

$$|\xi| \le c_1|t|^{p-1} \text{ for some } c_1 > 0,$$

for a.a. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial j(x, t)$. Therefore, from [27, Lemma 2.5], we obtain that $u_0 \in L^\infty(\Omega)$, hence, $w \in L^\infty(\Omega)$. By [27, Lemma 2.7] there exist $\alpha \in (0, s]$, $C > 0$ such that $u_0 \in C^\alpha(\overline{\Omega})$ with $||u_0||_{C^\alpha(\overline{\Omega})} \le C(1 + ||u_0||)$. In particular, by [26, Theorem 1.1], if $p \ge 2$ then $u_0 \in C_\delta^\alpha(\overline{\Omega})$ and it holds the following estimate $||u_0||_{C_\delta^\alpha(\overline{\Omega})} \le C(1+||u_0||)$, where $C_\delta^\alpha(\overline{\Omega}) = \{u \in C^\alpha(\overline{\Omega}) : u/\delta^s \in C^\alpha(\overline{\Omega})\}$ with $\alpha \in (0, 1)$ and $\delta(x) = \mathrm{dist}(x, \Omega^c)$.

A fundamental result for the sequel is a generalization of Amann's theorem to operators which are the sum of a Clarke generalized subdifferential and a subdifferential in the sense of convex analysis, that allow us to know the degree in an isolated local minimum (see [1, Theorem 8]). In order to do this, it is better clarifying some important facts. First of all, we observe that A is the Fréchet derivative of $u \mapsto \frac{||u||^p}{p}$, viewed as a functional on $W_0^{s,p}(\Omega)$, moreover we know by Lemma 3.2 that A is a bounded, $(S)_+$ operator. We set $J = \widehat{J}\,|_{W_0^{s,p}(\Omega)}$ and $\widehat{\varphi} = \frac{||u||^p}{p} - J$, then it makes sense to talk about the degree of $\partial\widehat{\varphi} = A - N$ with

$$N = \partial J = \partial\widehat{J}$$

(see [21, Proposition 1.3.17], for the last equality). Now we can state the extension of Amann's theorem for our problem.

Proposition 4.7. *Let $\widehat{\varphi} : W_0^{s,p}(\Omega) \to \mathbb{R}$ $\widehat{\varphi}(u) = \frac{||u||^p}{p} - J(u)$ be locally Lipschitz and $\psi \in \Gamma_0(W_0^{s,p}(\Omega))$, $\psi \ge 0$. If $u_0 \in W_0^{s,p}(\Omega)$ is an isolated minimizer of $\widehat{\varphi} + \psi$, then there exists $r > 0$ such that*

$$\deg(\partial\widehat{\varphi} + \partial\psi, B_r(u_0), 0) = 1.$$

Now, exploiting the hypothesis **(H)** *(iii)*, we prove that for small balls the degree map of $\partial\widehat{\varphi} + \partial\psi$ is equal to -1.

Proposition 4.8. *Let **(H)** hold. Then there exists $\rho_0 > 0$ such that for all $0 < \rho \le \rho_0$, we obtain*

$$\deg(\partial\widehat{\varphi} + \partial\psi, B_\rho(0), 0) = -1.$$

Proof. Let $m \in L^\infty(\Omega)_+$ be such that $\eta(x) \le m(x) \le \widehat{\eta}(x)$ a.e. on Ω. Let look at the homotopy $h : [0,1] \times W_0^{s,p}(\Omega) \to 2^{W^{-s,p'}(\Omega)} \setminus \{\varnothing\}$ defined by

$$h(t,u) = A(u) - tN(u) - (1-t)K_m(u) + t\partial\psi(u).$$

From Proposition 4.3 and Lemma 3.2 (i)-(ii), we obtain that $h_1(t,u) = A(u) - (1-t)K_m(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a $(S_+)-$ homotopy, $h_2(t,u) = -tN(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a $(P)-$ homotopy and $h_3(t,u) = t\partial\psi(u)$ for $(t,u) \in [0,1] \times W_0^{s,p}(\Omega)$ is a pseudomonotone homotopy (see [1]), hence h is an admissible homotopy (see Section 2).

