Equilibrium selection in an environmental growth model with a S-shaped production function

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June 14, 2019

Abstract

This paper presents a purely dynamic economic-environmental growth model with a S-shaped production function and a pollution externality. We first outline the role of the saving rate in inducing/suppressing the multiplicity of equilibria. The second part of the paper is devoted to global analysis. We prove that the system of dynamic laws implied by our model undergoes a Bogdanov-Takens (BT) singularity in specific regions of the parameter space. Assuming social preferences are in favor of the green steady state, there are two qualitatively separated regions in the parameter space, one at which the economy in *laissez-faire* is able to reach the green steady state and one where only appropriately devised choices of the saving rate and of the fraction of abated pollutants can put the economy on a path converging to the green steady state.

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Keywords: S-shaped production function; Bogdanov-Takens bifurcation; pollution externality; *laissez-faire*; policy-induced heteroclinic orbits

1 Introduction

This paper considers a purely dynamical growth model *à la* Solow where pollution creates a negative production externality. Our model differs from existing literature in that the production function is convex-concave S-shaped, instead of being of the standard Cobb-Douglas type. Production functions of this kind can be regularly found in growth models (cf., *inter al.*, Skiba, 1978), where the fading effects of economies of scale has to be taken into account.

We show that the system, in specific regions of the parameter space, gives rise to two steady states, one with high capital and pollution (the "dirty" steady state) and one with low capital and pollution (the "clean" or "green" steady state). Assuming that social preferences are in favor of the green steady state irrespectively of economic cost, we shed light on characteristics of policy action able to push the economy out of the dirty steady state and put it on a path converging to the green steady state.

Of course, information on this issues can be achieved only if local analysis is abandoned in favor of a more global perspective. In this regard, we show that our model, in specific regions of the parameter space, undergoes a Bogdanov-Takens bifurcation, a powerful mathematical tool for simplifying highly non-linear dynamical systems. What is interesting for us in this phenomenon is that a given dynamic system, undergoing a BT singularity, can be placed in correspondence with a simple planar system whose global unfolding is known in every aspect.

It is not the first time in the economic literature that the properties of the BT singularity are exploited. In a $\mathbb{R}^3$ ambient space, we can cite Bella and Mattana (2014) and Bosi (2019). An application in $\mathbb{R}^2$ can be found in Benhabib *et al.* (2001). However, we are not aware of contributions where the BT singularity is used to derive detailed policy implications.

The rest of the paper develops as follows. In Section 2, we present the model and obtain the associated 2-dimensional vector field. In Section 3, we study the long-run properties of this vector field. We particularly focus on the region of the parameters space such that the dynamics admits multiple steady states. Section 4 is devoted to the conditions required for the system to undergo the BT bifurcation. Moreover, we address the main point of the paper and derive policy implications from our analysis. Numerical examples are throughout provided. A brief conclusion reassesses the main findings of the paper. The Appendix provides all calculations and necessary proofs.

2 The Model

Consider the purely dynamical economic-environmental growth model

$$\begin{aligned} \dot{k}_t &= sf(k_t)\, d(p_t)(1-\tau) - \delta_k k_t \\ \dot{p}_t &= \theta(1-u) f(k_t) - \delta_p p_t \end{aligned} \qquad (\mathcal{S})$$

The first equation in system $\mathcal{S}$ models the evolution of physical capital, $\dot{k}_t$, where no exogenous technological progress is introduced. $s \in (0,1)$ represents the (fixed) fraction of the produced output that goes to capital investments, namely the saving rate; $f(k_t)$ is an implicit production function with $f'(k_t) > 0$; $d(p_t)$ is an implicit-form damage function with $d'(p_t) < 0$, which depends on the level of the pollutant; τ is the share of production that is paid out to the Government as an environmental tax, which is aimed at abate emissions. Finally, $\delta_k \in (0,1)$ measures the physical capital linear depreciation rate.

The second equation in system $\mathcal{S}$ describes the evolution of pollution, $\dot{p}_t$.[1] Production generates emissions which increase linearly the stock of pollution. $\theta > 0$ measures the degree of environmental inefficiency of economic activities. The abatement activities reduce a share $u \in (0,1)$ of emissions, thus $1-u$ represents unabated emissions. For the sake of simplicity, in a model that already includes non-linearities and non-convexities, the depreciation of pollution, that is the self-cleaning capacity of the environment, is assumed to be proportional to the stock of pollution, $\delta_p p_t$.[2]

The formulation of the model is in line with the literature reviewed in

[1] In the integrated models literature the word "pollution" generally means greenhouse gases. We instead interpret it in the broader sense of "undesired by-product" of economic activity.

[2] The linear depreciation of pollution, that is the self-cleaning capacity of the environment, is a gross simplification, since, as remarked in some literature (see, for example, Mäler *et al.*, 2003) depreceation is likely to be stock dependant, that is $\delta_p = \delta_p(p_t)$. As an example, the oceans absortion rate of carbon dioxide, depending strongly on the stock of carbon dioxide itself, can be mentioned.

Xepapadeas (2005) and, on another perspective, can also be considered a stripped-down variant of Nordhaus (1992, 2008) DICE model.

To proceed with our analysis, two further steps are required. We first notice that system $\mathcal{S}$ can be written in a more convenient form. Environmental taxation has been taken into account via the parameter τ: the tax revenue is $T_t = \tau s f(k_t)$, therefore proportional to saved production. We assume that the Government maintains a balanced budget at any point in time, so T_t is totally devoted to sustain abatement activities, $A(t)$ which aimed to abate a share u of pollution. The associated cost is $A_t = \mathcal{C}(u) s f(k_t)$, where $\mathcal{C}(u)$ is the cost function. By equating $T_t = A_t$, and assuming that the cost function takes the suitable form

$$\mathcal{C}(u) = 1 - (1-u)^{\epsilon}$$

proposed by Bartz and Kelly (2008), we conclude that

$$\tau = 1 - (1-u)^{\epsilon}$$

where $\epsilon \geq 1$.

To the purposes our this paper, we now provide explicit forms for the production and damage functions $f(k_t)$ and $d(p_t)$. As for the production function $f(k_t)$, we notice that, in the majority of the contributions in the economic-environmental literature proposes standard Cobb-Douglas forms.

This paper shall conversely treat the case of a convex-concave S-shaped production function (cf. Skiba, 1978) of the form

$$f(k_t) = \tfrac{\alpha_1 k_t^q}{1+\alpha_2 k_t^q} \tag{2}$$

where $\alpha_1 > 0,\ \alpha_2 > 0$ and $q > 1$ are parameters determining the position of the inflection point and the width of the transition from low and high value of k_t.

Many explicit damage functions have been proposed in the economic-environmental literature (see Bretschger and Pattakou, 2019, for a recent survey on the topic). Henceforth, we shall assume that the damage function $d(p_t)$ takes the form

$$d(p_t) = (1 + b p_t)^{-1}; \qquad b > 0 \tag{3}$$

as in La Torre *et al.* (2015). The formulation implies that the (negative) pollution externality on production is null when pollution is absent. It also implies that production falls non linearly when pollution increases.

