Containment logics: Algebraic Counterparts and Reduced Models

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Abstract

The containment companion of a logic $\vdash$ consists of the consequence relation $\vdash^r$ which satisfies all the inferences of $\vdash$, where the variables of the conclusion are *contained* into those of the set of premises, in case this is not inconsistent. Following the algebraic analysis started in Bonzio and Pra Baldi (2021, *Studia Logica*, 109, 969–994), this paper characterizes the algebraic counterpart of a finitary containment logic $\vdash^r$ and investigates the structure of the Leibniz and Suszko reduced models. The analysis is carried within the framework of abstract algebraic logic.

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1 Introduction

Logics that can be defined according to some constraints on the inclusion of variables in inferences have been recently studied at a very general level under the name of *logics of variable inclusion* [10, 11, 42, 43]. Prototypical examples are weak Kleene logics, namely Bochvar and Paraconsistent weak Kleene, that can be defined from (propositional) classical logic by imposing different constraints on the inclusion of variables. A particularly fruitful treatment of the semantics of logics of variable inclusion is provided by the theory of Płonka sums. Originally arisen in universal algebra due to the work of Płonka [37, 38], this theory has been extended to logical matrices in [10] and [11] to provide an algebraic treatment of logics of variable inclusion.

Containment logics occupy a prominent place in the realm of the logics of variable inclusion. Indeed, they share the distinctive feature of verifying inferences in which the variables occurring in the conclusion are contained into the ones occurring in the premises. This syntactic requirement consists in an abstraction of what Parry [36] coined as *Proscriptive Principle* (PP), in the context of logics of analytic implication [see also 20, 25, 32]. Quoting Parry, the principle can be summarized as follows.

> No formula is valid that has [analytic implication] as a main relation and has a variable [...] which occurs in the consequent but not in the antecedent (pp. 170–171).

Following the idea of (PP), it is possible to associate with every logic $\vdash$, a *containment companion* $\vdash^r$, namely the consequence relation which—roughly speaking—satisfies all those inferences whose conclusion's variables are included in the (set of) premises' variables. A general and preliminary

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investigation of containment companions has been conducted in [11]. The present paper aims at completing the initiated algebraic analysis. A detailed study of the algebraic counterparts of containment companions and a full investigation of the structure of their Leibniz and Suszko reduced models is the main aim of the present work. Such a task is particularly meaningful when dealing with logics that are neither protoalgebraic nor truth-equational, as in the case of containment logics. Under these circumstances, indeed, the connection between a logic and its intended algebraic semantic counterpart is weak, and its characterization is far from being an obvious task to reach. At least for an algebraic logician, the investigation of the semantic counterpart of a logic has an intrinsic value. Moreover, these models find fruitful applications in several areas of logic and its philosophy. In the case of Belnap–Dunn logic and the logic of paradox, a semantic analysis of their algebraic counterparts has led to different proof-theoretic formulations (Hilbert-style calculi and sequent calculi), as witnessed by [27, 45]. Similar considerations hold for paraconsistent weak Kleene logic [35]. Furthermore, the toolbox of second-order AAL, in particular the theory of generalized matrices, has recently found natural applications in belief revision theory [21]. Another remarkable application of the algebraic reducts of the Suszko reduced models consists in characterizing the extensions of the logic in question. A beautiful general result from AAL states that if a logic $\vdash$ is algebraizable with the prevariety $\mathcal{K}$ as equivalent algebraic semantics, there is a dual isomorphism between the poset of extensions of $\vdash$ and the poset of subprevarieties of $\mathcal{K}$ [28, Thm. 3.33]. When $\vdash$ is truth-equational, but possibly non-protoalgebraic, a result was found by Alexej Pynko [47]: in this case, there is a Galois retraction of the poset of subprevarieties of the prevariety generated by $\mathsf{Alg}^*(\vdash)$ onto the poset of extensions of $\vdash$. Building on this result, Pynko shows that the logic of paradox has exactly one proper nontrivial extension, other than classical logic. A similar result is obtained for paraconsistent weak Kleene logic in [34]. With respect to strong Kleene logics, the knowledge of the various classes of Leibniz and Suszko reduced models is crucial in determining their relationships with other extensions first degree entailment, as detailed in [6] and in [49].

The paper is structured as follows.

In Section 2, we introduce all the preliminary notions needed to go through the reading of the whole paper. They basically consist of abstract algebraic logic (the tool on which our analysis in based), the theory of Płonka sums and a brief recap of the general properties of containment logics.

In Section 3, we address the study of the algebraic counterpart of an arbitrary containment companion $\vdash^r$ of a logic $\vdash$. The analysis in carried on by distinguishing whether the initial logic $\vdash$ possesses or not a set of anti-theorems (a generalization of the classical notion of inconsistent set of formulas).

Sections 4 and 5 are devoted to singling out the structure of Leibniz and Suszko reduced models, respectively, of a containment logic $\vdash^r$. The main finding is that the property of a model to be (Leibniz or Suszko) reduced is actually rendered by some conditions on the semilattice structure of the system of the matrix models involved in the construction of the Płonka sums.

Finally, the paper is closed by Section 6, where the results obtained insofar are applied to some examples of containment logics appeared in literature: the containment companions of classical logic, Belnap–Dunn and the logic of paradox (LP).

2 Preliminaries

2.1 Abstract algebraic logic

For standard background on universal algebra and abstract algebraic logic, we refer the reader, respectively, to [5, 13] and [7, 16, 28]. In this paper, algebraic languages are assumed not to contain constant symbols. Moreover, unless stated otherwise, we work within a fixed but arbitrary algebraic language. We denote algebras by $\mathbf{A}, \mathbf{B}, \mathbf{C} \ldots$ respectively with universes $A, B, C \ldots$. We denote by $\mathrm{S}, \mathrm{P}, \mathrm{P}_{SD}$ the class operators of subalgebras, direct products and subdirect products. The same notation applies to logical matrices.

Let $\mathbf{Fm}$ be the algebra of formulas built up over a countably infinite set Var of variables. Given a formula $\varphi \in Fm$, we denote by $Var(\varphi)$ the set of variables really occurring in φ. Similarly, given $\Gamma \subseteq Fm$, we set

$$Var(\Gamma) = \bigcup \{Var(\gamma) \colon \gamma \in \Gamma\}.$$

A *logic* is a substitution invariant consequence relation $\vdash \subseteq \mathcal{P}(Fm) \times Fm$ meaning that for every substitution $\sigma \colon \mathbf{Fm} \to \mathbf{Fm}$,

$$\text{if } \Gamma \vdash \varphi, \text{ then } \sigma[\Gamma] \vdash \sigma(\varphi).$$

Given formulas φ, ψ, we write $\varphi \dashv\vdash \psi$ as a shorthand for $\varphi \vdash \psi$ and $\psi \vdash \varphi$. A logic $\vdash$ is *finitary* when for all $\Gamma \cup \varphi \subseteq Fm$,

$$\Gamma \vdash \varphi \iff \exists \Delta \subseteq \Gamma \text{ such that } \Delta \text{ is finite and } \Delta \vdash \varphi.$$

A *matrix* is a pair $\langle \mathbf{A}, F \rangle$ where $\mathbf{A}$ is an algebra and $F \subseteq A$. In this case, $\mathbf{A}$ is called the *algebraic reduct* of the matrix $\langle \mathbf{A}, F \rangle$.

Every class of matrices M defines a logic as follows:

$$\Gamma \vdash_{\mathsf{M}} \varphi \iff \text{for every } \langle \mathbf{A}, F \rangle \in \mathsf{M} \text{ and hom. } h \colon \mathbf{Fm} \to \mathbf{A},$$
$$\text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

We say that a logic $\vdash$ is *complete* w.r.t. a class of matrices M when $\vdash_{\mathsf{M}} = \vdash$. Sometimes, we will refer to such homomorphisms h as *evaluations*.

A matrix $\langle \mathbf{A}, F \rangle$ is a *model* of a logic $\vdash$ when

$$\text{if } \Gamma \vdash \varphi, \text{ then for every hom. } h \colon \mathbf{Fm} \to \mathbf{A},$$
$$\text{if } h[\Gamma] \subseteq F, \text{ then } h(\varphi) \in F.$$

A set $F \subseteq A$ is a (deductive) *filter* of $\vdash$ on $\mathbf{A}$, or simply a $\vdash$-*filter*, when the matrix $\langle \mathbf{A}, F \rangle$ is a model of $\vdash$. We denote by $\mathcal{F}i_{\vdash}\mathbf{A}$ the set of all filters of $\vdash$ on $\mathbf{A}$, which turns out to be a closure system. Moreover, we denote by $\mathrm{Fg}^{\mathbf{A}}_{\vdash}(\cdot)$ the closure operator of $\vdash$-filter generation on $\mathbf{A}$.

Let $\mathbf{A}$ be an algebra and $F \subseteq A$. A congruence θ of $\mathbf{A}$ is *compatible* with F when for every $a, b \in A$,

$$\text{if } a \in F \text{ and } \langle a, b \rangle \in \theta, \text{ then } b \in F.$$

The largest congruence of $\mathbf{A}$ which is compatible with F always exists and is called the *Leibniz congruence* of F on $\mathbf{A}$. It is denoted by $\Omega^{\mathbf{A}} F$.

Given an $\mathbf{A}$ an algebra, $F \subseteq A$ and a logic $\vdash$ the *Suszko congruence* of F on $\mathbf{A}$ is defined as

$$\widetilde{\Omega}^{\mathbf{A}}_{\vdash} F := \bigcap \{\Omega^{\mathbf{A}} G : F \subseteq G \text{ and } G \in \mathcal{F}i_{\vdash}\mathbf{A}\}.$$

The *Suszko operator* of $\vdash$ on an algebra $\mathbf{A}$ is the function $\widetilde{\Omega}^{\mathbf{A}}_{\vdash}$ with domain $\mathcal{F}i_{\vdash}\mathbf{A}$ defined as $F \to \widetilde{\Omega}^{\mathbf{A}}_{\vdash} F$ for all $F \in \mathcal{F}i_{\vdash}\mathbf{A}$.

Let $\mathbf{A}$ be an algebra. A function $p\colon A^n \to A$ is a *polynomial function* of $\mathbf{A}$ if there are a natural number m, a formula $\varphi(x_1,\dots,x_{n+m})$, and elements $b_1,\dots,b_m \in A$ such that

$$p(a_1,\dots,a_n) = \varphi^{\mathbf{A}}(a_1,\dots,a_n,b_1,\dots,b_m)$$

for every $a_1,\dots,a_n \in A$.

The following lemmas provide very useful criteria to establish whether a pair of elements (of an algebra) belongs to the Leibniz (Suszko, respectively) congruence of a given filter.

