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Inverse scattering transform and soliton solutions for a square matrix nonlinear Schrödinger equation with nonzero boundary conditions

Barbara Prinari

Department of Mathematics, University of Colorado Colorado Springs, Colorado Springs, CO 80918

Francesco Demontis

Dipartimento di Matematica, Università di Cagliari, 09124 Cagliari, Italy

Sitai Li

Department of Mathematics, State University of New York at Buffalo, Buffalo, NY 14260

Theodoros P. Horikis

Department of Mathematics, University of Ioannina, 45110 Ioannina, Greece

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Abstract

The inverse scattering transform (IST) with non-zero boundary conditions at infinity is presented for a matrix nonlinear Schrödinger-type equation which has been proposed as a model to describe hyperfine spin $F = 1$ spinor Bose-Einstein condensates with either repulsive interatomic interactions and anti-ferromagnetic spin-exchange interactions (self-defocusing case), or attractive interatomic interactions and ferromagnetic spin-exchange interactions (self-focusing case). Both the direct and the inverse problems are formulated in terms of a suitable uniformization variable which allows to develop the IST on the standard complex plane, instead of a two-sheeted Riemann surface or the cut plane with discontinuities along the cuts. Analyticity of the scattering eigenfunctions and scattering data, symmetries, properties of the discrete spectrum, and asymptotics are derived. The inverse problem is posed as a Riemann-Hilbert problem for the eigenfunctions, and the reconstruction formula of the potential in terms of eigenfunctions and scattering data is provided. In addition, the general behavior of the one-soliton solutions is analyzed in details in the self-focusing case, including some special cases not previously discussed in the literature.

1 Introduction

In recent years there has been an increasing focus on the study of multicomponent Bose-Einstein condensates (BECs), and particularly spinor condensates, within the context of atomic and nonlinear wave physics. Multicomponent ultracold atomic gases and BECs may be composed by two or more atomic gases, and may have the form of various different mixtures, which have been observed in experiments [15]. Such multicomponent systems support various types of matter-wave soliton complexes, with the type of soliton in one species being the same or different to that in the other species. Unlike what happens in multicomponent nonlinear optics, where Kerr-type nonlinearities only depend on the squared modulii of the components, the equations describing spinor condensates exhibit nonlinear terms reflecting the $SU(2)$ symmetry of the spins: the spin-exchange interactions that are the sources of the spin-mixing within condensates deviate from the above mentioned intensity-coupled nonlinearity, and they have no analogue in conventional nonlinear optics.

A number of theoretical works have been dealing with multicomponent vector solitons in $F = 1$ spinor BECs, which are described by a 3-component macroscopic wave function. Bright and dark solitons, as well as gap solitons, have been predicted in this context (the latter type requires the presence of an optical lattice). In this work we will be concerned with the matrix nonlinear Schrödinger equation:

$$iQ_t + Q_{xx} - 2\sigma QQ^\dagger Q = 0\,, \qquad \sigma = \pm 1\,, \tag{1.1}$$

where $Q(x,t)$ is a 2×2 matrix valued function, $Q^\dagger$ is the Hermitian conjugate of Q, and subscripts x, t denote partial differentiation throughout. The choice $\sigma = -1$ and $\sigma = +1$ distinguishes between the self-focusing and self-defocusing regimes, respectively. When the 2×2 matrix potential $Q(x,t)$ is chosen to be a symmetric matrix

$$Q(x,t) = \begin{pmatrix} q_1(x,t) & q_0(x,t) \\ q_0(x,t) & q_{-1}(x,t) \end{pmatrix}\,,$$

the system (1.1) has been proposed as a model to describe hyperfine spin $F = 1$ spinor Bose-Einstein condensates with either repulsive interatomic interactions and anti-ferromagnetic spin-exchange interactions ($\sigma = +1$), or attractive interatomic interactions and ferromagnetic spin-exchange interactions ($\sigma = -1$); q_1, q_0, q_{-1} are related to the vacuum expectation values of the three components of the quantum field operator in the three possible spin configurations $1, 0, -1$ [18, 26].

We are interested in developing the inverse scattering transform (IST) for the above system under nonzero boundary conditions (NZBCs) as $x \to \pm\infty$, as a tool to solve the initial-value problem, and also obtain explicit soliton solutions as a byproduct of the IST. It should be pointed out that in general the boundary conditions for the matrix $Q(x,t)$ will have to be time-dependent, but their time-independence can be easily achieved by means of the gauge transformation: $Q(x,t) = \hat{Q}(x,t)\, e^{-2i\sigma k_o^2 t}$. Then the equation becomes

$$iQ_t + Q_{xx} + 2\sigma \left(k_o^2 I_2 - QQ^\dagger\right) Q = 0 \tag{1.2}$$

where I_2 denotes the 2×2 identity matrix and where we have dropped the $\hat{}$ in the dependent variable for shortness.

In the following we will then consider the system (1.2) under constant NZBCs:

$$Q(x,t) \to Q_\pm \qquad \text{as} \quad x \to \pm\infty\,, \tag{1.3}$$

and assume that the following constraints on the boundary conditions hold:

$$Q_\pm^\dagger Q_\pm = Q_\pm Q_\pm^\dagger = k_o^2 I_2\,, \tag{1.4}$$

where k_o is a real, positive constant. The latter is equivalent to the following constraints on the boundary values of the individual entries of the potential matrix Q as $x \to \pm\infty$:

$$|q_{1,\pm}|^2 = |q_{-1,\pm}|^2\,, \qquad |q_{0,\pm}|^2 = k_o^2 - |q_{1,\pm}|^2 \equiv k_o^2 - |q_{-1,\pm}|^2\,, \qquad q_{1,\pm} q_{0,\pm}^* + q_{0,\pm} q_{-1,\pm}^* = 0\,.$$

Note that the boundary condition Q_+ can be chosen without loss of generality as $Q_+ = k_o I_2$, since an arbitrary boundary condition can be reduced to it by a change of dependent variable that exploits the invariance of (1.2) under multiplication on either side by an appropriate unitary matrix (see Appendix A for details).

The IST for the system (1.2) with NZBCs was first presented in [19], following the approach proposed by Kawata and Inoue for the scalar defocusing NLS equation (cf. [13, 14]), with the introduction of a smooth potential matrix $\tilde{Q}(x,t)$ with the same boundary conditions as $Q(x,t)$ as $x \to \pm\infty$, but which satisfies the constraint (1.4) for all $x \in \mathbb{R}$ (and not just asymptotically). The IST is then carried out in [19] both for the focusing ($\sigma = -1$) and for the defocusing ($\sigma = +1$) regimes, but only in the latter case soliton solutions are discussed in detail (cf. [26, 27]). Two snapshots of one soliton solutions for the focusing case were presented in [27], with no explicit formula and no details provided other than the fact that it "gives an oscillating profile of soliton". As typical within the IST framework, the soliton solutions are expressed in terms of discrete eigenvalues and norming constants (real/complex constants $\{\lambda_j\}_{j=1}^n$ and real/complex 2×2 symmetric matrices $\{\Pi_j\}_{j=1}^n$, respectively, in the notation of [19]), but while in the direct problem such norming constants are found, for simple eigenvalues, to have to satisfy the constraint $\det \Pi_j = 0$, this condition is then relaxed to obtain more general soliton solutions, without clarifying what this implies from a spectral point view. Moreover, choosing Q to be a symmetric matrix induces an additional symmetry in the scattering data (reflection coefficients and norming constants), which is instrumental to guarantee that the solution reconstructed from the inverse problem is itself a symmetric matrix, and which has not been taken into account in [19, 26, 27].

As a matter of fact, there are several very compelling reasons for re-deriving the IST: (i) formulate the direct problem entirely in terms of the original potential $Q(x,t)$, without the need of resorting to the introduction of $\tilde{Q}(x,t)$; (ii) develop the IST in terms of a uniformization variable, which allows to define the direct and inverse problems on the complex plane, instead of a two-sheeted Riemann surface or the cut plane with discontinuities along the cuts; (iii) account for the symmetry $Q^T = Q$, and obtain the corresponding symmetries in the scattering data (reflection coefficients and norming constants); (iv) clarify the definition of the norming constants, explain the condition $\det \Pi_j = 0$ vs $\det \Pi_j \neq 0$, and provide a clear spectral characterization of the corresponding solutions; (v) formulate the inverse problem as a Riemann-Hilbert problem, instead of in terms of Marchenko equations; (vi) derive novel soliton solutions for the focusing equation corresponding to special choices of the discrete eigenvalues (these solutions are the analog, in this spinor model, of the Akhmediev and Kuznetsov-Ma solutions of the focusing NLS equation with nonzero boundary conditions). All of these issues will be addressed in this paper.

As mentioned above, one of the boundary conditions can be assumed without loss of generality to be proportional to the identity matrix. Assuming a boundary condition $Q_+ = k_o I_2$ corresponds to having $q_1(x,t)$ and $q_{-1}(x,t)$ with the same modulus k_o as $x \to +\infty$, and their phases chosen to be zero; $q_0(x,t) \to 0$ as $x \to +\infty$. As $x \to -\infty$, the constraint (1.4) yields

$$\lim_{x\to-\infty} |q_{-1}(x,t)|^2 = \lim_{x\to-\infty} |q_1(x,t)|^2 =: \mu_o^2\,, \tag{1.5a}$$

$$\lim_{x\to-\infty} |q_0(x,t)|^2 = k_o^2 - \mu_o^2 \quad \Rightarrow \quad \mu_o^2 \leqslant k_o^2\,. \tag{1.5b}$$

As far as the asymptotic phases as $x \to -\infty$ are concerned, if we denote them as θ_j for $j = -1, 0, 1$, the off-diagonals entries of (1.4) imply that the asymptotic phases as $x \to -\infty$ should satisfy the constraint $e^{i(\theta_1-\theta_0)}+e^{i(\theta_0-\theta_{-1})}=0$, which is equivalent to

$$\theta_0=\frac{(\theta_1+\theta_{-1})}{2}+n\frac{\pi}{2} \qquad n \text{ odd integer} \qquad (\mathrm{mod}\, 2\pi)\,.$$

If, on the other hand, we start with $Q_+=k_o\sigma_1$, this corresponds to having the modulus of $q_1(x,t)$ and $q_{-1}(x,t)$ transitioning from 0 as $x\to+\infty$ to $\mu_o\leqslant k_o$ as $x\to-\infty$, and the modulus of $q_0(x,t)$ from k_o as $x\to+\infty$ to $\sqrt{k_o^2-\mu_o^2}$ as $x\to-\infty$. The conditions on the asymptotic phases are the same as in the previous case.

In the development of the IST, we will follow the construction of [4] for the scalar focusing NLS with non-zero boundary conditions, while implementing the necessary generalizations to account for both focusing and defocusing dispersion regimes, as well as for the matrix nature of the nonlinear equation at hand. It is worth mentioning that the IST for the spinor system shares some of the features of the one developed in [23, 16, 6, 24, 5] for the vector NLS (aka Manakov system [21]), with a most notable difference in that the Jost eigenfuctions for the spinor system are all analytic: in this respect, the system is more similar to the scalar case, provided one accounts for the degeneracy of the associated eigenvalues and eigenspaces, which significantly simplifies the construction.

The paper is organized as follows. In Section 2 we formulate the direct scattering problem, introduce the uniformization variable, define eigenfunctions and scattering data and establish their analyticity as functions of the uniformization variable; we also derive the symmetries in the scattering data induced by all the symmetries imposed on the potential, and discuss in detail the properties of discrete eigenvalues and norming constants. In Section 3 we formulate the inverse scattering problem for the eigenfunctions as a Riemann-Hilbert problem with poles, provide the formal solution of the latter and the reconstruction formula of the potential in terms of eigenfunctions and scattering data. In Section 4 we derive and plot the one-soliton solutions for the focusing equation, and in particular we analyze the soliton solutions obtained when the discrete eigenvalue is purely imaginary (analog of the Kuznetsov-Ma breather in the scalar case [17, 20, 4]), which corresponds to a solution that is periodic in t and homoclinic in x, as well as the limiting value of the soliton solution when the discrete eigenvalue is taken on the cut. In this case, which is the analog in the spinor system of the Akhmediev breather [2, 4], the solution is homoclinic in t and periodic in x (and hence, strictly speaking, outside of the class considered for the IST, where $Q(x,t)$ approaches some constant values $Q_\pm$ as $x\to\pm\infty$ for any fixed t.) Section 5 is devoted to a few concluding remarks, while Appendix A provides a brief discussion of the boundary condition Q_+; in Appendix B a weak version of the "trace formula" is derived, and in Appendix C the asymptotic behavior as $x\to\pm\infty$ is analyzed for the soliton solutions, both in the case when the norming constant has rank 1 and when it is full rank.

2 Direct scattering

2.1 Lax pair, Riemann surface and uniformization coordinate

The matrix NLS equation (1.2) can be written as the compatibility condition of the Lax pair

$$\varphi_x=U\,\varphi\,,\qquad \varphi_t=V\,\varphi\,, \tag{2.1}$$

with

$$U(x,t,z) = -ik\underline{\sigma}_3 + \underline{Q}\,, \qquad V(x,t,z) = -2ik^2\underline{\sigma}_3 + 2k\underline{Q} + i\underline{\sigma}_3\left[\underline{Q}_x + \sigma k_o^2 I_4 - \sigma \underline{Q}\,\underline{Q}^\dagger\right], \tag{2.2a}$$

$$\underline{\sigma}_3 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \qquad \underline{Q} = \begin{pmatrix} 0 & Q \\ \sigma Q^\dagger & 0 \end{pmatrix}. \tag{2.2b}$$

The first equation in the Lax pair (2.1) is usually referred to as the scattering problem. Asymptotically as $x \to \pm\infty$ the scattering problem becomes:

$$\varphi_x = U_\pm\,\varphi, \qquad U_\pm = -ik\underline{\sigma}_3 + \underline{Q}_\pm\,, \qquad \text{with} \quad \underline{Q}_\pm = \begin{pmatrix} 0 & Q_\pm \\ \sigma Q_\pm^\dagger & 0 \end{pmatrix}. \tag{2.3}$$

The eigenvalues of $U_\pm$ are $\pm i\sqrt{k^2 - \sigma k_o^2}$, each with multiplicity 2. To properly account for the branching of the eigenvalues, one can introduce the two-sheeted Riemann surface defined by

$$\lambda^2 = k^2 - \sigma k_o^2\,, \tag{2.4}$$

so that $\lambda(k)$ is a single-valued function on this surface. The branch points are the values of k for which $k^2 - \sigma k_o^2 = 0$, i.e., $k = \pm ik_o$ in the focusing case ($\sigma = -1$), and $k = \pm k_o$ in the defocusing case ($\sigma = 1$).

In the focusing case, letting $k + ik_o = r_1\,\mathrm{e}^{i\theta_1}$ and $k - ik_o = r_2\,\mathrm{e}^{i\theta_2}$, one can assume $\lambda(k) = \sqrt{r_1 r_2}\,\mathrm{e}^{i(\theta_1+\theta_2)/2}$ on sheet $\mathbb{C}_I$, and $\lambda(k) = -\sqrt{r_1 r_2}\,\mathrm{e}^{i(\theta_1+\theta_2)/2}$ on sheet $\mathbb{C}_{II}$. Choosing the local angles $-\pi/2 \leqslant \theta_j < 3\pi/2$ for $j = 1,2$ corresponds to placing the discontinuity of λ (i.e., the location of the branch cut) on the segment $i[-k_o, k_o]$ on the imaginary k-axis. The Riemann surface is then obtained by gluing the two copies of the complex plane along the cut.