3 Long-run equilibria and local stability

In this subsection we study the long-run equilibrium and present local analysis results. Consider first the following version of system $\mathcal{S}$

$$\begin{aligned} \dot{k}_t &= \frac{sk_t^2(1-u)}{\left(1+k_t^2\right)(1+bp_t))} - \delta_k k_t \\ \dot{p}_t &= \frac{\theta(1-u)k_t^2}{1+k_t^2} - \delta_p p_t \end{aligned} \qquad (\mathcal{M})$$

obtained by considering the explicit forms in (2) and (3) and the simplifying parametric assumptions

$$\alpha_1 = \alpha_2 = 1; \quad q = 2; \quad \varepsilon = 1 \qquad (4)$$

A steady state of the system is any solution of the pair (k, p) that satisfies the following relationships

$$\dot{k}_t = 0 \qquad (5.1)$$
$$\dot{p}_t = 0 \qquad (5.2)$$

We first observe that the pair $(k^*, p^*)_o = (0, 0)$ is an equilibrium point.

Remark 1 *Since we are only interested in interior solutions, we shall not consider this equilibrium in the rest of the paper.*

The following statement can be proved.

Proposition 1 (*Conditions for fold bifurcation*). *Let s, the saving rate, be the bifurcation parameter. Then, there exists a critical value*

$$\hat{s} = \frac{2\delta_k\sqrt{\delta_p[b\theta(1-u)+\delta_p]}}{\delta_p(1-u)} \qquad (6)$$

such that, if

1. *$s > \hat{s}(\delta_k, \delta_p, \theta, b, u)$, system $\mathcal{M}$ possesses two steady states, one with "low" capital and pollution*

$$(k^*, p^*)_{low} \equiv P^*_{low} = \left(\frac{1}{2}\frac{\delta_p s(1-u)-\sqrt{\Delta}}{\delta_k[b\theta(1-u)+\delta_p]}, \frac{\theta(1-u)k^{*2}_{low}}{\left(k^{*2}_{low}+1\right)\delta_p}\right) \qquad (7.1)$$

and one with "high" capital and pollution

$$(k^*,p^*)_{high} \equiv P^*_{high} = \left(\frac{1}{2}\frac{\delta_p\, s(1-u)+\sqrt{\Delta}}{\delta_k\left[b\theta(1-u)+\delta_p\right]}, \frac{\theta(1-u)k^{*2}_{high}}{\left(k^{*2}_{high}+1\right)\delta_p} \right) \tag{7.2}$$

2. $s = \hat{s}(\delta_k, \delta_p, \theta, b, u)$*, system* $\mathcal{M}$ *possesses two coincident steady states* (*coalescence equilibrium*)

$$(k^*,p^*)_{ce} \equiv P^*_{ce} = \left(\frac{1}{2}\frac{\delta_p\, s(1-u)}{\delta_k\left[b(1-u)+\delta_p\right]}, \frac{\theta(1-u)k^{*2}_{ce}}{(k^{*2}_{ce}+1)\delta_p} \right) \tag{7.3}$$

3. $s < \hat{s}(\delta_k, \delta_p, \theta, b, u)$*, system* $\mathcal{M}$ *does not admit a steady state.*

Proof. Solving (5.1) and (5.2) gives the coordinates of the steady states in (7.1), (7.2) and (7.3) depending on the value of the discriminant

$$\Delta = \delta_p \left[\delta_p s^2 (1-u)^2 - 4b\delta_k^2\theta(1-u) - 4\delta_k^2\delta_p\right]$$

which vanishes for the critical value of the saving rate in (6). Since $\frac{d\Delta}{ds} = 2\delta_p^2(1-u)^2 s > 0$ the statements in proposition are implied. ■

For a more immediate policy identification, we shall name P^*_{low} as the "green" or "clean" steady state, whereas P^*_{high} shall be referred to as the "dirty" steady state.

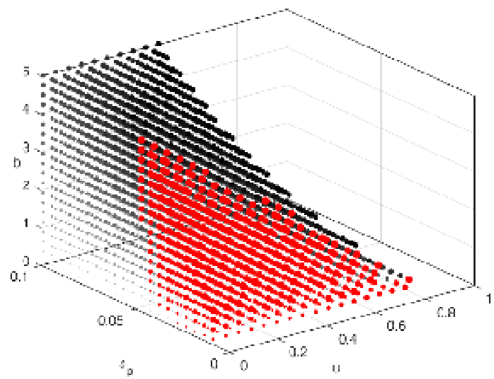
It is interesting here to observe that in our model the following occurs.

Remark 2 *Consider the case the economy has two steady states. Then, in the long-run, capital and pollution are complements.*

We provide now a parametric example giving rise to two steady states, starting from baseline parameter values.

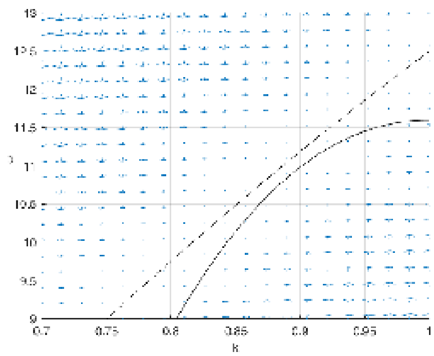
Example 1 *Set* δ_k *at the standard value of* 0.05*. In Figure* 1*, we depict the set of the remaining parameters* (δ_p, b, u) *such that* $\hat{s}$*, namely the critical value of the saving rate giving rise to the coalescence equilibria, is between zero and one. In Figure* 1*, the bigger the dots the higher* $\hat{s}$*. Notice that the region with plausible values of* $\hat{s}$ *is mainly located bottom left in the Figure,* *namely in correspondence of small* b*, small* u *and high* δ_p*.*

Figure 1*. Parameters combinations implying* s=$\hat{s}$ *with* δ_k=0.05



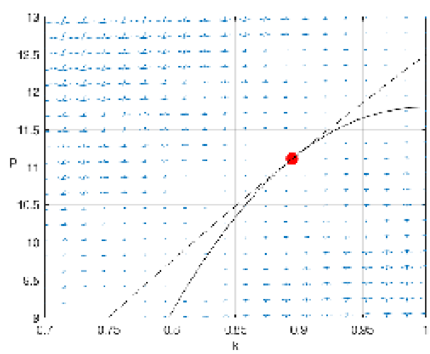
Set now $(\delta_p, b, u) = (0.02, 0.01, 0.5)$ *as in La Torre et al.* (2015)*. Thus,* $\hat{s} \cong 0.2236$*. At* $\hat{s} \cong 0.2236$*, the coalescence equilibrium has coordinates* $P^*_{ce} \cong (0.8944, 11.1111)$*. In Figure* 2*, we depict the nullclines in the phase space* (k, p) *for* $s = \hat{s}$ *in panel* (b)*, and for small deviations of the saving rate around* $\hat{s} \cong 0.2236$*. Specifically, in panel* (a) *we use* $s = 0.2232$*, whereas in* *panel* (c) *we have* $s = 0.2240$

Figure 2 (a)



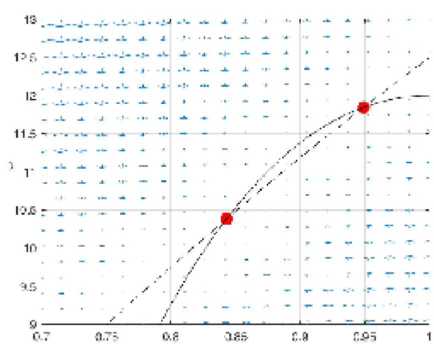
s=0.2232

Figure 2 (b)



s=0.2236

Figure 2 (c)



s=0.2240

Notice that the very high sensitivity of the bifurcation to deviations in the *saving rate. Namely, very small deviations of this parameter with regard to its critical value imply large distances of the two equilibria, both in terms of k^* and p^*. We leave the comments regarding the direction of the flow in Figure 1 to the next Subsection.*

3.1 Stability Analysis

The local stability properties of a given steady state can be described in terms of the signs of the invariants of the Jacobian matrix, $\mathbf{J}$, evaluated at a hyperbolic equilibrium point.

