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# Shilnikov chaos, low interest rates, and New Keynesian macroeconomics ☆

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ABSTRACT

The paper shows that in a New Keynesian (NK) model, an active interest rate feedback monetary policy, when combined with a Ricardian passive fiscal policy, à la Leeper-Woodford, may induce the onset of a Shilnikov chaotic attractor in the region of the parameter space where uniqueness of the equilibrium prevails locally. Implications, ranging from long-term unpredictability to global indeterminacy, are discussed in the paper. We find that throughout the attractor, the economy lingers in particular regions, within which the emerging aperiodic dynamics tend to evolve for a long time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, produced by the existence of a chaotic attractor, and not by the influence of an unintended steady state or the Central Bank's intentional choice of a steady state nominal interest rate at its lower bound. In addition, our finding of Shilnikov chaos can provide an alternative explanation for the controversial "loanable funds" over-saving theory, which seeks to explain why interest rates and, to a lesser extent, inflation rates have declined to current low levels, such that the real rate of interest may be below the marginal product of capital. Paradoxically, an active interest rate feedback policy can cause nominal interest rates, inflation rates, and real interest rates unintentionally to drift downwards within a Shilnikov attractor set. Our results are robust to whether money is in the production function, in the utility function, or not in the model at all. But our results do depend upon the existence of sticky prices.



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## 1. Introduction

A long literature exists on the search for policy-relevant chaos in economics. The earliest literature used tests developed by physicists for detecting chaos in data produced by controlled experiments with thousands of replications, thereby providing very large sample sizes. Those tests focused primarily on measuring the Hausdorf dimension of the attractor set and testing for positive dominant Liapunov exponent. Using those tests, Barnett and Chen (1988a, 1988b) found chaos in monetary aggregate data. Many relevant published papers followed. But since the data were not produced from a controlled experiment and the tests did not condition on an economic model, the tests had no way of attributing the source of the chaos to the nonlinear dynamics of the economy. The source of the chaos, for example, could be from the weather or the climate impacting the economy. As a result, the policy relevance of chaos has remained limited in economics.

The available sample size is the root of the difference between fields where chaos has made significant progress, such as the physical sciences, and fields where it has not, such as economics. The dependence on large samples is based upon the ability numerically to produce the geometry of the fractal attractor set that characterizes chaos. Demonstrating the existence of chaos and its fractal attractor set is not equivalent to the ability to map out the geometry of the fractal attractor set. The geometry of that attractor set, on the other hand, is what creates the potential policy relevance of chaos. The inability to map out the geometry of the fractal attractor set with the moderate sample size typically available to economists has hampered research in economic chaos.

Economists working in chaos theory next transitioned to exploring the possibility of formal statistical testing of the null hypothesis of chaos within dynamical macroeconomic models. This approach, however, was found to be prohibitively difficult. With more than three parameters, analytical solution for the boundaries of the chaotic subset of the parameter space was not available. Economists also considered exploration of the theoretical properties of macroeconomic models that can produce chaos. In a famous paper, Grandmont (1985) found that a classical model's parameter space is stratified into an infinite number of subsets separated by period-doubling bifurcation boundaries. Based upon the subset within which the parameters are located, the solution of the model could be monotonically stable, damped stable, periodic unstable, multiperiodic, or – after a converged infinite number of bifurcations – chaotic. But since the classical model contains no market imperfections, all solutions are Pareto optimal. Hence no clear reason exists for governmental intervention. If the parameters are in the chaotic region, the chaos is Pareto optimal, and governmental attempts to control the chaos could produce a Pareto loss, harming welfare. In addition, it has been speculated that the parameter settings leading to chaos in a classical model may not be plausible (see, e.g., Blanchard and Fischer, 1989, p. 261).

The focus of the research therefore turned to exploring the theoretical properties of NK models, within which governmental intervention can be justified. Findings of chaos in a NK model at plausible settings of parameters, when policy is based on active interest rate feedback rules, such as the Taylor Rule, have been reported by Benhabib et al. (2002). Note that the policy relevance of chaos is determined not only by the existence of chaos and its fractal attractor set at plausible parameter values, but also by the geometry of that attractor set. Because findings of "general" chaos cannot determine the geometry of the fractal attractor set without sample sizes far exceeding those available in macroeconomics, the policy implications of chaos found in NK literature have remained unclear.

Fortunately, in other fields a few important special classes of chaotic dynamics have been found to produce fractal attractor sets that characterize the majority of the chaotic attractor sets produced by nature. If found, the known geometry of the fractal attractor set characterizing the identified class of chaos is immediately available, allowing information to be extracted from the fractal attractor set without requiring the use of large samples. A particularly productive class of chaotic dynamics is Shilnikov chaos. We chose to investigate Shilnikov chaos for multiple reasons. One reason is that it can be detected directly from the Shilnikov criterion, allowing the properties of the resulting fractal attractor set to be determined. Moreover, as explained by Champneys (2010), "Over the years, Shilnikov's mechanism of chaos has proven to be one of the most robust and frequently occurring mechanisms chosen by nature."

As pointed out by Afraimovich et al. (2014, p. 19):

"Only starting from mid 70–80 s, when researchers became interested in computer studies of chaotic behavior in nonlinear models, it became clear that the Shilnikov saddle-focus is a pivotal element of chaotic dynamics in a broad range of real-world applications. In general, the number of various models from hydrodynamics, optics, chemical kinetics, biology etc., which demonstrated the numerically or experimentally strange attractors with the characteristic spiral structure suggesting the occurrence of a saddle focus homoclinic loop, was overwhelming. Indeed, this scenario has turned out to be typical for a variety of systems and models of very diverse origins."

Surprisingly, this highly productive approach, which has swept through the physical sciences and mathematics, has largely gone unnoticed by economists, who have been discouraged by the sample size issues in earlier research on economic chaos. Bella et al. (2017) recently discovered the relevance of this general observation to economics in a classical economic growth model. However, as previously discussed, the existence of such chaos in classical models with no rigidity has limited policy relevance.

We find potentially high relevance of Shilnikov chaos to current problems in the world's macroeconomies, when active Taylor rule monetary feedback policy is adjoined to a NK dynamic macroeconomic model. As stated by Christiano and Takahashi (2018), "Monetary models are notorious for having multiple equilibria. The standard NK model, which assumes that fiscal policy is passive and monetary policy is set by a Taylor rule, is no exception." In fact, a large literature exists on complicated dynamics produced by NK models with Taylor rule interest rate feedback policies. Our research introduces new problems that we believe are potentially highly relevant to policy challenges in recent years. Although this is not the first paper to find chaos in NK models, the policy implications of previous findings have been unclear. In the absence of access to the geometry of the fractal attractor set, findings of "general" chaos in the literature thus far imply only knowledge that deterministically intrinsic stochasticity exists, although with unknown properties. The finding of Shilnikov chaos in this paper allows us to determine the properties of the fractal attractor and thereby the system's intrinsic stochasticity.

The qualitative "dimensions" of the chaotic attractor are of great interest and highly relevant to recent policy. The relative frequency with which an orbit visits different regions of the attractor is heterogeneous. Throughout the attractor, the economy lingers on regions with higher densities. This is precisely what occurs in the numerical simulations presented in this paper. If the initial conditions of the jump variables are chosen far enough away from the targeted steady state, the emerging aperiodic dynamics evolve over time around lower-than-targeted inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon which, in our case, will depend on the presence of a chaotic attractor and not on the influence of an unintended steady state as found by Benhabib et al. (2001a, 2001b). The low inflation rate and low interest rate phenomenon arising in our research, as a consequence of density heterogeneity in the Shilnikov chaotic attractor, is disconnected from the liquidity trap that can emerge because of the influence of an unintended steady state, as in Benhabib et al. (2001a, 2001b). In fact, the two types of liquidity trap may even co-exist for a while, depending on the initial conditions of the economy.

The downward drift in interest rates in recent decades towards the lower bound has been very puzzling. The Taylor rule is a short-run countercyclical policy with no intended long-run consequences; it is not designed to produce any long-run interest rate drift. Furthermore, firms have continued to believe that the marginal product of capital remains high, implying that capital investment is profitable. This observation, reflecting actual behavior of firms, is difficult to reconcile with the controversial "loanable funds" oversaving theory, which seeks to explain why interest rates and, to a lesser extent, inflation rates have fallen to current low levels, such that the real rate of interest may be lower than the marginal product of capital. We believe that the downward trend was unintentionally caused by a change in the economic system's dynamics in the presence of Shilnikov chaos.

We now present the plan of the paper. Section 2 presents the model. While the implied three-dimensional system of first-order differential equations is derived in Section 2.1, the stability results for the intended steady state, when the monetary policy is active, is discussed in Section 2.2. Section 3 presents the analysis of Shilnikov chaos in the NK model, beginning with a discussion of the Shilnikov Theorem in the first Section 3.1, and demonstrating that the three-dimensional dynamics that characterize the model's solution can satisfy the requirements of the Shilnikov (1965) theorem under plausible NK model calibration settings in the following Section 3.3. This is established numerically because the system is highly nonlinear and heavily parameterized, making a general characterization of the parametric region that supports Shilnikov chaos nearly impossible. The Sections 3.4 and 3.5, which discusses the properties of the Shilnikov chaotic attractor and its economic implications, respectively, establishes the policy relevance of our findings of Shilnikov chaos in a NK macroeconomy. Section 4 concludes.