A similar construction applies in the defocusing case, with the branch cut chosen on the real k axis, and precisely for $k \in (-\infty, -k_o) \cup (k_o, +\infty)$. Specifically, on sheet I one can introduce the local polar coordinates $k - k_o = r_1 e^{i\theta_1}$ and $k + k_o = r_2 e^{i\theta_2}$, with the magnitudes r_1, r_2 uniquely fixed by the location of the point k, and angles $0 \leqslant \theta_1 < 2\pi$ and $-\pi \leqslant \theta_2 < \pi$. Then one can define $\lambda(k) = \sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2}$ on sheet $\mathbb{C}_I$, and $\lambda(k) = -\sqrt{r_1 r_2}e^{i(\theta_1+\theta_2)/2}$ on sheet $\mathbb{C}_{II}$. In this case, $(\theta_1 + \theta_2)/2$ varies continuously between 0 and π both in the upper and in the lower k-planes, with a cut on the semilines $(-\infty, -k_o) \cup (k_o, +\infty)$. The upper branches of the cuts on sheet $\mathbb{C}_I$ are then glued with the lower branches on sheet $\mathbb{C}_{II}$ and vice versa, so that $\lambda(k)$ is again continuous through the cut.

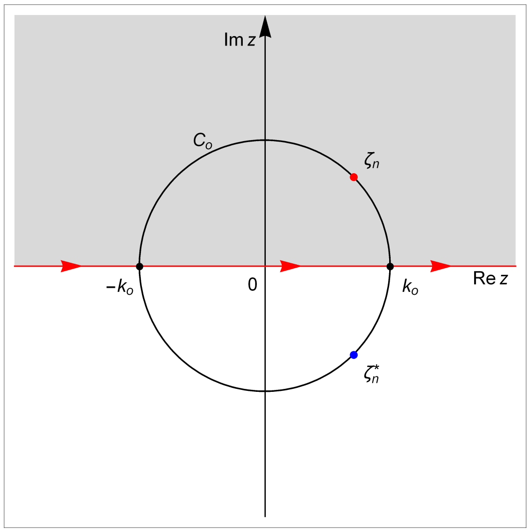
Following [12, 23, 8, 4], we introduce a uniformization variable:

$$z = k + \lambda \tag{2.5a}$$

whose inverse transformation is given by

$$k = \tfrac{1}{2}\left(z + \sigma k_o^2/z\right), \qquad \lambda = \tfrac{1}{2}\left(z - \sigma k_o^2/z\right). \tag{2.5b}$$

With these definitions, in the defocusing case the branch cut on either sheet is mapped onto the real z axis; the two sheets $\mathbb{C}_{\mathrm{I}}$ and $\mathbb{C}_{\mathrm{II}}$ of the Riemann surface are, respectively, mapped onto the upper and lower half-planes of the complex z-plane; a neighborhood of $k = \infty$ on either sheet is mapped onto a neighborhood of $z = \infty$ or $z = 0$ depending on the sign of $\mathrm{Im}\,k$.



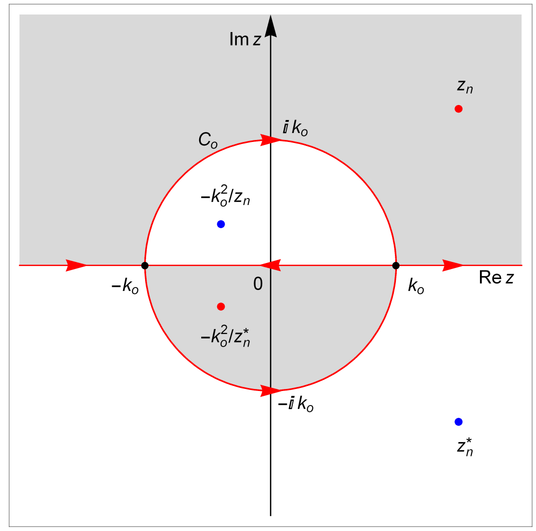


Figure 1: Left/Right: The complex z-plane, showing the regions $D^\pm$ where $\operatorname{Im}\lambda > 0$ (grey) and $\operatorname{Im}\lambda < 0$ (white), respectively, in the defocusing/focusing case. Also shown in the figures are the oriented contours for the Riemann-Hilbert problem (red), and the symmetries of the discrete spectrum of the scattering problem.

Conversely, in the focusing case the branch cut on either sheet is mapped onto the circle C_o (i.e., the circle centered at $z = 0$ and of radius k_o in the complex z-plane), $\mathbb{C}_\mathrm{I}$ is mapped onto the exterior of C_o; $\mathbb{C}_\mathrm{II}$ is mapped onto the interior of C_o; $z(\infty_I) = \infty$ and $z(\infty_\mathrm{II}) = 0$.

As a consequence, in the defocusing case, $\operatorname{Im}\lambda > 0$ in the upper-half plane (UHP) and $\operatorname{Im}\lambda < 0$ in the lower-half plane (LHP) of z:

$$\sigma = 1: \qquad D^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}\,, \qquad D^- = \{z \in \mathbb{C} : \operatorname{Im} z < 0\}\,. \tag{2.6a}$$

On the other hand, in the focusing case $\operatorname{Im}\lambda$ is not sign-definite in either half-plane; instead, one has $\operatorname{Im}\lambda > 0$ in D^+ and $\operatorname{Im}\lambda < 0$ in D^-, where, in this case:

$$\sigma = -1: \quad D^+ = \{z \in \mathbb{C} : (|z|^2 - k_o^2)\operatorname{Im} z > 0\}\,, \quad D^- = \{z \in \mathbb{C} : (|z|^2 - k_o^2)\operatorname{Im} z < 0\}\,. \tag{2.6b}$$

The complex z plane and the two domains $D^\pm$ are shown in Fig. 1 (defocusing case on the left, focusing case on the right). As will be discussed in the next section, it is precisely the sign of $\operatorname{Im}\lambda$ that determines the regions of analyticity of the Jost eigenfunctions. Note that with some abuse of notation we will rewrite all the k dependence as dependence on z wherever appropriate.

2.2 Jost solutions and analyticity

The Jost solutions are usually defined in terms of the asymptotic eigenvectors of the scattering problem. Taking into account (2.3), it follows that on either sheet of the Riemann surface we can write the asymptotic eigenvector matrix as

$$X_\pm(k) = I_4 - \frac{i}{k+\lambda}\sigma_3\underline{Q}_\pm \equiv I_4 - \frac{i}{z}\sigma_3\underline{Q}_\pm\,, \tag{2.7}$$

where I_4 denotes the 4×4 identity matrix, such that

$$U_\pm X_\pm = -i\lambda\, X_\pm\, \underline{\sigma}_3 \,. \tag{2.8}$$

Note that

$$\det X_\pm = \left(\frac{2\lambda}{\lambda + k}\right)^2 = \gamma^2(z)\,, \qquad \gamma(z) := 1 - \sigma \frac{k_o^2}{z^2}\,, \tag{2.9a}$$

$$X_\pm^{-1} = \frac{1}{\gamma(z)}\,[I_4 + \frac{i}{z}\underline{\sigma}_3 \underline{Q}_\pm]\,, \tag{2.9b}$$

where the inverse matrices $X_\pm^{-1}$ are defined for all values of z such that $\gamma(z) \neq 0$, i.e., away from the branch points: $z \neq \pm i k_o$ is the focusing case ($\sigma = -1$), and $z \neq \pm k_o$ in the defocusing case ($\sigma = 1$).

Let us now account for the time dependence of the eigenfunctions. As $x \to \pm\infty$, the time evolution of the solutions of (2.1) satisfy $\varphi_t = V_\pm\, \varphi$, with $V_\pm = -2ik^2\underline{\sigma}_3 + 2k\underline{Q}_\pm$, where we have used the boundary conditions (1.3), (1.4) and hence assumed $\underline{Q}_x \to 0$. Note that $V_\pm = 2k\, U_\pm$, hence $U_\pm$ and $V_\pm$ share the same eigenvectors, and specifically:

$$V_\pm X_\pm = -2ik\lambda\, X_\pm\, \underline{\sigma}_3 \,. \tag{2.10}$$

It is useful to observe that in terms of the uniformization variable one has $2k\lambda = \frac{1}{2}(z^2 - k_o^4/z^2)$.

The continuous spectrum Σ_k consists of all values of k (on either sheet) such that $\lambda(k) \in \mathbb{R}$; i.e., $\Sigma_k = \mathbb{R} \cup i[-k_o, k_o]$ in the focusing case, and $\Sigma_k = \mathbb{R} \setminus (-k_o, k_o)$ in the defocusing case. The corresponding sets in the complex z-plane are $\Sigma_z = \mathbb{R} \cup C_o$ and $\Sigma_z = \mathbb{R}$, respectively, C_o being the circle of radius k_o centered at the origin (see Fig. 1). Hereafter we will omit the subscripts on Σ, as the intended meaning will be clear from the context. For all $z \in \Sigma$, we can now define the Jost eigenfunctions $\Phi(x,t,z)$ and $\Psi(x,t,z)$ as the *simultaneous* solutions of both parts of the Lax pair such that

$$\Phi(x,t,z) \equiv (\phi(x,t,z)\ \bar\phi(x,t,z)) = X_-(z)\, \mathrm{e}^{-i\theta(x,t,z)\underline{\sigma}_3} + o(1) \qquad \text{as } x \to -\infty\,, \tag{2.11a}$$

$$\Psi(x,t,z) \equiv (\bar\psi(x,t,z)\ \psi(x,t,z)) = X_+(z)\, \mathrm{e}^{-i\theta(x,t,z)\underline{\sigma}_3} + o(1) \qquad \text{as } x \to +\infty\,, \tag{2.11b}$$

where

$$\theta(x,t,z) = \lambda(z)(x + 2k(z)t)\,, \tag{2.12}$$

and $\phi(x,t,z)$ and $\bar\phi(x,t,z)$ (resp. $\bar\psi(x,t,z)$ and $\psi(x,t,z)$) are 4×2 matrices which group the first two and last two column vectors of the 4×4 matrix solutions $\Phi(x,t,z)$ (resp. $\Psi(x,t,z)$). It is convenient to introduce modified eigenfunctions:

$$(M(x,t,z)\ \bar M(x,t,z)) = \Phi(x,t,z) e^{i\theta(x,t,z)\underline{\sigma}_3} \qquad (\bar N(x,t,z)\ N(x,t,z)) = \Psi(x,t,z) e^{i\theta(x,t,z)\underline{\sigma}_3}\,. \tag{2.13}$$

Following the same procedure as in [4], one can then write down integral equations for the modified eigenfunctions:

$$(M(x,t,z)\ \bar M(x,t,z)) = X_- + \int_{-\infty}^{x} X_- e^{i\lambda\underline{\sigma}_3(y-x)} X_-^{-1}(\underline{Q}(y,t) - \underline{Q}_-)(M(y,t,z)\ \bar M(y,t,z)) e^{i\lambda\underline{\sigma}_3(x-y)} dy \tag{2.14a}$$

$$(\bar{N}(x,t,z)\ N(x,t,z)) = X_+ - \int_x^\infty X_+ e^{i\lambda\underline{\sigma}_3(y-x)} X_+^{-1}(\underline{Q}(y,t) - \underline{Q}_+)(\bar{N}(y,t,z)\ N(y,t,z)) e^{i\lambda\underline{\sigma}_3(x-y)} dy \tag{2.14b}$$

and prove that under mild integrability conditions on the potential (essentially, one needs $Q(x,t) - Q_+$ in $L^1([a,+\infty())$ and $Q(x,t) - Q_-$ in $L^1((-\infty,b])$ as matrix functions of x for all t, for some $a,b \in \mathbb{R}$), the modified eigenfunctions $M(x,t,z)$ and $N(x,t,z)$ can be analytically extended in the complex z-plane where $\operatorname{Im}\lambda(z) \geqslant 0$, and $\bar{M}(x,t,z)$ and $\bar{N}(x,t,z)$ can be analytically extended in the complex z-plane where $\operatorname{Im}\lambda(z) \leqslant 0$. This means one has the following regions of analyticity for the modified eigenfunctions:

$$M(x,t,z),\ N(x,t,z): \quad D^+, \qquad \bar{M}(x,t,z),\ \bar{N}(x,t,z): \quad D^-$$

where $D^\pm$ are defined in (2.6) for the defocusing/focusing cases. The analyticity properties of the columns of Φ and Ψ follow trivially from the above.

2.3 Scattering coefficients

As a consequence of Jacobi's fomula, any matrix solution $\varphi(x,t,z)$ of (2.1) satisfies $\partial_x(\det\varphi) = \partial_t(\det\varphi) = 0$, given that both U and V in (2.1) are traceless. Thus, since for all $z \in \Sigma$ one has $\lim_{x\to-\infty}\Phi(x,t,z)\,\mathrm{e}^{i\theta\underline{\sigma}_3} = X_-$ and $\lim_{x\to+\infty}\Psi(x,t,z)\,\mathrm{e}^{i\theta\underline{\sigma}_3} = X_+$, it follows that

$$\det\Phi(x,t,z) = \det\Psi(x,t,z) = \det X_\pm(z) = \gamma^2(z) \qquad x,t \in \mathbb{R}\,, \qquad z \in \Sigma\,. \tag{2.16}$$

Letting Σ_o denote the continuous spectrum minus the branch points, i.e., $\Sigma_o = \Sigma \setminus \{\pm ik_o\}$ for the focusing case, and $\Sigma_o = \Sigma \setminus \{\pm k_o\}$ for the defocusing case, we then have that $\forall z \in \Sigma_o$ both Φ and Ψ are two fundamental matrix solutions of the scattering problem. Hence there exists a constant 4×4 scattering matrix $S(z)$ such that

$$\Phi(x,t,z) = \Psi(x,t,z)\,S(z)\,, \qquad S(z) = \begin{pmatrix} a(z) & \bar{b}(z) \\ b(z) & \bar{a}(z) \end{pmatrix}\,, \qquad x,t \in \mathbb{R}, \qquad z \in \Sigma_o\,. \tag{2.17}$$

In terms of the analytic groups of columns introduced in (2.11) one can write:

$$\phi = \psi b + \bar{\psi} a\,, \qquad \bar{\phi} = \psi\bar{a} + \bar{\psi}\bar{b}\,, \tag{2.18}$$

where $a(z), b(z)\bar{a}(z), \bar{b}(z)$ are the 2×2 blocks of the scattering matrix arranged as in (2.17).

Note that since Φ and Ψ are simultaneous solutions of both parts of the Lax pair, the entries of $S(z)$ are independent of time, and the same will be true for the norming constants (see Sections 2.5 and 2.6). Moreover, (2.16) and (2.17) also imply that $\det S(z) = 1$.