Lemma 1

[28, Thm. 4.23] Let $\mathbf{A}$ be an algebra, $F \subseteq A$ and $a, b \in A$.

$$\begin{aligned}\langle a,b\rangle \in \Omega^{\mathbf{A}}F \iff &\text{for every unary pol. function } p\colon \mathbf{A}\to\mathbf{A},\\ &p(a)\in F \text{ if and only if } p(b)\in F.\end{aligned}$$

Lemma 2

[28, Thm. 5.32] Let $\vdash$ be a logic, $\mathbf{A}$ be an algebra, $F \subseteq A$ and $a, b \in A$.

$$\begin{aligned}\langle a,b\rangle \in \widetilde{\Omega}^{\mathbf{A}}_{\vdash}F \iff &\text{for every unary pol. function } p\colon \mathbf{A}\to\mathbf{A},\\ &\mathrm{Fg}^{\mathbf{A}}_{\vdash}(F\cup\{p(a)\}) = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(F\cup\{p(b)\}).\end{aligned}$$

The Leibniz and Suszko congruence singles out two distinguished classes of models of a logic. More precisely, given a logic $\vdash$, we set

$$\mathsf{Mod}(\vdash) := \{\langle\mathbf{A},F\rangle : \langle\mathbf{A},F\rangle \text{ is a model of } \vdash\},$$

$$\mathsf{Mod}^{*}(\vdash) := \{\langle\mathbf{A},F\rangle \in \mathsf{Mod}(\vdash) : \Omega^{\mathbf{A}}F \text{ is the identity}\},$$

$$\mathsf{Mod}^{\mathrm{Su}}(\vdash) := \{\langle\mathbf{A},F\rangle \in \mathsf{Mod}(\vdash) : \widetilde{\Omega}^{\mathbf{A}}_{\vdash}F \text{ is the identity}\}.$$

The above classes of matrices are called, respectively, the classes of models, *Leibniz reduced models*, and *Suszko reduced models* of $\vdash$. Given a logic $\vdash$ and an algebra $\mathbf{A}$, the *Suszko operator* is the function $F \mapsto \widetilde{\Omega}^{\mathbf{A}}_{\vdash}F$ that maps every $\vdash$-filter to its Suszko congruence on $\mathbf{A}$. We say that a matrix $\langle\mathbf{A},F\rangle$ is *trivial* if $F = A$. We denote by $\langle\mathbf{1},\{1\}\rangle$ the trivial matrix, where $\mathbf{1}$ is the trivial algebra. Observe that the latter matrix is a model (resp. Leibniz and Suszko reduced model) of every logic. Moreover, if $\vdash$ is a logic and $\langle\mathbf{A},F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ is a trivial matrix, then $\langle\mathbf{A},F\rangle = \langle\mathbf{1},\{1\}\rangle$.

Given a logic $\vdash$, we set

$$\mathsf{Alg}^{*}(\vdash) = \{\mathbf{A} : \text{ there is } F\subseteq A \text{ s.t. } \langle\mathbf{A},F\rangle \in \mathsf{Mod}^{*}(\vdash)\},$$

and

$$\mathsf{Alg}(\vdash) = \{\mathbf{A} : \text{ there is } F\subseteq A \text{ s.t. } \langle\mathbf{A},F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)\}.$$

In other words, $\mathsf{Alg}^{*}(\vdash)$ ($\mathsf{Alg}(\vdash)$, respectively) is the class of algebraic reducts of matrices in $\mathsf{Mod}^{*}(\vdash)$ ($\mathsf{Mod}^{\mathrm{Su}}(\vdash)$, respectively). It is well known that $\mathsf{Alg}(\vdash) = \mathrm{P}_{SD}(\mathsf{Alg}^{*}(\vdash))$ and that $\mathsf{Mod}^{\mathrm{Su}}(\vdash^{r}) = \mathrm{P}_{SD}\mathsf{Mod}^{*}(\vdash^{r})$ [see 28, Theorem 5.70].

The class $\mathsf{Alg}(\vdash)$ is called the *algebraic counterpart* of $\vdash$. For the vast majority of logics $\vdash$, the class $\mathsf{Alg}(\vdash)$ is the class of algebras intuitively associated with $\vdash$. The so-called *Leibniz hierarchy* measures how strong is the relationship between a logic and its algebraic counterpart. A logic is called *truth-equational* when, intuitively, the logical filter of a Leibniz reduced model is equationally definable. Among others, an equivalent characterization of truth-equationality is the following.

THEOREM 3 ([28, Theorem 6.106]).
A logic $\vdash$ is truth-equational if and only if the Suszko operator is injective over the set of its filters, for any algebra.

A logic is *protoalgebraic* when its signature contains a connective which behaves as a suitable implication. A remarkable consequence of this demand allows for a model-theoretic characterization of protolagebraicity.

THEOREM 4 ([28, Theorem 6.17]).
A logic $\vdash$ is protoalgebraic if and only if the class $\mathsf{Mod}^{*}(\vdash)$ is closed under formation of subdirect products.

A stronger requirement defines the notion of *equivalential* logic. More precisely, a logic is equivalential if and only if the class of Leibniz reduced models is closed under submatrices and direct products [see 28, Cor. 6.74]. A common feature of any truth-equational or non-almost-inconsistent protoalgebraic logic is to have theorems (i.e. a formula φ such that $\emptyset \vdash \varphi$).

The following lemma will be applied in several proofs.

LEMMA 5
Let $\vdash$ be a logic and $\varepsilon, \delta \in Fm$. The following are equivalent:

1. $\mathsf{Alg}(\vdash) \vDash \varepsilon \approx \delta$;
2. $\varphi(\varepsilon, \vec{z}\,) \dashv\vdash \varphi(\delta, \vec{z}\,)$, for every formula $\varphi(v, \vec{z}\,)$.

PROOF. See [28, Lemma 5.74(1)] and [28, Theorem 5.76]. □

2.2 Containment logics

The notion of *antitheorem* is a generalization for arbitrary logics of the classical notion of inconsistent set of formulas.

The definition originates in [31], but see also [14, 48].

DEFINITION 6
A set of formulas Σ is an antitheorem of a logic $\vdash$ if $\sigma[\Sigma] \vdash \varphi$ for every substitution σ and formula φ.

Observe that, if the set $\Sigma(y_1, \ldots, y_n)$, where the variables $y_1, \ldots, y_n$ really occur, is an antitheorem for $\vdash$, then, by substitution, also $\Sigma(x)$ (where only x occurs) is an antitheorem for $\vdash$. In other words, if a logic $\vdash$ possesses an antitheorem Σ, then it possesses an antitheorem in one variable only. When referring to this fact, we will write $\Sigma(x)$.

EXAMPLE 7
For any formula φ, the set $\{\varphi, \neg\varphi\}$ is an antitheorem of intuitionistic, classical and both local and global modal logics. ⊠

The following theorem can be inferred from [48, Thm. 3.6], and it discloses fundamental properties of antitheorems for finitary protoalgebraic logics.

THEOREM 8
Let $\vdash$ be a finitary protoalgebraic logic and $\Sigma(x) = \{x, \psi_1(x), \ldots, \psi_n(x)\}$ be an antitheorem for $\vdash$.

Let F be a $\vdash$-filter on an algebra $\mathbf{A}$ and $a \in A$. Then,

$$A = \mathrm{Fg}^{\mathbf{A}}_{\vdash}(\{a\} \cup F) \iff \{\psi^{\mathbf{A}}_1(a) \ldots \psi^{\mathbf{A}}_n(a)\} \subseteq F.$$

Containment logics are consequence relations satisfying a variable inclusion constraint: every propositional variable in a formula which is the conclusion of an antitheorem-free inference shall be included in the set of variables of the formulas in the premises. In particular, to any (propositional) logic $\vdash$ a new consequence relation $\vdash^r$ can be associated, according to the following.

Definition 9
Let $\vdash$ be a logic. $\vdash^r$ is the logic defined as

$$\Gamma \vdash^r \varphi \iff \begin{cases} \Gamma \vdash \varphi \text{ and } Var(\varphi) \subseteq Var(\Gamma) & \text{or} \\ \Gamma \text{ is an antitheorem of } \vdash. \end{cases}$$

We will refer to $\vdash^r$ as the *containment companion* (or, *right variable inclusion companion*) of the logic $\vdash$. It is immediate to check that a containment logic is theoremless (as we assume constant-free signatures); therefore, it can be neither protoalgebraic nor truth-equational.

The best-known example of containment logic is Bochvar three-valued logic, which is the containment companion of classical logic [this follows from 51, Theorem 2.3.1]. Semantically, Bochvar is defined by a single matrix featuring the presence of an infectious truth-value, which makes it suitable for modelling different kind of situations such as computer programs affected by errors [22], non-sensical information [15], the notion on/off topic [3] and severe ignorance [8].

Containment companions have been studied also for other logics, such as Belnap–Dunn, the logic of paradox and paraconsistent weak Kleene. The respective containment companions are known as $\mathbf{FDE}_{\varphi}$, introduced by Priest [44] and, independently by Daniels [17] [the fact is proved in 23, Theorem 28]; the logic $\mathbf{S}_{fde}$, introduced by Deutsch [18] (the result that has firstly been shown in [24, Observation 9]).

A first study of containment logics under the perspective of abstract algebraic logic can be found in [11].

Remark 10
[11, Lemma 7] If $\vdash$ is a finitary logic then $\vdash^r$ is also finitary.

Płonka sums

As standard references on Płonka sums we mention [37, 38, 41, 50]. A *semilattice* is an algebra $\mathbf{A} = \langle A, \vee \rangle$, where $\vee$ is a binary associative, commutative and idempotent operation. Given a semilattice $\mathbf{A}$ and $a, b \in A$, we set

$$a \leq b \iff a \vee b = b.$$

It is easy to see that $\leq$ is a partial order on A.

Definition 11
A *direct system of algebras* consists of

1. a semilattice $I = \langle I, \vee \rangle$;
2. a family of similar algebras $\{\mathbf{A}_i : i \in I\}$ with pair-wise disjoint universes;

3. a homomorphism $f_{ij}\colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i,j \in I$ such that $i \leq j$;

moreover, f_{ii} is the identity map for every $i \in I$, and if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$.

Let X be a direct system of algebras as defined above. The *Płonka sum* of X, in symbols $\mathcal{P}_{\text{ł}}(X)$ or $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$[1] , is the algebra in the same type defined as follows: the universe of $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ is the union $\bigcup_{i\in I} A_i$. Moreover, for every n-ary basic operation f and $a_1, \ldots, a_n \in \bigcup_{i\in I} A_i$, we set

$$g^{\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}}(a_1, \ldots, a_n) \coloneqq g^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n)),$$

where $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$ and $j = i_1 \vee \cdots \vee i_n$.

Observe that if in the above display we replace g by any complex formula φ in n-variables, we still have that

$$\varphi^{\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}}(a_1, \ldots, a_n) = \varphi^{\mathbf{A}_j}(f_{i_1 j}(a_1), \ldots, f_{i_n j}(a_n)).$$

The theory of Płonka sums is strictly related with a special kind of binary operation, called partition function.

Definition 12

Let $\mathbf{A}$ be an algebra of type ν. A function $\cdot\colon A^2 \to A$ is a *partition function* in $\mathbf{A}$ if the following conditions are satisfied for all $a, b, c \in A$, $a_1, ..., a_n \in A^n$ and for any operation $g \in \nu$ of arity $n \geqslant 1$.

P1. $a \cdot a = a$;
P2. $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;
P3. $a \cdot (b \cdot c) = a \cdot (c \cdot b)$;
P4. $g(a_1, \ldots, a_n) \cdot b = g(a_1 \cdot b, \ldots, a_n \cdot b)$;
P5. $b \cdot g(a_1, \ldots, a_n) = b \cdot a_1 \cdot_{\ldots} \cdot a_n$.

Different definitions of partition function have appeared in literature. We adopted the one from [41], which uses the minimal number of definitional conditions.