Claim: There exists $\rho_0 > 0$ such that for all $t \in [0,1]$, all $0 < \rho \le \rho_0$ and all $u \in \partial B_\rho(0) \subseteq W_0^{s,p}(\Omega)$ we get

$$0 \notin h(t,u).$$

By contradiction, we can find $(t_n)_n \subseteq [0,1]$ and $u_n \in C$, $n \ge 1$, such that

$$t_n \to t \text{ in } [0,1], \quad ||u_n|| \to 0$$

and

$$0 \in A(u_n) - t_nN(u_n) - (1-t_n)K_m(u_n) + t_n\partial\psi(u_n), \quad n \ge 1. \tag{4.6}$$

We set

$$v_n = \frac{u_n}{||u_n||}, \quad n \ge 1,$$

hence, passing to a suitable subsequence, we can deduce that

$$v_n \rightharpoonup v \text{ in } W_0^{s,p}(\Omega),\ v_n \to v \text{ in } L^p(\Omega) \text{ and } v_n(x) \to v(x) \text{ a.e. in } \Omega,$$

hence $v \ge 0$ a.e. in Ω.

From (4.6), we have that there exists $w_n \in N(u_n)$ such that

$$-A(u_n) + t_nw_n + (1-t_n)K_m(u_n) \in t_n\partial\psi(u_n),$$

therefore,

$$\langle A(u_n), \bar{v} - u_n\rangle - t_n\int_\Omega w_n(\bar{v} - u_n)\,dx - (1-t_n)\int_\Omega m|u_n|^{p-2}u_n(\bar{v} - u_n)\,dx \ge 0.$$

for all $\bar{v} \in W_0^{s,p}(\Omega)_+$. Dividing the last inequality with $||u_n||^p$, we have

$$\langle A(v_n), \widehat{v} - v_n\rangle - t_n\int_\Omega \frac{w_n}{||u_n||^{p-1}}(\widehat{v} - v_n)\,dx - (1-t_n)\int_\Omega m|v_n|^{p-2}v_n(\widehat{v} - v_n)\,dx \ge 0, \tag{4.7}$$

for all $\widehat{v} \in W_0^{s,p}(\Omega)_+$.

By **(H)** *(iii)*, there exists $\delta > 0$ such that for a.e. $x \in \Omega$, all t with $|t| < \delta$ and all $\xi \in \partial j(x,t)$, we obtain

$$|\xi| \le (\widehat{\eta}(x) + 1)|t|^{p-1}. \tag{4.8}$$

While, from **(H)** *(i)*, for a.e. $x \in \Omega$, and all $t \in \mathbb{R}$ with $|t| \ge \delta$ and all $\xi \in \partial j(x,t)$ we get

$$|\xi| \le a(x) + c|t|^{p-1} \le \left(\frac{a(x)}{\delta^{p-1}} + c\right)|t|^{p-1}. \tag{4.9}$$

The expressions (4.8) and (4.9) imply that for a.e. $x \in \Omega$, all $t \in \mathbb{R}$ and all $\xi \in \partial j(x,t)$, we obtain

$$|\xi| \le c_1|t|^{p-1} \text{ for some } c_1 > 0. \tag{4.10}$$

Therefore, from (4.10), we deduce that $\left(\frac{w_n}{||u_n||^{p-1}}\right)_n \subseteq L^{p'}(\Omega)$ is bounded and, passing to a subsequence, we can state that

$$\frac{w_n}{||u_n||^{p-1}} \rightharpoonup f_0 \text{ in } L^{p'}(\Omega).$$

For every $\epsilon > 0$ and $n \ge 1$, we define the set

$$C^+_{\epsilon,n} = \left\{x \in \Omega : u_n(x) > 0, \eta(x) - \epsilon \le \frac{w_n(x)}{(u_n(x))^{p-1}} \le \widehat{\eta}(x) + \epsilon\right\}.$$

