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KÄHLER IMMERSIONS OF KÄHLER–RICCI SOLITONS INTO DEFINITE OR INDEFINITE COMPLEX SPACE FORMS

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ABSTRACT. Let (g, X) be a Kähler–Ricci soliton (KRS) on a complex manifold M. We prove that if the Kähler manifold (M, g) can be Kähler immersed into a definite or indefinite complex space form then g is Einstein. Notice that there is no topological assumptions on the manifold M and the Kähler immersion is not required to be injective. Our result extends the result obtained in Bedulli and Gori [Proc. Amer. Math. Soc. 142 (2014), pp. 1777–1781] asserting that a KRS on a compact Kähler submanifold $M \subset \mathbb{C}\mathrm{P}^N$ which is a complete intersection is Kähler-Einstein (KE).

CONTENTS

1. INTRODUCTION

A *Kahler–Ricci soliton* (KRS) on a complex manifold M is a pair (g, X) consisting of a Kähler metric g and a holomorphic vector field X, called the *solitonic vector field*, such that

$$\mathrm{Ric}_g = \lambda g + L_X g \tag{1}$$

for some $\lambda \in \mathbb{R}$, where Ric_g is the Ricci tensor of the metric g and $L_X g$ denotes the Lie derivative of g with respect to X, i.e.

$$(L_X g)(Y, Z) = X\left(g(Y, Z)\right) - g([X, Y], Z) - g(Y, [X, Z]),$$

for Y and Z vector fields on M. A Kähler metric g satisfying (1) gives rise to special solutions of the Kähler–Ricci flow (see e.g. [9]), namely they evolve under

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biholomorphisms. KRS generalize Kähler–Einstein (KE) metrics. Indeed any KE metric g on a complex manifold M gives rise to a trivial KRS by choosing $X = 0$ or X Killing with respect to g. Obviously if the automorphism group of M is discrete then a Kähler–Ricci soliton (g, X) is nothing but a KE metric g.

The first examples of non-Einstein compact KRS go back to the constructions of N. Koiso [17] and independently H.-D. Cao [6] of Kähler metrics on certain $\mathbb{C}P^1$-bundles over $\mathbb{C}P^n$. After that, X.-J. Wang and X. Zhu [29] proved the existence of a KRS on any compact toric Fano manifold, and this result was later generalized by F. Podestà and A. Spiro [20] to toric bundles over generalized flag manifolds. The reader is referred to [7], [23], [24] for the existence and uniqueness of Kähler–Ricci solitons on compact manifolds and to [11] for the noncompact case.

In this paper we address the problem of studying KRS which can be Kähler immersed into a definite or indefinite (finite dimensional) complex space form (S, g_c) of constant holomorphic sectional curvature $2c$. The main result of the paper is Theorem 1.1 which asserts that such a KRS is trivial.

Theorem 1.1. *Let (g, X) be a KRS on complex manifold M. If (M, g) can be Kähler immersed into a definite or indefinite complex space form (S, g_c) then g is KE. Moreover, its Einstein constant is a rational multiple of c.*

It is worth pointing out that in our theorem there are no topological assumptions on the manifold M and the Kähler immersion is not required to be injective. Notice also that our result is new even if we are in the realm of algebraic geometry, namely when one assumes that M is compact, the ambient complex space form is the complex projective space (equipped with the Fubini–Study metric of constant holomorphic sectional curvature 4) and that the immersion is an embedding. Indeed our result thereby extends the result obtained by A. Gori and L. Bedulli [3] asserting that a KRS on a compact Kähler submanifold $M \subset \mathbb{C}\mathrm{P}^N$ which is a complete intersection is KE (and hence by a deep result of Hano [12] M turns out to be the quadric or a complex projective space totally geodesically embedded in $\mathbb{C}P^N$). The reader is also referred to [4] where the condition on complete intersection is replaced by the more general assumption that the Kähler embedding has rational Gauss map.