The next result underlines the connection between Płonka sums and partition functions.

Theorem 13

[37, Thm. II] Let $\mathbf{A}$ be an algebra of type ν with a partition function $\cdot$. The following conditions hold.

1. A can be partitioned into $\{A_i : i \in I\}$ where any two elements $a, b \in A$ belong to the same component A_i exactly when

$$a = a \cdot b \text{ and } b = b \cdot a.$$

   Moreover, every A_i is the universe of a subalgebra $\mathbf{A}_i$ of $\mathbf{A}$.
2. The relation $\leq$ on I given by the rule

$$i \leq j \iff \text{there exist } a \in A_i, b \in A_j \text{ s.t. } b \cdot a = b$$

   is a semilattice order.

[1] When no confusion shall occur, we will write $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)$ instead of $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$.

3. For all $i,j \in I$ such that $i \leq j$ and $b \in A_j$, the map $f_{ij}\colon A_i \to A_j$, defined by the rule $f_{ij}(x) = x \cdot b$ is a homomorphism. The definition of f_{ij} is independent from the choice of b, since $a \cdot b = a \cdot c$, for all $a \in A_i$ and $c \in A_j$.
4. $Y = \langle\langle I, \leq\rangle, \{\mathbf{A}_i\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$ is a direct system of algebras such that $\mathcal{P}_{\nmid}(Y) = \mathbf{A}$.

The statement of Theorem 13 displayed above relies on the assumption that the algebraic language contains no constant symbols.[2]

It is worth remarking that the construction of Plonka sums preserves the validity of the *regular identities*. An identity $\varphi \approx \psi$ (of a given type) is *regular* provided that $Var(\varphi) = Var(\psi)$. Recall that a variety $\mathcal{V}$ is called *regular* if it satisfies regular identities only. Examples of regular varieties include semigroups, monoids and semilattices. A variety which is not regular is called *irregular*.

DEFINITION 14
A variety $\mathcal{V}$ (of type τ) is *strongly irregular* if there is formula $\phi(x,y)$ such that $\mathcal{V} \models \phi(x,y) \approx x$.

Examples of strongly irregular varieties include the variety of lattices (more in general, any variety having a lattice reduct), the variety of Heyting algebras, BCK algebras, groups and rings.

Given a strongly irregular variety $\mathcal{V}$, it is possible to associate with it a variety $R(\mathcal{V})$ which satisfies all and only the regular identities holding in $\mathcal{V}$. $R(\mathcal{V})$ is called the *regularization* of $\mathcal{V}$. Elements of the regularization of a strongly irregular variety can always be represented as Płonka sums.

THEOREM 15
[41, Thm. 7.1] Let $\mathcal{V}$ be a strongly irregular variety of type τ. Then $\mathbf{A} \in R(\mathcal{V})$ iff $\mathbf{A}$ is decomposable as a Płonka sum over a direct system of algebras in $\mathcal{V}$.

2.3 Płonka sums of matrices

The construction of the Płonka sum has been extended to logical matrices in [10] and in [11] for the algebraic study of containment logics. We recall the results from [11] that will be used in the present work.

DEFINITION 16
An *r-direct system* of matrices consists of

1. a semilattice $I = \langle I, \vee\rangle$;
2. a family of matrices $\{\langle \mathbf{A}_i, F_i\rangle : i \in I\}$ such that $I^+ := \{i \in I : F_i \neq \emptyset\}$ is a sub-semilattice of I;
3. a homomorphism $f_{ij}\colon \mathbf{A}_i \to \mathbf{A}_j$, for every $i,j \in I$ such that $i \leq j$, satisfying also that
    - f_{ii} is the identity map, for every $i \in I$;
    - if $i \leq j \leq k$, then $f_{ik} = f_{jk} \circ f_{ij}$;
    - if $F_j \neq \emptyset$ then $f_{ij}^{-1}[F_j] = F_i$, for any $i \leq j$.

[2] In presence of constants, the construction of the Płonka sum shall be slightly modified, as shall the definition of a partition function. In particular, the semilattice of indexes shall be equipped with a bottom element: the constant operations in the Płonka sum correspond to the constants of the algebra whose index is such minimum. For more details, we refer the reader directly to [40].

Given a r-direct system of matrices X, we define a new matrix as

$$\mathcal{P}_{\nmid}(X) \coloneqq \langle \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}, \bigcup_{i\in I} F_i\rangle.$$

We will refer to the matrix $\mathcal{P}_{\nmid}(X)$ as the *Płonka sum* over the r-direct system of matrices X. Given a class M of matrices, $\mathcal{P}_{\nmid}(\mathsf{M})$ will denote the class of all Płonka sums of r-direct systems of matrices in M.

Partitions functions, which have been originally introduced for algebras (see Definition 12), can be defined also for logics.

Definition 17

A logic $\vdash$ has an *r-partition function* if there is a formula $x * y$, in which the variables x and y really occur, such that

1. $x, y \vdash x * y$,
2. $x * y \vdash x$,
3. $\varphi(\varepsilon, \vec{z}) \dashv\vdash \varphi(\delta, \vec{z})$,

for every formula $\varphi(v, \vec{z})$ and every identity of the form $\varepsilon \approx \delta$ in Definition 12.[3]

Condition (iii) in the definition of r-partition function is actually equivalent to saying that the term operation $*$ is a partition function in every algebra $\mathbf{A} \in \mathsf{Alg}(\vdash)$. We will denote both the formula $x * y$ and the term operation $*$ as r-partition functions with respect to a logic $\vdash$.

Example 18

Logics with an r-partition function abound in the literature. Indeed, the term $x * y \coloneqq x \wedge (x \vee y)$ is a partition function for every logic $\vdash$ such that $\mathsf{Alg}(\vdash)$ has a lattice reduct. Such examples include all modal and substructural logics [29]. On the other hand, the term $x * y \coloneqq (y \to y) \to x$ as an r-partition function for all the logics $\vdash$ whose class $\mathsf{Alg}(\vdash)$ possesses a Hilbert algebra (see [19]) or a BCK algebra (see [30]) reduct.

Remark 19

It is easily checked that a logic $\vdash$ has r-partition function $*$ if and only if $\vdash^r$ has r-partition function $*$.

Płonka representation theorem can be proved for r-direct systems of logical matrices.

Theorem 20

[11, Theorem 24] Let $\vdash$ be a logic with a r-partition function $*$ and $\langle \mathbf{A}, F\rangle$ be a model of $\vdash$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash)$. Then Theorem 13 holds for $\mathbf{A}$. Moreover, by setting $F_i \coloneqq F \cap A_i$ for every $i \in I$, the triple

$$X = \langle \langle I, \leq\rangle, \{\langle \mathbf{A}_i, F_i\rangle\}_{i\in I}, \{f_{ij} : i \leq j\}\rangle$$

is an r-direct system of matrices such that $\mathcal{P}_{\nmid}(X) = \langle \mathbf{A}, F\rangle$.

[3] We deliberately adopt a different notation ($*$ instead of the previously introduced $\cdot$) to highlight that this definition applies to a logic $\vdash$ (not to a class of algebras).

The construction of the Płonka sum over r-direct systems of matrices is useful to provide algebraic semantics to containment logics. In particular, given a finitary logic $\vdash$ complete with respect to a class of nontrivial matrices M, then $\vdash^r$ is (sound and) complete with respect to $\mathcal{P}_{\nmid}(\mathsf{M}^{\emptyset})$ [11, Theorem 13], where $\mathsf{M}^{\emptyset} = \mathsf{M} \cup \langle \mathbf{A}, \emptyset \rangle$, for any arbitrary $\mathbf{A} \in \mathsf{Alg}(\vdash)$. We now state a lemma which can be directly inferred from [11, Cor. 16,18].

LEMMA 21
Let $\langle \mathcal{P}_{\nmid}(\mathbf{A}_i), \bigcup F_i \rangle_{i \in I}$ be a r-direct system of matrices with $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ for every $i \in I$. Suppose, moreover, that $\langle \mathbf{A}_j, F_j \rangle \in \mathsf{Mod}(\vdash)$ for every $j \in I^+$.

1. If $\vdash$ has no antitheorem, then $\langle \mathcal{P}_{\nmid}(\mathbf{A}_i), \bigcup F_i \rangle_{i \in I}$ is a model of $\vdash^r$.
2. If $\vdash$ has an antitheorem and $A_j \neq F_j$ for every $j \in I^+$, then $\langle \mathcal{P}_{\nmid}(\mathbf{A}_i), \bigcup F_i \rangle_{i \in I}$ is a model of $\vdash^r$.

Convention: from now on, we will assume (with no explicit mention) that all the logics $\vdash$ considered are finitary, possess an r-partition function $*$ and do not possess constants in their language.

3 The algebraic counterpart of containment logics

In this section, we describe the structure of the class Alg of a containment logic $\vdash^r$. Such a characterization depends on specific properties of the initial logic $\vdash$. We now recall a result that is instrumental in many of the subsequent proofs.

LEMMA 22
[28, Prop. 2.24] Let $\langle \mathbf{A}, F \rangle, \langle \mathbf{B}, G \rangle$ be two matrices, and let $h \colon \mathbf{A} \to \mathbf{B}$ a homomorphism such that $F = h^{-1}[G]$. If G is a $\vdash$-filter on $\mathbf{B}$ then F is a $\vdash$-filter on $\mathbf{A}$.

Let us now state a result which will be proved as a corollary of Theorem 40.

COROLLARY 23
For any logic $\vdash$, $\mathsf{Alg}^*(\vdash^r) \subseteq \mathcal{P}_{\nmid}(\mathsf{Alg}^*(\vdash))$.

The following lemma will be applied multiple times in this section.

LEMMA 24
Let $\mathsf{Alg}(\vdash)$ be a class closed under subalgebras. Then $\mathsf{Alg}(\vdash^r) \subseteq \mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash))$.

PROOF. We have

$$\mathsf{Alg}(\vdash^r) = \mathrm{P}_{SD}(\mathsf{Alg}^*(\vdash^r)) \subseteq \tag{1}$$

$$\mathrm{P}_{SD}(\mathcal{P}_{\nmid}(\mathsf{Alg}^*(\vdash))) \subseteq \tag{2}$$

$$\mathrm{SP}(\mathcal{P}_{\nmid}(\mathsf{Alg}^*(\vdash))) \subseteq \tag{3}$$

$$\mathcal{P}_{\nmid}(\mathrm{SP}(\mathsf{Alg}^*\vdash) \subseteq \tag{4}$$

$$\mathcal{P}_{\nmid}(\mathrm{SP}(\mathsf{Alg}\vdash) = \mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash)), \tag{5}$$

where (2) holds in virtue of Corollary 23 (a result that can be proven independently (see below Theorem 40)), while the last equality holds since $\mathsf{Alg}(\vdash)$ is closed under subalgebras. ☐

REMARK 25
Observe that any equivalential logic $\vdash$ falls under the scope of Lemma 24, as well as all the non-equivalential logics whose class Alg is a quasi-variety.