Since $||u_n|| \to 0$, we may suppose (at least for a subsequence) that

$$u_n(x) \to 0 \text{ a.e. on } \Omega \text{ as } n \to \infty.$$

Hence, by **(H)** *(iii)*, we get

$$\chi_{C^+_{\epsilon,n}}(x) \to 1 \text{ a.e. on } \{v > 0\}.$$

We observe that

$$\left\| \left(1 - \chi_{C^+_{\epsilon,n}}\right) \frac{w_n(x)}{||u_n||^{p-1}} \right\|_{L^{p'}(\{v>0\})} \to 0,$$

then

$$\chi_{C^+_{\epsilon,n}} \frac{w_n(x)}{||u_n||^{p-1}} \rightharpoonup f_0 \text{ in } L^{p'}(\{v > 0\}).$$

Recalling the definition of the set $C^+_{\epsilon,n}$, we obtain

$$\chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{||u_n||^{p-1}} = \chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{(u_n(x))^{p-1}} (v_n(x))^{p-1},$$

therefore

$$\chi_{C^+_{\epsilon,n}}(x)(\eta(x) - \epsilon)(v_n(x))^{p-1} \le \chi_{C^+_{\epsilon,n}}(x) \frac{w_n(x)}{||u_n||^{p-1}} \le \chi_{C^+_{\epsilon,n}}(x)(\widehat{\eta}(x) + \epsilon)(v_n(x))^{p-1} \text{ a.e. on } \Omega.$$

Passing to weak limits in $L^{p'}(\{v > 0\})$ and applying Mazur's lemma, we have

$$(\eta(x) - \epsilon)(v(x))^{p-1} \le f_0(x) \le (\widehat{\eta}(x) + \epsilon)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

Since $\epsilon > 0$ is arbitrary, we let $\epsilon \to 0$ and get

$$\eta(x)(v(x))^{p-1} \le f_0(x) \le \widehat{\eta}(x)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}. \tag{4.11}$$

Further, from (4.10), we get that

$$f_0(x) = 0 \text{ a.e. on } \{v = 0\}. \tag{4.12}$$

Hence, the conditions (4.11) and (4.12) imply that

$$f_0(x) = g_0(x)|v(x)|^{p-2}v(x) \text{ a.e. on } \Omega, \tag{4.13}$$

with $g_0 \in L^\infty(\Omega)_+$ such that $\eta(x) \le g_0(x) \le \widehat{\eta}(x)$ a.e. on Ω. In addition, if we set $\widehat{v} = v$ in (4.7), then since

$$\int_\Omega \frac{w_n(x)}{||u_n||^{p-1}} (v_n(x) - v(x))\, dx \to 0$$

and

$$\int_\Omega m(x)|v_n(x)|^{p-2}v_n(x)(v(x) - v_n(x))\, dx \to 0,$$

from (4.7) we deduce

$$\limsup_{n\to\infty} \langle A(v_n), v_n - v\rangle \le 0,$$

then

$$v_n \to v \text{ in } W^{s,p}_0(\Omega)$$

(We are using the fact that A is a $(S)_+$-map). Hence, if n goes to ∞ in (4.7), we have

$$\langle A(v), \widehat{v} - v\rangle - t\int_\Omega g_0|v|^{p-2}v(\widehat{v} - v)\, dx - (1 - t)\int_\Omega m|v|^{p-2}v(\widehat{v} - v)\, dx \ge 0$$

for all $\widehat{v} \in W^{s,p}_0(\Omega)_+$. We set

$$\widehat{g}_t = tg_0 + (1 - t)m,$$

hence we can rephrase the last inequality as

$$\langle A(v), \widehat{v} - v\rangle - \int_\Omega \widehat{g}_t(x)(v(x))^{p-1}(\widehat{v}(x) - v(x))\, dx \ge 0 \text{ for all } \widehat{v} \in W^{s,p}_0(\Omega)_+. \tag{4.14}$$

Let $w \in W_0^{s,p}(\Omega)_+$ and set $\widehat{v} = v + w$, then we can rewrite (4.14) as

$$\langle A(v), w\rangle \geq \int_\Omega \widehat{g}_t(x)(v(x))^{p-1} w(x)\, dx \text{ for all } w \in W_0^{s,p}(\Omega)_+.$$