## 2. Formulation and stability properties of a cashless NK model

We base our settings on the NK model studied by Benhabib et al. (2001a, 2001b), still a standard reference point for ongoing theoretical research and policy design (cf., *inter al.*, Benhabib et al., 2014; Le Riche et al., 2017). Furthermore, Benhabib et al. (2001a, 2001b) is a three-equation continuous-time system that is suitable for our analysis. We shall start with the cashless economy and then address the addition of money, both in the utility function and in the production function, later in the paper.

### *2.1. The formulation of the model*

Household-firm $i$ faces the following optimization problem (**Decision P**, henceforth):

$$\underset{c_i,l_i}{Max} \int_0^{\infty} \left[u(c_i) - f(l_i) - \frac{\eta}{2}(\pi_i - \pi^*)^2\right] e^{-\rho t} dt$$

*subject to*

$$\dot{a}_i = (R - \pi_i)a_i + \frac{p_i}{p} y(l_i) - c_i - \tau$$

$$\dot{p}_i = \pi_i p_i$$

$$a_i(0) = a_{i0}$$
$$p_i(0) = p_{i0}.$$

The objective of the household-firm is to maximize the discounted sum of a net utility stream, where $u(c_i)$ measures utility derived from consumption $(c_i)$, under the time discount rate, $\rho$. It is assumed that $u_c(c_i) > 0$ and $u_{cc}(c_i) < 0$. The function $f(l_i)$ measures the disutility of labor, where $f(l_i)$ is twice continuously differentiable, with $f_l > 0$ and $f_{ll} < 0$. The term $\frac{\eta}{2}(\pi_i - \pi^*)^2$ is standard to account for deviations of the price change, $\pi_i = \frac{\dot{p}_i}{p_i}$, with regard to the intended inflation rate $\pi^*$, where $p_i$ is the price charged by individual $i$ on the good it produces, and where the parameter $\eta$ measures the degree to which household-firms dislike to deviate in their price-setting behavior from the intended rate of inflation, $\pi^*$. Prices are sticky in the sense of Rotemberg (1982).

In the household-firm budget constraint, $a_i$ denotes real financial wealth, consisting of interest-bearing government bonds, where $R$ is the nominal interest rate and $y(l_i)$ is the amount of perishable goods, produced according to a production function using labor, $l_i$, as the only input. Real lump-sum taxes are denoted by $\tau$. Therefore, the instantaneous budget constraint says that the change in the firm-household real wealth equals real interest earnings on wealth, plus disposable income minus consumption expenditure.

Before applying the Maximum Principle, it is important to recall that sales of good $i$ are demand determined, so that $y(l_i) = (\frac{p_i}{p})^{-\phi} y^d$, where $\phi > 1$ is the elasticity of substitution across varieties, and $p$ is the aggregate price level. Hence, the discounted Hamiltonian can be set as:

$$H = u(c_i) - f(l(p_i)) - \frac{\eta}{2}(\pi_i - \pi^*)^2 + \mu_1 \left[ (R - \pi_i)a_i + \frac{p_i}{p}\left(\frac{p_i}{p}\right)^{-\phi} y^d - c_i - \tau \right] + \mu_2 \pi_i p_i \tag{1}$$

where $\mu_1$ and $\mu_2$ are the costate variables; $c_i$ and $\pi_i$ are control variables; and $p_i$ and $a_i$ are the state variables.

The necessary first order conditions are:

$$\frac{\partial H}{\partial c_i} = u_c(c_i) - \mu_1 = 0 \tag{2.a}$$

$$\frac{\partial H}{\partial \pi_i} = -\mu_1 a_i + \mu_2 p_i - \eta(\pi_i - \pi^*) = 0 \tag{2.b}$$

$$\dot{\mu}_1 = \rho\mu_1 - \frac{\partial H}{\partial a_i} = \rho\mu_1 - (R - \pi_i)\mu_1 \tag{2.c}$$

$$\dot{\mu}_2 = \rho\mu_2 - \frac{\partial H}{\partial p_i} = \rho\mu_2 + f'(l(p_i))l'(p_i) - (1 - \phi)\frac{y_i l(p_i)}{p}\mu_1 - \mu_2\pi_i. \tag{2.d}$$

Since $u_{cc}(c_i) < 0$ the Hamiltonian satisfies the second-order Arrow sufficient condition.

Consider now a symmetric equilibrium in which all household-firm units' behaviors are based on the same equations. Then, since the equilibrium in the goods market requires that $c = y(l)$, we are able to transform the first order conditions into the following three-dimensional system of differential equations (system $M$ henceforth**):**

$$\dot{\mu}_1 = (\rho - R + \pi)\mu_1$$
$$\eta\dot{\pi} = \rho(\pi - \pi^*)\eta - c(\mu_1)\left[(1 - \phi)\mu_1 + \phi c(\mu_1)^{\psi}\right]$$
$$\dot{a} = (R - \pi)a - \tau,$$

where the subscripts are dropped to simplify notation. In system $M$, the first equation denotes the time evolution of the Lagrange multiplier associated with the budget constraint (or shadow price of the real value of aggregate per capita government liabilities, real balances, and bonds) at instant of time $t$. The second equation is the NK Phillips Curve. The third equation is the budget constraint.

Solutions of system $M$ are admissible paths if the Transversality Condition (TVC),

$$0 = \lim_{t\to\infty} e^{-\int_0^t [R(s) - \pi(s)]ds} a(t) \tag{3}$$

is also satisfied.[1]

We now turn our attention to the behavior of the public authorities. Following Benhabib et al. (2001a, 2001b), we assume the following.

**Assumption 1.** (Zero lower bound on nominal rates and Taylor principle). Monetary authorities set $R$ as an increasing function of $\pi$, $R = R(\pi) > 0$. It is further assumed that $R'(\pi) > 0$ and $R''(\pi) > 0$. Moreover, at $\pi^*$, the steady-state Fisher equation is satisfied, $R(\pi^*) = \bar{R}$.

We now need the following definition.

**Definition 1.** Monetary policy is said to be active if $R'(\pi) > 1$ and passive otherwise.

Let us now turn our attention to fiscal policy. We assume that taxes are tuned according to fluctuations in total real government liabilities, $a$, so that $\tau = \tau(a)$. Similarly, for monetary policy, it is further assumed that there exists a tax rate corresponding to the steady-state level of real government liabilities,$\tau(a^*) = \bar{\tau}$. Let us consider the responses of $a$ to its own variations, $\frac{\partial \dot{a}}{\partial a} = R(\pi) - \pi - \tau'(a)$. Then, the dynamic path of total government liabilities is locally stable or unstable, according

[1] Notice that the TVC consists of a borrowing limit, preventing households from engaging in Ponzi games.

to the magnitude of the marginal tax rate, $\tau'(a)$. Therefore, as in Leeper (1991), Woodford (2003), and Kumhof et al. (2010), we have the following definition.

**Definition 2.** Fiscal policy is passive, if $\tau'(a) > R(\pi) - \pi$, and active otherwise.

Notice that adopting a passive fiscal policy is tantamount to assuming that the public authorities commit to fiscal solvency under all circumstances.

### *2.2. Steady states and local stability properties*

The long-run properties of system $M$ can be easily established. If Assumption 1 holds, our cashless economy presents two steady states, one where inflation is at the intended level, $\pi^*$, and one where inflation is higher or lower, $\hat{\pi}$, according to whether monetary policy is passive or active. In the latter case, the unintended steady-state may be labeled as a liquidity trap, in which the interest rate is zero, or near-zero, and inflation is below the target level, possibly negative.

For notational convenience, let us define $\mathrm{P}^* \equiv (\mu_1^*, \pi^*, a^*)$ and $\hat{\mathrm{P}} \equiv (\hat{\mu}_1, \hat{\pi}, \hat{a})$, the two vectors of triplets of $\mu_1$, $\pi$ and $a$ such that $\dot{\mu}_1 = \dot{\pi} = \dot{a} = 0$. Simple algebra shows that:

$$\mathrm{P}^* = \left( \frac{\phi}{\phi - 1} c(\mu_1^*)^{\psi}, \bar{R} - \rho, \frac{\bar{\tau}}{\rho} \right) \tag{4.1}$$

$$\hat{\mathrm{P}} = \left( \frac{\phi}{\phi - 1} c(\hat{\mu}_1)^{\psi} - \frac{\rho\left(\hat{\pi} - \pi^*\right)}{c(\hat{\mu}_1)(\phi - 1)}, R(\hat{\pi}) - \rho, \frac{\hat{\tau}}{\rho} \right). \tag{4.2}$$

Notice that, as in the economy with money (cf. Benhabib et al., 2001a, 2001b)), since $c(\mu_1^*)$ is positive and decreasing, the steady state value of the co-state variable in $\mathrm{P}^*$ exists and is unique. Conversely, the expression for $\hat{\mu}_1$ in $\hat{\mathrm{P}}$ may not be monotone when monetary policy is active and $(\hat{\pi} - \pi^*) < 0$. Therefore, multiple values of $\hat{\mu}_1$ may exist.

We now focus on the dynamics around the intended steady state. We can prove the following.

**Proposition 1.** (Local stability properties of the intended steady state). Consider first the case of an active monetary policy. Then, if fiscal policy is also active, there are no equilibrium paths converging to the steady state. The equilibrium is conversely locally unique, if fiscal policy is passive. Consider now the case of a passive monetary policy. Then, if fiscal policy is active, there is a continuum of equilibria that converge to the steady state. If fiscal policy is passive, the equilibrium is again locally unique.