The following lemma establishes how a logical filter of a logic $\vdash$ can be extended to a $\vdash^r$-filter by means of pre-images of a Płonka homomorphism. In order to simplify the notation, given a Płonka sum of algebras $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ and $G_j \subseteq A_j$ (for some $j \in I$), we set

$$\downarrow G_j = \bigcup_{k\leq j} (f_{kj}^{-1}(G_j)).$$

LEMMA 26
Let $\vdash^r$ be a logic possessing a r-partition function $*$ and $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ with $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ for each $i \in I$. If $G_i \neq A_i$ is a non-empty $\vdash$-filter, then $\langle \mathbf{A}, \downarrow G_i\rangle \in \mathsf{Mod}(\vdash^r)$.

PROOF. At first, observe that, by Lemma 22, $\langle \mathbf{A}_k, f_{ki}^{-1}(G_i)\rangle \in \mathsf{Mod}(\vdash)$, for each $k \leq i$. Since $\mathbf{A} \cong \mathcal{P}_{\nmid}(\mathbf{A}_i)$, by construction it is immediate to check that $\langle \mathbf{A}, \downarrow G_i\rangle$ is isomorphic to a Płonka sum over a r-direct system of matrices. We shall consider two possibilities: (i) $\vdash$ does not have an antitheorem, (ii) $\Sigma(x)$ is an antitheorem of $\vdash$. In the case of (i), our conclusion follows by Lemma 21. If (ii), we firstly show that for each $k \leq i$, $f_{ki}^{-1}(G_i) \neq A_k$. Suppose the contrary towards a contradiction. Consider an arbitrary homomorphism $h\colon \mathbf{Fm} \to \mathbf{A}_k$. Clearly, $h(\Sigma(x)) \in A_k = f_{ki}^{-1}(G_i)$ and so $f_{ki}(h(\Sigma(x))) \in G_i$. This entails $f_{ki} \circ h$ is an evaluation that maps $\Sigma(x)$ into a subset of G_i. Consider now $d \in A_i \smallsetminus G_i$ and an evaluation $v\colon \mathbf{Fm} \to \mathbf{A}_i$ such that $v(x) = f_{ki} \circ h(x)$ and $v(y) = d$ for all the variables $y \neq x$. Clearly, we have $v(\Sigma(x)) \in G_i$ and $v(y) \notin G_i$ against the assumption that $\langle \mathbf{A}_i, G_i\rangle$ is a model of $\vdash$. This proves that for each $k \leq i$ $f_{ki}^{-1}(G_i) \neq A_k$. So, by Lemma 21, we conclude that $\langle \mathbf{A}, \downarrow G_i\rangle$ is a $\vdash^r$ model. ☐

Before moving on, let us state one last preliminary lemma.

LEMMA 27
Let $\mathbf{A} \in \mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash))$, $a \in A_i, b \in A_j$ with $j \nleq i$. If for some unary polynomial function $\varphi(x, \vec{z})$ and $\vec{c} \in A_i$, $F = \downarrow \mathrm{Fg}_{\vdash}^{\mathbf{A}_i}(\varphi(a, \vec{c}))$ is a $\vdash^r$-filter on $\mathbf{A}$, then $\langle a, b\rangle \notin \Omega^{\mathbf{A}} \downarrow F$.

PROOF. It suffices to note that $\varphi(b, \vec{c}) \notin F$, so the statement follows by Lemma 1. ☐

We now investigate the structure of the class Alg of a logic without antitheorems.

3.1 Logics without antitheorems

REMARK 28
Observe that, when $\vdash$ has no antitheorem, the assumption of $G_i \neq A_i$ can be safely dropped from the statement of Lemma 26.

A natural task is to determine under which conditions the class $\mathsf{Alg}(\vdash^r)$ corresponds to the Płonka sums over $\mathsf{Alg}(\vdash)$. A sufficient condition for this to happen is stated in the following.

PROPOSITION 29
Let $\vdash$ be a logic possessing no antitheorem. Then, $\mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash)) \subseteq \mathsf{Alg}(\vdash^r)$.

PROOF. Consider an antitheorems-free logic $\vdash$, and set $\mathbf{A} = \mathcal{P}_{\text{ł}}(\mathbf{A}_i)_i$ with $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ for each $i \in I$. It is immediate to check that $\langle \mathbf{A}, \emptyset \rangle \in \mathsf{Mod}(\vdash^r)$. Let $a \in A_i$, $b \in A_j$ (for some $i,j \in I$). If $i = j$, then clearly there exists $F_i \subseteq A_i$ such that $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$, i.e. for some $\vec{c} \in A_i$ and unary polynomial function $\varphi(x, \vec{z})$ it holds $\varphi^{\mathbf{A}}(a, \vec{c}) \in F_i$ and $\varphi^{\mathbf{A}}(b, \vec{c}) \notin F_i$. Lemma 26 (and Remark 28) entails that $\downarrow F_i$ is a $\vdash^r$-filter on $\mathbf{A}$ and clearly $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} \downarrow F_i$, as desired.

Let now $i \neq j$; set $k = i \vee j$, and consider $b * c$ for $c = f_{ik}(a) \in A_k$. By Lemma 26 (and Remark 28), $\downarrow F_i$ is a $\vdash^r$-filter on $\mathbf{A}$. Since $a * a = a \in G$, while $b * a = b * c \notin \downarrow F_i$, we conclude $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} \downarrow F_i$, as desired. □

REMARK 30
The assumption on the lack of antitheorems is crucial in Proposition 29, which is indeed false in case $\vdash$ has antitheorems, as witnessed by the following example. Let $\vdash$ be classical logic. Clearly, the trivial algebra belongs to $\mathsf{Alg}(\vdash)$, while the Płonka sum $n \oplus m$ whose fibers are two trivial algebras does not belong to $\vdash^r$, i.e. Bochvar logic. It is indeed immediate to verify that the only $\vdash^r$-filters on $n \oplus m$ are the empty and the total filters.

A full description of the class $\mathsf{Alg}(\vdash^r)$ can be obtained provided that the starting logic $\vdash$ possesses no antitheorems and $\mathsf{Alg}(\vdash)$ is closed under subalgebras.

THEOREM 31
Let $\vdash$ be a logic without antitheorems and such that $\mathsf{Alg}(\vdash)$ is closed under subalgebras. Then $\mathcal{P}_{\text{ł}}(\mathsf{Alg}(\vdash)) = \mathsf{Alg}(\vdash^r)$.

PROOF. Combine Proposition 29 and Lemma 24. □

Since every equivalential logic $\vdash$ satisfies $\mathsf{Alg}(\vdash) = \mathbb{S}(\mathsf{Alg}(\vdash))$, we obtain the following.

COROLLARY 32
Let $\vdash$ be an equivalential logic without antitheorems. Then $\mathsf{Alg}(\vdash^r) = \mathcal{P}_{\text{ł}}(\mathsf{Alg}(\vdash))$.

Examples of equivalential logics with no antitheorems include the implicative fragment of classical logic, as well as Da Costa and D'Ottaviano's three-valued paraconsistent logic [see 28, p.482].

3.2 *Logics with antitheorems*

We now address the study of the class $\mathsf{Alg}(\vdash^r)$ in case $\vdash$ has antitheorems. The following lemma established the necessary conditions under which Płonka sums of algebras belonging to $\mathsf{Alg}(\vdash)$ actually belong to $\mathsf{Alg}(\vdash^r)$.

LEMMA 33
Let $\vdash$ be a logic with antitheorems, $\mathbf{A} \in \mathcal{P}_{\text{ł}}(\mathsf{Alg}(\vdash))$, and suppose that $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. Then, in case $\{a_i\}, \{a_j\}$ are universes of trivial fibers $\mathbf{A}_i, \mathbf{A}_j$ (of $\mathbf{A}$), then there exists a nontrivial fiber $\mathbf{A}_k$, with $i < k$ or $j < k$ such that $a_i * b \neq a_j * b$, for some $b \in A_k$.

PROOF. Assume $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$, and let $A_i = \{a_i\}, A_j = \{a_j\}$ be universes of two distinct trivial fibers. Since $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$, there exists a unary polynomial function $\varphi(x, \vec{z})$, elements $\vec{c} \in A$ and $\vdash^r$ filter F on $\mathbf{A}$ such that $\varphi^{\mathbf{A}}(a_i, \vec{c}) \in F$ and $\varphi^{\mathbf{A}}(a_j, \vec{c}) \notin F$. Let us set $p = i \vee q$, where q is the join of the indexes of fibers each element of $\vec{c}$ belongs to. Observe that the fact that $\vdash^r$ has antitheorems (since $\vdash$ has antitheorems) implies that $\mid A_p \mid > 1$. Indeed, suppose that $\mathbf{A}_p$ is trivial, then $A_p = F_p$; thus,

$A = F$ (otherwise, it is immediate to have a counterexample to an antitheorem), against the fact that $\varphi^{\mathbf{A}}(a_j, \vec{c}) \notin F$. Observe that we also have $i < p$ (as otherwise, we would get $F_i = A_i$, thus $A = F$). Let now $b \in A_p$. If $a_j * b \in A_p$, clearly, $a_i * b \neq a_j * p$ because $\varphi(a_i, \vec{c}) = \varphi(a_j, \vec{c})$. If, on the other hand, $a_j * b \notin A_p$, we obtain $a_i * b \neq a_j * p$. This concludes the proof. $\square$

The converse of the above result can be proved assuming the logic $\vdash$ is protoalgebraic, as shown by the next proposition.

PROPOSITION 34
Let $\vdash$ be a protoalgebraic logic with antitheorems and $\mathbf{A} \in \mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash))$. Moreover, assume that, if $\{a_i\}, \{a_j\}$ are the universes of distinct trivial fibers $\mathbf{A}_i, \mathbf{A}_j$, then there exists a nontrivial fiber $\mathbf{A}_k$, with $i < k$ or $j < k$ such that $a_i * b \neq a_j * b$, for some $b \in A_k$. Then $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$.

PROOF. Let a, b be two arbitrary distinct elements in A. If a, b belongs to the same fiber $\mathbf{A}_i$, or neither $\mathbf{A}_i$ nor $\mathbf{A}_j$ are trivial, let F_i be a Suszko $\vdash$-filter on A_i (assume without loss of generality that $j \nless i$). By Lemma 26, $\downarrow F_i$ is a $\vdash^r$-filter and our conclusion follows by Lemma 27. So consider $a \in A_i, b \in A_j$ with $i \neq j$, for some $i, j \in I$ and let $\{x, \psi_1(x), \dots, \psi_n(x)\}$ be an antitheorem of $\vdash$.