Let $\mathbf{Tr(J)}$ and $\mathbf{Det(J)}$, be the Trace and Determinant of $\mathbf{J}$, respectively. Explicit formulas can be found in Appendix 1. The following statements can be proved.

Proposition 2 *Recall Proposition 1. Assume $s > \hat{s}$. Then system $\mathcal{M}$ has a dual steady state. P^*_{low}, the green steady state, is always a saddle equilibrium, whereas P^*_{high}, the dirty steady state, is always a non saddle equilibrium.*

Proof. By standard arguments, the results on stability are obtained by evaluating the invariants of $\mathbf{J}$ at the two equilibrium points. In Appendix A.1, we show that at P^*_{low}, $\mathbf{Det(J)} < \mathbf{0}$. Therefore, P^*_{low} is always a saddle equilibrium. In the case of P^*_{high}, we conversely have $\mathbf{Det(J)} > \mathbf{0}$. The statements in Proposition are therefore implied. ■

4 Global analysis

In this section, we discuss the application of the Bogdanov-Takens (BT) bifurcation to system $\mathcal{M}$. The theorem allows us to detect a particular type of global phenomenon, namely, the homoclinic bifurcation, by which orbits growing around the non-saddle steady state collide with the saddle steady state. The interesting point is that a given dynamic system, undergoing a BT singularity, can be placed in correspondence with a simple planar system whose global unfolding is known in every aspect. Implications are very relevant for us: as shown hereafter, when the economy is in proximity of a BT singularity, the details of appropriate policies able to push the economy away from the dirty steady state towards the green steady state can be fully devised.

Consider the following preliminary result.

Proposition 3 (*System $\mathcal{M}$ may undergo the BT bifurcation*). *Recall Proposition* 1*. Let* $s = \hat{s}(\delta_k, \delta_p, \theta, b, u)$*. Then* $\mathbf{Det}(\mathbf{J}) = 0$*. Let furthermore*

$$\tilde{u} = \tilde{u}(\delta_k, \delta_p, \theta, b, \hat{s}(\delta_k, \delta_p, \theta, b, u)) = \frac{2{\delta_p}^2}{b\theta(\delta_k - \delta_p)}$$

be the value of the abated emissions which satisfies $\mathbf{Tr}(\mathbf{J}) = 0$*. Substituting* $\tilde{u}$ *into* $\hat{s}(\delta_k, \delta_p, \theta, b, u)$*, we obtain*

$$\tilde{s} = \frac{b\delta_k\theta(\delta_k - \delta_p)}{\delta_p^2}\sqrt{1 + \frac{2\delta_p}{\delta_k - \delta_p}}$$

Then, since technical non-degeneracy conditions are also met, at $(s, u) = (\tilde{s}, \tilde{u})$ *system* $\mathcal{M}$ *undergoes the BT bifurcation. The set of the parameters at which* $(\tilde{s}, \tilde{u}) \in (0, 1)^2$ *is not empty.*

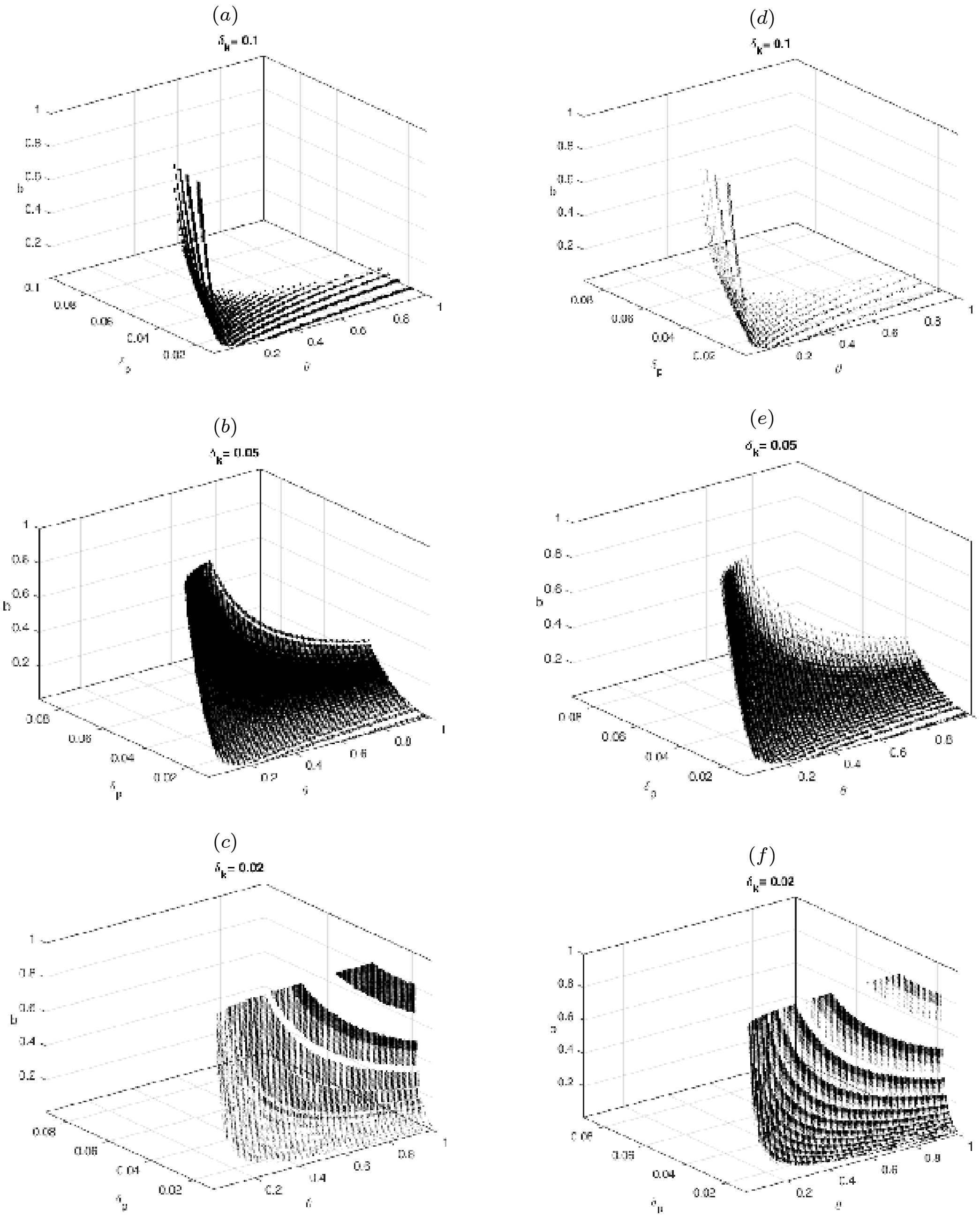
Proof. The application of the Bogdanov-Takens theorem to system $\mathcal{M}$ follows the detailed discussion in Kuznetsov (2004). For the satisfaction of the non-degeneracy conditions see Appendix $A.2$. ■