Using (2.17) one can easily verify that:

$$\det a(z) = \mathrm{Wr}(\phi,\psi)/\mathrm{Wr}(\bar{\psi},\psi) \equiv \det(\phi\ \psi)/\gamma^2\,, \tag{2.19a}$$

$$\det \bar{a}(z) = \mathrm{Wr}(\bar{\psi},\bar{\phi})/\mathrm{Wr}(\bar{\psi},\psi) \equiv \det(\bar{\psi}\ \bar{\phi})/\gamma^2\,, \tag{2.19b}$$

where $\mathrm{Wr}(f,g)$ denotes the Wronskian determinant of the 4×2 matrices f and g, i.e., the determinant of the 4×4 matrix whose columns are given by the 2 columns of f and the 2 columns of

g. While in the scalar case the analog of such Wronskian representations allow one to establish the analyticity properties of the diagonal entries of the scattering matrix, in this case they only provide a proof of analyticity for the determinants of the diagonal blocks $a(z)$ and $\bar{a}(z)$. One can, however, follow the approach introduced in [8, 9] for the defocusing and focusing scalar NLS with NZBCs, and use the integral equations (2.14) for the modified eigenfunctions to obtain the following integral representations for the scattering matrix:

$$S(z) = \int_0^\infty e^{i\lambda(z)y\underline{\sigma}_3} X_+^{-1}(z) \left[\underline{Q}(y,t) - \underline{Q}_+ \right] \Phi(y,t,z) dy \tag{2.20}$$
$$+ X_+^{-1}(z) X_-(z) \left\{ I_4 + \int_{-\infty}^0 e^{i\lambda(z)y\underline{\sigma}_3} X_-^{-1}(z) \left[\underline{Q}(y,t) - \underline{Q}_- \right] \Phi(y,t,z) dy \right\}.$$

The above integral representation allows one to show that $a(z)$ is analytic in D^+, and $\bar{a}(z)$ is analytic in D^- (cf (2.6)). As usual, however, the off-diagonal blocks of the scattering matrix $b(z)$ and $\bar{b}(z)$ are only defined on the continuous spectrum Σ, and in general nowhere analytic. An alternative proof of the analyticity of $a(z)$ and $\bar{a}(z)$ that uses the symmetries in the scattering data will be provided in Section 2.4.

Note also that $\det\Phi(x,t,z) = \det\Psi(x,t,z) = 0$ at the branch points ($z = \pm k_o$ in the defocusing case, and $z = \pm ik_o$ in the focusing case). As a result, generically speaking the scattering coefficients have a pole at the branch points. The behavior of the eigenfunctions and scattering matrix at the branch points can be established in the same way as was done in [4] for the scalar case.

Finally, for $z \in \Sigma$ we can use (2.13) and (2.18) to obtain

$$M(x,t,z)a^{-1}(z) = \bar{N}(x,t,z) + e^{2i\theta(x,t,z)} N(x,t,z)\rho(z), \tag{2.21a}$$
$$\bar{M}(x,t,z)\bar{a}^{-1}(z) = N(x,t,z) + e^{-2i\theta(x,t,z)} \bar{N}(x,t,z)\bar{\rho}(z), \tag{2.21b}$$

where $M(x,t,z)a^{-1}(z)$ and $\bar{M}(x,t,z)\bar{a}^{-1}(z)$ are meromorphic in D^+ and D^-, and we introduced (matrix) reflection coefficients

$$\rho(z) = b(z)a^{-1}(z), \qquad \bar{\rho}(z) = \bar{b}(z)\bar{a}^{-1}(z), \qquad z \in \Sigma. \tag{2.22}$$

2.4 Symmetries

In solving an initial-value problem by IST, one has to take into account that symmetries in the potentials in the Lax pair usually directly induce symmetries in the eigenfunctions, which in turn induce corresponding symmetries in the scattering data. The symmetries for the IST with NZBCs are complicated by the fact that $\lambda(k)$ changes sign from one sheet of the Riemann surface to the other, i.e., $\lambda_{\mathrm{II}}(k) = -\lambda_{\mathrm{I}}(k)$. In terms of the uniformization variable z, one needs to consider the following involutions: 1. $z \mapsto z^*$ (UHP/LHP), implying $(k,\lambda) \mapsto (k^*,\lambda^*)$ (same sheet); 2. $z \mapsto \sigma k_o^2/z$ (outside/inside C_o), implying $(k,\lambda) \mapsto (k,-\lambda)$ (opposite sheets). Both these transformations correspond to symmetries of the scattering problem; the first one is related to the usual conjugation symmetry in the potential, $\underline{Q}^\dagger = \sigma\underline{Q}$, while the second one is a direct consequence of the branching in the plane of the scattering parameter k, or, equivalently, the fact that the uniformization variable z provides a double covering of the k plane. In addition, we will also have to consider a third symmetry that corresponds to assuming $Q^T = Q$, which in terms of $\underline{Q}$ can be written as

$$\underline{Q} = -\underline{\sigma}_2 \underline{Q}^T \underline{\sigma}_2 \qquad \text{where} \qquad \underline{\sigma}_2 = \begin{pmatrix} 0 & iI_2 \\ -iI_2 & 0 \end{pmatrix}, \tag{2.23}$$

$\underline{\sigma}_2$ being a 4×4 generalization of the 2×2 Pauli matrix σ_2. In the remainder of this section we will discuss all three symmetries, taking into account that: (i) unlike the case of ZBCs, after removing the asymptotic oscillations, the Jost eigenfunctions do not tend to the identity matrix; (ii) the matrix nature of the equation implies that in some cases the symmetries one obtains for the eigenfunctions are bilinear symmetry relations.

First symmetry - conjugation: $\underline{Q}^\dagger = \sigma \underline{Q}$, corresponding to $z \mapsto z^*$ (UHP/LHP). We will follow the same approach as in [1] for the matrix NLS with zero boundary conditions to determine how the eigenfunctions and the scattering data are related when the above involution is considered. Let us introduce for $z \in \Sigma$

$$f(x,t,z) = \Phi^\dagger(x,t,z^*) J_\sigma \Phi(x,t,z), \qquad g(x,t,z) = \Psi^\dagger(x,t,z^*) J_\sigma \Psi(x,t,z),$$

where

$$J_\sigma = \begin{pmatrix} I_2 & 0 \\ 0 & -\sigma I_2 \end{pmatrix}. \tag{2.24}$$

[J_σ is the 4×4 identity in the focusing case, and it coincides with $\underline{\sigma}_3$ in the defocusing case.] Taking into account that Φ, Ψ are solutions of the scattering problem (2.1), it is easy to verify that f, g are independent of x, and evaluating the limits as $x \to \pm\infty$ one obtains

$$\Phi^\dagger(x,t,z^*) J_\sigma \Phi(x,t,z) = \Psi^\dagger(x,t,z^*) J_\sigma \Psi(x,t,z) = \gamma(z) J_\sigma. \tag{2.25}$$

On one hand, the above relationships can be written as:

$$\Psi^{-1}(x,t,z) = \frac{1}{\gamma(z)} J_\sigma \Psi^\dagger(x,t,z^*) J_\sigma, \qquad \Phi^{-1}(x,t,z) = \frac{1}{\gamma(z)} J_\sigma \Phi^\dagger(x,t,z^*) J_\sigma. \tag{2.26}$$

Then one has the following representation for the scattering matrix:

$$S(z) = \Psi^{-1}(x,t,z)\Phi(x,t,z) \equiv \frac{1}{\gamma(z)} J_\sigma \Psi^\dagger(x,t,z^*) J_\sigma \Phi(x,t,z). \tag{2.27}$$

It is convenient to introduce the following notation for the upper/lower blocks of the eigenfunctions:

$$\Phi(x,t,z) = \begin{pmatrix} \phi^{\mathrm{up}} & \bar\phi^{\mathrm{up}} \\ \phi^{\mathrm{dn}} & \bar\phi^{\mathrm{dn}} \end{pmatrix}, \qquad \Psi(x,t,z) = \begin{pmatrix} \bar\psi^{\mathrm{up}} & \psi^{\mathrm{up}} \\ \bar\psi^{\mathrm{dn}} & \psi^{\mathrm{dn}} \end{pmatrix},$$

where each block $^{\mathrm{up}}, ^{\mathrm{dn}}$ is a 2×2 matrix. Computing the diagonal 2×2 blocks of $S(z)$ in (2.27) and comparing with (2.17) then gives:

$$\gamma(z) a(z) = \left(\bar\psi^{\mathrm{up}}(x,t,z^*)\right)^\dagger \phi^{\mathrm{up}}(x,t,z) - \sigma \left(\bar\psi^{\mathrm{dn}}(x,t,z^*)\right)^\dagger \phi^{\mathrm{dn}}(x,t,z), \tag{2.28a}$$

$$\gamma(z) \bar a(z) = \left(\psi^{\mathrm{dn}}(x,t,z^*)\right)^\dagger \bar\phi^{\mathrm{dn}}(x,t,z) - \sigma \left(\psi^{\mathrm{up}}(x,t,z^*)\right)^\dagger \bar\phi^{\mathrm{up}}(x,t,z). \tag{2.28b}$$

Incidentally, the above expressions are the analog of the Wronskian representations for the scattering coefficients in the scalar case, and provide an alternative proof that $a(z)$ can be analytically continued in D^+, and $\bar{a}(z)$ can be analytically continued in D^-. For future reference, note that (2.25) implies that the off-diagonal blocks of $\Phi^\dagger(z^*)J_\sigma\Phi(z) = \Psi^\dagger(z^*)J_\sigma\Psi(z)$ are zero, and the explicit computation in terms of upper and lower blocks of the eigenfunctions yields:

$$(\phi^{\mathrm{up}}(x,t,z^*))^\dagger\,\bar{\phi}^{\mathrm{up}}(x,t,z) = \sigma\left(\phi^{\mathrm{dn}}(x,t,z^*)\right)^\dagger\bar{\phi}^{\mathrm{dn}}(x,t,z)\,, \tag{2.29a}$$

$$(\psi^{\mathrm{up}}(x,t,z^*))^\dagger\,\bar{\psi}^{\mathrm{up}}(x,t,z) = \sigma\left(\psi^{\mathrm{dn}}(x,t,z^*)\right)^\dagger\bar{\psi}^{\mathrm{dn}}(x,t,z)\,. \tag{2.29b}$$

On the other hand, (2.25) in turn implies

$$S^\dagger(z^*)J_\sigma S(z) = J_\sigma\,, \qquad z\in\Sigma\,. \tag{2.30}$$

Explicitly, in terms of the blocks of the scattering matrix $S(z)$ in (2.17) one obtains the same symmetries as in the case of zero boundary conditions, i.e.:

$$a^\dagger(z^*)a(z) - \sigma b^\dagger(z^*)b(z) = I_2 \tag{2.31a}$$

$$a^\dagger(z^*)\bar{b}(z) - \sigma b^\dagger(z^*)\bar{a}(z) = 0 \tag{2.31b}$$

$$\bar{b}^\dagger(z^*)a(z) - \sigma\bar{a}^\dagger(z^*)b(z) = 0 \tag{2.31c}$$

$$\bar{b}^\dagger(z^*)\bar{b}(z) - \sigma\bar{a}^\dagger(z^*)\bar{a}(z) = -\sigma I_2 \tag{2.31d}$$

and as a consequence also the following symmetry for the reflection coefficients:

$$\bar{\rho}(z) = \sigma\rho^\dagger(z^*) \qquad z\in\Sigma\,, \tag{2.32}$$

as well as

$$a(z)a^\dagger(z^*) = \left[I_2 - \sigma\rho^\dagger(z^*)\rho(z)\right]^{-1}\,, \qquad \bar{a}(z)\bar{a}^\dagger(z^*) = \left[I_2 - \sigma\bar{\rho}^\dagger(z^*)\bar{\rho}(z)\right]^{-1}\,. \tag{2.33}$$

It is also worth noticing that from (2.30) it follows

$$S^{-1}(z) = J_\sigma S^\dagger(z^*)J_\sigma\,, \qquad S^{-1}(z) = \begin{pmatrix} \bar{c}(z) & d(z) \\ \bar{d}(z) & c(z) \end{pmatrix}\,, \tag{2.34}$$

which provides a relationship between the blocks of $S(z)$ and those of its inverse for $z\in\Sigma$, namely:

$$\bar{c}(z) = a^\dagger(z^*)\,, \qquad c(z) = \bar{a}^\dagger(z^*)\,, \tag{2.35a}$$

$$d(z) = -\sigma b^\dagger(z^*)\,, \qquad \bar{d}(z) = -\sigma\bar{b}^\dagger(z^*)\,. \tag{2.35b}$$

Eqs. (2.35a) can be extended to $D^\pm$ by Schwartz reflection principle, while (2.35b) are in general only valid for $z\in\Sigma$.

In turn, the analogues of (2.19a) for $\Psi(x,t,z) = \Phi(x,t,z)S^{-1}(z)$, namely

$$\det c(z) = \mathrm{Wr}(\phi,\psi)/\mathrm{Wr}(\phi,\bar{\phi}) \equiv \det(\phi\ \psi)/\gamma^2\,, \tag{2.36a}$$

$$\det \bar{c}(z) = \mathrm{Wr}(\bar{\psi}, \bar{\phi}) / \mathrm{Wr}(\phi, \bar{\phi}) \equiv \det(\bar{\psi}\ \bar{\phi}) / \gamma^2 \,, \tag{2.36b}$$

allow one to conclude that

$$\det c(z) = \det a(z) \quad \text{for } z \in D^+ \,, \qquad \det \bar{c}(z) = \det \bar{a}(z) \quad \text{for } z \in D^- \,. \tag{2.37}$$

Taking into account (2.35), one finally obtains

$$\det \bar{a}(z) = \det a^\dagger(z^*) \equiv (\det a(z^*))^* \qquad \text{for } z \in D^- \,. \tag{2.38}$$

Second symmetry: $(k, \lambda) \mapsto (k, -\lambda)$ **or equivalently** $z \mapsto \sigma k_o^2/z$**.** While the first involution, $z \mapsto z^*$, is the same as for ZBCs (corresponding, in both cases, to the conjugation symmetry of the potential), this second involution simply expresses the switch from one sheet to the other ($\lambda \mapsto -\lambda$), and the fact that the scattering problem is independent of the choice of the branch, i.e., of the sign chosen for $\lambda = \sqrt{k^2 - \sigma k_o^2}$. In the uniformization variable, this involution corresponds to $z \mapsto \sigma k_o^2/z$, i.e., a reflection with respect to the circle C_o.

Since Jost eigenfunctions and scattering coefficients explicitly depend on $\lambda = \lambda(z) = (z - \sigma k_o^2/z)/2$, the second symmetry should relate their values on opposite sheets of the Riemann surface, i.e., inside and outside the circle C_o. Indeed, one can easily check that

$$X_\pm(z) = -\frac{i}{z} X_\pm(\sigma k_o^2/z)\, \underline{\sigma}_3\, \underline{Q}_\pm \,,$$

which, taking into account that $\theta(\sigma k_o^2/z) = -\theta(z)$ and $\underline{Q}_\pm e^{-i\theta(z)\underline{\sigma}_3} = e^{i\theta(z)\underline{\sigma}_3} \underline{Q}_\pm$, gives

$$\Phi(x,t,z) = \frac{1}{iz} \Phi(x,t,\sigma k_o^2/z)\, \underline{\sigma}_3\, \underline{Q}_- \,, \qquad \Psi(x,t,z) = \frac{1}{iz} \Psi(x,t,\sigma k_o^2/z)\, \underline{\sigma}_3\, \underline{Q}_+ \,, \qquad z \in \Sigma. \tag{2.39}$$

We then have, for the 4×2 blocks of columns:

$$\phi(x,t,z) = \frac{i\sigma}{z} \bar{\phi}(x,t,\sigma k_o^2/z) Q_-^\dagger \,, \qquad \bar{\phi}(x,t,z) = -\frac{i}{z} \phi(x,t,\sigma k_o^2/z) Q_- \,, \tag{2.40a}$$

$$\bar{\psi}(x,t,z) = \frac{i\sigma}{z} \psi(x,t,\sigma k_o^2/z) Q_+^\dagger \,, \qquad \psi(x,t,z) = -\frac{i}{z} \bar{\psi}(x,t,\sigma k_o^2/z) Q_+ \,. \tag{2.40b}$$

Then from (2.17) and using (2.39) we have, for all $z \in \Sigma$,

$$S(\sigma k_o^2/z) = \underline{\sigma}_3\, \underline{Q}_+\, S(z)\, \underline{Q}_-^{-1} \underline{\sigma}_3 \equiv \frac{\sigma}{k_o^2} \underline{\sigma}_3\, \underline{Q}_+\, S(z)\, \underline{Q}_-\, \underline{\sigma}_3 \,, \tag{2.41}$$

where for the last equality we have used (1.4), so that $\underline{Q}_\pm^{-1} = \underline{Q}_\pm^\dagger / k_o^2 = \sigma \underline{Q}_\pm / k_o^2$. Recalling (2.17) we then have, from (2.41) block-wise:

$$a(\sigma k_o^2/z) = \frac{1}{k_o^2} Q_+ \bar{a}(z) Q_-^\dagger \,, \qquad \bar{a}(\sigma k_o^2/z) = \frac{1}{k_o^2} Q_+^\dagger a(z) Q_- \,, \tag{2.42a}$$

$$b(\sigma k_o^2/z) = -\frac{\sigma}{k_o^2} Q_+^\dagger \bar{b}(z) Q_-^\dagger \,, \qquad \bar{b}(\sigma k_o^2/z) = -\frac{\sigma}{k_o^2} Q_+ b(z) Q_- \,. \tag{2.42b}$$

Finally, the above relations yield the corresponding symmetry for the reflection coefficients in (2.22):

$$\rho(\sigma k_o^2/z) = -\sigma Q_+^\dagger \bar{\rho}(z) Q_+^{-1} \equiv -\frac{\sigma}{k_o^2} Q_+^\dagger \bar{\rho}(z) Q_+^\dagger \qquad \forall z \in \Sigma\,. \tag{2.43}$$

Note that: (i) Even though the above symmetries are only valid for $z \in \Sigma$, whenever the individual columns and scattering coefficients involved are analytic, they can be extended to the appropriate regions of the z-plane using the Schwartz reflection principle. (ii) In the focusing case even the symmetries of the non-analytic scattering coefficients involve the map $z \mapsto z^*$, because, unlike what happens in the defocusing case, the continuous spectrum is not just a subset of the real z-axis.