Hence, applying the strong maximum principle [25, Proposition 2.2], we obtain that $v > 0$ a.e. in Ω. Let $z \in W_0^{s,p}(\Omega)$, $\epsilon > 0$ and consider $(v+\epsilon z)^+ = v + \epsilon z + (v+\epsilon z)^-$. We take $\widehat{v} = (v+\epsilon z)^+ \in W_0^{s,p}(\Omega)_+$ in (4.14) and we get

$$\left\langle A(v) - K_{\widehat{g}_t}(v), (v+\epsilon z)^+ - v\right\rangle \geq 0,$$

hence

$$\langle A(v) - K_{\widehat{g}_t}(v), \epsilon z\rangle \geq -\left\langle A(v) - K_{\widehat{g}_t}(v), (v+\epsilon z)^-\right\rangle. \tag{4.15}$$

We observe that

$$-\left\langle A(v) - K_{\widehat{g}_t}(v), (v+\epsilon z)^-\right\rangle = -\left\langle A(v), (v+\epsilon z)^-\right\rangle + \int_\Omega \widehat{g}_t(x) v(x)(v(x)+\epsilon z(x))^-\, dx \tag{4.16}$$

and, since $\widehat{g}_t$, $v \geq 0$, we have that

$$\int_\Omega \widehat{g}_t v (v+\epsilon z)^-\, dx \geq 0. \tag{4.17}$$

Now, we want to study the sign of $-\langle A(v), (v+\epsilon z)^-\rangle$. In order to do this, we introduce the sets

$$\Omega_\epsilon^- = \{v + \epsilon z < 0\} \text{ and } Q_\epsilon = \{(x,y) \in \Omega \times \mathbb{R}^\epsilon : v(x) + \epsilon z(x) < 0 \leq v(y) + \epsilon z(y), v(x) > v(y)\}.$$

By applying definition of A (3.2), we have that

$$\begin{aligned}
&-\left\langle A(v), (v+\epsilon z)^-\right\rangle = -\int_{\mathbb{R}^{2N}} \frac{(v(x)-v(y))^{p-1}((v+\epsilon z)^-(x) - (v+\epsilon z)^-(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x) - v(y) - \epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \int_{\Omega_\epsilon^- \times (\Omega\setminus\Omega_\epsilon^-)} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy - \int_{(\Omega\setminus\Omega_\epsilon^-)\times\Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \int_{\Omega_\epsilon^- \times \Omega^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy - \int_{\Omega^c\times\Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy + \epsilon \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ \int_{\Omega_\epsilon^- \times (\Omega_\epsilon^-)^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy - \int_{(\Omega_\epsilon^-)^c\times\Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(v(y)+\epsilon z(y))}{|x-y|^{N+ps}}\, dxdy \\
&= \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{|v(x)-v(y)|^p}{|x-y|^{N+ps}}\, dxdy + \epsilon \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy \\
&+ 2\int_{\Omega_\epsilon^- \times (\Omega_\epsilon^-)^c} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&\geq \epsilon \int_{\Omega_\epsilon^- \times \Omega_\epsilon^-} \frac{(v(x)-v(y))^{p-1}(z(x)-z(y))}{|x-y|^{N+ps}}\, dxdy + 2\int_{Q_\epsilon} \frac{(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))}{|x-y|^{N+ps}}\, dxdy \\
&= o(1)\epsilon \text{ as } \epsilon \to 0^+.
\end{aligned}$$

In the last passage we use the fact that $|\Omega_\epsilon^-| \to 0$ as $\epsilon \to 0^+$ for the first integral, while for the second integral we note that for every $(x,y) \in Q_\epsilon$

$$0 < v(x) - v(y) < \epsilon(z(y) - z(x))$$

and

$$0 > v(x) + \epsilon z(x) \geq v(x) + \epsilon z(x) - (v(y) + \epsilon z(y)) = (v(x) - v(y)) + \epsilon(z(x) - z(y)) > \epsilon(z(x) - z(y)).$$

Then,

$$|(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))| \leq \epsilon^p |z(x)-z(y)|^p,$$

integrating,

$$\int_{Q_\epsilon} \frac{|(v(x)-v(y))^{p-1}(v(x)+\epsilon z(x))|}{|x-y|^{N+ps}}\,dxdy \leq \epsilon^p \int_{\mathbb{R}^{2N}} \frac{|(z(x)-z(y)|^p}{|x-y|^{N+ps}}\,dxdy = o(\epsilon).$$