Interestingly, we observe that

Remark 3 *The BT singularity can only occur for* $\delta_k - \delta_p > 0$*.*

We study now the region of the parameters (δ_p, θ, b) such that $(\tilde{s}, \tilde{u}) \in (0, 1)^2$ for standard values of δ_k. In Figure 3, panels (a), (b), (c), the bigger the dots the higher $\tilde{s}$, whereas in panels (d), (e), (f), the bigger the dots the higher $\tilde{u}$.

Figure 3. Regions of the parameter space giving rise to $(\tilde{s},\tilde{u})\in(0,1)^2$



Some interesting characteristics of the subregion of the parameter space at which the BT bifurcation orccurs are the following. We first notice that high δ_k typically means relative high values of $\tilde{s}$ with regard to $\tilde{u}$; the converse applies for low values of δ_k.

Consider now the following Corollary of Proposition 3.

Corollary 1 (*Existence of a simple planar system topologically equivalent to system $\mathcal{M}$). Recall Proposition* 3. *Assume the economy is sufficiently close to the BT singularity. Then system $\mathcal{M}$ is topologically equivalent to the system*

$$\begin{aligned} \dot{w}_1 &= w_2 \\ \dot{w}_2 &= \eta_1 + \eta_2 w_2 + w_1^2 + \sigma w_1 w_2 \quad \sigma = \pm 1 \end{aligned} \tag{$\mathcal{Z}$}$$

The unfolding parameters, η_1 and η_2, are function of $\mu = s - \tilde{s}$ and $\nu = u - \tilde{u}$ satisfying

$$\eta_1 = D_1\mu - D_2\nu \tag{8.1}$$

$$\eta_2 = D_3\mu - D_4\nu \tag{8.2}$$

where D_i, with $i = 1...4$, are intricate combinations of the original parameters of the model.

Proof. In Appendix A.3 we have detailed all necessary steps. See also Wiggins (1990), p. 321-330. ■

The results in Corollary 1 are of great help for our analysis: the global analysis of system $\mathcal{M}$ can be based on a simple planar system whose unfolding (depending on the sign of σ) is completely known.

Consider the following details of the dynamics implied by system $\mathcal{Z}$.

Proposition 4 (*Unfolding of system $\mathcal{Z}$). Recall Proposition* 3 *and Corollary* 1. *Assume $\sigma = +1$. Then, for μ and ν sufficiently small, there exists a smooth curve $\mathcal{N}$ that originates at $(\eta_1, \eta_2) = (0, 0)$ and has a local representation given by*

$$\mathcal{N} \equiv \{(\eta_1, \eta_2) : \eta_1 = -\eta_2^2, \eta_2 > 0\}$$

such that:

- *above the curve $\mathcal{N}$, the non-saddle steady state is a source. There also exists a family of heteroclinic connections leading from the non-saddle steady state to the saddle steady state;*

- *below the curve $\mathcal{N}$, the non-saddle steady state is a sink.*

Proof. See Wiggins (1990), p. 321-330, for details. ■

To use the knowledge of system $\mathcal{Z}$ for our purposes, we need first to establish the sign of the coefficients. Our computations lead to the following result.

Lemma 1 (*Signs of the coefficients of the planar system $\mathcal{Z}$*). *Our computations show that $\sigma = +1$. Furthermore, for all combinations of the triplet (δ_p, θ, b) (for realistic δ_k) implying $(\tilde{s}, \tilde{u}) \in (0,1)^2$, $sign(D_1) < 0$ whereas $sign(D_2) = sign(D_3) = sign(D_4) > 0$.*

Proof. The sign of the coefficient σ can be easily obtained by applying the algorithm of Borisov and Dimitrova (2011) to our case. As shown in Appendix A.2, we obtain $sign(\sigma) = +1$. To establish the signs of the various D_i, we have repeated the numerical simulations made in Figure 3 and found that $D_1 < 0$ for all combinations of the parameters (δ_p, θ, b) such that $(\tilde{s}, \tilde{u}) \in (0,1)^2$ for standard values of δ_k and $sign(D_2) = sign(D_3) = sign(D_4) > 0$. ■

4.1 Policy induced heteroclinic orbits

The results provide an innovative framework in which it is possible to discuss several crucial policy issues. First of all, we are in the position of understanding conditions under which an equilibrium path approaching the green steady state exists.

Secondly, in the case an equilibrium path approaching the green steady state exists, we can distinguish economic conditions for which $laissez\text{-}faire$ is sufficient for a market economy to reach the green steady state, from those economic conditions at which this result can be only achieved by appropriate policy interventions.

Finally, in the case policy interventions are required to push the economy on the equilibrium path approaching the green steady state, we can uncover important details on the characteristics of these policies.

It is important to point out the following limitation of our analysis.

Remark 4 *Since system $\mathcal{Z}$ derives from a purely dynamical economic-environmental growth model, we cannot develop welfare considerations and only assume that social preferences are in favor of the green steady state whatever is the economic cost.*

Assume again $\mu = s - \tilde{s}$ and $\nu = u - \tilde{u}$ are sufficiently small.

Proposition 5 (*Heteroclinic orbits in laissez-faire*). *Assume the economy belongs to the region above the $\mathcal{N}$ curve. Assume furthermore the economy suddenly changes its preferences and is willing to reach the green steady state. Then, the perfect-foresight representative agent identifies a pair (k_0, p_0) on the unidimensional stable manifold of the green steady state such that the economy escapes the dirty steady state and approaches the green steady state. The set of parameters at which this phenomenon occurs is not empty.*

Proof. Recall Proposition 4. Consider the case s and u are chosen so that (η_1, η_2) are above the critical $\mathcal{N}$ curve. Then, by Proposition 4, we know that there is a heteroclinic connection going from the non-saddle to the saddle steady state. The rest of the Proposition follows. Example 2 below will show that this phenomenon can happen for parameter values inside the admissible region. ■

These statements are complemented by the following.