Firstly, suppose that $\mathbf{A}_i, \mathbf{A}_j$ are trivial fibers. Then, by assumption there exists a nontrivial fiber $\mathbf{A}_k$ with $i < k$ or $j < k$ such that $a * c \neq b * c$, for some $c \in A_k$. Without loss of generality, consider $i < k$ (the case $j < k$ is analogous). We have $a_k = a * c = f_{ik}(a) \neq b_p = b * c = f_{jp}(b)$ where $p = k \vee j$. Clearly, if $k = p$, our conclusion directly follows by the fact that $\mathbf{A}_k \in \mathsf{Alg}(\vdash)$ and Lemmas 26 and 27. So let $k \neq p$, which entails $k < p$. Consider a $\vdash$-filter F_k on $\mathbf{A}_k$ such that $F_k \neq \mathbf{A}_k$, which exists because $\mathbf{A}_k$ is nontrivial. Two cases may arise. If $A_k = \mathrm{Fg}_{\vdash}^{\mathbf{A}_k}(\{a_k\} \cup F_k)$, then, since $\vdash$ is protoalgebraic and finitary, we can apply Theorem 8 getting $\{\psi_1(a_k), \dots, \psi_n(a_k)\} \subseteq F_k$. So, by Lemma 26, $\downarrow F_k$ is a $\vdash^r$-filter on $\mathbf{A}$ and $\psi_1(a_k) \in \downarrow F_k$ while $\psi_1(b_p) \notin \downarrow F_k$, i.e. $\langle a, b \rangle \notin \widetilde{\Omega}_{\vdash^r}^{\mathbf{A}} \downarrow F_k$. Otherwise, let $H_i = \mathrm{Fg}_{\vdash}^{\mathbf{A}_k}(\{a_k\} \cup F_k) \neq A_k$. A further application of Lemma 26 guarantees that $\downarrow H_k$ is a $\vdash^r$-filter and clearly $a_k \in \downarrow H_k, b_p \notin \downarrow H_k$, i.e. $\langle a, b \rangle \notin \widetilde{\Omega}_{\vdash^r}^{\mathbf{A}} \downarrow H_k$, as desired. Finally, suppose $a \in A_i, b \in A_j$ with $i \neq j$ and at least one among $\mathbf{A}_i, \mathbf{A}_j$ is nontrivial. Without loss of generality, assume $\mathbf{A}_i$ is nontrivial, and consider a Suszko filter F_i over A_i. Observe that, by Lemma 26, we can claim that if $G_i \neq A_i$ is a $\vdash$-filter on A_i, then $\downarrow G_i$ cannot contain the universe of a trivial fiber. Now, if $\mathrm{Fg}_{\vdash}^{\mathbf{A}_i}(\{a\} \cup F_i) = A_i$, by Theorem 8, we have $\{\psi_1(a) \dots \psi_n(a)\} \subseteq F_i$ and clearly $\psi_1(b) \notin \downarrow F_i$. Indeed, since $b \in A_j$, we have $\psi_1(b) = b$, so $\psi_1(b) \in F$ if and only if $b \in F_j$, which is impossible by the above claim. This entails $\langle a, b \rangle \notin \Omega^{\mathbf{A}} \downarrow F_i$, as desired. The only case left is $G_i = \mathrm{Fg}_{\vdash}^{\mathbf{A}_i}(\{a\} \cup F_i) \neq A_i$. In this case, the above claim ensures $b \notin \downarrow G_i$. Since $a \in \downarrow G_i, b \notin \downarrow G_i$, we obtain again the desired conclusion that $\langle a, b \rangle \notin \Omega^{\mathbf{A}} \downarrow G_i$. This concludes the proof. $\square$

The following corollary provides a full characterization of the algebraic counterpart of an equivalential logic with antitheorems.

COROLLARY 35
Let $\vdash$ be an equivalential logic with antitheorems and $\mathbf{A}$ be an algebra. The following are equivalent.

1. $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$
2. $\mathbf{A} \in \mathcal{P}_{\nmid}(\mathsf{Alg}(\vdash))$ and if $\{a_i\}, \{a_j\}$ are the universes of trivial fibers $\mathbf{A}_i, \mathbf{A}_j$, then there exists a nontrivial fiber $\mathbf{A}_k$, with $i < k$ or $j < k$ such that $a_i * b \neq a_j * b$, for some $b \in A_k$.

PROOF. Since in an equivalential logic the class Alg is closed under subalgebras, our statement follows by Lemmas 24 and 33 and Proposition 34. $\square$

For a logic $\vdash$ possessing antitheorems, a simpler characterization of the class $\mathsf{Alg}(\vdash^r)$ can be given in case every nontrivial member of $\mathsf{Alg}(\vdash)$ lacks trivial subalgebras. A quasivariety of algebras with the property that each of its nontrivial members lacks trivial subalgebras is called *Kollár* in [33]. Coherently, we extend this terminology also to an arbitrary class of algebras. More precisely, a class of algebras is *Kollár* if every nontrivial member of the class lacks trivial subalgebras. We now proceed with a full description of the class $\mathsf{Alg}(\vdash^r)$, which is available provided that $\vdash$ also enjoys protoalgebraicity.

THEOREM 36
Let $\vdash$ be a logic with antitheorems such that $\mathsf{Alg}(\vdash)$ is Kollár, and let $\mathbf{A} \in \mathcal{P}_{\text{ł}}(\mathsf{Alg}(\vdash))$. If $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$ then $\mathbf{A}$ has at most one trivial fiber.

PROOF. Let $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$, and assume, by contradiction, that there are two trivial fibers $\mathbf{A}_i, \mathbf{A}_j$ in $\mathbf{A}$, with $A_i = \{a\}, A_j = \{b\}$. Let $k \in I$ with $i \vee j \leq k$; then, $f_{ik}(\mathbf{A}_i)$ (and $f_{jk}(\mathbf{A}_j)$) is the universe of a subalgebra of $\mathbf{A}_k$. Since each nontrivial member of $\mathsf{Alg}(\vdash)$ lacks trivial subalgebras, then $\mathbf{A}_k$ is trivial. This contradicts Lemma 33, so there exists at most one trivial fiber. □

COROLLARY 37
Let $\vdash$ be a protoalgebraic logic with antitheorems such that $\mathsf{Alg}(\vdash)$ is Kollár. Let $\mathbf{A} \in \mathcal{P}_{\text{ł}}(\mathsf{Alg}(\vdash))$, then the following are equivalent:

1. $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$;
2. $\mathbf{A}$ has at most one trivial fiber.

PROOF. (i) $\Rightarrow$ (ii) follows from Theorem 36.
(ii) $\Rightarrow$ (i) follows from Proposition 34. □

REMARK 38
It follows from the proof of Theorem 36 that, in case $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$ contains a (unique) trivial subalgebra, then the lattice of indexes of the Płonka sum representation has a top element, which coincides with the index of the trivial algebra.

4 Leibniz reduced models of a containment logic

This short section contains a description of the Leibniz reduced models of a containment logic. Some concrete applications of the results are provided in Section 6.

We start by an auxiliary lemma. It provides information concerning the Płonka sum representation of the filter of a Leibniz reduced model of $\vdash^r$.

LEMMA 39
Let $\vdash$ a logic, $\mathbf{A} \neq \mathbf{1}$ and $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. Suppose $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^r)$, then I^+ is a singleton.

PROOF. Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\vdash^r)$. Since $\mathsf{Alg}^*(\vdash^r) \subseteq \mathsf{Alg}(\vdash^r)$, then, by applying Theorem 20, the matrix $\langle \mathbf{A}, F\rangle$ is a Płonka sum over a r-direct system X of matrices. Suppose, towards a contradiction, that I^+ is not a singleton. Clearly, $I^+ \neq \emptyset$; otherwise, $\langle \mathbf{A}, F\rangle$ is a Płonka sum of matrices with empty filters, any of which can not be Leibniz reduced as we have assumed that $\mathbf{A} \neq \mathbf{1}$. Suppose now there are elements $i, j \in I^+$ such that $i < j$ (this is justified by the fact that I^+ is a semilattice). Since $F_i \neq \emptyset$, let $a \in F_i$ and $f_{ij}(a) = b \in F_j$. We claim that $\langle a, b\rangle \in \Omega^{\mathbf{A}} F$.

In order to show this, we use the characterization provided in Lemma 1. Let $\varphi(v, \vec{z})$ be an arbitrary unary polynomial function, and assume $\varphi^{\mathbf{A}}(a, \vec{c}) \in F$, with $\vec{c} \in A_s$, for some $s \in I$. Clearly, $\varphi^{\mathbf{A}}(a, \vec{c}) \in F_k$, where $k = i \vee s$. Observe that $j, k \in I^+$, hence also $k \vee j = p \in I^+$ (as I^+ is a sub-semilattice of I). In particular,

$$\varphi^{\mathbf{A}}(b, \vec{c}) = \tag{6}$$

$$\varphi^{\mathbf{A}}(f_{ij}(a), \vec{c}) = \tag{7}$$

$$\varphi^{\mathbf{A}_p}(f_{jp}(f_{ij}(a)), f_{sp}(\vec{c})) = \tag{8}$$

$$\varphi^{\mathbf{A}_p}(f_{kp}(f_{ik}(a)), f_{kp}(f_{sk}(\vec{c}))) = \tag{9}$$

$$f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c})) = \tag{10}$$

$$f_{kp}(\varphi^{\mathbf{A}}(a, \vec{c})) \in F_p. \tag{11}$$

In particular, (9) holds as $s \vee j = p$; (10) by observing that $f_{ip} = f_{jp} \circ f_{ij} = f_{kp} \circ f_{ik}$ and $s \leq k \leq p$; (11) since $\varphi^{\mathbf{A}}(a, \vec{c}) \in F_k$ implies that $f_{kp}(\varphi^{\mathbf{A}}(a, \vec{c})) \in F_p$.

Similarly, assume $\varphi(b, \vec{c}) \in F$, i.e. $\varphi(b, \vec{c}) \in F_p$. Suppose, towards a contradiction that $\varphi(a, \vec{c}) \notin F$, which means $\varphi^{\mathbf{A}}(a, \vec{c}) = \varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c})) \notin F_k$, whence $f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c}))) \notin F_p$. However,

$$\begin{aligned} f_{kp}(\varphi^{\mathbf{A}_k}(f_{ik}(a), f_{sk}(\vec{c}))) &= \\ \varphi^{\mathbf{A}_p}(f_{kp}(f_{ik}(a)), f_{kp}(f_{sk}(\vec{c}))) &= \\ \varphi^{\mathbf{A}_p}(f_{jp}(f_{ij}(a)), f_{kp}(f_{sk}(\vec{c}))) &= \\ \varphi^{\mathbf{A}_p}(f_{jp}(b), f_{sp}(\vec{c})) &= \\ \varphi^{\mathbf{A}}(b, \vec{c}) &\in F_p. \end{aligned}$$

This is a contradiction, so $\varphi(a, \vec{c}) \in F_k \subseteq F$. This established our claim that $\langle a, b \rangle \in \Omega^{\mathbf{A}} F$. Therefore $a = b$, which implies that $i = j$, i.e. I^+ does not possess two different elements. Then I^+ is a singleton. □

We can now proceed to prove the following characterization of the Leibniz reduced models of a containment logic. Given an algebra $\mathbf{A}$, we denote by $\mathbf{A} \oplus \mathbf{1}$ the unique Płonka sum with two-fibers such that $\mathbf{A}$ is indexed by the lowest index and the trivial fiber $\mathbf{1}$ is indexed by the greatest index.

THEOREM 40
Let $\vdash$ a logic, $\mathbf{A} \neq \mathbf{1}$ and $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$. The following are equivalent:

1. $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^*(\vdash^r)$;
2. I^+ is a singleton and, either $\mathbf{A} = \mathbf{A}_i$ or $\mathbf{A} = \mathbf{A}_i \oplus \mathbf{1}$, with $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^*(\vdash)$.