Third symmetry: $\underline{Q} = -\underline{\sigma}_2 \underline{Q}^T \underline{\sigma}_2$, corresponding to $Q^T = Q$. In analogy to what was done to account for the first symmetry in the potential, let us introduce for $z \in \Sigma$:

$$\tilde{f}(x,t,z) = \Phi^T(x,t,z)\underline{\sigma}_2\Phi(x,t,z)\,, \qquad \tilde{g}(x,t,z) = \Psi^T(x,t,z)\underline{\sigma}_2\Psi(x,t,z)\,,$$

where $\underline{\sigma}_2$ is as defined in (2.23). Again, it is easy to verify that $\tilde{f}, \tilde{g}$ are independent of x: for instance, from the scattering problem (2.1) it follows that

$$\partial_x \tilde{f} = \Phi^T \left[-ik\underline{\sigma}_3\underline{\sigma}_2 + \underline{Q}^T\underline{\sigma}_2 - ik\underline{\sigma}_2\underline{\sigma}_3 + \underline{\sigma}_2\underline{Q} \right] \Phi = 0\,,$$

since $\underline{\sigma}_3\underline{\sigma}_2 = -\underline{\sigma}_2\underline{\sigma}_3$ and $\underline{Q}^T\underline{\sigma}_2 = -\underline{\sigma}_2\underline{Q}$. Evaluating the limits as $x \to \pm\infty$ one obtains

$$\Phi^T(x,t,z)\underline{\sigma}_2\Phi(x,t,z) = \Psi^T(x,t,z)\underline{\sigma}_2\Psi(x,t,z) = \gamma(z)\underline{\sigma}_2\,, \tag{2.44}$$

which in turn implies

$$S^T(z)\underline{\sigma}_2 S(z) = \underline{\sigma}_2\,, \qquad z \in \Sigma\,. \tag{2.45}$$

In terms of the blocks of the scattering matrix the latter yields:

$$b^T(z)a(z) = a^T(z)b(z)\,, \qquad \bar{b}^T(z)\bar{a}(z) = \bar{a}^T(z)\bar{b}(z)\,,$$
$$a^T(z)\bar{a}(z) - b^T(z)\bar{b}(z) = I_2\,,$$

which then in particular imply:

$$\rho^T(z) = \rho(z)\,, \qquad \bar{\rho}^T(z) = \bar{\rho}(z)\,, \tag{2.46}$$

showing that the reflection coefficients need to be symmetric matrices themselves, as well as

$$a(z)\bar{a}^T(z) = [I_2 - \bar{\rho}(z)\rho(z)]^{-1} \qquad z \in \Sigma. \tag{2.47}$$

Finally, from (2.45) it also follows that $S^{-1}(z) = \underline{\sigma}_2 S^T(z)\underline{\sigma}_2$ for $z \in \Sigma$, i.e.,

$$\bar{c}(z) = \bar{a}^T(z)\,, \qquad c(z) = a^T(z)\,, \qquad d(z) = -\bar{b}^T(z)\,, \qquad \bar{d}(z) = -b^T(z)\,, \tag{2.48}$$

which, combined with (2.35), yield

$$\bar{a}(z) = a^*(z^*)\,, \qquad \bar{b}(z) = \sigma b^*(z^*) \qquad z \in \Sigma\,. \tag{2.49}$$

2.5 Discrete spectrum and residue conditions

The discrete spectrum of the scattering problem is the set of all values $z \in \mathbb{C} \setminus \Sigma$ such that the scattering problem admits eigenfunctions in $L^2(\mathbb{R})$. We next show that these values are the zeros of $\det a(z)$ in D^+ and those of $\det \bar{a}(z)$ in D^-. Note that in the defocusing case it is easy to show that for any $z \in \Sigma$ and for any $\xi \in \mathbb{C}^2$ Eqs. (2.31) imply:

$$\|a(z)\xi\|^2 = \|\xi\|^2 + \|b(z)\xi\|^2 \,, \qquad \|\bar{a}(z)\xi\|^2 = \|\xi\|^2 + \|\bar{b}(z)\xi\|^2 \,.$$

As a consequence, $a(z)\xi = 0$ implies $\xi = 0$, and therefore $\det a(z) \neq 0$ for all $z \in \Sigma$. The same of course holds for $\bar{a}(z)$. In the focusing case, however, one cannot exclude the existence of zeros of $\det a(z)$ and $\det \bar{a}(z)$ for $z \in \Sigma$, which in the case of ZBCs give rise to the so-called real spectral singularities [32]. In the following, when dealing with the focusing case, we will assume that there are no spectral singularities, namely that $\det a(z) \neq 0$ for all $z \in \Sigma$. This also implies $\det \bar{a}(z) \neq 0$ for all $z \in \Sigma$, because of (2.38).

For all $z \in D^+$, $\phi(x,t,z) \to 0$ as $x \to -\infty$, and $\psi(x,t,z) \to 0$ as $x \to +\infty$. Recalling the first of (2.19a), if $\det a(z) = 0$ at $z = z_n$ the two eigenfunctions $\phi(x,t,z_n)$ and the two eigenfunctions $\psi(x,t,z_n)$ become linearly dependent. While in the scalar case this amounts to a simple proportionality relation between two vectors, the situation is more complicated in the matrix case, since in general the linear dependence should be expressed as:

$$\psi(x,t,z_n)\xi_n = \phi(x,t,z_n)\eta_n \,, \tag{2.50a}$$

for some non-zero complex vectors $\xi_n, \eta_n \in \mathbb{C}^2 \setminus \{0\}$ [10, 11]. Such vectors are of course not uniquely defined, since one would always have the freedom to divide both sides by any of the nonzero component of either ξ_n or η_n. In any case, for any such z_n the linear combination above would provide an $L^2(\mathbb{R})$ eigenfunction, decaying exponentially at both space infinities in light of (2.50). Because of the symmetry (2.38), corresponding to any zero $z_n \in D^+$ of $\det a(z)$, one also has $z_n^* \in D^-$ such that $\det \bar{a}(z_n^*) = 0$, and hence

$$\bar{\psi}(x,t,z_n^*)\bar{\xi}_n = \bar{\phi}(x,t,z_n^*)\bar{\eta}_n \,, \tag{2.50b}$$

for some non-zero complex vectors $\bar{\xi}_n, \bar{\eta}_n \in \mathbb{C}^2 \setminus \{0\}$.

In the existing literature on matrix NLS systems the assumption is generally made that

$$\phi(x,t,z_n) = \psi(x,t,z_n)\, b_n \,, \qquad \bar{\phi}(x,t,z_n^*) = \bar{\psi}(x,t,z_n^*)\, \bar{b}_n \,, \tag{2.51}$$

where $b_n, \bar{b}_n$ are 2×2 nonzero constant matrices.This is also the approach followed in [19], where, with no details provided, a relation deduced from the analog of (2.18) is used to conclude that the analog of the Jost eigenfunctions $\phi(x,t,z_n)$ and $\psi(x,t,z_n)$ are “proportional”. Note that (2.51) is a stronger assumption than (2.50), since the former implies that *each* of the two columns of ϕ (resp. $\bar{\phi}$) is a linear combination of the two columns of ψ (resp. $\bar{\psi}$). Under this assumption, we obviously still obtain L^2 eigenfunctions; and it corresponds to assuming that at z_n (2.50) holds both with $\eta_n = (1\ 0)^T$ and ξ_n given by the first column of b_n, and with $\eta_n = (0\ 1)^T$ and ξ_n given by the second column of b_n (and similarly for z_n^*). For the remainder of this section we will assume (2.51) hold. In Section 2.6 we will discuss what happens if instead one only assumes (2.50).

Suppose that $\det a(z)$ has a finite number $\mathcal{N}$ of simple zeros $z_1, \ldots, z_{\mathcal{N}}$ in $D^+ \cap \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. That is, let $\det a(z_n) = 0$ and $(\det a)'(z_n) \neq 0$, with $|z_n| > k_o$ and $\operatorname{Im} z_n > 0$ for $n = 1, \ldots, \mathcal{N}$, and where the prime denotes differentiation with respect to z. Taking into account the symmetries (2.38) and (2.42) we have that

$$\det a(z_n) = 0 \;\Leftrightarrow\; \det \bar{a}(z_n^*) = 0 \;\Leftrightarrow\; \det \bar{a}(\sigma k_o^2/z_n) = 0 \;\Leftrightarrow\; \det a(\sigma k_o^2/z_n^*) = 0. \tag{2.52}$$

For each $n = 1, \ldots, \mathcal{N}$ we therefore have a quartet of discrete eigenvalues, which means that the discrete spectrum is given by the set

$$Z = \{z_n, z_n^*, \sigma k_o^2/z_n, \sigma k_o^2/z_n^*\}_{n=1}^{\mathcal{N}}.$$

Note, however, that in the defocusing case, the self-adjointness of the scattering problem implies that the discrete eigenvalues in the k plane are real. Since spectral singularities (i.e., zeros of $\det a(z)$ and $\det \bar{a}(z)$ on Σ) are excluded, then eigenvalues must be such that $-k_o \leqslant k_j \leqslant k_o$, which means that in the z plane the eigenvalues are all located on the circle of radius k_o, i.e., $z_j \in C_o$ for $j = 1, \ldots, \mathcal{N}$, and they come in complex conjugate pairs. Therefore, in the defocusing case the second symmetry does not give rise to a quartet of eigenvalues, because if $\sigma = 1$ and $z_j \in C_o$, then $k_o^2/z_n^* \equiv z_n$ and, obviously, $k_o^2/z_n \equiv z_n^*$. The discrete spectrum is hence given by:

$$\sigma = -1 \quad \text{(focusing case):} \qquad Z = \{z_n, -k_o^2/z_n^*, z_n^*, -k_o^2/z_n,\}_{n=1}^{\mathcal{N}} \tag{2.53a}$$

$$\sigma = 1 \quad \text{(defocusing case):} \qquad Z = \{\zeta_n, \zeta_n^*\}_{n=1}^{\mathcal{N}} \tag{2.53b}$$

where in the focusing case each first pair is in D^+ (and we assume without loss of generality $\operatorname{Im} z_n > 0$) and each second pair is in D^-; in the defocusing case the eigenvalues are on the circle C_o (cf Fig. 1).

Next we derive the residue conditions that will be needed for the inverse problem. We can write (2.51) equivalently as $M(x,t,z_n) = \mathrm{e}^{2i\theta(x,t,z_n)} N(x,t,z_n)\, b_n$. Thus, in the case of a simple zero of $\det a(z)$

$$\operatorname*{Res}_{z=z_n} \left[M(x,t,z)a^{-1}(z)\right] = \mathrm{e}^{2i\theta(x,t,z_n)} N(x,t,z_n)\, C_n\,, \qquad C_n = \frac{1}{(\det a)'(z_n)} b_n\, \alpha(z_n)\,, \tag{2.54a}$$

where $\alpha(z) := \operatorname{cof} a(z)$ is the cofactor (or adjugate) matrix of $a(z)$, i.e., such that $a(z)\alpha(z) = \alpha(z)a(z) = \det a(z) I_2$. Similarly, if $z_n^* \in D^-$ is a simple zero of $\det \bar{a}(z)$ we obtain

$$\operatorname*{Res}_{z=z_n^*} \left[\bar{M}(x,t,z)\bar{a}^{-1}(z)\right] = \mathrm{e}^{-2i\theta(x,t,z_n^*)} \bar{N}(x,t,z_n^*)\bar{C}_n\,, \qquad \bar{C}_n = \frac{1}{(\det \bar{a})'(z_n^*)} \bar{b}_n\, \bar{\alpha}(z_n^*)\,, \tag{2.54b}$$

where $\bar{\alpha}(z)$ is the cofactor matrix of $\bar{a}(z)$.

It is well-known that for an $m \times m$ matrix A one has $\det(\operatorname{cof} A) = (\det A)^{m-1}$; since in our case $m = 2$, we have

$$\det \alpha(z) = \det a(z)\,,$$

and therefore $\det \alpha(z)$ has a zero of the same order as $\det a(z)$ for each $z_n \in D^+ \cap Z$. The same of course holds for $\det \bar{\alpha}(z)$, which will have a zero of the same order as $\det \bar{a}(z)$ for each $z_n^* \in D^- \cap Z$.

In the case of simple eigenvalues one also has

$$\tau_n := \operatorname*{Res}_{z=z_n} a^{-1}(z) = \frac{\alpha(z_n)}{(\det a)'(z_n)}, \qquad \bar{\tau}_n := \operatorname*{Res}_{z=z_n^*} \bar{a}^{-1}(z) = \frac{\bar{\alpha}(z_n^*)}{(\det \bar{a})'(z_n^*)}. \tag{2.55}$$

and $\det \tau_n = \det \bar{\tau}_n = 0$, so the residues are always rank 1 matrices in the case of simple eigenvalues. We can then express the norming constants defined in (2.54) in terms of the above residues:

$$C_n = b_n \tau_n, \qquad \bar{C}_n = \bar{b}_n \bar{\tau}_n, \tag{2.56}$$

and we conclude that for simple discrete eigenvalues one necessarily has:

$$\det C_n = \det \bar{C}_n = 0, \tag{2.57}$$

so for simple discrete eigenvalues (simple zeros of $\det a(z)$), the associated norming constants they are rank 1 matrices.

It is also worth mentioning that since for any $z \in D^+ \setminus Z$ one has $a^{-1}(z) = \alpha(z)/(\det a(z))$, and since $\alpha(z)$ is analytic in D^+, it follow that $a^{-1}(z)$ will be meromorphic is D^+, with poles at each of the discrete eigenvalues, and the order of the pole at each z_n is at most equal to the order of z_n as a zero of $\det a(z)$. Obviously the same holds for $\bar{a}^{-1}(z)$ in D^-.