Going back to (4.15), we have that

$$\epsilon\,\langle A(v)-K_{\widehat{g}_t}(v), z\rangle \geq o(1)\epsilon,$$

hence, taking the limit when $\epsilon \to 0$, we get

$$\langle A(v)-K_{\widehat{g}_t}(v), z\rangle \geq 0.$$

Since $z \in W_0^{s,p}(\Omega)$ is arbitrary, it follows that $A(v)-K_{\widehat{g}_t}(v)=0$, hence

$$A(v)=K_{\widehat{g}_t}(v),$$

therefore

$$\begin{cases} (-\Delta)_p^s\, v(x) = \widehat{g}_t(x)|v(x)|^{p-2}v(x) & \text{in } \Omega \\ v(x)=0 & \text{on } \Omega^c. \end{cases} \tag{4.18}$$

Since $||v||=1$, we deduce that $v \neq 0$ and hence v is an eigenfunction of the weighted eigenvalue problem (4.18), with weight $\widehat{g}_t \in L^\infty(\Omega)_+$. Exploiting these facts

$$\widehat{g}_t(x) \geq \eta(x) \text{ a.e. on } \Omega$$

and

$$\lambda_1(\widehat{g}_t) \leq \lambda_1(\eta) < \lambda_1(\lambda_1) = 1,$$

we discover that v cannot be the principal eigenfunction of the weighted eigenvalue problem with weight $\widehat{g}_t \in L^\infty(\Omega)_+$, hence, v must be nodal, but $v \in C$, a contradiction. Therefore, the claim is true.

Applying the homotopy invariance property of the degree map, we deduce that

$$\deg(A-N+\partial\psi, B_\rho(0), 0) = \deg_{S_+}(A-K_m, B_\rho(0), 0)$$

for all $0<\rho\leq\rho_0$.

But from Proposition 3.5, we know that

$$\deg_{(S)_+}(A-mK, B_\rho(0), 0) = -1.$$

Therefore, we get

$$\deg(A-N+\partial\psi, B_\rho(0), 0) = -1$$

for all $0<\rho\leq\rho_0$. □

Analogously, we show a corresponding result for big balls.

Proposition 4.9. *Let **(H)** hold. Therefore there exists $R_0>0$ such that for all $R\geq R_0$, we obtain*

$$\deg(A-N+\partial\psi, B_R(0), 0) = 1.$$

Proof. We take into account the homotopy

$$h(t,u) = A(u) - tN(u) + t\partial\psi(u) \text{ for } (t,u) \in [0,1]\times W_0^{s,p}(\Omega).$$

From Proposition 4.3 and Lemma 2.5, we have that $\widehat{h}(t,u) = -tN(u)$ for $(t,u) \in [0,1]\times W_0^{s,p}(\Omega)$ is a $(P)-$homotopy and $\tilde{h}(t,u) = A(u)+t\partial\psi(u)$ for $(t,u)\in[0,1]\times W_0^{s,p}(\Omega)$ is a pseudomonotone homotopy, hence $h(t,u)$ is an admissible homotopy.

Claim: There exists $R_0 \geq 0$ such that for all $t\in[0,1]$, all $R\geq R_0$ and all $u\in\partial B_R(0)$, we have

$$0 \notin h(t,u).$$

By contradiction, we can find $(t_n)_n \subseteq [0,1]$ and $u_n \in W_0^{s,p}(\Omega)_+$, $n\geq 1$, such that

$$t_n \to t \text{ in } [0,1], ||u_n|| \to \infty \text{ and } 0 \in h(t_n,u_n), \quad n\geq 1.$$

Hence, there exists $w_n \in N(u_n)$ such that

$$-A(u_n) + t_n w_n \in t_n \partial\psi(u_n), \quad \forall n \geq 1,$$

then

$$\langle A(u_n), \bar{v} - u_n\rangle - t_n \int_\Omega w_n(x)(\bar{v}(x) - u_n(x))\, dx \geq 0 \text{ for all } \bar{v} \in W_0^{s,p}(\Omega)_+. \tag{4.19}$$