Proposition 6 (*Policy-induced heteroclinic orbits*). *Assume the economy belongs to the region below the $\mathcal{N}$ curve. Assume furthermore that the economy suddenly changes its preferences and is willing to reach the green steady state. Then, a sufficiently high increase of the abatement share u pushes the economy above the $\mathcal{N}$ curve. Then, again, the perfect-foresight representative agent identifies a pair (k_0, p_0) on the unidimensional stable manifold of the green steady state such that the economy escapes the dirty steady state and approaches the green steady state. The set of parameters at which this phenomenon occurs is not empty.*

Proof. Consider the case s and u are chosen so that (η_1, η_2) are below the critical $\mathcal{N}$ curve. Then, by Proposition 4, we know that there are no paths leading to the green steady state. Consider however, substituting (8.1) and (8.2) into the $\mathcal{N}$ curve, solving for μ and substituting back into the formula for η_2.[3] We obtain

$$\bar{\eta}_2 = \frac{1}{2}\frac{\sqrt{D_1^2+4D_3(D_2D_3-D_1D_4)\nu}-D_1}{D_3}$$

[3] Notice that the solutions for μ are two, one positive and one negative. Since, in our case, the dual steady states requires $s > \hat{s}$, and therefore a positive μ we have neglected the negative solution.

whose derivative with regard to ν gives

$$\frac{\partial \bar{\eta}_2}{\partial \nu} = \frac{D_2 D_3 - D_1 D_4}{\sqrt{D_1^2 + 4D_3(D_2 D_3 - D_1 D_4)\nu}}$$

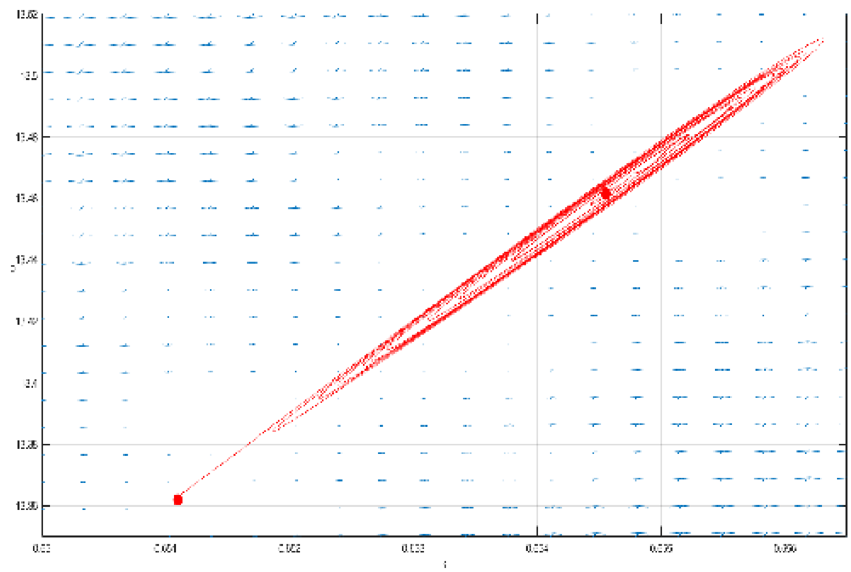
Therefore, recalling from Lemma 2 the signs of the various D_i, we see that $\frac{\partial \bar{\eta}_2}{\partial \nu} > 0$. An example below will show that this phenomenon can happen for parameter values inside the admissible region. ■

These results are innovative in the field: public intervention can appropriately calibrate its instruments (in our model, the saving rate and the abatement share) to induce market responses such that the policy objective is reached.

We close this Section by discussing a numerical example giving rise to a heteroclinic orbit.

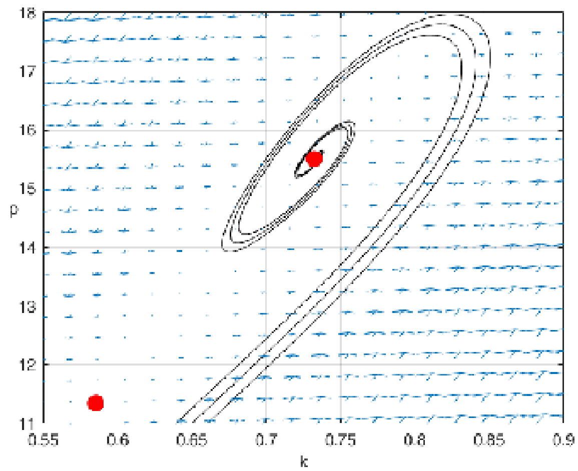
Example 2 *Set* $(\delta_k, \delta_p, \theta, b) = (0.05, 0.02, 0.93, 0.03)$*. Then* $(\tilde{s}, \tilde{u}) \in (0.1598, 0.0442)$ *and* $D_1 = -12.4865$*,* $D_2 = 848.5521$*,* $D_3 = 0.4380$*,* $D_4 = 0.0732$*. By Proposition* 1*, we know that for a dual steady state,* $\mu = s - \tilde{s} > 0$*. Consider the case of* $\mu = s - \tilde{s} = 0.0011$*. Using the formula of the local represention of the curve* $\mathcal{N}$ *in Proposition* 4*, we know that if* $\nu = u - \tilde{u} = 0.0002$*, the economy is exactly on the curve* $\mathcal{N}$*. Let now set* $\nu = 0.0094$*. Then, since* $\frac{\partial \bar{\eta}_2}{\partial \nu} > 0$*, we know that the economy belongs to the region above the curve* $\mathcal{N}$*. In Figure* 4*, we depict the heteroclinic orbit going from the dirty steady state to the green* *steady state.*

Figure 4. *The heteroclinic connection*



We complement the analysis by showing that a wrong policy may push the economy below the curve $\mathcal{N}$. If this happens, P^*_{high} becomes a sink and the economy remains trapped around the dirty steady state, as shown in Figure 5 for the same parameter values used in the Example above.

Figure 5. P^*_{high} when the economy is below the curve $\mathcal{N}$



5 Concluding remarks

This paper sheds light on the role of fiscal policy in models where multiple long-run equilibria emerge because of the presence of pollution externalities. Results are based on a purely dynamical growth model *à la* Solow with a S-shaped production function and where pollution is modeled through a non linear damage function. The saving ratio and the ratio of abated emissions are naturally used as policy variable.

We first outline the role of the saving rate in inducing/suppressing the multiplicity of equilibria. In particular, we find that there exists a specific level of the saving rate such that the system of dynamic laws implied by our model either has: (i) no steady states, (ii) one steady state, and (iii) two steady states, one with lower levels of capital and pollution (the "green" or "clean" steady state) with regard to the other (the "dirty" steady state). By means of the local analysis, we also find that, in the case of a dual equilibrium, the green steady state is always a saddle, whereas the dirty steady state is always a non-saddle rest point.

The second part of the paper is devoted to global analysis. We first prove that the system of dynamic laws implied by our model undergoes a Bogdanov-Takens (BT) singularity in specific regions of the parameter space. The usefulness of this phenomenon lays on the possibility of putting in correspondence a highly non linear system with a simple planar system whose global unfolding is known in every aspect. The main results are as follows. Assuming social preferences are in favor of the green steady state, there are two qualitatively separated regions in the parameter space, one at which the economy in $laissez\text{-}faire$ is able to reach the green steady state and one where only appropriately devised choices of the saving rate and of the fraction of abated pollutants can put the economy on a path converging to the green steady state.