The arguments above can be generalized to zeros of higher order. For instance, if z_n is a second order zero of $\det a(z)$, then $\det \alpha(z)$ has a second order zero at z_n as well. In this case, however, in a neighborhood of z_n one has

$$a^{-1}(z) = \frac{1}{(z-z_n)^2}\tau_{n,2} + \frac{1}{z-z_n}\tau_{n,1} + \tilde{a}(z)$$

where $\tilde{a}(z)$ is analytic at z_n, and

$$\tau_{n,2} = \lim_{z\to z_n} (z-z_n)^2 a^{-1}(z) \equiv \frac{2}{(\det a)''(z_n)}\alpha(z_n), \tag{2.58}$$

$$\tau_{n,1} = \lim_{z\to z_n} \frac{d}{dz}\Big[(z-z_n)^2 a^{-1}(z)\Big] \equiv \frac{2}{(\det a)''(z_n)}\alpha'(z_n) - \frac{4}{3}\frac{(\det a)'''(z_n)}{((\det a)''(z_n))^2}\alpha(z_n). \tag{2.59}$$

For a genuine second order zero (double eigenvalue), one would expect that $\tau_{n,2} \neq 0$ and $\det \tau_{n,2} = 0$ (because $\det \alpha(z_n) = 0$, as explained above); on the other hand, $\tau_{n,1}$ might or might not be zero, and if it is non-zero it is possible to have $\det \tau_{n,1} \neq 0$. On the other hand, it is also possible to have $\tau_{n,2} = 0$, which is equivalent to $\alpha(z_n) = 0$; in this case one would still have a first order pole for $a^{-1}(z)$, with residue

$$\tau_{n,1} = \frac{2}{(\det a)''(z_n)}\alpha'(z_n).$$

Importantly, in this case in general $\det(\alpha')(z_n)$ needs not to be zero, so $\tau_{n,1}$ needs not to be rank 1. In this situation, z_n is a double zero of $\det a(z)$, and 0 is an eigenvalue of $a(z_n)$ with algebraic and geometric multiplicity equal to 2 (since $\alpha(z_n) = a(z_n) = 0$). Nonetheless, because $\tau_{n,2} = 0$, $a^{-1}(z)$ has only a pole of first order at z_n, and (2.54) become:

$$\operatorname*{Res}_{z=z_n}\Big[M(x,t,z)a^{-1}(z)\Big] = \mathrm{e}^{2i\theta(x,t,z_n)} N(x,t,z_n)\, C_n, \qquad C_n = \frac{2}{(\det a)''(z_n)} b_n\, \alpha'(z_n), \tag{2.60a}$$

$$\operatorname*{Res}_{z=z_n^*}\left[\bar{M}(x,t,z)\bar{a}^{-1}(z)\right] = \mathrm{e}^{-2i\theta(x,t,z_n^*)}\bar{N}(x,t,z_n^*)\bar{C}_n\,, \qquad \bar{C}_n = \frac{2}{(\det\bar{a})''(z_n^*)}\bar{b}_n\,\bar{\alpha}'(z_n^*)\,, \tag{2.60b}$$

and in this case $C_n, \bar{C}_n$ need not be rank 1.

The above norming constants are related by the symmetries. In Section 3.2 we will prove that the norming constants $C_n, \bar{C}_n$ satisfy the same symmetry as in the matrix NLS with zero boundary conditions [1], namely:

$$\bar{C}_n = \sigma C_n^\dagger\,. \tag{2.61a}$$

Moreover, the third symmetry also requires that C_n and $\bar{C}_n$ be symmetric matrices:

$$C_n = C_n^T, \qquad \bar{C}_n = \bar{C}_n^T\,. \tag{2.61b}$$

Finally, in the focusing case we need to discuss the remaining two points of the eigenvalue quartet, where, in analogy to (2.51) one has:

$$\phi(x,t,\hat{z}_n) = \psi(x,t,\hat{z}_n)\hat{b}_n\,, \qquad \hat{z}_n = \sigma k_o^2/z_n^*\,, \tag{2.62a}$$

$$\bar{\phi}(x,t,\hat{z}_n^*) = \bar{\psi}(x,t,\hat{z}_n^*)\hat{\bar{b}}_n\,, \qquad \hat{z}_n^* = \sigma k_o^2/z_n\,, \tag{2.62b}$$

and $\hat{b}_n$, $\hat{\bar{b}}_n$ are constant 2×2 matrices. As discussed above, this second symmetry for the discrete eigenvalues and associated norming constants only applies to the focusing case, so for the remainder of this section one can assume $\sigma = -1$.

Using (2.40a) and (2.51) we have on one hand

$$\phi(x,t,z_n) = \frac{i\sigma}{z_n}\bar{\phi}(x,t,\hat{z}_n^*)Q_-^\dagger = \frac{i\sigma}{z_n}\bar{\psi}(x,t,\hat{z}_n^*)\hat{\bar{b}}_n Q_-^\dagger\,,$$

and on the other hand using (2.40b) and (2.51)

$$\phi(x,t,z_n) = \psi(x,t,z_n)\,b_n = -\frac{i}{z_n}\bar{\psi}(x,t,\hat{z}_n^*)Q_+ b_n\,.$$

Comparing the two we finally obtain:

$$\hat{\bar{b}}_n = -\sigma Q_+ b_n (Q_-^\dagger)^{-1} \equiv -\frac{\sigma}{k_o^2}Q_+ b_n Q_-\,. \tag{2.63a}$$

Similarly, from (2.40b) in (2.51) it follows that

$$\hat{b}_n = -\sigma Q_+^\dagger \bar{b}_n (Q_-)^{-1} \equiv -\frac{\sigma}{k_o^2}Q_+^\dagger \bar{b}_n Q_-^\dagger\,. \tag{2.63b}$$

Moreover, differentiating (2.42) and evaluating at $z = z_n$ or $z = z_n^*$, we have

$$(\det a)'(\sigma k_o^2/z_n^*) = -\sigma\left(\frac{z_n^*}{k_o}\right)^2 \frac{\det Q_+ \det Q_-^\dagger}{k_o^4}(\det\bar{a})'(z_n^*)\,, \tag{2.64a}$$

$$(\det\bar{a})'(\sigma k_o^2/z_n) = -\sigma\left(\frac{z_n}{k_o}\right)^2 \frac{\det Q_+^\dagger \det Q_-}{k_o^4}(\det a)'(z_n)\,. \tag{2.64b}$$

From (2.42) it also follows that

$$\alpha(\sigma k_o^2/z_n^*) = \frac{1}{k_o^2}\mathrm{cof}\,(Q_-^\dagger)\bar{\alpha}(z_n^*)\mathrm{cof}\,(Q_+)\,, \qquad \bar{\alpha}(\sigma k_o^2/z_n) = \frac{1}{k_o^2}\mathrm{cof}(\,Q_+^\dagger)\alpha(z_n)\mathrm{cof}(\,Q_-)\,. \tag{2.65}$$

Combining these relations, we then have

$$\operatorname*{Res}_{z=\hat{z}_n\equiv\sigma k_o^2/z_n^*}\left[M(x,t,z)\,a^{-1}(z)\right] = \mathrm{e}^{2i\theta(x,t,\hat{z}_n)}N(x,t,\hat{z}_n)\,\hat{C}_n\,, \tag{2.66a}$$

$$\operatorname*{Res}_{z=\hat{z}_n^*\equiv\sigma k_o^2/z_n}\left[\bar{M}(x,t,z)\,\bar{a}^{-1}(z)\right] = \mathrm{e}^{-2i\theta(x,t,\hat{z}_n^*)}\bar{N}(x,t,\hat{z}_n^*)\,\hat{\bar{C}}_n\,, \tag{2.66b}$$

where the norming constants $\hat{C}_n$ satisfy the symmetry relations:

$$\hat{C}_n = \frac{1}{(z_n^*)^2}Q_+^\dagger\bar{C}_nQ_+^\dagger\,, \qquad \hat{\bar{C}}_n = \frac{1}{z_n^2}Q_+C_nQ_+\,. \tag{2.67}$$

Note that $\hat{\bar{C}}_n = \sigma\hat{C}_n^\dagger$, consistently with (2.61a).

2.6 Generalized norming constants

Let us define the 4×4 matrix solutions of (2.1):

$$P(x,t,z) = (\phi(x,t,z)\ \psi(x,t,z))\,, \qquad \bar{P}(x,t,z) = (\bar{\psi}(x,t,z)\ \bar{\phi}(x,t,z))\,. \tag{2.68}$$

Clearly, $P(x,t,z)$ is analytic for $z\in D^+$ and $\bar{P}(x,t,z)$ is analytic for $z\in D^-$. Then one can consider the bilinear combinations $A_\sigma(z) = \bar{P}^\dagger(x,t,z^*)J_\sigma P(x,t,z)$ and $A_\sigma^\dagger(z^*) = P^\dagger(x,t,z^*)J_\sigma\bar{P}(x,t,z)$, which are analytic in D^- and D^+, respectively, and independent of x (by virtue of the same argument used when discussing the first and the third symmetry in Section 2.4). Explicitly computing the 2×2 blocks of $P^\dagger(x,t,z^*)J_\sigma\bar{P}(x,t,z)$ and $\bar{P}^\dagger(x,t,z^*)J_\sigma P(x,t,z)$ in terms of the blocks of the eigenfunctions, and taking into account (2.28) and (2.29), one finds:

$$A_\sigma(z) := \bar{P}^\dagger(x,t,z^*)J_\sigma P(x,t,z) \equiv \begin{pmatrix}\gamma(z)a(z) & 0\\ 0 & -\sigma\gamma^*(z^*)\bar{a}^\dagger(z^*)\end{pmatrix} \qquad z\in D^+\,, \tag{2.69a}$$

$$A_\sigma^\dagger(z^*) = P^\dagger(x,t,z^*)J_\sigma\bar{P}(x,t,z) \equiv \begin{pmatrix}\gamma^*(z^*)a^\dagger(z^*) & 0\\ 0 & -\sigma\gamma(z)\bar{a}(z)\end{pmatrix} \qquad z\in D^-\,. \tag{2.69b}$$

We then have the following:

i) From (2.19a) and (2.68) it follows $\det P(x,t,z) = \mathrm{Wr}(\phi(x,t,z),\psi(x,t,z)) = \gamma^2(z)\det a(z)$, and $\det\bar{P}(x,t,z) = \mathrm{Wr}(\bar{\psi}(x,t,z),\bar{\phi}(x,t,z)) = \gamma^2(z)\det\bar{a}(z)$; therefore the zeros of $\det a(z)$ in D^+ (resp. those of $\det\bar{a}(z)$ in D^-) are precisely the points where $\phi(x,t,z)$ and $\psi(x,t,z)$ (resp. $\bar{\phi}(x,t,z)$ and $\bar{\psi}(x,t,z)$) become linearly dependent. It is worth noticing that the Wronskians may also vanish at the zeros of $\gamma(z)$, i.e., at the branch points ($z=\pm k_o$ in the defocusing case, and $z=\pm ik_o$ in the focusing case); generically, all eigenfunctions are continuous at the branch points, with $\mathrm{Wr}(\psi,\bar{\psi})$ and $\mathrm{Wr}(\phi,\bar{\phi})$ vanishing. Also, generically the scattering coefficients have poles the branch points. But if $\mathrm{Wr}(\phi,\psi)=0$ at either (or both) of the branch points, then the corresponding branch point is referred to as "virtual level" [12], and in this case the scattering coefficients do not have a pole there.

ii) For any $z_n \in D^+ \cap Z$ one has

$$\operatorname{rank} P(x,t,z_n) \leqslant 3\,, \qquad \operatorname{rank} A_\sigma(z_n) \leqslant 2\,, \tag{2.70}$$

since $A_\sigma(z)$ is block diagonal and the determinants of both diagonal blocks vanish at $z = z_n$ (cf. (2.69)).

iii) If $z_n \in D^+ \cap Z$ then $\det P(x,t,z_n) = 0$, and since the two columns in $\phi(x,t,z_n)$ are linearly independent, and so are the two columns in $\psi(x,t,z_n)$, there exist nonzero vectors $\xi_n, \eta_n \in \mathbb{C}^2 \setminus \{0\}$ such that

$$\phi(x,t,z_n)\eta_n = \psi(x,t,z_n)\xi_n\,, \tag{2.71a}$$

and similarly, at $z_n^* \in D^- \cap Z$

$$\bar{\phi}(x,t,z_n^*)\bar{\eta}_n = \bar{\psi}(x,t,z_n^*)\bar{\xi}_n\,, \tag{2.71b}$$

for some $\bar{\xi}_n, \bar{\eta}_n \in \mathbb{C}^2 \setminus \{0\}$. Obviously ξ_n, η_n and $\bar{\xi}_n, \bar{\eta}_n$ are not uniquely defined. Since each of the vectors is nonzero, one can without loss of generality assume that one of the two components of, say, ξ_n (correspondingly, $\bar{\xi}_n$) is equal to one, leaving then three free parameters in the other component of ξ_n and the two components of η_n.

iv) For any $z_n \in D^+ \cap Z$ one then has

$$\psi(x,t,z_n)\xi_n \sim \begin{pmatrix} -\frac{i}{z_n}Q_+\xi_n \\ \xi_n \end{pmatrix} e^{i\lambda(z_n)(x+2k(z_n)t)} \qquad \text{as } x \to +\infty$$

$$\psi(x,t,z_n)\xi_n = \phi(x,t,z_n)\eta_n \sim \begin{pmatrix} \eta_n \\ \frac{i\sigma}{z_n}Q_-^\dagger\eta_n \end{pmatrix} e^{-i\lambda(z_n)(x+2k(z_n)t)} \qquad \text{as } x \to -\infty$$

and since $\operatorname{Im}\lambda(z_n) > 0$ for $z_n \in D^+$ the linear combination of eigenfunctions in $\psi(x,t,z_n)$ is then a bound state (exponentially decaying as $x \to \pm\infty$). The same obviously holds for each $z_n^* \in D^- \cap Z$.

v) In the general situation, the linear combinations (2.71) correspond to having $\operatorname{rank} P(x,t,z_n) \leqslant 3$ and $\operatorname{rank} \bar{P}(x,t,z_n^*) \leqslant 3$ (with the rank being 3 if only one independent condition such as (2.71) holds at each discrete eigenvalue in D^+ and in each in D^-). Under the assumption (2.51) in Section 2.5 one has $\operatorname{rank} P(x,t,z_n) = 2$ and $\operatorname{rank} \bar{P}(x,t,z_n^*) = 2$. And this is equivalent to having *two* independent conditions such as (2.71) holding at the same z_n and z_n^*.

vi) $z_n \in D^+ \cap Z$ is a zero of $\det a(z)$ iff it is a zero of $\det P(x,t,z)$, and obviously of the same order [note D^+ excludes the continuous spectrum, Σ, as well as the branch points]. If $\det P(x,t,z_n) = \det a(z_n) = 0$, then 0 is an eigenvalue of $P(x,t,z_n)$ as well as of $a(z_n)$, and since $a(z_n)$ is a 2×2 matrix, the algebraic multiplicity of 0 as a eigenvalue of $a(z_n)$ can either be 1 (if $\operatorname{trace}(a(z_n)) \neq 0$) or 2 (if $\operatorname{trace}(a(z_n)) = 0$). Since $2 \leqslant \operatorname{rank} P(x,t,z_n) \leqslant 3$, the null spaces of $P(x,t,z_n)$ and $a(z_n)$ can either both be one-dimensional (if $\operatorname{rank} P(x,t,z_n) = 3$), or two-dimensional (if $\operatorname{rank} P(x,t,z_n) = 2$). As far as $a(z_n)$ is concerned, in the first case $\operatorname{rank} a(z_n) = 1$, and $a(z_n)$ is reducible to a Jordan block; and in the second case $a(z_n) = 0$.