By combining well-known results on KE immersions into the complex projective space in codimension one and two due to S.-S. Chern [9] and K. Tsukada [25] respectively, we obtain the following corollary of Theorem 1.1.

Corollary 1.2. *Let (g, X) be a KRS on a n-dimensional complex manifold M. If (M, g) can be Kähler immersed into $\mathbb{C}P^{n+k}$ with $k \leq 2$, then M is either an open subset of the complex quadric or an open subset of $\mathbb{C}P^n$ totally geodesically embedded in $\mathbb{C}P^{n+k}$.*

Since a rotation invariant KE metric on a complex manifold of dimension ≥ 3, which admits a Kähler immersion into a complex projective space in such a way that the codimension is ≤ 3, is forced to be the Fubini–Study metric (see [22, Theorem 1.2] for a proof) we also get:

Corollary 1.3. *Let (g, X) be a KRS on a n-dimensional complex manifold M with $n \geq 3$. Assume that g is rotation invariant and that (M, g) can be Kähler immersed into $\mathbb{C}P^{n+k}$ with $k \leq 3$. Then (M, g) is an open subset of $\mathbb{C}P^n$ totally geodesically embedded in $\mathbb{C}P^{n+k}$.*

Theorem 1.1 also yields the following result which can be deduced by D. Hulin's theorem [15, Prop. 4.5] on the extension of germs of KE projectively induced metrics.

Corollary 1.4. *Let (g, X) be a projectively induced shrinking KRS (i.e. $\lambda > 0$ in (1)) on a complex manifold M. Then g extends to a projectively induced KE metric $\hat{g}$, with positive and rational Einstein constant λ, on a compact complex manifold $\hat{M}$.*

Notice that D. Hulin [16] shows that all the compact KE submanifolds of the complex projective space have necessary positive (rational) Einstein constant and it is conjecturally true (see e.g. [18]) that all such manifolds are flag manifolds.

By combining Theorem 1.1 with M. Umehara [27] results on KE manifolds immersed into the flat or complex hyperbolic space we get:

Corollary 1.5. *Let (g, X) be a KRS on a complex manifold M. A Kähler immersion of (M, g) into a definite complex space form of nonpositive holomorphic sectional curvature is totally geodesic.*

As a special case of the last part of Theorem 1.1 in the indefinite case we get:

Corollary 1.6. *The Einstein constant of a KE submanifold of an indefinite complex space form (S, g_c) is a rational multiple of c.*

When the ambient space is the indefinite complex projective space this corollary can be considered an extension of D. Hulin result [15, Prop. 5.1] on the rationality of the Einstein constant of a projectively induced KE metric. Moreover, to the best of authors' knowledge, the classification of the KE submanifolds of the indefinite complex hyperbolic space is missing. Even in the codimension one case (where such classification is known ([21, Theorem 3.2.4]) Umehara's theorem does not hold (there exists non totally geodesic KE submanifolds of the indefinite complex hyperbolic space). Thus, Corollary 1.6 seems to be a novelty also in the case of KE submanifolds of the indefinite complex hyperbolic space.

The proof of Theorem 1.1 is based on Theorem 2.1 (see next section) which describes some properties of Umehara algebra and its field of fractions. Section 3 is dedicated to the proof of Theorem 1.1.

2. Umehara algebra and its field of fractions

Let M be a complex manifold. Fix a point $p \in M$ and let $\mathcal{O}_p$ be the algebra of germs of holomorphic functions around p. Denote by $\mathbb{R}_p$ the germs of real numbers. The Umehara algebra (see [28]) is defined to be the $\mathbb{R}$-algebra Λ_p generated by the elements of the form $h\bar{k}+\bar{h}k$, for $h, k \in \mathcal{O}_p$. Umehara algebra has been an important tool in the study of relatives Kähler manifolds (see [8, 10, 26–28]).