**Proof.** These results are obtained by applying the Routh-Hurwitz stability criterion to system $M$, evaluated at the steady state. Cf. Appendix A. ∎

## 3. Shilnikov chaos

Consider a scenario in which the policymaker runs an active fiscal-monetary regime. Then, by Proposition 1, the policymaker may be pressured to increase the marginal tax rate above the real interest rate to induce uniqueness of the equilibrium path. In this Section, we show that following this prescription may bring in another class of policy difficulties, namely the emergence of chaotic dynamics. Before proceeding with the analysis, we first lay down the mathematical underpinnings behind this prediction, and their relevance to system $M$.

### *3.1. The Shilnikov bifurcation theorem*

Consider the following Theorem (Chen and Zhou, 2011), which is a generalized version of the original result of Shilnikov (1965).

**Theorem 1.** Consider the dynamical system

$$\frac{dY}{dt} = f(Y, \alpha), \qquad Y \in \mathbb{R}^3, \qquad \alpha \in \mathbb{R}^1$$

with f sufficiently smooth. Assume f has a hyperbolic saddle-focus equilibrium point, $Y_0 = 0$ at $\alpha = 0$, implying that eigenvalues of the Jacobian, $J = Df$, are of the form $\gamma$ and $\chi \pm \xi i$, where $\gamma$, $\chi$, and $\xi$ are real constants with $\gamma\chi < 0$. Assume that the following conditions also hold:

(H.1) The saddle quantity, $\sigma \equiv |\gamma| - |\chi| > 0$;
(H.2) There exists a homoclinic orbit, $\Gamma_0$, based at $Y_0$.

Then the following results hold:

(1) The Shilnikov map, defined in the neighborhood of the homoclinic orbit of the system, possesses an infinite number of Smale horseshoes in its discrete dynamics;[2]
(2) For any sufficiently small $C^1$-perturbation, $g$, of $f$, the perturbed system has at least a finite number of Smale horseshoes in the discrete dynamics of the Shilnikov map, defined in the neighborhood of the homoclinic orbit;
(3) Both the original and the perturbed system exhibit horseshoes chaos.

We now use the explicit forms of the functions in order to determine whether system $M$ satisfies conditions H.1 and H.2 in the Theorem, at least in some regions of the parameter space.

### 3.2. An explicit variant of the model

We first assume that the utility function has the following standard form:

$$u(c) = \frac{c^{1-\Phi}}{1-\Phi}, \tag{5}$$

where $\Phi > 0$ is the inverse of the intertemporal elasticity of substitution. Moreover, it is also standard to assume that the disutility of labor is captured by the functional,

$$f(l) = \frac{l^{1+\psi}}{1+\psi}, \tag{6}$$

where $\psi > 0$ measures the preference weight of leisure in utility. Furthermore, following Carlstrom and Fuerst (2001), we also assume that production is linear in labor:

$$y(l) = Al, \tag{7}$$

where $A$, which denotes the productivity level in composite goods production, can be set to 1 without losing generality.

Additionally, we use the specification of the Taylor principle in Benhabib et al. (2001a, 2001b) and assume that monetary authorities observe the inflation rate and conduct market operations to ensure that

$$R(\pi) = \bar{R}e^{\left(C/\bar{R}\right)(\pi-\pi^*)}, \tag{8}$$

where $C$ is a positive constant. Notice that, our chosen functional form implies:

$$R(\pi^*) = \bar{R};\, R'(\pi^*) = C. \tag{9}$$

Finally, in order to avoid violating the Transversality Condition, we assume that the economy follows a Ricardian monetary-fiscal regime. More specifically, Eq. (8) is complemented by a fiscal rule setting:

$$\tau(a) = \alpha a \tag{10}$$

where the marginal tax rate, $\alpha \equiv \tau'(a) \in (0, 1)$.

### 3.3. Existence of Shilnikov chaos in the cashless NK model

As is clear, system $M$ is highly non-linear and heavily parameterized. Therefore, our effort to derive a general characterization of the parametric region at which the Shilnikov phenomenon can occur has been frustrated by frequent numerical anomalies. We have therefore resorted to a numerical strategy. We mainly ground our discussion on the Lubik and Schorfheide (2004) calibration of a prototypical monetary DSGE NK model for the US (see also King, 2000; Woodford, 2003). We shall concentrate on parameter estimates of the post-1982 sample, for which Lubik and Schorfheide (2004) report a vector of annualized implied steady-state values, $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$, provide direct evidence for the subset $(\rho, \Phi) \cong (0.0075, 1.86)$ of the deep parameters[3], and also estimate the slope coefficient, $\kappa = 0.58$, linking inflation and output in the expectational Phillips curve. We know that this slope coefficient is linked to the deep parameters of the NK model according to the relationship:[4]

$$\kappa = (\phi - 1)\frac{\psi}{\eta}. \tag{11}$$

[2] Consider a Poincaré map, which is the intersection of a periodic orbit in the state space of a continuous dynamic system with a certain lower-dimensional subspace, known as the Poincaré section, transversal to the flow of the system. The Shilnikov map is another name for the 2-dimensional Poincaré section.

[3] Lubik and Schorfheide (2004) approximate as follows the discrete-time quarterly discount rate: $\beta = (1 + \frac{r}{100})^{-1/4} \cong 0.9925$. Using the standard formula $\rho = -\ln(\beta)$, we obtain the continuous-time analog $\cong 0.0075$.

[4] King (2000) shows that this slope coefficient is linked to the deep parameters of the NK model according to the formula: $\kappa = \frac{(1-\delta^h)(1-\beta\delta^h)}{\delta^h}\psi$, where $\delta^h$ is the fraction of Calvo firms not able to change their price during the quarter and $\beta$ is the discrete-time discount factor. Taking into account the reduced form implications for the linearized Phillips curve of Rotemberg and Calvo models of costly and sticky price adjustments (for details, see Benhabib et al. (2014)), we get, $\eta = (\phi - 1)\frac{\delta^h}{(1-\delta^h)(1-\beta\delta^h)}$. It follows that $\kappa = (\phi - 1)\frac{\psi}{\eta}$.

As a consequence, since $\tau$ cancels out in the calculations, the characteristic equation (A.2 in Appendix A) can be expressed as a function only of the deep parameters, $(\phi, \psi, \eta(\phi, \psi, \kappa)) = (\phi, \psi)$, and policy coefficients, $(C, \tau')$.

Section 3.3.1 in the following section is devoted to the proving that there are regions in the parameter space, such that system $M$ satisfies condition $H.1$ in Theorem 1. Section 3.3.2 discusses the onset of the homoclinic orbit linking P* to itself in forward and backward time.

### *3.3.1. A saddle-focus steady-state with positive saddle quantity*

To locate the region in the parameter space where P* satisfies the requirements in Theorem 1, we follow Bella et al. (2017) and study the critical surface:

$\Theta \equiv \mathrm{B(J)} + \mathrm{Tr(J)}^2 = 0$, where B(J) and Tr(J) are the Sum of Principal Minors and the Trace of the linearizing matrix associated with the intended steady-state, respectively. In a generic three-dimensional system, the boundary $\Theta = 0$ separates the region giving rise to three real eigenvalues, from the region where we have one real and a pair of complex conjugate eigenvalues (with positive saddle quantity).

Based on Lubik and Schorfheide's (2004) calibration of the economy, we obtain a graphical representation of this critical surface. In Fig. 1, we depict the parametric surface, $\Theta(\phi, \psi, C) = 0$, for several choices of the marginal tax rate. Panel (a) reports the cases of $\tau' = 0.015$ (gray color) and $\tau' = 0.025$ (red color). In panel (b) we depict the surface, $\Theta(\phi, \psi, C) = 0$, when $\tau' = 0.05$ (gray color) and $\tau' = 0.1$ (red color).

As can be seen, in all depicted cases, $\Theta(\phi, \psi, C)$ vanishes for values of $C$ in the neighborhood of 1 across all the plausible domain of the $(\phi, \psi)$ deep parameters. Only for extreme values of the $(\phi, \psi)$ parameters, the four surfaces significantly separate from the $C = 1$ plane. Furthermore, while the surface is located above the $C = 1$ plane in case of small marginal tax rates, it slips below the $C = 1$ plane, when fiscal policy is characterized by higher tax rates.

Now, Examples 1 and 2 below show clearly that the parametric region giving rise to complex-conjugate eigenvalues is above the surface. Economic implications are interesting. Limiting the discussion to the case of an active monetary-fiscal regime, we see that, if the marginal tax rate is sufficiently high (see panel (b)), we always have one real negative eigenvalue and two complex-conjugate eigenvalues with positive real parts (also see Proposition 1); therefore, convergence to the intended steady-state in our cashless NK economy typically occurs through (damped) oscillations. If the marginal tax rate is low (see panel (a)), there is an interesting exception to this rule, if parameters belong to the (small) subregion above the $C = 1$ plane but below the critical surface. In this case, convergence to P* occurs along generically monotonic paths.

We are now ready to prove the following preliminary result.

**Lemma 1.** (Fulfillment of pre-condition H.1 in Theorem 1). Assume the cashless NK economy is calibrated according to the estimates obtained for the U.S. (post-1982 sample period) in Lubik and Schorfheide (2004). Then, if $\tau'$ is set sufficiently high, then an active monetary regime paired with a passive fiscal regime always implies that the intended steady state satisfies condition H.1 in Theorem 1. The H.1 condition is not satisfied in a small part of the domain of the $(\phi, \psi, C)$ parameters, when $\tau'$ is low.