Appendix

A.1. Local stability analysis

The Jacobian matrix of the system $\mathcal{M}$, evaluated at the steady state is

$$\mathbf{J} = \begin{bmatrix} \frac{2s(1-u)k^*}{(1+k^{*2})^2(bp^*+1)} - \delta_k & -\frac{sbk^{*2}(1-u)}{(1+k^{*2})(bp^*+1)^2} \\ \frac{2\theta\,(1-u)k^*}{(1+k^{*2})^2} & -\delta_p \end{bmatrix} \tag{A.1}$$

Therefore

$$\mathbf{Tr}(\mathbf{J}) = \frac{2s(1-u)k^*}{(1+k^{*2})^2(bp^*+1)} - \delta_k - \delta_p \tag{A.2}$$

$$\mathbf{Det}(\mathbf{J}) = \frac{2s(1-u)k^*}{(1+k^{*2})^2(1+bp^*)}\left[-\delta_p + \frac{\theta\, b(1-u)k^{*3}}{(1+k^{*2})(1+bp^*)}\right] + \delta_k\delta_p \tag{A.3}$$

We can now proceed to evaluate the signs of $\mathbf{Det}(\mathbf{J})$ at the green and dirty steady states. We report here only the final stage of a number of algebraic manipulation performed with Maple 18. The complete sequence of commented Maple is available from the authors upon request.

We start by considering the green steady state. We find that

$$\begin{aligned} \mathbf{Det}(\mathbf{J})_{P^*_{low}} &= C[s\,(1-u)\,\delta_p\left\{\delta_p s^2(1-u)^2 - 4\delta_k^2\left[b\theta\,(1-u)+\delta_p\right]\right\} + \\ &\quad -\sqrt{\Delta}\left\{\delta_p s^2(1-u)^2 - 2\delta_k^2\left[b\theta\ (1-u)+\delta_p\right\}\right] \end{aligned}$$

Where the positive constant C_{low} reads as

$$C_{low} = \frac{2\delta_k\delta_p}{s\,(1-u)\,[\sqrt{\Delta} - \delta_p s(1-u)]^2}$$

In the formula for $\mathbf{Det(J)}_{P^*_{low}}$, if

$$\sqrt{\Delta}\left\{\delta_p s^2(1-u)^2 - 2\delta_k^2\left[b\theta\ (1-u)+\delta_p\right\} > s\,(1-u)\,\delta_p[\delta_p s^2(1-u)^2 - 4\delta_k^2\left[b\theta\,(1-u)+\delta_p\right] \tag{A.4}$$

then $\mathbf{Det(J)}_{P^*_{low}} < 0$.

By recalling from the proof in Proposition 1 in the main text that

$$\Delta = \delta_p\left[\delta_p s^2(1-u)^2 - 4b\delta_k^2\theta(1-u) - 4\delta_k^2\delta_p\right]$$

(A.4) reduces to

$$\delta_p s^2(1-u)^2 - 2\delta_k^2[b\theta\,(1-u)+\delta_p] > s\ (1-u)\sqrt{\Delta}$$

Further, we now divide both members by $s^2(1-u)^2\delta_p$. After some simple algebra we get

$$1 - \frac{2\delta_k^2[b\theta\,(1-u)+\delta_p]}{s^2(1-u)^2\delta_p} > \sqrt{1-\frac{4\delta_k^2[b\theta\,(1-u)+\delta_p]}{s^2(1-u)^2\delta_p}} \tag{A.5}$$

Now, consider the following scaling

$$t = \frac{\delta_k^2[b\theta(1-u)+\delta_p]}{s^2(1-u)^2\delta_p}$$

(A.5) can be written as

$$1-2t > \sqrt{1-4t}$$

Taking the square of both members it is easy to show that the last inequality is true whenever $0 < t < 1/4$. Reversing the substitution we get

$$0 < \frac{\delta_k^2\,[b\theta\,(1-u)+\delta_p]}{s^2(1-u)^2\delta_p} < \frac{1}{4}$$

which is exactly the condition for the existence of a dual steady state. As a consequence, when there exist two steady states, $\mathbf{Det(J)}_{P^*_{low}} < 0$.

Now it is the turn of $\mathbf{Det(J)}_{P^*_{high}}$. Repeating the steps above, we find now that

$$\begin{aligned}\mathbf{Det(J)}_{P^*_{high}} \quad &= \quad C_{high}[s\,(1-u)\,\delta_p\left\{s^2(1-u)^2\delta_p - 4\delta_k^2[b\theta\ (1-u)+\delta_p]\right\} + \\ &\quad +\sqrt{\Delta}\left\{s^2(1-u)^2\delta_p - 2\,\delta_k^2\,[b\theta\ (1-u)+\delta_p]\right\}]\end{aligned}$$

Where the positive constant C^{high} reads as

$$C^{high} = \frac{2\delta_k\delta_p}{s\,(1-u)\,[\sqrt{\Delta}+\delta_p s(1-u)]^2}$$

Contrarily to what happens in P^*_{low}, $\mathbf{Det}(\mathbf{J})_{P^*_{high}}$ is now positive. Indeed, the term

$$s\,(1-u)\,\delta_p\left\{s^2(1-u)^2\delta_p - 4\delta_k^2[b\theta\ (1-u)+\delta_p]\right\} = s(1-u)\Delta > 0$$

while the term

$$s^2(1-u)^2\delta_p - 2\,\delta_k^2\,[b\theta\ (1-u)+\delta_p] = \Delta + 2\delta_k^2\,[b\theta\ (1-u)+\delta_p] > 0$$

when $\Delta > 0$, namely when there exists a dual steady state.

A.2. Existence of the BT singularity

In order to detect the BT bifurcation in planar systems, Borisov and Dimitrova (2011) outline an easy-to-check algorithm based on Kuznetsov (2004). We need first to perform the coordinate change

$$\begin{aligned}
\tilde{k} &= k - k^*_{ce} \\
\tilde{p} &= p - p^*_{ce} \\
\mu &= s - \tilde{s} \\
\nu &= u - \tilde{u}
\end{aligned}$$

System $\mathcal{M}$ becomes

$$\begin{aligned}
\dot{\tilde{k}} &= \frac{(\tilde{s}+\mu)(1-\tilde{u}-\nu)\left(\tilde{k}_t+k^*_{ce}\right)^2}{[1+b(\tilde{p}_t+p^*_{ce})]\left[1+\left(\tilde{k}_t+k^*_{ce}\right)^2\right]} - \delta_k\left(\tilde{k}_t+k^*_{ce}\right) \\
\dot{\tilde{p}} &= \frac{\theta(1-\tilde{u}-\nu)\left(\tilde{k}_t+k^*_{ce}\right)^2}{1+\left(\tilde{k}_t+k^*_{ce}\right)^2} - \delta_p\left(\tilde{p}_t+p^*_{ce}\right)
\end{aligned} \tag{A.6}$$

Now the system is in the form

$$\dot{x} = f(x,\alpha), \quad x\in\mathbb{R}^2, \quad \alpha\in\mathbb{R}^2$$

where $x = \left(\tilde{k}, \tilde{p}\right)$ and $\alpha = (\mu, \nu)$. In primis the algorithm checks if there are regions in the parameter space such that the point $(\tilde{k}_t, \tilde{p}_t) = (0, 0)$ is a bifurcation point with a double zero eigenvalue at $(\mu, \nu) = (0, 0)$. As shown in Proposition 3, this happens at

$$u = \tilde{u} = \tfrac{2\delta_p^2}{b\theta(\delta_k - \delta_p)}; \quad s = \tilde{s} = \tfrac{b\delta_k\theta(\delta_k - \delta_p)}{\delta_p^2}\sqrt{1 + \tfrac{2\delta_p}{\delta_k - \delta_p}}$$

which may occur inside the admissible parameter space.