Let

$$\hat{\mathcal{O}}_p = \{\alpha = (\alpha_1, \dots, \alpha_m) \mid \alpha_j \in \mathcal{O}_p, \alpha_j(p) = 0, \forall j = 1, \dots, m, m \geq 1\}.$$

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \hat{\mathcal{O}}_p$ and $\ell \in \mathbb{N}$ such that $\ell \leq |\alpha| := m$ we set

$$\langle \alpha, \alpha \rangle_\ell(z) = \sum_{j=1}^{\ell} |\alpha_j(z)|^2 - \sum_{k=\ell+1}^{|\alpha|} |\alpha_k(z)|^2.$$

Since $h\bar{k}+\bar{h}k=|h+k|^2-|h|^2-|k|^2$ it is not hard to see (see [28] for details) that each $f\in\Lambda_p$ can be written as

$$f=h+\bar{h}+\langle\alpha,\alpha\rangle_\ell$$

for some $h\in\mathcal{O}_p$, $\alpha=(\alpha_1,\dots,\alpha_m)\in\hat{\mathcal{O}}_p$, $\ell\leq|\alpha|$ and such that $\alpha_1,\dots,\alpha_m$ are linearly independent over $\mathbb{C}$.

Consider the $\mathbb{R}$-algebra $\tilde{\Lambda}_p\subset\Lambda_p$ given by

$$\tilde{\Lambda}_p=\left\{a+\langle\alpha,\alpha\rangle_\ell \mid a\in\mathbb{R}_p,\ \alpha\in\hat{\mathcal{O}}_p,\ \ell\leq|\alpha|\right\}. \tag{2}$$

Notice that the germ of the real part of a nonconstant holomorphic function $h\in\mathcal{O}_p$ belongs to Λ_p but not to $\tilde{\Lambda}_p$.

The key element in the proof of Theorem 1.1 is the following Theorem 2.1 whose proof is inspired by the work X. Huang and Y. Yuan [13] (see also [30]).

Theorem 2.1. *Let $\tilde{K}_p$ be the field of fractions of $\tilde{\Lambda}_p$. Let μ be a real number and $g=\frac{b+\langle\beta,\beta\rangle_q}{c+\langle\gamma,\gamma\rangle_r}\in\tilde{K}_p$, then*

$$e^g\notin\tilde{\Lambda}_p^\mu\tilde{K}_p\setminus\mathbb{R}_p \tag{3}$$

where $\tilde{\Lambda}_p^\mu\tilde{K}_p=\left\{f^\mu h \mid f\in\tilde{\Lambda}_p,\ h\in\tilde{K}_p\right\}$.

Remark 1. Theorem 2.1 extends [8, Theorem 2.1, part (ii)] which asserts that $e^g\notin\tilde{K}_p\setminus\mathbb{R}_p$, for $g\in\tilde{\Lambda}_p$. Moreover [8, Theorem 2.1, part (iii)] shows that if f^α belongs to $\tilde{K}_p\setminus\mathbb{R}_p$ for some real number α and $f\in\tilde{\Lambda}_p$, then α is rational. This result will be used to prove the rationality of the Einstein constant in Theorem 1.1.

Lemma 2.2 is a key ingredient in the proof of Theorem 2.1.

Lemma 2.2 ([14, Lemma 2.2])**.** *Let $V\subset\mathbb{C}^\kappa$ be a connected open set. Let $H_1(\xi_1,\dots,\xi_\kappa)$, …, $H_K(\xi_1,\dots,\xi_\kappa)$ and $H(\xi_1,\dots,\xi_\kappa)$ be holomorphic Nash algebraic functions on V. Assume that*

$$\exp^{H(\xi_1,\dots,\xi_\kappa)}=\prod_{\alpha=1}^{K}\left(H_k(\xi_1,\dots,\xi_\kappa)\right)^{\mu_\alpha},\xi\in V$$

for certain real numbers $\mu_1,\dots,\mu_K$. Then $H(\xi_1,\dots,\xi_\kappa)$ is constant.