Set $v_n = \frac{u_n}{||u_n||}$, $n \geq 1$ and, passing to a subsequence, we can suppose that

$$v_n \rightharpoonup v \text{ in } W_0^{s,p}(\Omega),\ v_n \to v \text{ in } L^p(\Omega) \text{ and } v_n(x) \to v(x) \text{ a.e. in } \Omega,$$

hence $v \geq 0$ a.e. in Ω. Dividing (4.19) by $||u_n||^p$, we have

$$\langle A(v_n), \widehat{v} - v_n\rangle - t_n \int_\Omega \frac{w_n(x)}{||u_n||^{p-1}}(\widehat{v}(x) - v_n(x))\, dx \geq 0 \tag{4.20}$$

for all $\widehat{v} \in W_0^{s,p}(\Omega)_+$. Using (4.10), we obtain that $\left(\frac{w_n}{||u_n||^{p-1}}\right)_n \subseteq L^{p'}(\Omega)$ is bounded, hence, we can suppose that

$$\frac{w_n}{||u_n||^{p-1}} \rightharpoonup f_\infty \text{ in } L^{p'}(\Omega), \text{ as } n \to \infty.$$

For every $\epsilon > 0$ and $n \geq 1$, we define the set

$$D^+_{\epsilon,n} = \left\{x \in \Omega : u_n(x) > 0, -\epsilon \leq \frac{w_n(x)}{(u_n(x))^{p-1}} \leq \theta(x) + \epsilon\right\}.$$

From **(H)** *(ii)*, we get

$$\chi_{D^+_{\epsilon,n}}(x) \to 1 \text{ a.e. on } \{v > 0\}.$$

We observe that

$$\left\|\left(1 - \chi_{D^+_{\epsilon,n}}(x)\right)\frac{w_n}{||u_n||^{p-1}}\right\|_{L^{p'}(\{v>0\})} \to 0,$$

therefore,

$$\chi_{D^+_{\epsilon,n}}(x)\frac{w_n}{||u_n||^{p-1}} \rightharpoonup f_\infty \text{ in } L^{p'}(\{v > 0\}).$$

By the definition of $D^+_{\epsilon,n}$, we get that

$$\begin{aligned}\chi_{D^+_{\epsilon,n}}(x)(-\epsilon)(v_n(x))^{p-1} \leq \chi_{D^+_{\epsilon,n}}(x)\frac{w_n(x)}{||u_n||^{p-1}} &= \chi_{D^+_{\epsilon,n}}(x)\frac{w_n(x)}{(u_n(x))^{p-1}}(v_n(x))^{p-1}\\ &\leq \chi_{D^+_{\epsilon,n}}(x)(\theta(x) + \epsilon)(v_n(x))^{p-1} \text{ a.e. on } \Omega.\end{aligned}$$

Passing to weak limits in $L^{p'}(\{v > 0\})$ and applying Mazur's lemma, we have

$$-\epsilon(v(x))^{p-1} \leq f_\infty(x) \leq (\theta(x) + \epsilon)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

Let $\epsilon \to 0$, we obtain

$$0 \leq f_\infty(x) \leq \theta(x)(v(x))^{p-1} \text{ a.e. on } \{v > 0\}.$$

While, by (4.10), we obtain that

$$f_\infty(x) = 0 \text{ a.e. on } \{v = 0\}.$$

Since $\Omega = \{v > 0\} \cup \{v = 0\}$ (recalling that $v \in W_0^{s,p}(\Omega)_+$), we get

$$0 \leq f_\infty(x) \leq \theta(x)(v(x))^{p-1} \text{ a.e. on } \Omega,$$

hence

$$f_\infty = g_\infty v^{p-1} \text{ with } g_\infty \in L^\infty(\Omega)_+,\ g_\infty(x) \leq \theta(x) \text{ a.e. on } \Omega.$$