Secondly, the algorithm checks if $a(0)b(0) \neq 0$, where $a(0)$ and $b(0)$ are certain quadratic coefficients (generated by the algorithm itself). The algorithm shows that

$$\begin{aligned} a(0) &= -\tfrac{\delta_k(\delta_p+\delta_k)\delta_p^2}{\delta_k-\delta_p}\tfrac{1}{\sqrt{\frac{\delta_p^2(\delta_p+\delta_k)}{\delta_k-\delta_p}}} \\ b(0) &= -\tfrac{\delta_k^2-\delta_p^2}{\delta_k\delta_p}\sqrt{\tfrac{\delta_p^2(\delta_p+\delta_k)}{\delta_k-\delta_p}} \end{aligned}$$

As a consequence, $a(0)b(0) \neq 0$.

Finally, the map $(x, \alpha) \to (f(x, \alpha), Tr(f_x(x, \alpha)), Det(f_x(x, \alpha)))$ must be regular at $(x, \alpha) = (0, 0)$. Since the determinant of the map is

$$-\tfrac{1}{2}\,\tfrac{\delta_p^2(\delta_k-\delta_p)^2}{\delta_p+\delta_k}\sqrt{\tfrac{\delta_p^2(\delta_p+\delta_k)}{\delta_k-\delta_p}}$$

the conditions is satisfied.

As a consequence of the steps above, system $\mathcal{M}$ is a candidate for the onset of a BT singularity.

Moreover the algorithm gives

$$sign(\sigma) = sign\left(\delta_p\left(\delta_p + \delta_k\right)^2\right) = +1$$

A.3. Transformation of system $\mathcal{M}$ into normal form

The procedure requires 5 steps.

Step 1. Consider system A.6 and perform the second order Taylor expansion on $\left(\tilde{k}, \tilde{p}, \mu, \nu\right)$. We obtain

$$\begin{bmatrix} \dot{\tilde{k}} \\ \dot{\tilde{p}} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \tilde{k} \\ \tilde{p} \end{bmatrix} + \mathbf{B}(\mu, \nu) \begin{bmatrix} \tilde{k} \\ \tilde{p} \end{bmatrix} + \begin{bmatrix} F_1\left(\tilde{k}, \tilde{p}, \mu, \nu\right) \\ F_2\left(\tilde{k}, \tilde{p}, \mu, \nu\right) \end{bmatrix} \tag{A.7}$$

where $\mathbf{J}$ contains the first order terms, while the matrix $\mathbf{B}$ and the vector $\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$ group the second order terms among which $\mathbf{B}$ collects the ones linearly dependent on $\left(\tilde{k}, \tilde{p}\right)$. In our case we have

$$\begin{aligned}
\mathbf{J} &= \begin{bmatrix} \frac{2\tilde{s}(1-\tilde{u})k^*_{ce}}{(1+k^{*2}_{ce})^2(1+bp^*_{ce})} - \delta_k & -\frac{\tilde{s}(1-\tilde{u})k^{*2}_{ce}b}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})} \\ \frac{2\theta(1-\tilde{u})k^*_{ce}}{(1+k^{*2}_{ce})^2} & -\delta_p \end{bmatrix} \\
\mathbf{B} &= \begin{bmatrix} \frac{2(\mu\tilde{u}+\nu\tilde{s})k^*_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})^2} & -\frac{(\mu\,\tilde{u}+\nu\,\tilde{s})bk^{*2}_{ce}}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})} \\ \frac{2\theta\nu k^*_{ce}}{(1+k^{*2}_{ce})^2} & 0 \end{bmatrix} \\
\mathbf{F}_1 &= \frac{1}{1+k^{*2}_{ce}}\left(\frac{\tilde{s}(1-\tilde{u})}{1+bp^*_{ce}} - \frac{\tilde{s}\,(1-\tilde{u})\,k^{*2}_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})} - \frac{4k^{*2}_{ce}(1-\tilde{u})\tilde{s}\,(bp^*_{ce}+\tilde{s})}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})^2}\right)\tilde{k}^2 + \\
&\quad \frac{1}{1+k^{*2}_{ce}}\left(\frac{-k^*_{ce}(2\,bp^*_{ce}\,\tilde{s}\,(1-\tilde{u})+2\,\tilde{s}\,(1-\tilde{u}))b}{(1+bp^*_{ce})^3} + \frac{2k^{*3}_{ce}(1-\tilde{u})\tilde{s}\,(b+k^*_{ce})}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})^2}\right)\tilde{k}\tilde{p} + \\
&\quad + \frac{\tilde{s}\,(1-\tilde{u})k^{*2}_{ce}b}{(1+bp^*_{ce})^3(1+k^{*2}_{ce})}\tilde{p}^2 - \frac{k^{*2}_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})}\mu\nu \\
\mathbf{F}_2 &= \frac{1}{1+k^{*2}_{ce}}\left(\theta\,(1-\tilde{u}) - \frac{\theta\,(1-\tilde{u})k^{*2}_{ce}}{1+k^{*2}_{ce}} - \frac{4\theta(1-\tilde{u})k^{*2}_{ce}}{(1+k^{*2}_{ce})^2}\right)\tilde{k}^2
\end{aligned}$$

Step 2. To get the normal form discussed in Corollary 1, we need to perform the similar transformation of variables

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \mathbf{T} \begin{bmatrix} \tilde{k} \\ \tilde{p} \end{bmatrix}$$

where

$$\mathbf{T} = \begin{bmatrix} 0 & 0 \\ \mathbf{J}(1,1) & \mathbf{J}(1,2) \end{bmatrix}$$

is the transformation matrix which transforms A.7 into

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \mathbf{M}(\mu, \nu) \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} \tilde{F}_1(w_1, w_2) \\ \tilde{F}_2(w_1, w_2) \end{bmatrix} \tag{A.8}$$

where

$$\mathbf{M} = \mathbf{T^{-1}BT}$$

and

$$\left[\begin{array}{c} \tilde{F}_1\left(w_1, w_2\right) \\ \tilde{F}_2\left(w_1, w_2\right) \end{array}\right] = \tilde{\mathbf{F}} = \mathbf{T^{-1}F}\left(\mathbf{T}\left[\begin{array}{c} \tilde{k} \\ \tilde{p} \end{array}\right]\right)$$