Proof of Theorem 2.1. Assume that there exists $f=a+\langle\alpha,\alpha\rangle_\ell\in\tilde{\Lambda}_p$ and $h=\frac{d+\langle\delta,\delta\rangle_u}{e+\langle\epsilon,\epsilon\rangle_v}\in\tilde{K}_p$ such that

$$e^{g(z)}=e^{\frac{b+\langle\beta(z),\beta(z)\rangle_q}{c+\langle\gamma(z)),\gamma(z))\rangle_r}}=f^\mu(z)h(z)=[a+\langle\alpha(z),\alpha(z)\rangle_p]^\mu\,\frac{d+\langle\delta(z),\delta(z)\rangle_u}{e+\langle\epsilon(z),\epsilon(z)\rangle_v}. \tag{4}$$

By renaming the functions involved in (4) we can write

$$S=\{\varphi_1,\dots,\varphi_s\}=\left\{\alpha_1,\dots,\alpha_{|\alpha|},\beta_1,\dots,\beta_{|\beta|},\dots,\epsilon_1,\dots,\epsilon_{|\epsilon|}\right\}.$$

Let D be an open neighborhood of the origin of $\mathbb{C}^n$ on which each φ_j is defined. Consider the field $\mathfrak{R}$ of rational function on D and its field extension $\mathfrak{F}=\mathfrak{R}(S)$, namely, the smallest subfield of the field of the meromorphic functions on D, containing rational functions and the elements of S. Let l be the transcendence degree of the field extension $\mathfrak{F}/\mathfrak{R}$. If $l=0$, then each element in S is holomorphic Nash algebraic and hence g is forced to be constant by Lemma 2.2. Assume then that $l>0$. Without loss of generality we can assume that $\mathcal{G}=\{\varphi_1,\dots,\varphi_l\}\subset S$ is a

maximal algebraic independent subset over $\mathfrak{K}$. Then there exist minimal polynomials $P_j(z, X, Y)$, $X = (X_1, \dots, X_l)$, such that

$$P_j(z, \Phi(z), \varphi_j(z)) \equiv 0, \ \forall j = 1, \dots, s,$$

where $\Phi(z) = (\varphi_1(z), \dots, \varphi_l(z))$.

Moreover, by the definition of minimal polynomial

$$\frac{\partial P_j(z, X, Y)}{\partial Y}(z, \Phi(z), \varphi_j(z)) \not\equiv 0, \ \forall j = 1, \dots, s.$$

on D. Thus, by the algebraic version of the existence and uniqueness part of the implicit function theorem, there exist a connected open subset $U \subset D$ with $p \in \overline{U}$ and Nash algebraic functions $\hat{\varphi}_j(z, X)$, defined in a neighborhood $\hat{U}$ of $\{(z, \Phi(z)) \mid z \in U\} \subset \mathbb{C}^n \times \mathbb{C}^l$, such that

$$\varphi_j(z) = \hat{\varphi}_j(z, \Phi(z)), \ \forall j = 1, \dots, s,$$

for any $z \in U$. Denoting $\hat{\varphi}(z, X) = (\hat{\varphi}_1(z, X), \dots, \hat{\varphi}_s(z, X))$ we can write

$$\varphi(z) = \hat{\varphi}(z, \Phi(z)) = (\hat{\alpha}(z, \Phi(z)), \dots, \hat{\epsilon}(z, \Phi(z))), \tag{5}$$

where $\varphi = (\varphi_1, \dots, \varphi_s)$ and $\hat{\alpha}(z, X), \dots, \hat{\epsilon}(z, X)$ are vector-valued holomorphic Nash algebraic functions on $\hat{U}$ such that $\alpha(z) = \hat{\alpha}(z, \Phi(z)), \dots, \epsilon(z) = \hat{\epsilon}(z, \Phi(z))$.