**Proof.** As represented in Fig. 1, panel (b), the surface $\Theta = 0$ is below the plane $C = 1$ for any sufficiently high $\tau'$. Therefore, all parametric combinations above the plane imply the existence of a saddle-focus with $\sigma > 0$. When the marginal tax rate is low, panel (a) in Fig. 1 applies. Then, there exists a (small) parametric region located above the $C = 1$ plane, but below the critical surface, $\Theta = 0$, giving rise to only real eigenvalues. ■

We now discuss two examples. To proceed, we need the calibrated values for the pair $(\phi, \psi)$, which Lubik and Schorfheide (2004) don't report. Bianchi and Melosi (2019), in a follow-up paper, set $\phi = 6$. In a similar framework, Benhabib et al. (2014) set $\psi = 1$. Therefore, recalling that $\kappa = 0.58$, and the formula in (11), we can also compute $\eta \cong 8.62$. Denote now, for notational convenience:

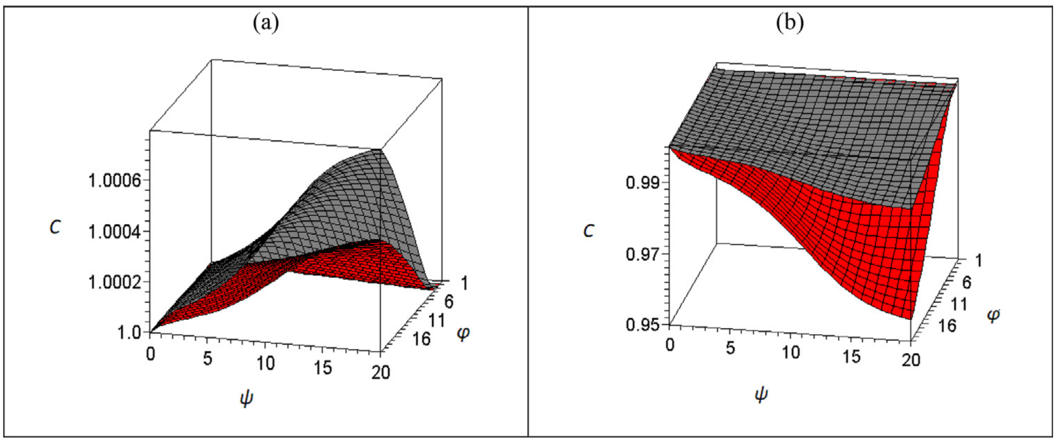


**Fig. 1.** *The surface $\Theta(\phi, \psi, C) = 0$ for different levels of $\tau'$.*

$\mathrm{D} \equiv \{(\eta, \Phi, \phi, \psi, \rho) \cong (8.62, 1.86, 6, 1, 0.0075)\} \equiv \bar{\mathrm{D}}$.

Examples 1 and 2 present the cases for active and passive monetary policy, respectively, and show that the condition *H*.1 can only be established for the case of active monetary policy.

**Example 1.** Assume $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$. Set furthermore $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$ and fix $\tau' = 0.025$. We consider the case of active monetary policy with $C = 2.19$[5]. Then, the solution of the characteristic equation gives:

$\lambda_1 = -0.06750317158$; $\lambda_{2,3} = 0.003748414221 \pm 1.170195840i$.

Since the saddle quantity $\sigma \equiv |\lambda_1| - |\mathrm{Re}(\lambda_{2,3})| = 0.0637$ is positive in this example, the condition H.1 for the onset of the Shilnikov phenomenon is met.

**Example 2.** Set $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$, $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$, and $\tau' = 0.025$. Consider now the case of passive monetary policy with $C = 0.9$. Then, the solution of the characteristic equation gives:

$\lambda_1 = 0.3429936909$; $\lambda_2 = -0.0175031715$; $\lambda_3 = -0.3354968625$.

*In this example, eigenvalues are all real and the pre-condition H.*1 *for the existence of Shilnikov chaos is clearly not satisfied.*

*3.3.2. A family of homoclinic orbits*

With Lemma 1 in hand, we must now determine under what conditions P* is connected to itself by a doubly asymptotic homoclinic orbit (condition *H*.2 in Theorem 1). Bella et al. (2017) describe the various steps required to accomplish this task.

System *M* must first be translated to the origin and then into its normal form:

$$\begin{pmatrix} \dot{w}_1 \\ \dot{w}_2 \\ \dot{w}_3 \end{pmatrix} = \begin{bmatrix} \chi & -\xi & 0 \\ \xi & \chi & 0 \\ 0 & 0 & \gamma \end{bmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} F_{1a}w_1w_2 + F_{1b}w_1w_3 + F_{1c}w_2w_3 + F_{1d}w_1^2 + F_{1e}w_2^2 + F_{1f}w_3^2 \\ F_{2a}w_1w_2 + F_{2b}w_1w_3 + F_{2c}w_2w_3 + F_{2d}w_1^2 + F_{2e}w_2^2 + F_{2f}w_3^2 \\ F_{3a}w_1w_2 + F_{3b}w_1w_3 + F_{3c}w_2w_3 + F_{3d}w_1^2 + F_{3e}w_2^2 + F_{3f}w_3^2 \end{pmatrix} \tag{12}$$

where $(w_1, w_2, w_3)^T$ is the vector of transformed coordinates, and where the $F_{i,j}$ coefficients, with $i = 1, 2, 3$ and $j = a, b, \ldots, f$, are combinations of the original parameters of the model, also depending on the values of three free constants, $\varphi_i$, with $i = 1, 2, 3$, arising in the computation of the eigen-basis.

Once the normal form (12) is obtained, the method of undetermined coefficients (Shang and Han, 2005) can be used to derive a polynomial approximation of the analytical expressions of both the two-dimensional unstable manifolds associated with $\lambda_2$ and $\lambda_3$, as well as the one-dimensional stable manifold associated with $\lambda_1$. The procedure leads to the following split function[6]

$$\Sigma = \Xi + \frac{F_{3f}\Xi^2}{\gamma} + (2\chi - \gamma)\frac{F_{3a}\Psi\Omega + F_{3d}\Psi^2 + F_{3e}\Omega^2}{(2\chi - \gamma)^2 + 4\xi^2} \tag{13}$$

where $(\Xi, \Psi, \Omega) \in (0, 1)^3$ are free constants, while $\gamma = \lambda_1$, $\chi = \mathrm{Re}(\lambda_{2,3})$, and $\xi = \mathrm{Im}(\lambda_{2,3})$. Then, conditions for the existence of the homoclinic loop, doubly asymptotic to the saddle-focus equilibrium point, rely on the existence of a triplet $(\Xi, \Psi, \Omega) \in (0, 1)^3$ satisfying $\Sigma = 0$.[7]

Consider the following example.

**Example 3.** Set $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$, $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$, and $C = 2.19$ as in Example 1. Let us now use the marginal tax rate, $\tau'$, as the bifurcation parameter. Running a numerical scan with a grid of 0.01, we observe that combinations $(\Xi, \Psi, \Omega) \in (0, 1)^3$ satisfying $\Sigma = 0$ exist in the entire domain of $\tau'$. Let therefore $\tau' = 0.025$, as in Example 1. Then, the valid combinations of $(\Xi, \Psi, \Omega) \in (0, 1)^3$ are depicted in Fig. 2.

As is clear from the Shang and Han's (2005) methodology, P* admits a family of doubly asymptotic homoclinic orbits, since there exist combinations of the $(\Xi, \Psi, \Omega) \in (0, 1)^3$.

The following statement is therefore implied.

**Lemma 2.** (Fulfillment of pre-condition H.2 in Theorem 1). Assume the cashless NK economy presents the standard calibration, $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$, of the deep parameters. Let furthermore $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$ and $C = 2.19$, as estimated by Lubik and Schorfheide (2004) for the U.S. economy in the post-1982 sample period. Then, for any $\tau' \in (0, 1)$, there exists a family of homoclinic orbits leaving P* and returning to it in backward and forward time.[8]

We now keep the vector of the deep parameters at the standard level of $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$ and conduct an extensive robustness analysis to see whether the results in Lemma 2 hold when the vector $(\bar{R}, \pi^*)$ and $C$ are not derived from Lubik and Schorfheide's (2004) calibration period. For a significant part of their empirically relevant domain of existence, parameters in all of the following additional numerical experiments satisfy *H*.1 and *H*.2 of the Shilnikov theorem. Given

[5] Lubik and Schorfheide (2004) find evidence that monetary authorities on average raise the nominal rate by 2.19 percent in response to a 1% discrepancy between actual and desired inflation.

[6] See Kuznetsov (1998, p. 198) for the geometrical interpretation of the split function in the context of homoclinic bifurcations.

[7] The reason why the three constants $(\Xi, \Psi, \Omega)$ are bound to belong to the cube $(0,1)^3$ is strictly related to the geometry of the stable and unstable manifolds, which intersect near the origin (in the transformed eigenspace) and forms the homoclinic loop. The issue is well discussed in Kuznetsov (1998, p. 259).

[8] Notice that, since $\frac{\partial \Sigma}{\partial \tau'} > 0$ for given values of $(\Xi, \Psi, \Omega) \in (0, 1)^3$, there exists a unique critical value of $\tau'$ solving the split function (13).