Now let $\chi_n \in \mathbb{C}^4 \setminus \{0\}$ be a right null vector of $P(x,t,z_n)$, i.e., $\chi_n \in \ker P(x,t,z_n)$. If we denote

$$\chi_n = \begin{pmatrix} \chi_n^{\text{up}} \\ \chi_n^{\text{dn}} \end{pmatrix} \qquad \chi_n^{\text{up}}, \chi_n^{\text{up}} \in \mathbb{C}^2 ,$$

from the definition (2.68) of $P(x,t,z_n)$ it follows that

$$\phi(x,t,z_n)\chi_n^{\text{up}} + \psi(x,t,z_n)\chi_n^{\text{dn}} = 0$$

showing that a right null vector (defined, of course, up to an arbitrary constant) of $P(x,t,z_n)$ yields (2.71a), with $\eta_n = \chi_n^{\text{up}}$ and $\xi_n = -\chi_n^{\text{dn}}$. Note that since the first two columns of $P(x,t,z_n)$ are linearly independent, and so are the last two columns, necessarily $\eta_n = \chi_n^{\text{up}} \neq 0$ and $\xi_n = -\chi_n^{\text{dn}} \neq 0$. Vice-versa, given ξ_n and η_n as in (2.71a), the 4×1 vector $\chi_n = (\eta_n, -\xi_n)^T$ is a right null vector of $P(x,t,z_n)$. Obviously the analog of all the above statements can be proved for $z_n^* \in D^- \cap Z$ and $\bar{P}(x,t,z)$.

vii) If $\xi_n, \eta_n \in \mathbb{C}^2 \setminus \{0\}$ satisfy (2.71a), then $\chi_n = (\eta_n, -\xi_n)^T$ is a right null vector of $A_\sigma(z_n) = \bar{P}^\dagger(x,t,z_n^*) J_\sigma P(x,t,z_n)$, and from (2.69) it then follows that

$$a(z_n)\eta_n = 0 , \qquad \bar{a}^\dagger(z_n^*)\xi_n = 0 , \tag{2.72a}$$

showing that η_n must be in the right null space of $a(z_n)$, and ξ_n must be in the right null space of $\bar{a}^\dagger(z_n^*)$. Conversely, right null vectors of $a(z_n)$ and $\bar{a}^\dagger(z_n^*)$ provide vectors that satisfy (2.71a). Repeating the same argument, we can show that the same holds for $\bar{\xi}_n, \bar{\eta}_n$

$$a^\dagger(z_n)\bar{\xi}_n = 0 , \qquad \bar{a}(z_n^*)\bar{\eta}_n = 0 , \tag{2.72b}$$

so that $\bar{\xi}_n$ is in the right null space of $a^\dagger(z_n)$ and $\bar{\eta}_n$ is in the right null space of $\bar{a}(z_n^*)$.

viii) Recalling the definitions of $\alpha(z)$ and $\bar{\alpha}(z)$ as cofactor matrices of $a(z)$ and $\bar{a}(z)$ respectively, from $a(z_n)\alpha(z_n) = \alpha(z_n)a(z_n) = 0$ and $\bar{a}(z_n^*)\bar{\alpha}(z_n^*) = \bar{\alpha}(z_n^*)\bar{a}(z_n^*) = 0$ it follows that each of the two columns of $\alpha(z_n)$ are both right and left null vectors of $a(z_n)$, and each of the two columns of $\bar{\alpha}(z_n^*)$ are both right and left null vectors of $\bar{a}(z_n^*)$. Of course the two columns of $\alpha(z_n)$ and $\bar{\alpha}(z_n^*)$ are proportional to each other, since $\det \alpha(z_n) = \det \bar{\alpha}(z_n^*) = 0$. Therefore, we can choose two right null vectors of $P(x,t,z_n)$ such that the first two components of each vector coincide with the first and the second column of $\alpha(z_n)$, and let $-C_n$ denote the 2×2 matrix that collects columnwise the remaining two components of said null vectors:

$$0 = P(x,t,z_n)\begin{pmatrix} \alpha(z_n) \\ -C_n \end{pmatrix} \qquad \Leftrightarrow \qquad \phi(x,t,z_n)\alpha(z_n) = \psi(x,t,z_n)C_n . \tag{2.73}$$

If the right null space of $P(x,t,z_n)$ is 1-dimensional, which happens if z_n is a simple zero of $\det a(z)$ (simple discrete eigenvalue), then the two columns of the matrix multiplying P must be proportional to each other (same as the two columns of $\alpha(z_n)$), which then implies C_n is a rank 1 matrix. Also, since $\alpha(z) = a^{-1}(z)/\det a(z)$, in the case of a simple zero we can write the above equation as

$$\operatorname*{Res}_{z=z_n} \frac{\phi(x,t,z)\alpha(z)}{\det a(z)} = \psi(x,t,z_n)\, C_n , \qquad \det C_n = 0 . \tag{2.74a}$$

which then provides the definition of the norming constant C_n for a simple discrete eigenvalue z_n. In a similar way, one can obtain

$$\operatorname*{Res}_{z=z_n^*} \frac{\bar{\phi}(x,t,z)\bar{\alpha}(z)}{\det \bar{a}(z)} = \bar{\psi}(x,t,z_n^*)\,\bar{C}_n\,, \qquad \det \bar{C}_n = 0\,. \tag{2.74b}$$

This shows how, in the case of a simple zero of $\det a(z)$, one arrives at the same relationships as (2.54) even if the less restrictive conditions (2.71) are assumed instead of (2.51). Note that in [19] the norming constants (Π_n, in their notations) are defined as residues of (the analog of) $b(z)a^{-1}(z)$ at z_n, which strictly speaking is not allowed since $b(z)$ is only defined for $z \in \Sigma$ and z_n is off Σ. In our construction no analytic continuation off Σ is necessary in order to define the norming constants. We also stress that the only constraint on the norming constants is that they have to be symmetric matrices; other than that, their entries can be arbitrary complex numbers.

ix) As discussed in Section 2.5, $\det \alpha(z)$ has a zero of the same order as $\det a(z)$ for each $z_n \in D^+ \cap Z$. The same holds for $\det \bar{\alpha}(z)$, which will have a zero of the same order as $\det \bar{a}(z)$ for each $z_n^* \in D^- \cap Z$. Moreover, for any $z \in D^+ \setminus Z$ one has $a^{-1}(z) = \alpha(z)/(\det a(z))$, and since $\alpha(z)$ is analytic in D^+, it follows that $a^{-1}(z)$ is meromorphic is D^+, with poles at each of the discrete eigenvalues, and the order of the pole at each z_n is at most equal to the order of z_n as a zero of $\det a(z)$. Obviously the same holds for $\bar{a}^{-1}(z)$ in D^-.

x) If z_n is a simple zero of $\det a(z)$ (hence, simple pole of $a^{-1}(z)$), one can easily show that $\ker a(z_n) = \{\tau_n \xi \,:\, \xi \in \mathbb{C}^2\} \equiv \operatorname{range} \tau_n$, where $\tau_n = \operatorname{Res} a^{-1}(z) \equiv \alpha(z_n)/((\det a)'(z_n))$.

2.7 Asymptotics as $z \to \infty$ and $z \to 0$

The asymptotic properties of the eigenfunctions and the scattering matrix are needed to properly define the inverse problem. Moreover, the next-to-leading-order behavior of the eigenfunctions will allow us to reconstruct the potential from the solution of the Riemann-Hilbert problem.

Note that the limit $k \to \infty$ corresponds to $z \to \infty$ in $\mathbb{C}_{\mathrm{I}}$ and to $z \to 0$ in $\mathbb{C}_{\mathrm{II}}$, and both limits will be necessary. The asymptotic expansion of the eigenfunctions in terms of the uniformization variable can be obtained via standard WKB expansions. Explicitly, the modified eigenfunctions $\mu = \varphi e^{i\theta\underline{\sigma}_3}$ satisfy

$$\partial_x \mu = (-ik\underline{\sigma}_3 + \underline{Q})\mu + i\lambda\mu\underline{\sigma}_3\,,$$

which we can write in terms of the uniformization variable z by means of (2.5b). Then, recalling for instance that $\Phi(x,t,z)\mathrm{e}^{i\theta(x,t,z)\underline{\sigma}_3} = (M(x,t,z)\ \bar{M}(x,t,z))$ we have:

$$\partial_x M^{\mathrm{up}} = -(i\sigma k_o^2/z)M^{\mathrm{up}} + Q\,M^{\mathrm{dn}}\,, \qquad \partial_x M^{\mathrm{dn}} = \sigma Q^\dagger M^{\mathrm{up}} + izM^{\mathrm{dn}}\,,$$
$$\partial_x \bar{M}^{\mathrm{up}} = -iz\bar{M}^{\mathrm{up}} + Q\,\bar{M}^{\mathrm{dn}}\,, \qquad \partial_x \bar{M}^{\mathrm{dn}} = \sigma Q^\dagger \bar{M}^{\mathrm{up}} + i(\sigma k_o^2/z)\bar{M}^{\mathrm{dn}}\,,$$

where, as before, $^{\mathrm{up}}$ and $^{\mathrm{dn}}$ denote the upper and lower 2×2 blocks of the corresponding 4×2 matrices $M, \bar{M}$. We can then anchor the WKB expansion as: $M^{\mathrm{up}} = I_2 + A_1/z + h.o.t.$, and $M^{\mathrm{dn}} = B_1/z + B_2/z^2 + h.o.t.$ (here and in the following $h.o.t.$ denotes higher order terms), where

$A_1, B_1, \ldots$ are functions of x, t to be determined. Inserting the WKB ansatz into the above differential equations, and matching equal powers of z yields: $B_1 = i\sigma Q^\dagger$ and $\partial_x A_1 = i\sigma(QQ^\dagger - k_o^2 I_2)$, which in turn gives

$$M(x,t,z) = \begin{pmatrix} I_2 + \dfrac{i\sigma}{z} \int\limits_{-\infty}^{x} [Q(x',t)Q^\dagger(x',t) - k_o^2 I_2]dx' + O(1/z^2) \\ \dfrac{i\sigma}{z} Q^\dagger(x,t) + O(1/z^2) \end{pmatrix} \qquad z \to \infty, z \in D^+, \quad (2.75a)$$

where we have taken the boundary conditions for M as $x \to -\infty$ into account (and implicitly assumed that the limits $z \to \infty$ and $x \to -\infty$ commute).

In a similar way one can find the asymptotic expansion for $\bar{M}$, as well as $N, \bar{N}$ as $z \to \infty$ in the appropriate region of analyticity:

$$\bar{M}(x,t,z) = \begin{pmatrix} -\dfrac{i}{z} Q(x,t) + O(1/z^2) \\ I_2 - \dfrac{i\sigma}{z} \int\limits_{-\infty}^{x} [Q^\dagger(x',t)Q(x',t) - k_o^2 I_2]dx' + O(1/z^2) \end{pmatrix} \qquad z \to \infty, z \in D^-, \quad (2.75b)$$

and

$$\bar{N}(x,t,z) = \begin{pmatrix} I_2 + \dfrac{i\sigma}{z} \int\limits_{x}^{\infty} [Q(x',t)Q^\dagger(x',t) - k_o^2 I_2]dx' + O(1/z^2) \\ \dfrac{i\sigma}{z} Q^\dagger(x,t) + O(1/z^2) \end{pmatrix} \qquad z \to \infty, z \in D^- \quad (2.75c)$$

$$N(x,t,z) = \begin{pmatrix} -\dfrac{i}{z} Q(x,t) + O(1/z^2) \\ I_2 - \dfrac{i\sigma}{z} \int\limits_{-\infty}^{x} [Q^\dagger(x',t)Q(x',t) - k_o^2 I_2]dx' + O(1/z^2) \end{pmatrix} \qquad z \to \infty, z \in D^+. \quad (2.75d)$$

Similarly, asymptotics as $z \to 0$ in the proper region $D^\pm$ yields:

$$M(x,t,z) = \begin{pmatrix} QQ_-^\dagger/k_o^2 + O(z) \\ i\sigma Q_-^\dagger/z + O(1) \end{pmatrix}, \qquad \bar{M}(x,t,z) \begin{pmatrix} -iQ_-/z + O(1) \\ Q^\dagger Q_-/k_o^2 + O(z) \end{pmatrix}, \quad (2.76a)$$

$$\bar{N}(x,t,z) = \begin{pmatrix} QQ_+^\dagger/k_o^2 + O(z) \\ i\sigma Q_+^\dagger/z + O(1) \end{pmatrix}, \qquad N(x,t,z) \begin{pmatrix} -iQ_+/z + O(1) \\ Q^\dagger Q_+/k_o^2 + O(z) \end{pmatrix}. \quad (2.76b)$$

The above equations will allow us to reconstruct the scattering potential $Q(x,t)$ from the solution of the inverse problem for the eigenfunctions.

Finally, inserting the above asymptotic expansions for the Jost eigenfunctions into (2.17) one shows that, as $z \to \infty$ in the appropriate regions of the complex z-plane,

$$S(z) = I_2 + O(1/z). \quad (2.77)$$

The above asymptotics holds with $\operatorname{Im} z \geqslant 0$ and $\operatorname{Im} z \leqslant 0$ for $a(z)$ and $\bar{a}(z)$, respectively, and with $z \in \Sigma$ for $b(z)$ and $\bar{b}(z)$. Similarly, one can show that as $z \to 0$

$$S(z) = \frac{1}{k_o^2} \begin{pmatrix} Q_+ Q_-^\dagger & 0 \\ 0 & Q_+^\dagger Q_- \end{pmatrix} + O(z), \quad (2.78)$$

where again the asymptotics for the block diagonal entries of $S(z)$ can be extended to D^+ for $a(z)$ and to D^- for $\bar{a}(z)$, while the asymptotics for the off-diagonal blocks hold for $z \in \Sigma$.