Consider the function

$$\Psi(z, X, w) := \frac{b + \langle \hat{\beta}(z, X), \beta(w) \rangle_q}{c + \langle \hat{\gamma}(z, X), \gamma(w) \rangle_r} - \mu \log [a + \langle \hat{\alpha}(z, X), \alpha(w) \rangle_\ell] - \log \left[\frac{d + \langle \hat{\delta}(z, X), \delta(w) \rangle_u}{e + \langle \hat{\epsilon}(z, X), \epsilon(w) \rangle_v} \right],$$

where, for $\alpha(z) = (\alpha_1(z), \dots, \alpha_m(z))$ and corresponding $\hat{\alpha}(z, X) = (\hat{\alpha}_1(z, X), \dots, \hat{\alpha}_m(z, X))$ we mean

$$\langle \hat{\alpha}(z, X), \alpha(w) \rangle_\ell = \sum_{j=1}^{\ell} \hat{\alpha}_j(z, X) \alpha_j(\overline{w}) - \sum_{k=\ell+1}^{|\alpha|} \hat{\alpha}_k(z, X) \alpha_k(\overline{w})$$

(and similarly with the other terms). By shrinking U if necessary we can assume $\Psi(z, X, w)$ is defined on $\hat{U} \times U$. We claim that $\Psi(z, X, w)$ vanishes identically on this set. Since $\varphi_j(p) = 0$ for all $j = 1, \dots, s$ and $p \in \overline{U}$, it follows by (4) that $\Psi(z, X, 0) \equiv 0$. Hence, in order to prove the claim, it is enough to show that $(\partial_w \Psi)(z, X, w) \equiv 0$ for all $w \in U$. Assume, by contradiction, that there exists $w_0 \in U$ such that $(\partial_w \Psi)(z, X, w_0) \not\equiv 0$. Since $(\partial_w \Psi)(z, X, w_0)$ is Nash algebraic in (z, X) there exists a holomorphic polynomial $P(z, X, t) = A_d(z, X) t^d + \cdots + A_0(z, X)$ with $A_0(z, X) \not\equiv 0$ such that $P(z, X, (\partial_w \Psi)(z, X, w_0)) = 0$. Since, by (4) and (5) we have $\Psi(z, \Phi(z), w) \equiv 0$ we get $(\partial_w \Psi)(z, \Phi(z), w) \equiv 0$. Thus $A_0(z, \Phi(z)) \equiv 0$ which contradicts the fact that $\varphi_1(z), \dots, \varphi_l(z)$ are algebraic independent over $\mathfrak{K}$. Hence $(\partial_w \Psi)(z, X, w_0) \equiv 0$ and the claim is proved.

Therefore

$$e^{\frac{b + \langle \hat{\beta}(z, X), \beta(w) \rangle_q}{c + \langle \hat{\gamma}(z, X), \gamma(w) \rangle_r}} = [a + \langle \hat{\alpha}(z, X), \alpha(w) \rangle_\ell]^\mu \left[\frac{d + \langle \hat{\delta}(z, X), \delta(w) \rangle_u}{e + \langle \hat{\epsilon}(z, X), \epsilon(w) \rangle_v} \right],$$

for every $(z, X, w) \in \hat{U} \times U$. By fixing $w \in U$ and applying Lemma 2.2 we deduce that $\frac{b + \langle \hat{\beta}(z, X), \beta(w) \rangle_q}{c + \langle \hat{\gamma}(z, X), \gamma(w) \rangle_r}$ is constant in (z, X). Thus, by evaluating at $X = \Phi(z)$ one obtains that $\frac{b + \langle \beta(z), \beta(w) \rangle_q}{c + \langle \gamma(z), \gamma(w) \rangle_r}$ is constant for fixed w forcing $g(z) = \frac{b + \langle \beta(z), \beta(z) \rangle_q}{c + \langle \gamma(z), \gamma(z) \rangle_r}$ to be constant for all z. The proof of the theorem is complete. □

3. Proof of Theorem 1.1

In the proof of Theorem 1.1 we also need the concept of diastasis function and Bochner's coordinates briefly recalled below. The reader is referred either to the celebrated work of Calabi [5] or to [19] for details.