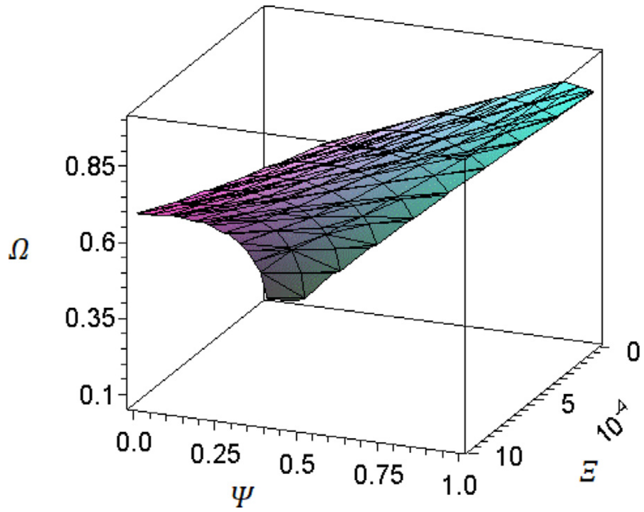


**Fig. 2.** *Combinations of* $(\Xi, \Psi, \Omega) \in (0,1)^3$ *satisfying* $\Sigma = 0$.

$(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$, pre-conditions *H*.1 and *H*.2 are satisfied for virtually the entire empirically relevant domain for the policy parameters, $(C, \tau', \tau)$. This is not quite the case for $(\bar{R}, \pi^*)$, given $(C, \tau') = (2.19, 0.025)$. If we keep $\bar{R} = 0.0644$, conditions *H*.1 and *H*.2 are only satisfied, when $\pi^* \in (-0.027, 0.051)$. Moreover, if we set $\pi^* = 0.0343$, the range of the nominal rate at which the Shilnikov conditions are confirmed is $\bar{R} \in (0.041, 0.135)$.

We are now ready to prove the main result. Consider the formal argument below. Let's start by choosing $\tau'$ as the bifurcation parameter. Assume there exists a critical level of the marginal tax rate, $\bar{\tau}'$, such that, given the parametric configuration and coordinates $(\Xi, \Psi, \Omega) \in (0,1)^3$, a specific homoclinic orbit leaves the origin and returns to it in backward and forward time. Define $\upsilon = \tau' - \bar{\tau}'$. Let finally $V \subset \mathbb{R}$ be a sufficiently small open neighborhood of the origin. Then, we can prove the following statement.

**Proposition 2.** (Existence of a Shilnikov chaotic attractor) Assume that the parametric conditions in Lemmas 1 and 2 are satisfied. Let $\upsilon \in V$. Then, a given triplet of initial conditions sufficiently close to the origin, system (12) admits perfect-foresight horseshoe chaotic equilibria. By topological equivalence, this result also applies to system M.

**Proof.** See Appendix B. ∎

As a check on robustness of our conclusions, we repeat our analysis for the model with money in the utility and production functions. Our robustness analysis reveals two interesting details. On the one hand, we find that the phenomenon is confirmed irrespective of whether money is included in the utility and/or production functions or whether it is removed completely. On the other hand, we find that a necessary ingredient for the onset of the Shilnikov scenario is price stickiness. A sketch of the mathematical underpinnings behind these findings is presented in Appendices D and E.

### *3.4. Properties of the chaotic attractor*

As emerges clearly from the literature, this is not the first paper detecting chaos in NK macro models (cf. *inter al.* Benhabib et al., 2002). However, prior findings are of general chaos and hence cannot determine the geometry of the fractal attractor set without sample sizes far exceeding those available in macroeconomics. As a result, the policy relevance of those prior findings has not been clear. Without access to the geometry of the fractal attractor set, findings of general chaos imply only the existence of deterministically intrinsic stochasticity, but with unknown properties. The interesting part is that, since we find Shilnikov chaos, the properties of the fractal attractor can be determined and thereby the properties of the intrinsic stochasticity.

Consider the following example.

**Example 4.** Set $(\eta, \Phi, \phi, \psi, \rho) = \bar{\text{D}}$ and $(\bar{R}, \pi^*) \cong (0.0644, 0.0343)$, as in Example 1. Assume furthermore $(C, \tau') = (2.19, 0.025)$, as in Example 3. Then, we know that there exists a family of homoclinic loops doubly asymptotic to the saddle-focus equilibrium point. Consider the case of initial conditions $(w_1(0), w_2(0), w_3(0)) = (0.01, 0.01, 0.01)$. Then the attractor has the form represented in Fig. 3.

The procedures we used to solve and simulate the model are explained in Appendix F. As appears clear from Fig. 3, along the Shilnikov attractor, the economy has short periods of relative quiescence, when the phase point approaches the steady state (the bullet point in Fig. 3). On the other hand, when the phase point starts to spiral away from the steady state, irregular episodes of oscillatory activity appear. Very interestingly, we also see that, in the very long-run, the economy

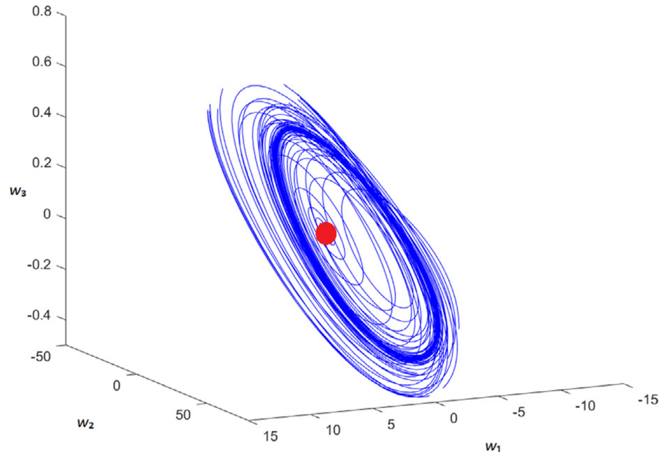


**Fig. 3.** The chaotic attractor around the intended steady state.

settles down to an irregular cycle (the dense orbit in Fig. 3), not centered on the coordinates of the steady state. This is a crucial characteristic of our simulations, as we shall discuss in greater detail in the next section.

### 3.5. Economic implications

Proposition 2 has a significant impact on the ability of the NK model to make useful predictions, given the fundamentals of the economy. First of all, the existence of a chaotic attractor implies that small changes in initial conditions can produce large changes in dynamics over time. Two economies, starting contiguously in the space of initial conditions, can follow completely different patterns over time. Since an initial condition is known only to a finite degree of precision, it is impossible to predict dynamics deterministically over extended periods of time.

Moreover, our results are in sharp contrast with the remark "our empirical results show that post-1982 monetary policy is sufficiently anti-inflationary to rule out any indeterminacy" (Lubik and Schorfheide, 2004). Within a chaotic attractor, given the initial value of the predetermined variable, there exists a *continuum* of initial conditions of the jump variables giving rise to admissible equilibria. Therefore, the policy options required to recover the uniqueness suggested by the local analysis are exactly those which may cause global indeterminacy of the equilibrium. In this regard, showing that the equilibrium is globally indeterminate requires demonstrating that, given an initial condition in terms of the predetermined state variable, $a(0)$, there exist multiple choices of the jump variables, $\mu_1(0)$ and $\pi(0)$, lying outside the small neighborhood relevant to the local analysis.

Consider the following formal argument. Recall Proposition 2. Let us first denote

$$\nu_h = \{\nu \in V : \textit{system } M \textit{ exhibits horseshoe chaos}\}$$

to be the set of deviations of the marginal tax rate from the critical threshold $\bar{\tau}'$ for which, given the parameters of the model and a specific combination $(\Xi, \Psi, \Omega) \in (0, 1)^3$, there exists a homoclinic orbit linking P* to itself in forward and backward time. Then, by Theorem 1, given $\nu \in \nu_h$, there exists a three-dimensional region in the phase space, around the homoclinic orbit, such that if the dynamics is initialized within this region, system $M$ exhibits horseshoe chaos. Let us denote by $T_\nu \subset \mathbb{R}^3$ the set of all points of this region, with Int$T_\nu$ and Bd$T_\nu$ being the set of all interior and boundary points of $T_\nu$, respectively. Let $E_\nu$ be a three-dimensional manifold containing the set of all possible paths starting on Int$T_\nu$. Then, by Theorem 1, all paths starting on the set $E_\nu$ are recurrent paths, bound to stay forever in $E_\nu$. We can therefore establish the following result.

**Corollary 1.** (Global indeterminacy of the equilibrium). Assume that, given a parametric configuration of the cashless NK model and a specific combination $(\Xi, \Psi, \Omega) \in (0, 1)^3$, there exists a critical threshold $\bar{\tau}'$ for which a homoclinic orbit departs from P* and returns to it in infinite time. Furthermore, let $\nu \in \nu_h$. Consider an initial value of the state variable, $a(0) \in E_h$. Then, the cashless NK model exhibits global indeterminacy of the equilibrium.

**Proof.** See Appendix C. ■

Finally, the qualitative "dimensions" of the chaotic attractor are of particular interest in the present context.[9] Assume that the relative frequency at which an orbit visits different region of the attractor is largely heterogeneous. Then, across the volume of all possible coordinates contained in the attractor, the economy lingers on particular regions with higher density. In the numerical simulations presented in the paper, it is evident that if the initial conditions of the jump variables are chosen far enough from the target steady-state, then the emerging dynamics are characterized for a long time by low

[9] Cf. Farmer et al. (1983) for a classical discussion on the relevant dimensions of a chaotic attractor.