Since $v \in W_0^{s,p}(\Omega)_+$, then in (4.20) we can set $\widehat{v} = v$ to obtain

$$\langle A(v_n), v_n - v\rangle \leq t_n \int_\Omega \frac{w_n(x)}{||u_n||^{p-1}}(v_n(x) - v(x))\, dx,$$

therefore

$$\limsup_{n\to\infty} \langle A(v_n), v_n - v\rangle \le 0,$$

and since A is of type $(S)_+$,

$$v_n \to v \text{ in } W_0^{s,p}(\Omega).$$

If n goes to ∞ in (4.20), we get

$$\langle A(v), \widehat{v} - v\rangle \ge t\int_\Omega g_\infty(x)(v(x))^{p-1}(\widehat{v}(x) - v(x))\,dx, \quad \forall \widehat{v} \in W_0^{s,p}(\Omega)_+. \tag{4.21}$$

Set $\widehat{g}_t = tg_\infty(x)$. Using the test function $\widehat{v} = (v + \epsilon z)^+$ for any $z \in W_0^{s,p}(\Omega)$ and $\epsilon > 0$, then, as in the proof of Theorem 4.8, we have

$$\langle A(v) - K_{\widehat{g}_t}(v), z\rangle \ge 0,$$

by the arbitrariety of z, it follows that

$$A(v) = K_{\widehat{g}_t}(v),$$

therefore

$$\begin{cases} (-\Delta)_p^s\, v = tg_\infty(x)|v|^{p-2}v & \text{in } \Omega, \\ v = 0 & \text{on } \Omega^c. \end{cases} \tag{4.22}$$

Since $||v|| = 1$, we deduce that $v \ne 0$ and hence v is an eigenfunction of the weighted eigenvalue problem (4.22), with weight $tg_\infty \in L^\infty(\Omega)_+$. Recalling the strict monotonicity on the weight of the principal eigenvalue and since

$$0 \le tg_\infty \le g_\infty \le \theta,$$

we obtain

$$\lambda_1(tg_\infty) \ge \lambda_1(g_\infty) \ge \lambda_1(\theta) > \lambda_1(\lambda_1) = 1.$$

Then from (4.22) we deduce that $v = 0$, a contradiction.
Therefore, from the homotopy invariance of the degree map, we obtain that

$$\deg(A - N + \partial\psi, B_R(0), 0) = \deg_{S_+}(A, B_R(0), 0) \text{for all } R \ge R_0. \tag{4.23}$$

We take the $(S)_+-$homotopy (see [33, Proposition 4.41])

$$h_1(t,u) = tA(u) + (1-t)\mathcal{F}(u) \text{ for all } (t,u) \in [0,1] \times W_0^{s,p}(\Omega).$$

We have that $\langle h_1(t,u), u\rangle \ne 0$ for all $u \ne 0$ and hence, by the homotopy invariance of $\deg_{(S)_+}$, we have

$$\deg_{(S)_+}(A, B_R(0), 0) = \deg_{(S)_+}(\mathcal{F}, B_R(0), 0) = 1. \tag{4.24}$$

(The last passage follows from the normalization property). From (4.23) and (4.24), we can state that

$$\deg(A - N + \partial\psi, B_R(0), 0) = 1$$

for all $R \ge R_0$. □

Finally, we can formulate our multiplicity result for problem (4.1).

Theorem 4.10. *Let **(H)** hold. Therefore the problem (4.1) admits at least two nontrivial solutions $u_0, \widehat{u} \in W_0^{s,p}(\Omega)$.*

Proof. By Proposition 4.5, there exists $u_0 \in W_0^{s,p}(\Omega)$ such that

$$\varphi(u_0) = \inf_{u \in W_0^{s,p}(\Omega)} \varphi(u). \tag{4.25}$$

Since u_0 is a minimizer, by applying Proposition 4.7, there exists $r > 0$ such that

$$\deg(A - N + \partial\psi, B_r(u_0), 0) = 1. \tag{4.26}$$

Therefore, (4.26) and Proposition 4.8 imply $u_0 \ne 0$. We choose $\rho_0 > 0$ small such that

$$B_r(u_0) \cap B_{\rho_0}(0) = \varnothing$$

and $R_0 > 0$ large such that

$$B_{\rho_0}(0), B_r(u_0) \subseteq B_{R_0}(0).$$

Exploiting the additivity of the domain property of the degree map and applying Proposition 4.7, Proposition 4.8 and Proposition 4.9, we get

$$\begin{aligned}\deg(A - N + \partial\psi, B_{R_0}(0), 0) &= \deg(A - N + \partial\psi, B_r(u_0), 0) + \deg(A - N + \partial\psi, B_{\rho_0}(0), 0)\\ &\quad + \deg(A - N + \partial\psi, B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0)), 0),\end{aligned}$$

therefore

$$1 = \deg(A - N + \partial\psi, B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0)), 0).$$