PROOF. (i)⇒(ii). The fact that the matrix is a Płonka sum over a right direct system of matrices follows by Theorem 20, while that I^+ is a singleton, say $\{i\}$, follows by Lemma 39. Now, the equivalence relation θ which collapses all the fibers j for $j \nleq i$ into a single point is a congruence which is compatible with F. Because the matrix $\langle \mathbf{A}, F \rangle$ is reduced, θ is the identity congruence and therefore either i is the top element of the semilattice I or it is the coatom, with the top fiber being $\mathbf{1}$. Similarly, the equivalence relation ψ which collapses each $a \in A_k$ for $k < i$ with $f_{ki}(a) \in A_i$ is also a congruence which is compatible with F. It follows that i is the bottom element of the semilattice. This proves the implication.[4]

(ii)⇒ (i). Let $\mathcal{P}_{\natural}(X) = \langle \mathbf{A}, F \rangle$ satisfying (*ii*). Since the Płonka sum over an r-direct system of matrices is a model of $\vdash^r$ [by 11, Lemma 10], $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$. Moreover, since $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{*}(\vdash)$, for any pair of elements $a, b \in A_i$, there exists a unary polynomial function $\varphi(x, \vec{z})$ such that, for some $\vec{c} \in A_i$,

$$\varphi^{\mathbf{A}}(a, \vec{c}) \in F_i \text{ if and only if } \varphi^{\mathbf{A}}(b, \vec{c}) \notin F_i.$$

In order to prove (i), we just need to show that $\langle d, 1 \rangle \notin \Omega^{\mathbf{A}} F$, for an arbitrary $d \in A_i$. To this end, let $e \in F_i$. Clearly, $e *^{\mathbf{A}} d = e \in F$, while $e *^{\mathbf{A}} 1 = 1 \notin F$. That is, the function $*$ is a unary polynomial function witnessing that $\langle d, 1 \rangle \notin \Omega^{\mathbf{A}} F$. This concludes our proof. □

From the above Theorem 40 it follows Corollary 23 (we have actually already used in the proof Lemma 26).

5 Suszko reduced models

We now turn our attention to the structure of the Suszko reduced models of a containment logic.

In what follows, given a logic $\vdash$ and an algebra $\mathbf{A} \in \mathsf{Alg}(\vdash)$, we say that, if $\langle \mathbf{A}, G \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$, then G is a *Suszko filter* over $\mathbf{A}$.

We start by proving that, in the Płonka sum representation of a Suszko reduced model of a containment logic, there can be at most one fiber with non-empty filter.

LEMMA 41
Let $\vdash^r$ be a logic. If $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$, then $| I^+ | \leq 1$.

PROOF. By [34, Lemma 6] and the definition of direct product of matrices, if $\{\langle \mathbf{A}_w, F_w \rangle\}$ is a family of Płonka sums of matrices such that $| I^+_w | \leq 1$ for each $w \in W$, then $\prod_{w \in W} \langle \mathbf{A}_w, F_w \rangle$ is a Płonka sum of matrices having at most one fiber with non-empty filter, too. By Theorem 40, every Leibniz reduced model of $\vdash^r$ has at most one fiber with non-empty filter. Therefore, because $\mathsf{Mod}^{\mathrm{Su}}(\vdash^r) = \mathsf{P}_{SD}\mathsf{Mod}^{*}(\vdash^r)$, our conclusion follows. □

THEOREM 42
Let $\vdash^r$ be a logic and $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}(\vdash^r)$ such that $\mathbf{A} \in \mathsf{Alg}(\vdash^r)$, $\langle \mathbf{A}_i, F_i \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ for every $i \in I^+$. Assume, moreover, that, for each $j \in I$, $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$ and there exists a Suszko filter G_j over $\mathbf{A}_j$ such that $F_i \subseteq f_{ij}^{-1}(G_j)$. The following are equivalent:

1. $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$;
2. (a) $I^+ = \emptyset$ or
   (b) $I^+ = \{i\}$ where i is the bottom element of I.

PROOF. (i)⇒(ii). By Lemma 41, we have that $| I^+ | \leq 1$, i.e. either $I^+ = \emptyset$, namely $F = \emptyset$, or I^+ is a singleton, say $\{i\}$, i.e. $F = F_i$. In order to prove (ii) we only need to show that if $I^+ = \{i\}$ then i is the bottom element in I. We reason by contradiction, so assume that i is not the bottom element of I, i.e. there exists $j \in I$ such that $i \not\leq j$.

[4] We thank an anonymous referee for suggesting the present version of the proof.

Let $a \in A_j$ and $s = i \vee j$; consider an element $b = f_{js}(a) \in A_s$. Since $F_j = \emptyset$ (as $j \notin I^+$), then, by Definition 16 $b \notin F_s$ (as if $b \in F_s$ then $F_j = f_{js}^{-1}[F_s] \neq \emptyset$). Moreover, as $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$ there exists a $\vdash^r$-filter $G \supseteq F$ and a unary polynomial function $\varphi(v, \vec{z})$ such that for $\vec{c} \in A_k$, it holds

$$\varphi(a, \vec{c}) \in G \iff \varphi(b, \vec{c}) \notin G.$$

Without loss of generality, assume $\varphi(a, \vec{c}) \in G_q \subseteq A_q$ (with $q = j \vee k$). Now, as $G_i \neq \emptyset$ and $G_q \neq \emptyset$, by Theorem 20, we have $G_p \neq \emptyset$ (with $p = s \vee k$). Observe also that this implies $f_{qp}(\varphi(a, \vec{c})) \in G_p$. Moreover, by applying the same strategy used in the proof of Lemma 41

$$f_{qp}(\varphi(a, \vec{c})) = \varphi(b, \vec{c}) \in G_p,$$

which is a contradiction. The same argument can be applied to the case $\varphi(b, \vec{c}) \in G$. This proves (ii).

(ii)⇒(i). We have to show that each of the conditions (a) and (b) implies (i).

(a)⇒(i). Assume the Płonka decomposition of $\langle \mathbf{A}, F \rangle$ is such that $I^+ = \emptyset$. Consider $a, b \in A$, with $a \neq b$. We aim at showing $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$. Consider first the case when $a \in A_i, b \in A_j$ for arbitrary $i \neq j$. We assume without loss of generality that if i, j are comparable then $i < j$. Now, as $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ consider a non-empty $\vdash$filter $G_i \neq A_i$. By Lemma 26, $\langle \mathbf{A}, \downarrow G_i \rangle$ is a model of $\vdash^r$. In particular, as $F = \emptyset$, $\downarrow G_i$ is a $\vdash^r$-filter extending F.

Now, fix $c \in G_i$. We have that $c * a = c \in \downarrow G_i$, while $c * b \notin \downarrow G_i$, proving $\langle a, b \rangle \notin \Omega^{\mathbf{A}} \downarrow G_i$, i.e. $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$. This proves $\langle \mathbf{A}, F \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$, as desired.

The only case left is $a, b \in A_i$. As $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$ there exists $\langle \mathbf{A}_i, G_i \rangle \in \mathsf{Mod}(\vdash)$ such that $\langle a, b \rangle \notin \Omega^{\mathbf{A}_i} G_i$, i.e. there exist $\vec{c} \in A_i$ and a unary polynomial function $\varphi(v, \vec{z})$ satisfying $\varphi(a, \vec{c}) \in G_i$ if and only if $\varphi(b, \vec{c}) \notin G_i$. Observe this implies $G_i \neq A_i$, for otherwise $\Omega^{\mathbf{A}_i} G_i = A_i \times A_i$ and, by Lemma 26, this entails $\langle \mathbf{A}, \downarrow G_i \rangle \in \mathsf{Mod}(\vdash^r)$. So, we obtain $\varphi(a, \vec{c}) \in \downarrow G_i$ if and only if $\varphi(b, \vec{c}) \notin \downarrow G_i$, proving $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$.

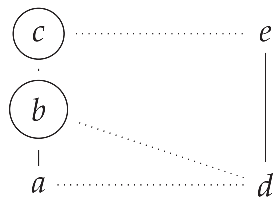
(b)⇒(i). Assume that $I^+ = \{i\}$ is the bottom of I, and consider arbitrary $a, b \in A$. Again, we aim at showing $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$. The case $a, b \in A_i$ is immediate, as $F = F_i$ and $\widetilde{\Omega}^{\mathbf{A}_i}_{\vdash} F_i = id$, for $\langle A_i, F_i \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$. So let $a \in A_j, b \in A_k$ assuming without loss of generality that if j, k are comparable then $j < k$. The argument of Lemma 26, together with the fact that there exists a Suszko filter G_j such that $F_i \subseteq f_{ij}^{-1}(G_j)$ for each $j \geqslant i$, imply that $\langle \mathbf{A}, \downarrow G_j \rangle$ is a model of $\vdash^r$ and $F \subseteq \downarrow G_j$. Moreover, as $G_j \neq \emptyset$, we can fix $c \in G_j$. Clearly, $c * a \in \downarrow G_j$ and $c * b \notin \downarrow G_j$, so $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$, as desired.

The only case left is $a, b \in A_j$. Again, consider $\langle \mathbf{A}_j, G_j \rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ such that $F_i \subseteq f_{ij}^{-1}(G_j)$, and let $H_j \supseteq G_j$ be the $\vdash$-filter on $\mathbf{A}_j$ such that $\langle a, b \rangle \notin \Omega^{\mathbf{A}_j} H_j$. This is to say that there exists a unary polynomial function $\varphi(v, \vec{z})$ and $\vec{c} \in A_j$ such that $\varphi(a, \vec{c}) \in H_j$ if and only if $\varphi(b, \vec{c}) \notin H_j$. As $H_j \supseteq G_j$ and $F_i \subseteq f_{ij}^{-1}(G_j)$, we have $F_i \subseteq f_{ij}^{-1}(H_j)$. This, as before, implies $\langle \mathbf{A}, \downarrow H_j \rangle$ is a model of $\vdash^r$, and therefore, we obtain $\varphi(a, \vec{c}) \in \downarrow H_j$ if and only if $\varphi(b, \vec{c}) \notin \downarrow H_j$. This proves $\langle a, b \rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$ and it concludes the proof. □

REMARK 43

Observe that the assumption concerning the existence of a specific Suszko filter in Theorem 42 is fundamental, as witnessed by the following example. Consider the Płonka sum of matrices $\langle \mathbf{A}, G \rangle$ represented in the diagram below. The algebraic reduct is a Płonka sum of two distributive lattices $\mathbf{D}_3, \mathbf{D}_2$, namely the three elements chain with universe $\{a, b, c\}$ and the two element chain with universe $\{d, e\}$. Dotted lines are Płonka homomorphisms and circled elements represent the logical filter $G = \{a, b\}$. It is immediate to verify that it is a model of $\mathrm{CL}^r_{\wedge,\vee}$. Moreover,

$\langle \mathbf{D}_3, \{c, b\}\rangle, \langle \mathbf{D}_2, \emptyset\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\mathrm{CL}_{\wedge,\vee})$, I^+ is the bottom of I but $\widetilde{\Omega}^{\mathbf{A}}_{\mathrm{CL}^r_{\wedge,\vee}} G \neq id$. However, there is no Suszko filter F over $\mathbf{D}_2$ such that c, b are contained in the pre-image of F.



5.1 *truth-equational logics*

If the logic $\vdash$ is truth-equational, the characterization of the Suszko reduced models can be significantly simplified. The reason relies on the fact that if a logic is truth-equational, then the Leibniz operator and the Suszko operators behaves in a suitable way.

The following technical lemma highlights the effect of the injectivity of the Suszko operator in a Płonka sum over a right direct system.