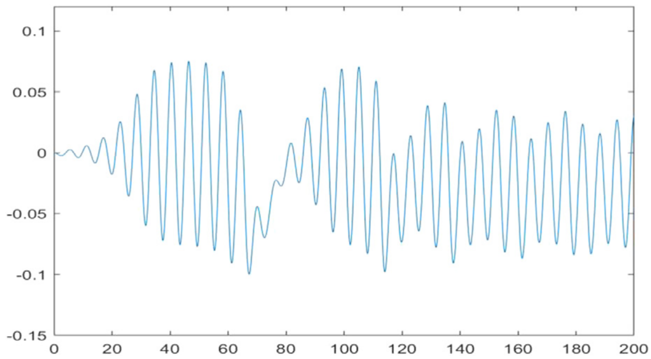


**Fig. 4.** The chaotic inflation rate.

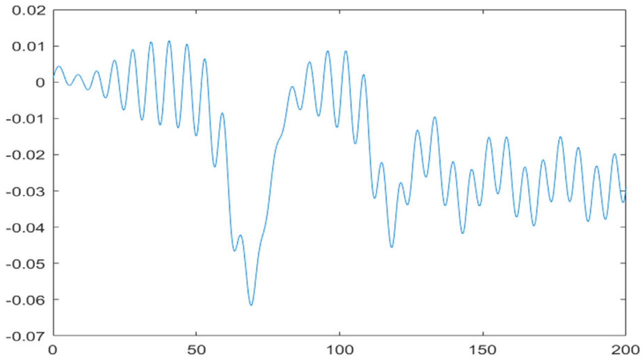


**Fig. 4a.** The moving average of the chaotic inflation rate.

inflation and nominal interest rates. This can be interpreted as a liquidity trap phenomenon, which now depends on the existence of a chaotic attractor and not on the influence of an unintended steady state.

Fig. 4 reports the reconstructed time profile of the chaotic inflation rate. The time profile is reported after subtracting the (annualized) value of 0.0343 estimated by Lubik and Schorfheide (2004) for the post-1982 sample period. Numbers in the abscissa are quarters. The simulation is constructed using parameters and initial conditions from Example 3. In order to filter out much of the high frequency oscillations, and to more clearly reveal long-run trends, we also compute the 10-quarters moving average (MA) transformation of the chaotic inflation rate (cf. Fig. 4a).[10]

The Figures provide interesting insights regarding the properties of the inflation rate evolving within the spiral attractor in this specific example. Consider the case of a shock pushing away inflation from its long-run equilibrium. Then, initially (for roughly 50 quarters), there is a burst of oscillatory activity, corresponding to the period when the phase point departs from the saddle-focus along the unstable manifold. Later, when the dynamics points again towards the long-run equilibrium, we observe a first sudden drop, which is recovered in roughly 30 quarters. Finally, around 110 quarters after the perturbation, inflation loses connection with the original steady state and settles down on an irregular cycle. In this experiment, inflation remains persistently low (akin to a deflationary equilibrium) for the whole final part of the simulation.[11]

We can therefore state the following.

**Corollary 2.** (Existence of persistent deflationary perfect-foresight equilibrium paths). Assume the parametric configuration in Example 4 applies. Then, persistently lower-than-intended inflation rates can emerge. By Assumption 1 and Eq. (8), along these paths the nominal interest rate evolves below the long run level.

Extensive numerical simulations, with different choices of the deep parameters and initial conditions, qualitatively confirm the tendency of the inflation and nominal interest rates to stay stubbornly below the intended steady state for a large part of the computed pattern. Thus, this phenomenon appears to be a distinct trait of the cashless NK model.[12]

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[10] The moving average operator is a standard in time series contexts to smooth out short-term fluctuations and highlight longer-term trends or cycles. Details on the construction of the MA filter are in Appendix **G**.

[11] A robustness analysis shows that the shape of the time profile of the inflation rate and the relevance of the oscillatory periods are sensitive to the calibration scheme. For instance, calibrating the parameters as in Benhabib et al. (2001a,b), we observe much longer periods of quiescence, greatly reducing the relevance of the oscillatory phase in Figure 4. From a more technical point of view, and recalling Theorem 1, the relevance of the oscillatory phase in the actual time profiles of the variables evolving within the chaotic attractor depends on the relative magnitudes of the imaginary part ($\xi$) with regard to the real one ($\chi$) in the complex conjugate eigenvalues.

[12] This feature survives the insertion of money either in the utility function or in the production function. See the discussion in Appendix D.

These features have naturally important implications for the debate regarding liquidity traps. As discussed in our introduction, this phenomenon has previously been linked mainly to the existence of a low-inflation steady state (cf., in particular, Benhabib et al., 2001a, 2001b) and to its basin of attraction. We offer an alternative explanation based on the long-run peculiarities of a chaotic attractor and the evolution of the dynamics within that attractor set, such that the economy drifts into the liquidity trap without any policy intent.[13]

A note of caution is due here. Notice that the phenomenon described above occurs after roughly 110 quarters (corresponding to more than 27 years) from the shock perturbing the economy. It is therefore evident that this stylized cashless NK economy is not able to engender a liquidity trap within the time horizon relevant to the conduct of short run monetary policy. More exploration is needed here to understand whether and to what extent a more sophisticated modeling of the economy may generate a lower-than-targeted inflation rate in the short or medium run.[14] However, our results remain potentially relevant to the long run downward drift in US interest rates, beginning in 1981 and ending in a liquidity trap attained in 2009, a 28-years downward drift in interest rates.

## 4. Conclusions

Using the Shilnikov criterion, we find bifurcation to chaos in a NK model at plausible settings of parameters with common NK policy design. The existence of chaos is consistent with the fact that economists who provide short term forecasts are rarely willing to provide long term forecasts. Within the Shilnikov chaos attractor set, we find a downward bias in the interest rate and inflation orbits producing a phenomenon similar to a liquidity trap. The problems associated with the zero-lower bound on nominal interest rates would thereby not be an intentional objective of central bank policy, but of the dynamics of the system within the attractor set. The existence of this downward bias, which has been evident for three decades of declining interest rates and inflation rates, has produced the puzzle of very low real rates of interest substantially below the marginal product of capital.

That puzzle has often been observed and frequently unconvincingly imputed to oversaving.[15] Our explanation is different and therefore has different policy implications. Paradoxically, an active interest rate feedback policy can cause nominal interest rates, inflation rate, and real interest rates unintentionally to drift downward, as a result of the dynamical response of the economy within a Shilnikov attractor set. Unlike an open loop interest rate rule, which would directly control an interest rate, active interest rate feedback rules are closed loop rules that link an interest rate to the dynamics of the rest of the economy.

In future research, we plan to explore robustness of our findings against different Taylor rules and alternatives to the Ricardian fiscal policy laws[16]. But we do not expect fundamental changes in our conclusions, which are systems theory properties of sticky price NK macroeconomic dynamics, when augmented by closed loop interest rate feedback rules, as in all Taylor rules.

## APPENDIX

### A. Proof of Proposition 1

Let J denote the Jacobian matrix of system $M$, evaluated at the long-run equilibrium, and let starred values denote steady-state levels. Simple algebra leads to the following $(3 \times 3)$ matrix:

$$\mathrm{J} = \begin{bmatrix} 0 & (1 - R'(\pi^*))\mu_1^* & 0 \\ j_{21}^* & \rho & 0 \\ 0 & 0 & \bar{R} - \pi^* - \tau'(a^*) \end{bmatrix}, \tag{A.1}$$

where $j_{21}^* = \frac{\phi-1}{\eta} c^* (\frac{\psi}{\sigma} + 1)$. The eigenvalues of J are the solutions of the characteristic equation:

$$\det(\lambda \mathrm{I} - \mathrm{J}) = \lambda^3 - Tr(\mathrm{J})\lambda^2 + B(\mathrm{J})\lambda - Det(\mathrm{J}), \tag{A.2}$$

[13] The differences in the qualitative dynamics, arising because of an unintended steady state or because of the existence of a chaotic attractor, are remarkable. The time profile for inflation featured in Benhabib et al. (2001a,b) presents higher and higher amplitude oscillations around the active steady state. Then inflation suddenly reaches the passive (lower) steady state value, when the saddle connection is established. This kind of predictable/regular behavior of the economy could be traced out by an econometric exercise. In our case, inflation begins to oscillate around a lower-than-intended average very quickly along the spiral attractor. This kind of pattern is highly unpredictable and cannot be inferred by conventional econometric tools, since such behavior violates the regularity conditions for available statistical inference methodologies, such as the usually assumed properties of the likelihood function and polyspectra (see, e.g., Barnett et al. (1997) and Geweke (1992)).

[14] While such extensions are beyond the scope of this paper, it is already evident from figures in Appendix D that when money is included in the production function, the liquidity trap is induced much earlier (around 30 quarters, i.e. approximately 7.5 years).

[15] Regarding that popular, but controversial, "loanable funds" explanation, see, e.g., Bofinger and Ries (2017). Paradoxically, that explanation is the opposite of the frequent overconsumption (under saving) explanation of the US balance of payments deficit with China. See, e.g., Hanke (2019) and Hanke and Li (2019).