Hence, by the existence property of the degree map we deduce that there exists

$$\widehat{u} \in B_{R_0}(0) \setminus (B_r(u_0) \cup B_{\rho_0}(0))$$

hence $\widehat{u} \neq u_0$, $\widehat{u} \neq 0$, such that

$$0 \in A(\widehat{u}) - N(\widehat{u}) + \partial\psi(\widehat{u}) = \partial\widehat{\varphi}(\widehat{u}) + \partial\psi(\widehat{u}),$$

namely, there exists $w \in N(\widehat{u})$ such that

$$-A(\widehat{u}) + w \in \partial\psi(\widehat{u}).$$

From the latter we deduce

$$\langle A(\widehat{u}), v - \widehat{u}\rangle - \int_\Omega w(x)(v(x) - \widehat{u}(x))\, dx \geq 0 \text{ for all } v \in W_0^{s,p}(\Omega)_+,$$

hence $\widehat{u} \in W_0^{s,p}(\Omega)$ is a nontrivial solution of (4.1).
Now we have to show that u_0 is a critical point of φ and it is a second nontrivial solution of (4.1). By (4.25), for all $\lambda > 0$ and all $v \in W_0^{s,p}(\Omega)$ one has

$$0 \leq \varphi(u_0 + \lambda v) - \varphi(u_0) = \widehat{\varphi}(u_0 + \lambda v) - \widehat{\varphi}(u_0) + \psi(u_0 + \lambda v) - \psi(u_0)$$

hence

$$\begin{aligned}0 &\leq \frac{1}{\lambda}(\widehat{\varphi}(u_0 + \lambda v) - \widehat{\varphi}(u_0)) + \frac{1}{\lambda}(\psi(u_0 + \lambda v) - \psi(u_0))\\ &\leq \frac{1}{\lambda}[\widehat{\varphi}(u_0 + \lambda v) - \widehat{\varphi}(u_0)] + (\psi(u_0 + v) - \psi(u_0))\end{aligned}$$

(since ψ is convex). When λ goes to 0, we get

$$0 \leq \widehat{\varphi}^0(u_0; v) + \psi(u_0 + v) - \psi(u_0). \tag{4.27}$$

Let $z \in W_0^{s,p}(\Omega)$, we set $v = z - u_0$ in (4.27) and we obtain

$$0 \leq \widehat{\varphi}^0(u_0; z - u_0) + \psi(z) - \psi(u_0). \tag{4.28}$$

Therefore, by Definition 2.2, $u_0 \in W_0^{s,p}(\Omega)$ is a critical point of $\varphi = \widehat{\varphi} + \psi$, hence, by Proposition 2.3

$$0 \in \partial\widehat{\varphi}(u_0) + \partial\psi(u_0).$$

Therefore we can deduce that there exists $w \in N(u_0)$ such that

$$-A(u_0) + w \in \partial\psi(u_0),$$

hence

$$\langle A(u_0), v - u_0\rangle - \int_\Omega w(x)(v(x) - u_0(x))\, dz \geq 0 \text{ for all } v \in W_0^{s,p}(\Omega)_+.$$

Consequently $u_0 \in W_0^{s,p}(\Omega)$ is a second nontrivial solution of (4.1). □

Remark 4.11. In the linear case ($p = 2$), a solution $\widehat{u}$ of problem (1.2) belongs to $C(\overline{\Omega})$, under the additional assumptions that Ω satisfies the exterior ball condition and $w \in N(\widehat{u})$ such that $w \in L^2(\Omega)$ with $N < 4s$ (see [4, Proposition 2.12]). Regularity results of solutions of (1.2) can be obtained by strengthening the assumptions of w, moreover, in the case of a general obstacle it is necessary that such obstacle has some regularity properties (see [4, Proposition 2.12] and the references therein).

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