LEMMA 44
Let $\vdash$ be a truth-equational logic. Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$ with $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$, for each $i \in I$. If, for some $k, j \in I$, $k \leq j$ and $\langle \mathbf{A}_j, G_j\rangle, \langle \mathbf{A}_k, G_k\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ then $G_k = f_{kj}^{-1}(G_j)$.

PROOF. Let $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$ such that $\mathbf{A}_i \in \mathsf{Alg}(\vdash)$, for each $i \in I$. Consider $k \leq j$, and let $\langle \mathbf{A}_j, G_j\rangle, \langle \mathbf{A}_k, G_k\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$. First, observe that $\vdash$ is truth-equational; therefore, $G_j, G_k \neq \emptyset$. By Lemma 22, $f_{kj}^{-1}(G_j)$ is a $\vdash$-filter on $\mathbf{A}_k$ hence $f_{kj}^{-1}(G_j) \neq \emptyset$. Consider now $G_k \cap f_{kj}^{-1}(G_j)$, which is again a non-empty $\vdash$-filter on $\mathbf{A}_k$. Clearly, $G_k \cap f_{kj}^{-1}(G_j) \subseteq G_k$ so, as the Suszko operator is monotone [see 28, Lemma 5.37], $\widetilde{\Omega}^{\mathbf{A}_k}_{\vdash} G_k \cap f_{kj}^{-1}(G_j) \subseteq \widetilde{\Omega}^{\mathbf{A}_k}_{\vdash} G_k = id$, which entails $\widetilde{\Omega}^{\mathbf{A}_k}_{\vdash} G_k \cap f_{kj}^{-1}(G_j) = id$. By Theorem 4, the Suszko operator is injective and this, together with $\widetilde{\Omega}^{\mathbf{A}_k}_{\vdash} G_k \cap f_{kj}^{-1}(G_j) = \widetilde{\Omega}^{\mathbf{A}_k}_{\vdash} G_k$, implies $G_k = f_{kj}^{-1}(G_j)$, as desired. □

The next theorem is a refinement of Theorem 42 that characterizes the Suszko reduced models of $\vdash^r$ in case $\vdash$ is a truth-equational logic.

THEOREM 45
Let $\vdash$ be a truth-equational logic. Let moreover $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}(\vdash^r)$ with $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$, for every $j \in I$ and $\langle \mathbf{A}_j, F_j\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$ for every $j \in I^+$. The following are equivalent:

1. $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$;
2. $I^+ = \emptyset$ or $I^+ = \{i\}$ with i the bottom element of I.

PROOF. (i)⇒(ii). The proof is analogous to that of Theorem 42 (it is immediate to verify that the additional assumption in Theorem 42 is not used in this direction).

(ii)⇒(i). We need to show that any of the two conditions ($I^+ = \emptyset$, $I^+ = \{i\}$ with i the bottom element in I) implies (i). If $I^+ = \emptyset$ then the argument is the same applied (in the proof of) Theorem 42.

Suppose it is the case that $I^+ = \{i\}$, where i is the bottom element of I, and consider two distinct elements $a, b \in A$. The case $a, b \in A_i$ is immediate, as $\langle a, b\rangle \notin \widetilde{\Omega}^{\mathbf{A}_i}_{\vdash} F_i$. So, suppose that $a \in A_j, b \in A_k$ (with $j \neq k$); we can consider a Suszko filter G_j on $\mathbf{A}_j$ and, by applying Lemmas 26 and 44, we obtain that $\langle \mathbf{A}, \downarrow G_j\rangle$ is a model of $\vdash^r$ and that $F_i = F \subseteq \downarrow G_j$. If $j \neq k$, then, as before, we can fix $c \in G_j$ and observe that $c * a \in \downarrow G_j$ while $c * b \notin \downarrow G_j$. Otherwise, if $j = k$, then, from the fact that $\langle \mathbf{A}_j, G_j\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$, we deduce that there exists a $\vdash$-filter $H_j \supseteq G_j$ such that $\langle a, b\rangle \notin \Omega^{\mathbf{A}_j} H_j$. This is equivalent to the fact that $\varphi(a, \vec{c}) \in H_j$ if and only if $\varphi(b, \vec{c}) \notin H_j$, for a unary polynomial function $\varphi(v, \vec{z})$ and $\vec{c} \in A_j$. Clearly, $H_j \supseteq G_j$ implies $\downarrow H_j \supseteq \downarrow G_j$, so by Lemma 26, $\downarrow H_j$ is a $\vdash^r$-filter extending $\downarrow G_j$. As $\downarrow H_j$ we have that $\varphi(a, \vec{c}) \in \downarrow H_j$ if and only if $\varphi(b, \vec{c}) \notin \downarrow H_j$.

This proves that in all the considered cases $\langle a, b\rangle \notin \widetilde{\Omega}^{\mathbf{A}}_{\vdash^r} F$, i.e. $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$. □

COROLLARY 46
Let $\vdash^r$ be a the containment companion of a logic. Then $\mathsf{Mod}^{*}(\vdash^r) \subsetneq \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$.

5.2 *Two significant cases*

We conclude this section by considering two representative cases in which a full characterization the Suszko reduced models of $\vdash^r$ is available. We begin with the case the logic $\vdash$ has antitheorems.

COROLLARY 47
Let $\vdash$ be an algebraizable logic with antitheorems and such that $\mathsf{Alg}(\vdash)$ is Kollár. Given a matrix $\langle \mathbf{A}, F\rangle$, the following are equivalent:

1. $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$;
2. $\mathbf{A} \cong \mathcal{P}_{\l}(\mathbf{A}_i)_{i\in I}$, $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$ for each $j \in I$, $\mathcal{P}_{\l}(\mathbf{A}_i)_{i\in I}$ has at most one trivial fiber indexed by the top of I and
   (a) $I^+ = \emptyset$ or
   (b) $I^+ = \{i\}$, where i is the bottom of I and $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$.

PROOF. The statement follows by Lemma 24 and Theorems 42 and 45, upon noticing that an algebraizable logic is both equivalential and truth-equational. □

COROLLARY 48
Let $\vdash$ be a truth-equational logic without antitheorems and such that $\mathsf{Alg}(\vdash)$ is closed under subalgebras. Given a matrix $\langle \mathbf{A}, F\rangle$, the following are equivalent:

1. $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r)$;
2. $\mathbf{A} \cong \mathcal{P}_{\l}(\mathbf{A}_i)_{i\in I}$, $\mathbf{A}_j \in \mathsf{Alg}(\vdash)$ for each $j \in I$ and
   (a) $I^+ = \emptyset$ or
   (b) $I^+ = \{i\}$ is the bottom of I and $\langle \mathbf{A}_i, F_i\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash)$.

PROOF. The statement follows by Lemma 24 and Theorems 31 and 45. □

6 Applications and examples

6.1 *Bochvar logic*

As already mentioned, Bochvar logic $\vdash_{\mathrm{B}_3}$ is the most well-known example of containment companion of a logic. Indeed, it is the containment companion of (propositional) classical logic. Since classical logic is protoalgebraic, possesses anti-theorems and its equivalent algebraic semantics (the variety of Boolean algebras) $\mathsf{Alg}(\vdash)$ is Kollár, Corollary 37 allows us to provide a description of the algebraic counterpart $\mathsf{Alg}(\vdash_{\mathrm{B}3})$ of Bochvar logic. This consists of the regularization of Boolean algebras (in the language without constants)[5] whose members are Płonka sums of Boolean algebras with at most one trivial Boolean fiber.

DEFINITION 49
A *generalized involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg \rangle$ of type (2, 2, 1) satisfying

I1. $x \vee x \approx x$;
I2. $x \vee y \approx y \vee x$;
I3. $x \vee (y \vee z) \approx (x \vee y) \vee z$;
I4. $\neg\neg x \approx x$;
I5. $x \wedge y \approx \neg(\neg x \vee \neg y)$;
I6. $x \wedge (\neg x \vee y) \approx x \wedge y$.

$\mathsf{Alg}(\vdash^r)$ is the quasi-variety of generalized involutive bisemilattices satisfying the quasi-identity

$$x \approx x' \ \&\ y \approx y' \Rightarrow x \approx y.$$

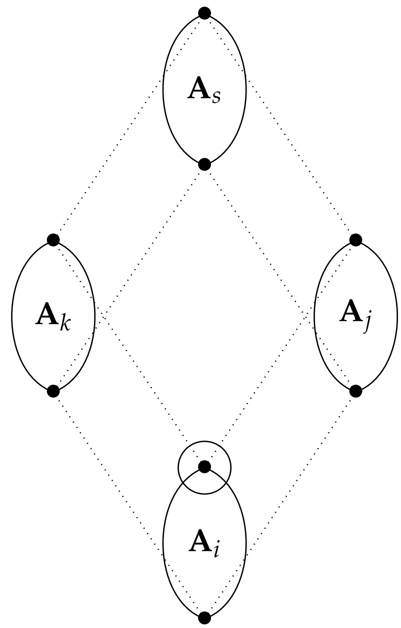
The quasi-variety of generalizes involutive bisemilattices satisfying the above quasi-identity is introduced in [34] and called $\mathcal{SGIB}$. Observe that $\mathsf{Alg}(\mathrm{B}_3)$ coincides with the $\mathsf{Alg}(\mathrm{PWK})$ [see 9, Theorem 63]. In words, the two logics in the weak Kleene family have the same algebraic counterpart. A characteristic featured also by the two (main) logics in the strong Kleene family, namely strong Kleene and the logic of paradox [see 1, Proposition 5.7, 5.8].

The results obtained so far allow to provide a full description of the Leibniz and Suszko reduced models of B_3. In particular, it follows from Theorem 40 that Leibniz reduced models are either a Boolean algebra with the top element only in the filter or a Płonka sum of a nontrivial Boolean algebra with a trivial one, with the top element of the former as unique element in the filter (in the following drawing $\mathbf{A}$ stands for a Boolean algebra with the singleton of the top element as filter, dotted arrow for homomorphism Płonka homomorphisms).

An example of a Suszko reduced model (which is not Leibniz reduced) of Bochvar logic B_3 $\mathbf{A}_i, \mathbf{A}_j\mathbf{A}_k, \mathbf{A}_s$ is the Płonka sum of the four Boolean algebras represented below, where circles indicate

[5]This variety has been introduced in [34] as a generalization of the regularization of Boolean algebras defined in the language containing constants (see [9, 39]).

filters and dotted lines Płonka homomorphisms. This follows from Corollary 47.



6.2 Belnap-Dunn logic

Belnap–Dunn logic B has been originally introduced, under the name *first degree entailment*, in the context of relevance and entailment logic [2, 4]. B has no anti-theorems and is not protoalgebraic [27, Theorem 2.11]. The containment companion of B—called $\mathbf{FDE}_{\varphi}$—has been introduced, independently, in [44] and [17] (the fact that indeed $\vdash_{\mathbf{FDE}_{\varphi}} = \vdash^{r}_{\mathsf{B}}$ is proven in [23]). The algebraic counterpart $\mathsf{Alg}(\mathsf{B})$ is the variety of De Morgan lattices [27] (in the language without constants).