[16] Robustness of our conclusions also could be investigated relative to other changes in our standard NK model. In Appendix D, we have done so by introducing money both in the utility and production function. Our conclusions about Shilnikov chaos, when a Taylor rule is adjoined to the model, are unchanged.

where I is the identity matrix and where:

$$Tr(J) = \rho + \bar{R} - \pi^* - \tau'(a^*) \tag{A.3}$$

$$Det(J) = \left[\bar{R} - \pi^* - \tau'(a^*)\right]\left[R'(\pi^*) - 1\right]\mu_1^* j_{21}^* \tag{A.4}$$

$$B(J) = \left[R'(\pi^*) - 1\right]\mu_1^* j_{21}^* + \left[\bar{R} - \pi^* - \tau'(a^*)\right]\rho \tag{A.5}$$

are Trace, Determinant, and Sum of principal minors of J, respectively. Also define:

$$G(J) = -B(J) + \frac{Det(J)}{Tr(J)}. \tag{A.6}$$

Hereafter, we study the local stability properties of system $M$, in the neighborhood of P*, for the case where monetary policy is active (i.e., $R'(\pi^*) > 1$). Fiscal policy can be either active ($\rho > \tau'(a^*)$) or passive ($\rho < \tau'(a^*)$). To this purpose, we apply the standard Routh-Hurwitz criterion to determine the number of eigenvalues with positive reals parts depending on the number of sign variations in the scheme:

$$-1, \qquad Tr(J), \qquad -B(J) + \frac{Det(J)}{Tr(J)}, \qquad Det(J), \tag{A.7}$$

where it is easy to derive that

$$-B(J) + \frac{Det(J)}{Tr(J)} = -\frac{(R'(\pi^*) - 1)\rho\mu_1^* j_{21}^*}{2\rho - \tau'(a^*)} - (\rho - \tau'(a^*))\rho. \tag{A.8}$$

Consider first the case with active monetary policy and active fiscal policy. In this case, we have $Tr(J) > 0$ and $Det(J) > 0$. Moreover, $-B(J) + \frac{Det(J)}{Tr(J)} < 0$. Therefore, the sequence in (A.6) always exhibits three changes of sign $(- + - +)$, which implies three roots with positive real parts of the characteristic equation in (A.2).

Conversely, in the case with active monetary policy and passive fiscal policy, we have that $Det(J) < 0$. Then, for $Tr(J) < 0$, necessarily $-B(J) + \frac{Det(J)}{Tr(J)} > 0$. Therefore, we obtain the sequence $(- - + -)$; whereas, if $Tr(J) > 0$, then $-B(J) + \frac{Det(J)}{Tr(J)}$ has no definite sign. The sequence now becomes $(- + \pm -)$. However, in all these different cases, we end up always with a sequence that produces only two changes of sign. Hence, the roots in (A.2) are one negative and two with positive real parts.

## B. Proof of Proposition 2

Assume that the conditions in Lemmas 1 and 2 are satisfied. If we start in the neighborhood of the origin, we know from Theorem 1 that, in the phase space of system (12), the solution trajectories are bounded to evolve forever in the neighborhood of the origin and are therefore valid equilibria. By construction, the results obtained for system (12) also apply to the original system of differential equations, $M$. ■

## C. Proof of Corollary 1

Recall that $a(t)$ is the predetermined variable of the system, and that $\mu_1(t)$ and $\pi(t)$ are jump variables. If we choose $a(0)$ to belong to the tubular neighborhood of the homoclinic orbit, as defined above, the Shilnikov Theorem assures that there must be a *continuum* of possible choices of $\mu_1(0)$ and $\pi(0)$ capable of giving rise to perfect-foresight solutions which are bound to stay forever within $E_v$ (recurrent paths). Since all these paths are bound to stay in a neighborhood of P* that can well exceed the small neighborhood valid for the local analysis, we have in fact global indeterminacy of the equilibrium. ■

## D. Robustness analysis

The results in Section 3 of this paper are based on a cashless NK model. As a check on the robustness of our results to that assumption, we now repeat our analysis for the case in which money enters the utility function or the production function (see Benhabib et al., 2001b; Stokes, 2016).[17]

[17] The money in the utility function approach implicitly uses the derived utility function shown to exist by Arrow and Hahn (1971), if money has positive value in equilibrium. A long literature has repeatedly confirmed this existence using models with various explicit motives for holding money, such as transactions or liquidity constraints (e.g., Feenstra (1986), Poterba and Rotemberg (1987), and Wang and Yip (1992). The mapping from explicit motives for holding money to the derived utility function does not have a unique inverse. Hence, money in the utility function models cannot reveal the explicit motive for holding money. But the ability to infer the explicit motive is not relevant to our research. Hence, for our purposes, we can assume that money has positive value in equilibrium, without the need to condition upon an explicit motive.

*D1. The NK model with money in the utility function*

Let us first assume that economic agents receive utility from consumption of the composite good, $c_i$, and from real money balances, $m_i$. Assume, in equilibrium, $c_i$ and $m_i$ contribute to utility according to the CES aggregator as follows:

$$u(c, m) = \frac{\left[xc^{1-\vartheta} + (1-x)m^{1-\vartheta}\right]^{\frac{1-\Phi}{1-\vartheta}}}{1-\Phi}$$

where $0 < x < 1$ is a share parameter, $\vartheta$ measures the intra-temporal elasticity of substitution between $c$ and $m$, and $\Phi > 0$ is the inverse of the intertemporal elasticity of substitution (as in the cashless economy in the main text). The application of the Maximum Principle leads to the following system of differential equations:

$$\begin{aligned} \dot{\mu}_1 &= (\rho - R + \pi)\mu_1 \\ \eta\dot{\pi} &= \rho(\pi - \pi^*)\eta - c(\mu_1, \pi)\left[(1-\phi)\mu_1 + \phi c(\mu_1, \pi)^{\psi}\right] \\ \dot{a} &= (R - \pi)a - Rm(c(\mu_1, \pi), R) - \tau. \end{aligned}$$

As discussed in Benhabib et al. (2001a), in the presence of an active-passive monetary-fiscal regime, the intended steady state of this model is a candidate for the application of the Shilnikov Theorem only when $c$ and $m$ are Edgeworth substitutes. This requires $\vartheta < \Phi$. Consider the same calibration of the economy discussed in Example 4. Then, $(\eta, \Phi, \phi, \psi, \rho) = \bar{\mathrm{D}}$, $(\bar{R}, \pi^*) = (0.0644, 0.0343)$, and $(C, \tau') = (2.19, 0.025)$. To locate values of the remaining utility function parameters, $(x, \vartheta)$, where the Shilnikov bifurcation occurs, we proceed with a numerical scan (grid of 0.01). The numerical experiment shows that conditions for the onset of Shilnikov chaos are only satisfied when $(x, \vartheta)$ are chosen close to the upper limit of their domain of existence, so that $(1-x)$ and $(\vartheta - \Phi)$ are close to zero. Consider the case of $x = 0.97$ and $\vartheta = 1.85$[18]. Then, Shilnikov conditions are satisfied. Consider furthermore initial conditions given by $(w_1(0), w_2(0), w_3(0)) = (0.01, 0.01, 0.01)$, as in Example 4. Then the reconstructed inflation rate and its MA are depicted in Figs. D.1 and D.1a, respectively. See footnote 15 and Appendix **G** for details on the construction of the MA filter.

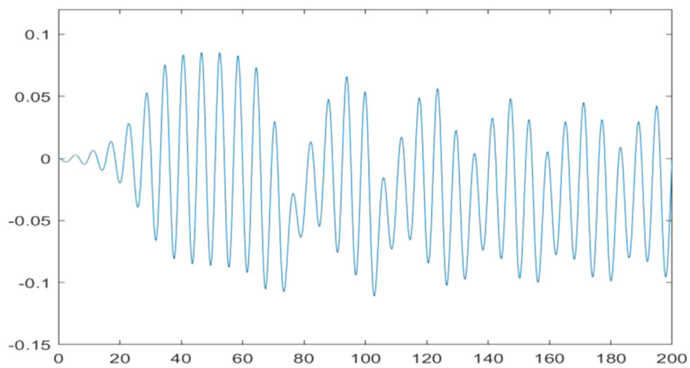


**Fig. D.1.** The chaotic inflation. The case with money in the utility function.

The MA filter, more clearly displaying the long run drift, is in Fig. D.1a.

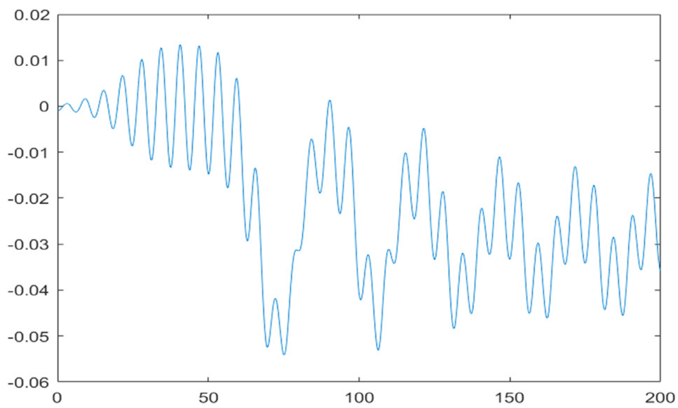


**Fig. D.1a.** The moving average of the chaotic inflation. Money in the utility function.

[18] Notice that the parameterization of $x$ is consistent with the literature where the weight of real balances in the utility function is significant but small (see, inter al. Holman (1998)).