Given a complex manifold M endowed with a real analytic Kähler metric g (notice that a Kähler metric induced by a complex space form is real analytic), Calabi introduced, in a neighborhood of a point $p \in M$, a very special Kähler potential D_p^g for the metric g, which he christened *diastasis*. Among all the potentials the diastasis is characterized by the fact that in every coordinate system $\{z_1, \dots, z_n\}$ centered in p

$$D_p^g(z, \bar{z}) = \sum_{|j|,|k| \geq 0} a_{jk} z^j \bar{z}^k,$$

with $a_{j0} = a_{0j} = 0$ for all multi-indices j. One of the main feature of Calabi's diastasis function is its hereditary property: *if $\varphi : M \to S$ is a Kähler immersion from a Kähler manifolds (M, g) into a definite or indefinite complex space form (S, g_c) of constant holomorphic sectional curvature $2c$, then $D_p^g = \varphi^*(D^{g_c}_{\varphi(p)})$ for all $p \in M$.*

More generally, in the rest of the paper we say that a real analytic function defined on a neighborhood U of a point p of a complex manifold M is of *diastasis-type* if in one (and hence any) coordinate system $\{z_1, \dots, z_n\}$ centered at p its expansion in z and $\bar{z}$ does not contains nonconstant purely holomorphic or anti-holomorphic terms (i.e. of the form z^j or $\bar{z}^j$ with $j > 0$). The following simple remarks will be used in the proof of Theorem 1.1.

Remark 2. The following facts holds true.

(a) A real-analytic function g is of diastasis-type if and only if e^g is of diastasis-type.

(b) A function $f \in \Lambda_p$ (resp. K_p) belong to $\tilde{\Lambda}_p$ (resp. $\tilde{K}_p$) if and only if f is of diastasis-type, where $\tilde{K}_p$ is the field of fractions of the algebra $\tilde{\Lambda}_p$ defined by (2).

In a neighborhood of $p \in M$ one can find local (complex) coordinates such that

$$D_p^g(z, \bar{z}) = |z|^2 + \sum_{|j|,|k| \geq 2} b_{jk} z^j \bar{z}^k,$$

where D_p^g is the diastasis relative to p. These coordinates, uniquely defined up to a unitary transformation, are called *the Bochner or normal coordinates* with respect to the point p (cfr. [1, 2, 5]).

Proposition 3.1, interesting on its own sake, will be used in the proof of Theorem 1.1.

Proposition 3.1. *Let (M, g) be a Kähler manifold which can be Kähler immersed into an N-dimensional definite or indefinite complex space form (S, g_c) of constant holomorphic sectional curvature $2c$. Let $p \in M$ and $\{z_1, \dots, z_n\}$ Bochner's coordinates in a neighborhood U of a point $p \in M$ where the diastasis D_p^g is defined. Then*

$$\det \left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta} \right] \in \tilde{K}_p, \tag{6}$$

$$D_p^g \in \tilde{\Lambda}_p, \ \textit{if } c = 0 \tag{7}$$

and

$$e^{\frac{c}{2}D_p^g} \in \tilde{\Lambda}_p, \ \textit{if } c \neq 0. \tag{8}$$

In the proof of the proposition we need the following result.