In particular, if we rename the entries of the Jacobian matrix as $\mathbf{J}(i,j) = J_{ij}$, for $i, j = 1..2$, then we have

$$\begin{array}{rcl}
\mathbf{M} & = & \left[\begin{array}{cc} \frac{(2\,\mu(1-\tilde{u})+2\,\tilde{s}\,(1-\nu))k^*_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})^2} - \frac{(\mu(1-\tilde{u})+\tilde{s}\,(1-\nu))bk^{*2}_{ce}J_{11}}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})} & -\frac{(\mu(1-\tilde{u})+\tilde{s}\,(1-\nu))bk^{*2}_{ce}J_{12}}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})} \\ -\frac{J_{11}\,(2\,\mu(1-\tilde{u})+2\,\tilde{s}\,(1-\nu))k^*_{ce}}{J_{12}\,(1+bp^*_{ce})(1+k^{*2}_{ce})^2} + \frac{2\theta\,(1-\nu)k^*_{ce}}{J_{12}\,(1+k^{*2}_{ce})^2} + \frac{J_{11}^2(\mu(1-\tilde{u})+\tilde{s}\,(1-\nu))bk^{*2}_{ce}}{J_{12}^2(1+bp^*_{ce})(1+k^{*2}_{ce})} & \frac{(\mu(1-\tilde{u})+\tilde{s}\,(1-\nu))bk^{*2}_{ce}J_{11}}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})} \end{array}\right] \\
 & & \frac{1}{(1+k^{*2}_{ce})}\left(\frac{\tilde{s}(1-\tilde{u})\,\alpha}{(1+bp^*_{ce})} - \frac{\tilde{s}\,(1-\tilde{u})\,k^{*2}_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})^2} + 2\;\frac{k^{*2}_{ce}(-2\,bp^*_{ce}\,\tilde{s}\,(1-\tilde{u})-2\,\tilde{s}\,(1-\tilde{u}))}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})^2}\right)w_1^2 + \\
\tilde{\mathbf{F}}_\mathbf{1} & = & \left(-\frac{k^*_{ce}\,(2bp^*_{ce}\,\tilde{s}\,(1-\tilde{u})+2\,\tilde{s}\,(1-\tilde{u}))b}{(1+bp^*_{ce})^3} + 2\,\frac{k^{*2}_{ce}\left(bk^{*3}_{ce}\tilde{s}\,(1-\tilde{u})+bk^*_{ce}\,\tilde{s}\,(1-\tilde{u})\right)}{(1+bp^*_{ce})^2(1+k^{*2}_{ce})^2}\right)\frac{\left(J_{11}w_1^2+J_{12}w_1w_2\right)}{(1+k^{*2}_{ce})} + \\
 & & \frac{\tilde{s}\,(1-\tilde{u})\,k^{*2}_{ce}b^2(J_{11}w_1+J_{12}w_2)^2}{(1+bp^*_{ce})^3(1+k^{*2}_{ce})} - \frac{k^{*2}_{ce}\mu\nu}{(1+bp^*_{ce})(1+k^{*2}_{ce})} \\
\tilde{\mathbf{F}}_\mathbf{2} & = & \frac{1}{1+k^{*2}_{ce}}\left(\theta\,(1-\tilde{u}) - \frac{\theta\,(1-\tilde{u})k^{*2}_{ce}}{1+k^{*2}_{ce}} - \frac{4\theta(1-\tilde{u})k^{*2}_{ce}}{(1+k^{*2}_{ce})^2}\right)w_1^2
\end{array}$$

Step 3. To simplify the nonlinear parts in A.8, we follow Gamero *et al.* (1991). Hence, A.8 can be rewritten as

$$\left[\begin{array}{c} \dot{w}_1 \\ \dot{w}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right]\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] + \mathbf{M}\left(\mu,\nu\right)\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] + \left[\begin{array}{c} 0 \\ a_2w_1^2 + b_2w_1w_2 \end{array}\right] \qquad \text{(A.9)}$$

where

$$\begin{array}{rcl} a_2 & = & -\frac{1}{2}\frac{\partial^2\tilde{F}_2(w_1,w_2)}{\partial w_1^2} \\ b_2 & = & \frac{\partial^2\tilde{F}_2(w_1,w_2)}{\partial w_1 \partial w_2} + \frac{\partial^2\tilde{F}_1(w_1,w_2)}{\partial w_1^2} \end{array}$$

Step 4. At this stage, Wiggins (1990) shows that system A.9 is similar to

$$\left[\begin{array}{c} \dot{w}_1 \\ \dot{w}_2 \end{array}\right] = \left[\begin{array}{cc} 0 & 1 \\ \eta_1 & \eta_2 \end{array}\right]\left[\begin{array}{c} w_1 \\ w_2 \end{array}\right] + \left[\begin{array}{c} 0 \\ a_2w_1^2 + b_2w_1w_2 \end{array}\right] \qquad \text{(A.10)}$$

where

$$\left[\begin{array}{c} \eta_1 \\ \eta_2 \end{array}\right] = \left[\begin{array}{cc} D_1 & D_2 \\ D_3 & D_4 \end{array}\right]\left[\begin{array}{c} \mu \\ \nu \end{array}\right]$$

given

$$\begin{aligned}
D_1 &= \frac{2\,J_{11}(1-\tilde{u})k^*_{ce}}{J_{12}(1+bp^*_{ce})(1+k^*_{ce})^2} - \frac{J^2_{11}(1-\tilde{u})\,bk^{*2}_{ce}}{J_{12}(1+bp^*_{ce})^2(1+k^*_{ce})}\\
D_2 &= \frac{2\,J_{11}\tilde{s}\,k^*_{ce}}{J_{12}(1+bp^*_{ce})(1+k^{*2}_{ce})^2} - \frac{2\theta k^{*2}_{ce}}{J^2_{12}(1+k^{*2}_{ce})} - \frac{J^2_{11}\tilde{s}\,bk^{*2}_{ce}}{J_{12}(1+bp^*_{ce})^2(1+k^{*2}_{ce})}\\
D_3 &= \frac{2\,(1-\tilde{u})k^*_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})^2}\\
D_4 &= \frac{2\tilde{s}k^*_{ce}}{(1+bp^*_{ce})(1+k^{*2}_{ce})^2}
\end{aligned}$$

Since

$$\begin{vmatrix} D_1 & D_2 \\ D_3 & D_4 \end{vmatrix} = \frac{4\theta(1-\tilde{u})\,k^{*2}_{ce}}{J_{12}(1+k^{*2}_{ce})^4(1+bp^*_{ce})}$$

does not vanish, the versal deformation satisfies the transversality condition.
Step 5. Finally, Wiggings (1991) shows that A.10 can be further transformed into

$$\begin{aligned}
\dot{w}_1 &= w_2\\
\dot{w}_2 &= \eta_1 + \eta_2 w_2 + a_2 w_1^2 + b_2 w_1 w_2 + \mathcal{O}\left(|\cdot|^3\right)
\end{aligned}$$

and rescaled to obtain the system

$$\begin{aligned}
\dot{w}_1 &= w_2\\
\dot{w}_2 &= \eta_1 + \eta_2 w_2 + w_1^2 \pm \sigma w_1 w_2
\end{aligned}$$

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