3 Inverse problem

3.1 Riemann-Hilbert problem

The starting point for the formulation of the inverse problem is (2.21), which we now regard as a relation between eigenfunctions analytic in D^+ and those analytic in D^-. Thus, we introduce the sectionally meromorphic matrices

$$\mu^+(x,t,z) = (M\,a^{-1}\;\; N)\,, \qquad \mu^-(x,t,z) = (\bar N\;\; \bar M\,\bar a^{-1})\,, \tag{3.1}$$

where superscripts $\pm$ distinguish between analyticity in D^+ and D^-, respectively. From (2.21) we then obtain the jump condition

$$\mu^-(x,t,z) = \mu^+(x,t,z)\,(I_4 - G(x,t,z)) \qquad z \in \Sigma\,, \tag{3.2}$$

where the jump matrix is

$$G(x,t,z) = \begin{pmatrix} 0 & -\mathrm{e}^{-2i\theta(x,t,z)}\bar\rho(z) \\ \mathrm{e}^{2i\theta(x,t,z)}\rho(z) & \rho(z)\bar\rho(z) \end{pmatrix}. \tag{3.3}$$

Equations (3.1)–(3.3) define a matrix, multiplicative, homogeneous Riemann-Hilbert problem (RHP). In order to complete the formulation of the RHP one needs a normalization condition, which in this case is the asymptotic behavior of $\mu^\pm$ as $z \to \infty$. Recalling the asymptotic behavior of the Jost eigenfunctions and scattering coefficients, it is easy to check that

$$\mu^\pm = I_4 + O(1/z)\,, \qquad z \to \infty\,. \tag{3.4a}$$

On the other hand,

$$\mu^\pm = -(i/z)\underline{\sigma}_3\,\underline{Q}_+ + O(1)\,, \qquad z \to 0\,. \tag{3.4b}$$

To solve the RHP, one needs to regularize it by subtracting out the asymptotic behavior and the pole contributions. Recall that in the focusing case discrete eigenvalues come in symmetric quartets [cf. (2.52)]. It is then convenient to define $\zeta_n = z_n$ and $\zeta_{n+\mathcal{N}} = \sigma k_o^2/z_n^*$ for $n = 1,\ldots,\mathcal{N}$ as well as $C_{n+\mathcal{N}} = \hat C_n$ and $\bar C_{n+\mathcal{N}} = \hat{\bar C}_n$ for $n = 1,\ldots,\mathcal{N}$ and rewrite (3.2) as

$$\begin{aligned} &\mu^- - I_4 + (i/z)\underline{\sigma}_3\,\underline{Q}_+ - \sum_{n=1}^{2\mathcal{N}} (\operatorname*{Res}_{\zeta_n^*}\mu^-)/(z-\zeta_n^*) - \sum_{n=1}^{2\mathcal{N}} (\operatorname*{Res}_{\zeta_n}\mu^+)/(z-\zeta_n) \\ &\quad = \mu^+ - I_4 + (i/z)\underline{\sigma}_3\,\underline{Q}_+ - \sum_{n=1}^{2\mathcal{N}} (\operatorname*{Res}_{\zeta_n}\mu^+)/(z-\zeta_n) - \sum_{n=1}^{2\mathcal{N}} (\operatorname*{Res}_{\zeta_n^*}\mu^-)/(z-\zeta_n^*) - \mu^+ G\,. \end{aligned} \tag{3.5}$$

The left-hand side (LHS) of (3.5) is now analytic in D^- and is $O(1/z)$ as $z \to \infty$ from D^-, while the sum of the first four terms of the right-hand side (RHS) is analytic in D^+ and is $O(1/z)$ as $z \to \infty$ from D^+. Finally, the asymptotic behavior of the off-diagonal scattering coefficients implies that $G(x,t,z)$ is $O(1/z)$ as $z \to \pm\infty$ and $O(z)$ as $z \to 0$ along the real axis. We then introduce the Cauchy projectors $P_\pm$ over Σ:

$$P_\pm[f](z) = \frac{1}{2\pi i}\int_\Sigma \frac{f(\zeta)}{\zeta - (z \pm i0)}\,\mathrm{d}\zeta\,,$$

where $\int_\Sigma$ denotes the integral along the oriented contours shown in Fig. 1, and the notation $z \pm i0$ indicates that when $z \in \Sigma$, the limit is taken from the left/right of it. Now recall Plemelj's formulae: if $f^\pm$ are analytic in $D^\pm$ and are $O(1/z)$ as $z \to \infty$, one has $P^\pm f^\pm = \pm f^\pm$ and $P^+ f^- = P^- f^+ = 0$. Applying P^+ and P^- to (3.5) we then have

$$\mu(x,t,z) = I_4 - (i/z)\underline{\sigma}_3\,\underline{Q}_+ + \sum_{n=1}^{2\mathcal{N}} \frac{\operatorname*{Res}_{\zeta_n} \mu^+}{z-\zeta_n} + \sum_{n=1}^{2\mathcal{N}} \frac{\operatorname*{Res}_{\zeta_n^*} \mu^-}{z-\zeta_n^*} \\ + \frac{1}{2\pi i}\int_\Sigma \frac{\mu^+(x,t,\zeta)}{\zeta - z}\,G(x,t,\zeta)\,\mathrm{d}\zeta\,, \qquad z \in \mathbb{C}\setminus\Sigma\,. \quad (3.6)$$

As usual, the expressions for μ^+ and μ^- are formally identical, except for the fact that the integral appearing in the right-hand side is a P^+ and a P^- projector, respectively. Also, in the defocusing case the sums are only for $n = 1,\ldots,\mathcal{N}$.

3.2 Residue conditions and reconstruction formula

To close the system we need to obtain an expression for the residues appearing in (3.6). From the definition (3.1) it follows that only the first two columns of μ^+ have a pole at $z = z_n$ and $z = \sigma k_o^2/z_n^*$ in D^+, and only the last two columns of μ^- have a pole at $z = z_n^*$ and $z = \sigma k_o^2/z_n$ in D^-. The residue relations (2.54) and (2.66) then imply that such residues are proportional, respectively, to the last two columns of μ^+ and the first two columns of μ^-. Explicitly:

$$\operatorname*{Res}_{\zeta_n} \mu^+ = \big(\mathrm{e}^{2i\theta(x,t,\zeta_n)} N(x,t,\zeta_n)\,C_n\,, 0\big)\,, \qquad n = 1,\ldots,2\mathcal{N}\,, \quad (3.7\text{a})$$

$$\operatorname*{Res}_{\zeta_n^*} \mu^- = \big(0\,, \mathrm{e}^{-2i\theta(x,t,\zeta_n^*)} \bar{N}(x,t,\zeta_n^*)\,\bar{C}_n\big)\,, \qquad n = 1,\ldots,2\mathcal{N}\,. \quad (3.7\text{b})$$

We can therefore evaluate the last two columns of (3.6) at $z = z_n$ and at $z = \sigma k_o^2/z_n^*$, obtaining:

$$N(x,t,\zeta_n) = \begin{pmatrix} -iQ_+/\zeta_n \\ I_2 \end{pmatrix} + \sum_{j=1}^{2\mathcal{N}} \frac{\mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}}{\zeta_n - \zeta_j^*}\,\bar{N}(x,t,\zeta_j^*)\,\bar{C}_j + \frac{1}{2\pi i}\int_\Sigma \frac{(\mu^+ G)_2(x,t,\zeta)}{\zeta - \zeta_n}\,\mathrm{d}\zeta\,, \quad (3.8\text{a})$$

for $n = 1,\ldots,2\mathcal{N}$, and where the subscript 2 in $\mu^+ G$ denotes the last two columns of the product, namely

$$(\mu^+ G)_2(x,t,\zeta) = -e^{-2i\theta(x,t,\zeta)} M(x,t,\zeta) a^{-1}(\zeta)\bar{\rho}(\zeta) + N(x,t,\zeta)\,\rho(\zeta)\bar{\rho}(\zeta) \\ \equiv -e^{-2i\theta(x,t,\zeta)}\bar{N}(x,t,\zeta)\bar{\rho}(\zeta)\,.$$

Similarly, we can evaluate the first two columns of (3.6) at $z = z_n^*$ and at $z = \sigma k_o^2/z_n$, obtaining:

$$\bar{N}(x,t,\zeta_n^*) = \begin{pmatrix} I_2 \\ i\sigma Q_+^\dagger/\zeta_n^* \end{pmatrix} + \sum_{j=1}^{2\mathcal{N}} \frac{\mathrm{e}^{2i\theta(x,t,\zeta_j)}}{\zeta_n^* - \zeta_j}\,N(x,t,\zeta_j)\,C_j + \frac{1}{2\pi i}\int_\Sigma \frac{(\mu^+ G)_1(x,t,\zeta)}{\zeta - \zeta_n^*}\,\mathrm{d}\zeta\,, \quad (3.8\text{b})$$

again for $n = 1,\ldots,2\mathcal{N}$, and with subscript 1 in $\mu^+ G$ indicating the first two columns of the product, namely:

$$(\mu^+ G)_1(x,t,\zeta) = e^{2i\theta(x,t,\zeta)} N(x,t,\zeta)\rho(\zeta)\,.$$

Finally, evaluating the last two columns of $\mu^+(x,t,z)$ (thus with a P^+ projector) and the first two columns of μ^- (thus with a P^- projector) via (3.6) for $z \in \Sigma$ we obtain:

$$N(x,t,z) = \begin{pmatrix} -iQ_+/z \\ I_2 \end{pmatrix} + \sum_{j=1}^{2\mathcal{N}} \frac{\mathrm{e}^{-2i\theta(x,t,\zeta_j^*)}}{z-\zeta_j^*} \bar{N}(x,t,\zeta_j^*)\,\bar{C}_j - \frac{1}{2\pi i}\int_\Sigma \frac{e^{-2i\theta(x,t,\zeta)}\bar{N}(x,t,\zeta)\bar{\rho}(\zeta)}{\zeta-(z+i0)}\,\mathrm{d}\zeta\,, \tag{3.8c}$$

$$\bar{N}(x,t,z) = \begin{pmatrix} I_2 \\ i\sigma Q_+^\dagger/z \end{pmatrix} + \sum_{j=1}^{2\mathcal{N}} \frac{\mathrm{e}^{2i\theta(x,t,\zeta_j)}}{z-\zeta_j} N(x,t,\zeta_j)\,C_j + \frac{1}{2\pi i}\int_\Sigma \frac{e^{2i\theta(x,t,\zeta)}N(x,t,\zeta)\rho(\zeta)}{\zeta-(z-i0)}\,\mathrm{d}\zeta\,, \tag{3.8d}$$

which, together with equations (3.8), yield a closed system of linear algebraic-integral equations for the solution of the RHP.

The last task is to reconstruct the potential from the solution of the RHP. From (3.6), one obtains the asymptotic behavior of $\mu^\pm(x,t,z)$ as $z\to\infty$ as

$$\mu(x,t,z) = I_4 + \frac{1}{z}\Big\{ -i\underline{\sigma}_3\,\underline{Q}_+ + \sum_{n=1}^{2\mathcal{N}} \big(\operatorname*{Res}_{\zeta_n}\mu^+ + \operatorname*{Res}_{\zeta_n^*}\mu^-\big) - \frac{1}{2\pi i}\int_\Sigma \mu^+(x,t,\zeta)\,G(x,t,\zeta)\,\mathrm{d}\zeta \Big\} + O(1/z^2)\,, \tag{3.9}$$

where the residues are given by (3.7). Taking $\mu = \mu^+$ and comparing the upper right 2×2 block of (3.9) of this expression with (2.75d) we then find:

$$Q(x,t) = Q_+ + i\sum_{n=1}^{2\mathcal{N}} \mathrm{e}^{-2i\theta(x,t,\zeta_n^*)}\,\bar{N}^{\mathrm{up}}(x,t,\zeta_n^*)\,\bar{C}_n + \frac{1}{2\pi}\int_\Sigma e^{-2i\theta(x,t,\zeta)}\bar{N}^{\mathrm{up}}(x,t,\zeta)\bar{\rho}(\zeta)\,\mathrm{d}\zeta\,. \tag{3.10a}$$

Similarly, taking $\mu = \mu^-$ and comparing the lower left 2×2 block of (3.9) of this expression with (2.75d) we then obtain the reconstruction formula for the potential:

$$Q^\dagger(x,t) = Q_+^\dagger - i\sigma\sum_{n=1}^{2\mathcal{N}} \mathrm{e}^{2i\theta(x,t,\zeta_n)}\,N^{\mathrm{dn}}(x,t,\zeta_n)\,C_n + \frac{\sigma}{2\pi}\int_\Sigma e^{2i\theta(x,t,\zeta)}N^{\mathrm{dn}}(x,t,\zeta)\rho(\zeta)\,\mathrm{d}\zeta\,. \tag{3.10b}$$

Recall that the time dependence of the solution is automatically taken into account by the fact that the Jost eigenfunctions are simultaneous solutions of both parts of the Lax pair.

The above reconstruction formulas allow us to identify the symmetries of the norming constants. In fact, recalling that $N^{\mathrm{dn}}(x,t,z)\sim I_2$ as $x\to\infty$ for any $z\in D^+$ and $\bar{N}^{\mathrm{up}}(x,t,z)\sim I_2$ as $x\to\infty$ for any $z\in D^-$, the comparison of the two equations in (3.10) yields

$$\bar{C}_n = \sigma C_n^\dagger \qquad n = 1,\dots,\in\mathcal{N}\,. \tag{3.11}$$

Similarly, in order for $Q(x,t)$ to be a symmetric matrix for all x,t (i.e., $Q^T(x,t) = Q(x,t)$), the norming constants have to satisfy the same symmetry (and the same symmetry as the reflection coefficients), namely:

$$C_n^T = C_n\,, \qquad \bar{C}_n^T = \bar{C}_n\,, \qquad n = 1,\dots,\in\mathcal{N}\,. \tag{3.12}$$

3.3 Reflectionless potentials

We now consider potentials $Q(x,t)$ for which the reflection coefficient $\rho(z)$ (and hence $\bar{\rho}(z)$, according to (2.32)) vanishes identically for $z \in \Sigma$. In this case there is no jump from μ^+ to μ^- across the continuous spectrum, and the inverse problem therefore reduces to an algebraic system, whose solution yields the soliton solutions of (1.2).

Below we will consider solutions of the focusing equations, $\sigma = -1$. Recall that in this case discrete eigenvalues appear in quartets, with $\zeta_{\mathcal{N}+j} = -k_o^2/z_j^*$ and from (2.67) $C_{\mathcal{N}+j} = Q_+^\dagger \bar{C}_j Q_+^\dagger/(z_j^*)^2$ for all $j = 1,\ldots,\mathcal{N}$; moreover, one can also easily check that $\theta(x,t,z^*) = \theta^*(x,t,z)$. Recall also that according to (2.61a) one has $\bar{C}_j = -C_j^\dagger$ for all $j = 1,\ldots,2\mathcal{N}$. It is convenient to introduce the quantities

$$c_j(x,t,z) = \frac{C_j}{z-\zeta_j}\,\mathrm{e}^{2i\theta(x,t,\zeta_j)}, \qquad j = 1,\ldots,2\mathcal{N}. \tag{3.13}$$

Note from (3.10) that only the upper block of the eigenfunction $\bar{N}(x,t,z)$ is needed in the reconstruction formula. The algebraic system obtained from the inverse problem for said upper blocks is then expressed as

$$N^{\mathrm{up}}(\zeta_j) = -\frac{i}{\zeta_j}Q_+ - \sum_{\ell=1}^{2\mathcal{N}} \bar{N}^{\mathrm{up}}(\zeta_\ell^*)c_\ell^\dagger(\zeta_j^*), \qquad j = 1,\ldots,2\mathcal{N}, \tag{3.14a}$$

$$\bar{N}^{\mathrm{up}}(\zeta_n^*) = I_2 + \sum_{j=1}^{2\mathcal{N}} N^{\mathrm{up}}(\zeta_j)c_j(\zeta_n^*), \qquad n = 1,\ldots,2\mathcal{N}, \tag{3.14b}$$

and substituting (3.14a) into (3.14b) yields

$$\bar{N}^{\mathrm{up}}(\zeta_n^*) = I_2 - iQ_+\sum_{j=1}^{2\mathcal{N}} c_j(\zeta_n^*)/\zeta_j - \sum_{j=1}^{2\mathcal{N}}\sum_{\ell=1}^{2\mathcal{N}} \bar{N}^{\mathrm{up}}(\zeta_\ell^*)\,c_\ell^\dagger(\zeta_j^*)c_j(\zeta_n^*), \qquad n = 1,\ldots,2\mathcal{N}, \tag{3.15}$$

where for brevity we omitted the x and t dependence. We now write this system in matrix form. Introducing $\mathbf{X} = (X_1,\ldots,X_{2\mathcal{N}})^t$ and $\mathbf{B} = (B_1,\ldots,B_{2\mathcal{N}})^T$, where

$$X_n = \bar{N}^{\mathrm{up}}(x,t,\zeta_n^*), \qquad B_n = I_2 - iQ_+\sum_{j=1}^{2\mathcal{N}} c_j(\zeta_n^*)/\zeta_j, \qquad n = 1,\ldots,2\mathcal{N}.$$

and defining the block matrix $\Gamma = (\Gamma_{n,\ell})$, where

$$\Gamma_{n,\ell} = \sum_{j=1}^{2\mathcal{N}} c_\ell^\dagger(\zeta_j^*)c_j(\zeta_n^*), \qquad n,\ell = 1,\ldots,2\mathcal{N},$$

the system (3.15) becomes simply $A\,\mathbf{X} = \mathbf{B}$, where $A = I_p + \Gamma$, and I_p is the identity matrix of size $p = 2^{2\mathcal{N}}$.