Lemma 3.2 ([27, Lemma 2.2])**.** *Let M be an n-dimensional complex manifold and $p \in M$. For any f in the Umehara algebra Λ_p and for any system of complex coordinates $\{z_1, \dots z_n\}$ around p one has:*

$$f^{n+1}\frac{\partial^2 \log f}{\partial z_\alpha \partial \bar{z}_\beta} \in \Lambda_p. \ \forall \alpha, \beta = 1, \dots, n.$$

Proof of Proposition 3.1. Let $\varphi : M \to S$ be a Kähler immersion into (S, g_c), i.e. φ is holomorphic and $\varphi^* g_c = g$. If one assumes that (S, g_c) is complete and simply-connected one has the corresponding three cases, depending on the sign of c:

- for $c = 0$, $S = \mathbb{C}^N$ and g_0 is the flat metric with associated Kähler form

$$\omega_0 = \frac{i}{2}\partial\bar{\partial}|z|_s^2, \tag{9}$$

where we set

$$|z|_s^2 = |z_1|^2 + \cdots + |z_s|^2 - |z_{s+1}|^2 - \cdots - |z_N|^2, \ 0 \leq s \leq N;$$

- for $c > 0$, $S = \mathbb{C}P_s^N$ is the open submanifold

$$\left\{[Z_0, \dots, Z_s, Z_{s+1}, \dots, Z_N] \in \mathbb{C}P^N \mid |Z_0|^2 + \cdots + |Z_s|^2 - |Z_{s=1}|^2 - \cdots - |Z_N|^2 > 0\right\}$$

of $\mathbb{C}P^N$ and g_c is the metric with associated Kähler form ω_c given in the affine chart $U_0 = \{[Z_0, \dots, Z_N] \mid Z_0 \neq 0\}$ with coordinates $z_j = \frac{Z_j}{Z_0}$ as:

$$\omega_c = \frac{i}{c}\partial\bar{\partial}\log(1 + |z|_s^2); \tag{10}$$

- for $c < 0$, $S = \mathbb{C}H_s^N$ is open subset of $\mathbb{C}^N$ given by $\{z \in \mathbb{C}^N \mid |z|_s < 1\}$ with the metric g_c with associated Kähler form

$$\omega_c = \frac{i}{c}\partial\bar{\partial}\log(1 - |z|_s^2). \tag{11}$$

Let

$$\varphi_{|U} : U \to \mathbb{C}^N, z = (z_1, \dots z_n) \mapsto (\varphi_1(z), \dots, \varphi_N(z)),$$

where $\varphi_j \in \mathcal{O}_p$ and $\varphi_j(p) = 0$, $j = 1, \dots, N$, be the local expression of φ in Bochner's coordinates $\{z_1, \dots, z_n\}$.

In order to prove (6) we first consider the case $c = 0$. By the hereditary property of the diastasis function and (9) we have

$$D_p^g = \sum_{i=1}^{N} |\varphi_i|_s^2 . \tag{12}$$

Thus the function $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right]$ is finitely generated by holomorphic or anti-holomorphic functions around 0. Furthermore it is real valued, since the matrix $\frac{\partial^2 D_p}{\partial z_a \partial z_\beta}$ is Hermitian. We conclude that $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \Lambda_p$. It is easy to check

that in Bochner's coordinates the function $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right]$ is of diastasis-type. Thus $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \tilde{\Lambda}_p \subset \tilde{K}_p$.

Let us now consider the case $c \neq 0$. Again by the hereditary property of the diastasis and by (10) and (11) we can write

$$D_p^g = \frac{2}{c} \log\left(1 + \frac{c}{|c|} \sum_{i=1}^{N} |\varphi_i|_s^2\right). \tag{13}$$

It follows by Lemma 3.2 applied to

$$e^{\frac{c}{2} D_p^g} = \left(1 + \frac{c}{|c|} \sum_{i=1}^{N} |\varphi_i|_s^2\right) \in \tilde{\Lambda}_p \subset \Lambda_p$$

that

$$e^{(n+1)\frac{c}{2} D_p^g} \det\left[\frac{\partial^2 (\frac{c}{2} D_p^g)}{\partial z_a \partial \bar{z}_\beta}\right] = \left(\frac{c}{2}\right)^n e^{(n+1)\frac{c}{2} D_p^g} \det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in \Lambda_p,$$

and hence

$$\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right] \in K_p.$$

Also in this case it not hard to see that in Bochner's coordinates $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar{z}_\beta}\right]$ is of diastasis-type and hence (6) readily follows.