*D2. The model with money in the production function*

We now introduce real balances into the production function according to the constant-returns Cobb-Douglas formula:

$$y(l, m) = Al^{\theta} m^{1-\theta},$$

where $0 < \theta < 1$ is the share of labor, so that $1 - \theta$ is the share of real balances in production. The application of the Maximum Principle leads to the following explicit system:

$$\dot{\mu}_1 = (\rho - R + \pi)\mu_1$$
$$\eta\dot{\pi} = \rho(\pi - \pi^*)\eta - c(\mu_1, \pi)\left[(1-\phi)\mu_1 + \frac{\phi}{\theta}c(\mu_1, \pi)^{\frac{1+\psi-\theta}{\theta}} m(c(\mu_1, \pi))^{\frac{\theta-1}{\theta}}\right]$$
$$\dot{a} = (R - \pi)a - Rm(c(\mu_1, \pi), \bar{R}) - \tau,$$

where the parameter $A$ has been set to one. We consider the same calibration of the economy discussed in Example 4. Therefore, $(\eta, \Phi, \phi, \psi, \rho) = \bar{D}$, $(\bar{R}, \pi^*) = (0.0644, 0.0343)$, and $(C, \tau') = (2.19, 0.025)$. To locate values of the remaining free parameter, $\theta$, where the Shilnikov bifurcation occurs, we proceed with a numerical scan (grid of 0.01). We find that only values of $\theta$ close the upper limit of its domain of existence guarantee the existence of a Shilnikov attractor. Consider therefore the case of $\theta = 0.97$ and initial conditions given by $(w_1(0), w_2(0), w_3(0)) = (0.05, 0.05, 0.05)$.[19] In this case, the time profile of the inflation rate is depicted in Fig. D.2.

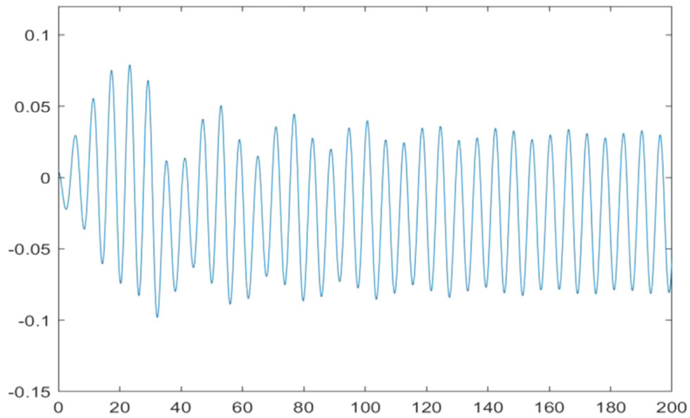


**Fig. D.2.** The chaotic inflation. The case with money in the production function.

Notice that now the inflation dynamics start very soon to irregularly cycle around a region of the attractor characterized by a lower-than-targeted inflation rate. The MA filter, more clearly displaying the long run drift, is in Fig. D.2a. See footnote 15 and Appendix G for details.

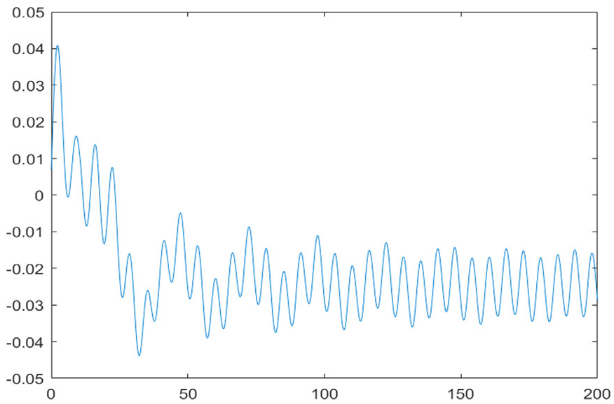


**Fig. D.2a.** The moving average of the chaotic inflation. Money in the production function.

[19] Small coefficients to real balances estimated with Cobb-Douglas production functions are reported in the recent literature (cf., *inter al.* Stokes (2016)). The finding of near constant returns to scale, when appropriate measures of labor and capital are used along with exogenous technical change, provides a useful benchmark for evaluating the plausibility of low elasticity of real balances.

*E. The case with flexible prices*

In this case, the first-order differential equation describing the equilibrium dynamics of inflation simplifies to

$$\dot{\pi} = -\frac{\mu_1(R(\pi))[R(\pi) - \pi - \rho]}{\mu_1{}'(R(\pi))R'(\pi)}.$$

(cf. also Eq. (6) in Benhabib et al., 2001b). In this case, it is straightforward to show that the element $j_{21}^*$ in (A.1) vanishes. Condition *H*.1 in Theorem 1 is thus never satisfied.

*F. How the model is solved and simulated*

Here we provide the steps that led us to obtain and simulate a time profile for chaotic inflation.

*F1. The three-dimensional autonomous system of differential equations*

Based on a cashless variant of the NK model studied by Benhabib et al. (2001a, 2001b), we first obtain the following three-dimensional system of differential equations (system ***M***):

$$\dot{\mu}_1 = (\rho - R + \pi)\mu_1$$
$$\eta\dot{\pi} = \rho(\pi - \pi^*)\eta - c(\mu_1)\Big[(1-\phi)\mu_1 + \phi c(\mu_1)^{\psi}\Big]$$
$$\dot{a} = (R - \pi)a - \tau.$$

The system is in implicit form. The first equation denotes the time evolution of the Lagrange multiplier associated with the budget constraint (or shadow price of the real value of aggregate per capita government liabilities, real balances, and bonds) at instant of time *t*. The second equation is the NK Phillips Curve. The third equation is the budget constraint.

*F2. Steady states*

We then study the dynamics around the intended steady state, $P^* = (\frac{\phi}{\phi-1}c(\mu_1^*)^{\psi}, \bar{R} - \rho, \frac{\bar{\tau}}{\rho})$.

*F3. An explicit setting of the model*

We consider the following explicit version of the model.

1. Utility has the following standard form: $u(c) = \frac{c^{1-\Phi}}{1-\Phi}$.
2. Disutility of labor is captured by: $f(l) = \frac{l^{1+\psi}}{1+\psi}$.
3. Production is linear in labor: $y(l) = Al$.
4. Taylor principle ensures nominal interest rate is: $R(\pi) = \bar{R}e^{(C/_{\bar{R}})(\pi - \pi^*)}$.
5. Fiscal rule sets: $\tau(a) = \alpha a$, where the marginal tax rate, $\alpha \equiv \tau'(a) \in (0, 1)$.

*F4. Shilnikov chaos*

Based on Lubik and Schorfheide's (2004) calibration of the economy, we verify that:

1) System *M* has a hyperbolic saddle-focus equilibrium point, implying that eigenvalues of the Jacobian Matrix, are of the form $\gamma$ and $\chi \pm \xi i$, where *i* is the imaginary unit.
2) The associated saddle quantity has to be positive: i.e., $\sigma \equiv |\gamma| - |\chi| > 0$.
3) There exists a homoclinic orbit, $\Gamma_0$, that connects the equilibrium $P^*$ to itself.

Therefore, Shilnikov chaos can occur in the model.

*F5. Simulation package*

We first obtain the normal form of system ***M*** in the proximity of the saddle-focus:
$$\begin{pmatrix}\dot{w}_1\\ \dot{w}_2\\ \dot{w}_3\end{pmatrix} = \begin{bmatrix}\chi & -\xi & 0\\ \xi & \chi & 0\\ 0 & 0 & \gamma\end{bmatrix}\begin{pmatrix}w_1\\ w_2\\ w_3\end{pmatrix} + \begin{pmatrix}F_{1a}w_1w_2 + F_{1b}w_1w_3 + F_{1c}w_2w_3 + F_{1d}w_1^2 + F_{1e}w_2^2 + F_{1f}w_3^2\\ F_{2a}w_1w_2 + F_{2b}w_1w_3 + F_{2c}w_2w_3 + F_{2d}w_1^2 + F_{2e}w_2^2 + F_{2f}w_3^2\\ F_{3a}w_1w_2 + F_{3b}w_1w_3 + F_{3c}w_2w_3 + F_{3d}w_1^2 + F_{3e}w_2^2 + F_{3f}w_3^2\end{pmatrix},$$ where $(w_1, w_2, w_3)^T$ is the vector of transformed coordinates, and where the $F_{i,j}$ represent the coefficients of the nonlinear terms.

Still using the Lubik and Schorfheide's (2004) calibration of the economy, we use the MatCont software (a routine of Matlab) to solve the normal form. The result is the chaotic attractor depicted in Fig. 3 of the paper. The MatCont routine uses an ode45 solver to approximate the solution of our (parametrized) system of odes (ordinary differential equations). Therefore, the solution can be obtained both in simulation time and in conventional time (quarters, in our case).

### G. Computation of MA filter

As discussed in Appendix **F5**, the MatCont routine obtains the time series for inflation in both simulation and conventional time. To compute the MA in the exercises above, we have proceeded as follows. The first element of the MA is obtained by taking the average of the first 100 observations provided by MatCont for the time series of inflation in simulation time. Then, the operator “shifts forward”, keeping a fixed set of 100 observations in the computation of the successive elements. Then, the MA time series is converted into conventional time. As a result, each point of the MA plotted in Figs. 5, D.1a and D.2a represents the rolling mean of 10 quarters observations.

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