We observe that, since $\mathsf{Alg}(\mathsf{B})$ is a variety, B satisfies the assumptions of Theorem 31; thus, $\mathsf{Alg}(\mathbf{FDE}_{\varphi})$ coincides with the class whose members are Płonka sums of De Morgan lattices; thus, by Theorem 15, $\mathsf{Alg}(\mathbf{FDE}_{\varphi})$ is the regularization of the variety of De Morgan lattices. This variety has been introduced in [26] (see also [12]), under the name of *De Morgan quasi-lattices*. A De Morgan quasi-lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, ' \rangle$ of type $(2, 2, 1)$ satisfying the following identities:

1. $x \wedge x \approx x$;
2. $x \wedge y \approx y \wedge x$;
3. $x \wedge (y \wedge z) \approx (x \wedge y) \wedge z$;
4. $x \vee x \approx x$;
5. $x \vee y \approx y \vee x$;
6. $x \vee (y \vee z) \approx (x \vee y) \vee z$;
7. $x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z)$;
8. $x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z)$;
9. $(x')' \approx x$;
10. $(x \wedge y)' \approx x' \vee y'$;
11. $(x \vee y)' \approx x' \wedge y'$.

Identities (1)–(8) remind us that the $\wedge, \vee$-reduct of a De Morgan quasi-lattice is a distributive bisemilattice (see [38]).

A characterization of Leibniz reduced models of $\mathbf{FDE}_\varphi$ can be given by applying Theorem 40 and the characterization of (Leibniz) reduced models of B in [27, Theorem 3.14] (since such description is not particuarly intellegible, we prefer not to recall it here).

6.3 The logic of paradox

The containment companion of the logic of paradox is the logic $\mathbf{S}_{fde}$, introduced by Deutsch [18]: the fact that $\vdash_{\mathbf{S}_{fde}} = \vdash^r_{\mathsf{LP}}$ has been shown by Ferguson [24] (see also [11]). LP is not protoalgebraic and has no anti-theorems (see [1, 46]). The class $\mathsf{Alg}(\mathrm{LP})$ is the variety of Kleene lattices ($\mathcal{KL}$), which is the subvariety of Kleene lattices axiomatized by adding $x \wedge x' \leq y \vee y'$. LP satisfies the assumptions of Theorem 31; thus, we can conclude in virtue of our analysis that $\mathsf{Alg}(\mathbf{S}_{fde})$ is the regularization of the variety of Kleene lattices ($R(\mathcal{KL})$).

Moreover, [1, Theorem 3.7] shows that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^*(\mathrm{LP})$ if and only if $\mathbf{A}$ is a Kleene lattice and $F = \{a \in A \mid a' \leq a\}$. It then follows from our analysis, in particular from Theorem 40 that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r_{\mathrm{LP}})$ if and only if $\mathbf{A}$ is a Kleene lattice or $\mathbf{A} = \mathbf{B} \oplus \mathbf{1}$ (for some Kleene lattice $\mathbf{B}$) and $F = \{a \in B \mid a' \leq a\}$. Since LP is truth-equational [see 1, Theorem 5.3], $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\mathrm{LP})$ if and only if $\mathbf{A} \in \mathcal{KL}$ and $F = \{a \vee a' \mid a \in A\}$. Applying Corollary 48, we get that $\langle \mathbf{A}, F\rangle \in \mathsf{Mod}^{\mathrm{Su}}(\vdash^r_{\mathrm{LP}})$ if and only if $\mathbf{A} \in R(\mathcal{KL})$ and $F = \emptyset$ or $F = b \vee b' \mid b \in B$, where $\mathbf{B}$ is the algebra with the lowest index in the Płonka sum representation of $\mathbf{A}$.

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References

[1] H. Albuquerque, A. Přenosil and U. Rivieccio. An algebraic view of super-Belnap logics. *Studia Logica*, **105**, 1051–1086, 2017.

[2] A. Anderson and N. Belnap. Tautological entailments. *Philosophical Studies*, **13**, 9–24, 1961.

[3] J. C. Beall. Off-topic: A new interpretation of weak-Kleene logic. *The Australasian Journal of Logic*, **13**, 2016.

[4] N. Belnap. A useful four-valued logic. In *Modern Uses of Multiple-Valued Logic*, J. M. Dunn and G. Epstein, eds, pp. 5–37. Springer, Dordrecht, Netherlands, 1977.

[5] C. Bergman. *Universal Algebra: Fundamentals and Selected Topics*. Chapman & Hall Pure and Applied Mathematics. Chapman and Hall/CRC, 2011.

[6] P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic*. Vol. 53 in Cambridge Tracts in Theoretical Computer Science. Cambridge University Press, Cambridge, 2001.

[7] W. J. Blok and D. Pigozzi. Algebraic semantics for universal Horn logic without equality. In *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith, eds, pp. 1–56. Heldermann, 1992.

[8] S. Bonzio, V. Fano and P. Graziani. A logical modeling of severe ignorance. Submitted, 2021.

[9] S. Bonzio, J. Gil-Férez, F. Paoli and L. Peruzzi. On Paraconsistent weak Kleene logic: Axiomatization and algebraic analysis. *Studia Logica*, **105**, 253–297, 2017.

[10] S. Bonzio, T. Moraschini and M. Pra Baldi. Logics of left variable inclusion and Płonka sums of matrices. *Archive for Mathematical Logic*, **60**, 49–76, 2021.

[11] S. Bonzio and M. Pra Baldi. Containment logics: Algebraic completeness and axiomatization. *Studia Logica*, **109**, 969–994, 2021.

[12] J. Brzozowski. De Morgan bisemilattices. In *30th IEEE International Symposium on Multiple-Valued Logic*, pp. 23–25. IEEE Press, 2000.

[13] S. Burris and H. P. Sankappanavar. *A course in Universal Algebra*. Available at https://www.math.uwaterloo.ca/snburris/htdocs/ualg.html, the millennium edition, 2012.

[14] M. A. Campercholi and J. G. Raftery. Relative congruence formulas and decompositions in quasivarieties. *Algebra Universalis*, **78**, 407–425, 2017.

[15] R. Ciuni, T. M. Ferguson and D. Szmuc. Logics based on linear orders of contaminating values. *Journal of Logic and Computation*, **29**, 631–663, 2019.

[16] J. Czelakowski. *Protoalgebraic Logics*. Vol. 10 of Trends in Logic—Studia Logica Library. Kluwer Academic Publishers, Dordrecht, 2001.

[17] C. Daniels. A note on negation. *Erkenntnis*, **32**, 423–429, 1990.

[18] H. Deutsch. Relevant analytic entailment. *The Relevance Logic Newsletter*, **1**, 26–44, 1977.

[19] A. Diego. *Sobre Álgebras de Hilbert*. Vol. 12 of Notas de Lógica Matemática. Universidad Nacional del Sur, Bahía Blanca (Argentina), 1965.

[20] R. L. Epstein. The semantic foundations of logic. In *The Semantic Foundations of Logic Volume 1: Propositional Logics*, pp. 315–321. Springer, 1990.

[21] D. Fazio and M. Pra Baldi. On a logico-algebraic approach to AGM belief contraction theory. *Journal of Philosophical Logic*, **50**, 911–938, 2021.

[22] T. M. Ferguson. A computational interpretation of conceptivism. *Journal of Applied Non-Classical Logics*, **24**, 333–367, 2014.

[23] T. M. Ferguson. Faulty Belnap computers and subsystems of FDE. *Journal of Logic and Computation*, **26**, 1617–1636, 2014.

[24] T. M. Ferguson. Logics of nonsense and Parry systems. *Journal of Philosophical Logic*, **44**, 65–80, 2015.

[25] T. M. Ferguson. *Meaning and Proscription in Formal Logic: Variations on the Propositional Logic of William T. Parry*. Trends in Logic. Springer, 2017.

[26] V. Finn and R. Grigolia. Nonsense logics and their algebraic properties. *Theoria*, **59**, 207–273, 1993.

[27] J. M. Font. Belnap's four-valued logic and De Morgan lattices. *Logic Journal of IGPL*, **5**, 1–29, 1997.

[28] J. M. Font. *Abstract Algebraic Logic: An Introductory Textbook*. College Publications, 2016.

[29] N. Galatos, P. Jipsen, T. Kowalski and H. Ono. *Residuated Lattices: An Algebraic Glimpse at Substructural Logics*. Vol. 151 of Studies in Logic and the Foundations of Mathematics. Elsevier, Amsterdam, 2007.

[30] K. Iseki. BCK-algebras. *Mathematical Seminar Notes*, **4**, 77–86, 1976.

[31] T. Lávička. *An Abstract Study of Completeness in Infinitary Logics*. PhD Thesis, Charles University, 2018.

[32] A. Ledda, F. Paoli and M. Pra Baldi. Algebraic analysis of demodalised analytic implication. *Journal of Philosophical Logic*, **48**, 957–979, 2019.

[33] T. Moraschini, J. G. Raftery and J. J. Wannenburg. Singly generated quasivarieties and residuated structures. *Mathematical Logic Quarterly*, **66**, 150–172, 2020.

[34] F. Paoli and M. Pra Baldi. Extensions of paraconsistent weak Kleene logic. *Logic Journal of the IGPL*, **29**, 798–822, 2021.

[35] F. Paoli and M. Pra Baldi. Proof theory of paraconsistent weak Kleene logic. *Studia Logica*, **108**, 779–802, 2020.

[36] W. T. Parry. *Implication*. PhD Thesis, Harvard University, 1932.

[37] J. Płonka. On a method of construction of abstract algebras. *Fundamenta Mathematicae*, **61**, 183–189, 1967.

[38] J. Płonka. On distributive quasilattices. *Fundamenta Mathematicae*, **60**, 191–200, 1967.

[39] J. Płonka. On sums of direct systems of boolean algebras. *Colloquium Mathematicum*, **21**, 209–214, 1969.

[40] J. Płonka. On the sum of a direct system of universal algebras with nullary polynomials. *Algebra Universalis*, **19**, 197–207, 1984.

[41] J. Płonka and A. Romanowska. Semilattice sums. In *Universal Algebra and Quasigroup Theory*, A. Romanowska and J. D. H. Smith, eds, pp. 123–158. Heldermann, 1992.

[42] M. Pra Baldi. *An Algebraic Study of Logics of Variable Inclusion and Analytic Containment*. PhD Thesis, University of Padua, 2018.

[43] M. Pra Baldi. Logics of variable inclusion and the lattice of consequence relations. *Journal of Applied Non-Classical Logics*, 2020. To appear.

[44] G. Priest. The logic of catuskoti. *Comparative Philosophy*, **1**, 24–54, 2010.

[45] A. Pynko. Characterizing Belnap's logic via De Morgan's laws. *Mathematical Logic Quarterly*, **41**, 442–454, 1995.

[46] A. Pynko. On Priest's logic of paradox. *Journal of Applied Non-Classical Logics*, **5**, 219–225, 1995.

[47] A. P. Pynko. Subprevarieties versus extensions. Application to the logic of paradox. *Journal of Symbolic Logic*, 756–766, 2000.

[48] J. G. Raftery. Inconsistency lemmas in algebraic logic. *Mathematical Logic Quarterly*, **59**, 393–406, 2013.

[49] U. Rivieccio. An infinity of super-Belnap logics. *Journal of Applied Non-Classical Logics*, **22**, 319–335, 2012.

[50] A. Romanowska and J. Smith. *Modes*. World Scientific, 2002.

[51] A. Urquhart. Basic many-valued logic. In *Handbook of Philosophical Logic*, D. M. Gabbay and F. Guenthner, eds, pp. 249–295. Springer, Dordrecht, 2001.

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