Finally, the proofs of (7) and (8) follow by (12) and (13). □

Proof of Theorem 1.1. Let us start to write down equation (1) in local complex coordinates $\{z_1, \dots, z_n\}$ in a neighborhood U of a point $p \in M$ where the diastasis D_p^g for the metric g is defined. Since the solitonic vector field X can be assumed to be the real part of a holomorphic vector field, we can write

$$X = \sum_{j=1}^{n} \left(f_j \frac{\partial}{\partial z_j} + \bar{f}_j \frac{\partial}{\partial \bar{z}_j}\right)$$

for some holomorphic functions $f_j, j = 1, \dots, n$, on U.

Thus, by the definition of Lie derivative, after a straightforward computation we can write on U

$$L_X \omega = \frac{i}{2} \partial\bar{\partial} f_X, \tag{14}$$

where ω is the Kähler form associated to g and

$$f_X = \sum_{j=1}^{n} f_j \frac{\partial D_p^g}{\partial z_j} + \bar{f}_j \frac{\partial D_p^g}{\partial \bar{z}_j}. \tag{15}$$

Notice that equation (1) is equivalent to

$$\rho_\omega = \lambda \omega + L_X \omega, \tag{16}$$

where ρ_ω the Ricci form of ω.

Since, $\omega = \frac{i}{2}\partial\bar\partial D_p^g$ and $\rho_\omega = -i\partial\bar\partial \log\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right]$ on U, the local expression of the KRS equation (16) is

$$-i\partial\bar\partial \log\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right] = \lambda\frac{i}{2}\partial\bar\partial D_p^g + \frac{i}{2}\partial\bar\partial f_X,$$

and by the $\partial\bar\partial$-Lemma one has

$$\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right] = e^{-\frac{\lambda}{2}D_p^g - \frac{f_X}{2} + h + \bar h}, \tag{17}$$

for a holomorphic function h on U.

We treat the two cases $c = 0$ and $c \neq 0$ separately. If $c = 0$, combining (7), (12) and (15) we get that

$$g_1 := -\frac{\lambda}{2}D_p^g - \frac{f_X}{2} + h + \bar h \in \Lambda_p.$$

By (6) in Proposition 3.1, $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right] \in \tilde K_p$ and hence (17) gives $e^{g_1} \in \tilde K_p$. In particular e^{g_1} and so, by (a) of Remark 2, g_1 is of diastasis-type. It follows then by (b) of Remark 2 that $g_1 \in \tilde\Lambda_p$. Thus Theorem 2.1 (with $\mu = 0$) forces g_1 to be a constant. Hence $\det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right]$ is a constant and so g is Ricci flat.

If $c \neq 0$, by combining (13) and (15) one easily sees that

$$g_2 := -\frac{f_X}{2} + h + \bar h \in K_p.$$

By (6), (8) and (17) one deduces that

$$e^{g_2} = \left[e^{\frac{c}{2}D_p^g}\right]^{\frac{\lambda}{c}} \det\left[\frac{\partial^2 D_p^g}{\partial z_a \partial \bar z_\beta}\right] \in \tilde\Lambda_p^\mu \tilde K_p,\ \mu = \frac{\lambda}{c}. \tag{18}$$

On the one hand (18) shows that e^{g_2} and hence (by (a) of Remark 2) g_2 is of diastasis-type and so, by (b) of Remark 2, $g_2 \in \tilde K_p$. On the other hand (18) together with Theorem 2.1 force g_2 to be a constant and so f_X is the real part of a holomorphic function. Therefore, by (14) and (16) the metric g is KE. Moreover $\frac{\lambda}{c}$ is forced to be rational by the second part of Remark 1, completing the proof of Theorem 1.1. □

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