Finally, upon substituting $X_1,\ldots,X_{2\mathcal{N}}$ into the reconstruction formula (3.10) (without the integral, of course, since we are assuming $\rho(\zeta) \equiv 0$), one obtains the corresponding reflectionless solution for $Q(x,t)$. Note that, even though the discrete eigenvalues appear in quartets in the NZBC case as opposed to pairs in the case of zero boundary conditions, the number of unknowns in the inverse problem is still the same. This is because the symmetry (2.40a) implies

$$N^{\mathrm{up}}(x,t,\zeta_j) = -\frac{i}{\zeta_j}\bar{N}^{\mathrm{up}}(x,t,\zeta_{j+\mathcal{N}}^*)Q_+, \qquad N^{\mathrm{up}}(x,t,\zeta_{j+\mathcal{N}}) = -\frac{i\sigma\zeta_j^*}{k_o^2}\bar{N}^{\mathrm{up}}(x,t,\zeta_j^*)Q_+, \tag{3.16}$$

for all $j = 1, \ldots, \mathcal{N}$. Therefore one can equivalently write the linear algebraic system (3.14) in terms of just $2\mathcal{N}$ unknowns, as in the case of ZBCs.

4 Soliton solutions

The focusing NLS equation with NZBC possesses a rich family of soliton solutions [2, 3, 4, 17, 20, 22, 25, 28, 29], and in this section we will derive their counterpart in the spinor system. Let us start by computing the one soliton solution in the focusing case, with one quartet of discrete eigenvalues. Introducing the notation $X_j := \bar{N}_j^{\mathrm{up}}(x,t,\zeta_j^*)$ for $j = 1, 2$, and using the symmetries (3.16) in (3.14), we find

$$X_1\, D_1 = I_2 - \frac{i}{\zeta_1} X_2 Q_+ c_1(\zeta_1^*)\,, \qquad X_2\, D_2 = I_2 + \frac{i\zeta_1^*}{k_o^2} X_1 Q_+ c_2(\zeta_2^*)\,,$$

where

$$D_1 = I_2 + \frac{i}{(\zeta_1^*)^2 + k_o^2} C_1^\dagger Q_+^\dagger e^{-2i\theta(\zeta_1^*)}\,, \qquad D_2 = D_1^\dagger\,, \tag{4.1}$$

where we have used the explicit expression of the matrices $c_j(\zeta_\ell^*) = C_j e^{2i\theta(\zeta_j)}/(\zeta_\ell^* - \zeta_j)$, as well as $\zeta_2 = -k_o^2/\zeta_1^*$ and the two combined symmetries for the norming constant: $C_2 = -Q_+^\dagger C_1^\dagger Q_+^\dagger/(\zeta_1^*)^2$. This is formally a linear algebraic system of two equations in the two unknowns X_1, X_2, but both the unknowns and the coefficients are 2×2 matrices. We can solve the system by back substitution, and obtain:

$$X_1 = \left[I_2 - \frac{i}{\zeta_1} D_2^{-1} Q_+ c_1(\zeta_1^*) \right] \left[D_1 - \frac{\zeta_1^*}{\zeta_1 k_o^2} Q_+ c_2(\zeta_2^*) D_2^{-1} Q_+ c_1(\zeta_1^*) \right]^{-1}, \tag{4.2a}$$

$$X_2 = \left[I_2 + \frac{i\zeta_1^*}{k_o^2} D_1^{-1} Q_+ c_2(\zeta_2^*) \right] \left[D_2 - \frac{\zeta_1^*}{\zeta_1 k_o^2} Q_+ c_1(\zeta_1^*) D_1^{-1} Q_+ c_2(\zeta_2^*) \right]^{-1}, \tag{4.2b}$$

where D_1, D_2 are given in (4.1) and

$$c_1(\zeta_1^*) = \frac{C_1 e^{2i\theta(x,t,\zeta_1)}}{\zeta_1^* - \zeta_1}\,, \qquad c_2(\zeta_2^*) = \frac{\zeta_1}{\zeta_1^* k_o^2 (\zeta_1^* - \zeta_1)} Q_+^\dagger C_1^\dagger Q_+^\dagger e^{-2i\theta(x,t,\zeta_1^*)}\,.$$

Recall that in the above formulas the eigenvalue $\zeta_1 \in D^+$ is assumed located in the upper half plane, i.e., such that $|\zeta_1| > k_o$ and $\operatorname{Im}\zeta_1 > 0$, and the associated norming constant C_1 is an arbitrary 2×2 symmetric matrix with $\det C_1 = 0$. As discussed in Section 2.6, one can also consider the case when C_1 is not rank 1, corresponding to having $a(\zeta_1) = 0$. Note that in general all entries of C_1 can be complex, and therefore the norming constant is defined in terms of two arbitrary complex numbers in the rank 1 case, and three in the rank 2 case:

$$C_1 = \begin{pmatrix} \gamma_1 & \gamma_0 \\ \gamma_0 & \gamma_{-1} \end{pmatrix} \tag{4.3}$$

with $\gamma_j \in \mathbb{C}$ for $j = -1, 0, 1$, and $\gamma_0^2 = \gamma_1 \gamma_{-1}$ in the rank 1 case. The potential is then reconstructed in terms of X_1, X_2 via (3.10), namely:

$$Q(x,t) = Q_+ - iX_1 e^{-2i\theta(x,t,\zeta_1^*)} C_1^\dagger + iX_2 e^{2i\theta(x,t,\zeta_1)} Q_+ C_1 Q_+/\zeta_1^2\,. \tag{4.4}$$

A generic $\zeta_1 \in D^+$ in the UHP gives the analog in the spinor model of the Tajiri-Watanabe breather solution of the scalar focusing NLS [25]; when ζ_1 in located on the imaginary axis ($\zeta_1 = iZ$ with $Z > k_o$) one has the analog of the Kuznetsov-Ma solution [17, 20]; the limit $\zeta_1 \to C_o$, i.e., $\zeta_1 = \sqrt{k_o^2 - Z^2} + iZ$ with $0 < Z < k_o$ gives the analog of the Akhmediev breather solution [2].

In order to more easily analyze the soliton solutions, it is useful to simplify the formulas for X_1, X_2 above. First, assume $Q_+ = k_o I_2$ for simplicity, and note that in this case

$$D_1^{-1} = \frac{1}{\Delta_1}\left[I_2 + \frac{ik_o}{(\zeta_1^*)^2 + k_o^2} e^{-2i\theta(\zeta_1^*)} \mathrm{cof}(C_1^\dagger)\right], \tag{4.5a}$$

$$D_2^{-1} = \frac{1}{\Delta_1^*}\left[I_2 - \frac{ik_o}{\zeta_1^2 + k_o^2} e^{2i\theta(\zeta_1)} \mathrm{cof}(C_1)\right], \tag{4.5b}$$

$$\Delta_1 = \det D_1 = 1 + \frac{ik_o}{(\zeta_1^*)^2 + k_o^2} e^{-2i\theta(\zeta_1^*)} \mathrm{trace}(C_1^\dagger) - \frac{k_o^2}{((\zeta_1^*)^2 + k_o^2)^2} e^{-4i\theta(\zeta_1^*)} \det C_1^\dagger . \tag{4.5c}$$

Taking these expressions into account, we then have

$$X_1 = A_1 B_1^{-1}, \qquad X_2 = A_2 B_2^{-1}, \tag{4.6a}$$

$$A_1 = I_2 - \frac{ik_o}{\zeta_1(\zeta_1^* - \zeta_1)} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1 - \frac{k_o^2 \det C_1}{\zeta_1(\zeta_1^* - \zeta_1)(\zeta_1^2 + k_o^2)} \frac{e^{4i\theta(\zeta_1)}}{\Delta_1^*} I_2, \tag{4.6b}$$

$$A_2 = I_2 + \frac{i\zeta_1}{k_o(\zeta_1^* - \zeta_1)} \frac{e^{-2i\theta(\zeta_1^*)}}{\Delta_1} C_1^\dagger - \frac{\zeta_1 \det C_1^\dagger}{(\zeta_1^* - \zeta_1)((\zeta_1^*)^2 + k_o^2)} \frac{e^{-4i\theta(\zeta_1^*)}}{\Delta_1} I_2, \tag{4.6c}$$

$$B_1 = I_2 + C_1^\dagger e^{-2i\theta(\zeta_1^*)}\left[\frac{ik_o}{(\zeta_1^*)^2 + k_o^2}\left(1 + \frac{(\zeta_1^*)^2 + k_o^2}{\zeta_1^2 + k_o^2} \frac{\det C_1}{(\zeta_1^* - \zeta_1)^2} \frac{e^{4i\theta(\zeta_1)}}{\Delta_1^*}\right) I_2 - \frac{1}{(\zeta_1^* - \zeta_1)^2} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1\right], \tag{4.6d}$$

$$B_2 = I_2 + C_1 e^{2i\theta(\zeta_1)}\left[-\frac{ik_o}{\zeta_1^2 + k_o^2}\left(1 + \frac{\zeta_1^2 + k_o^2}{(\zeta_1^*)^2 + k_o^2} \frac{\det C_1^\dagger}{(\zeta_1^* - \zeta_1)^2} \frac{e^{-4i\theta(\zeta_1^*)}}{\Delta_1}\right) I_2 - \frac{1}{(\zeta_1^* - \zeta_1)^2} \frac{e^{-2i\theta(\zeta_1^*)}}{\Delta_1} C_1^\dagger\right]. \tag{4.6e}$$

Clearly the above expressions significantly simplify when C_1 is rank 1, i.e., if $\det C_1 = 0$. In this case, one has:

$$\Delta_1 = 1 + \frac{ik_o}{(\zeta_1^*)^2 + k_o^2} e^{-2i\theta(\zeta_1^*)} \mathrm{trace}(C_1^\dagger), \tag{4.7a}$$

$$A_1 = I_2 - \frac{ik_o}{\zeta_1(\zeta_1^* - \zeta_1)} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1, \tag{4.7b}$$

$$A_2 = I_2 + \frac{i\zeta_1}{k_o(\zeta_1^* - \zeta_1)} \frac{e^{-2i\theta(\zeta_1^*)}}{\Delta_1} C_1^\dagger, \tag{4.7c}$$

$$B_1 = I_2 + C_1^\dagger e^{-2i\theta(\zeta_1^*)}\left[\frac{ik_o}{(\zeta_1^*)^2 + k_o^2} I_2 - \frac{1}{(\zeta_1^* - \zeta_1)^2} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1\right], \tag{4.7d}$$

$$B_2 = I_2 + C_1 e^{2i\theta(\zeta_1)}\left[-\frac{ik_o}{\zeta_1^2 + k_o^2} I_2 - \frac{1}{(\zeta_1^* - \zeta_1)^2} \frac{e^{-2i\theta(\zeta_1^*)}}{\Delta_1} C_1^\dagger\right]. \tag{4.7e}$$

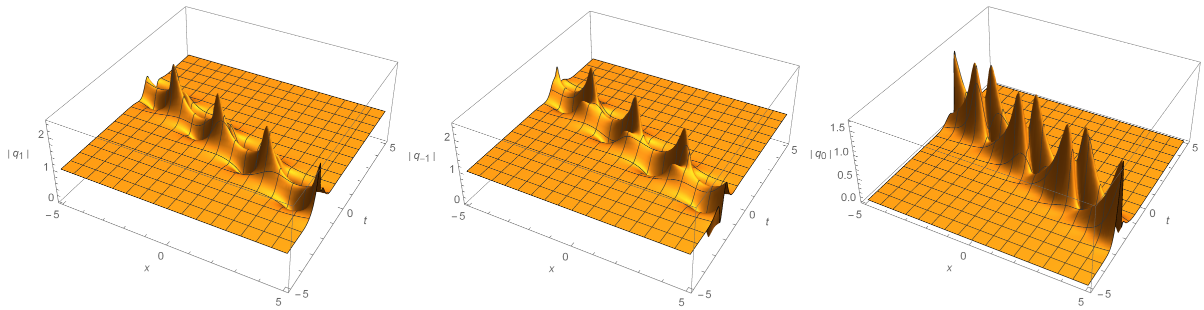


Figure 2: Three components (q_1, q_{-1} and q_0 from left to right) with $Q_+ = I_2$, $\zeta_1 = 1 + 2i$ and the entries of C_1 in (4.3) are chosen to be $\gamma_1 = 2, \gamma_{-1} = 1$, $\gamma_0 = 1$ (with $\det C_1 \neq 0$).

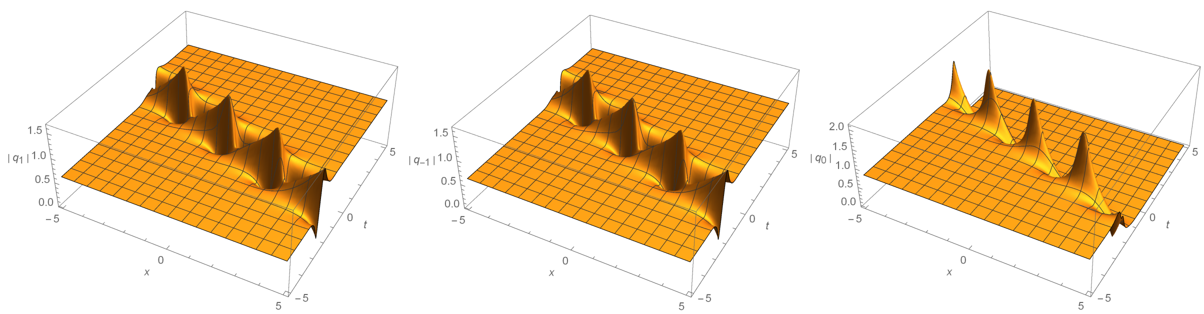


Figure 3: Three components (q_1, q_{-1} and q_0 from left to right) with $Q_+ = I_2$, $\zeta_1 = 1 + 2i$ and the entries of C_1 in (4.3) are chosen to be $\gamma_1 = \gamma_{-1} = \gamma_0 = 1 + i$ (obviously $\det C_1 = 0$).

The above expressions can then be used to obtain the various soliton solutions. Fig. 2 shows a one-soliton solution (analog of the Tajiri-Watanabe soliton for the scalar focusing NLS) corresponding to $\zeta_1 \in D^+$ in generic position, and when the norming constant is a rank 2 symmetric matrix [these type of solitons are referred to as "polar states" in [26] for the repulsive/antiferromagnetic (defocusing) spinor system, and correspond to states in which the total spin is 0]. In Fig. 3 the Tajiri-Watanabe soliton is plotted for the same $\zeta_1 \in D^+$ and a rank 1 norming constant [these types of solitons are referred to as "ferromagnetic states", and correspond to states in which the total spin is nonzero]. Note that in this second case the amplitudes of the individual entries of Q_- are different than those of Q_+, and this is always the case with ferromagnetic vs polar states, i.e., rank 1 vs rank 2 norming constants (the large x asymptotics are discussed in details in Appendix B).

When the discrete eigenvalue $\zeta_1 \in D^+$ is purely imaginary (so, $\zeta_1 = iZ$ with $Z > k_o$), the corresponding soliton solutions are stationary, and one obtains the analog for the spinor system of the Kuznetsov-Ma breather of the focusing NLS, namely a solution that is periodic in t and homoclinic in x, plotted in Fig. 4 for the polar state (rank 2 norming constant).

Finally, if one considers the limit as the discrete eigenvalue ζ_1 approaches the circle C_o, one obtains the analog of the Akhmediev breather. The corresponding polar and ferromagnetic states are plotted in Fig. 5 and Fig. 6, respectively. Note that these solutions are periodic in x and

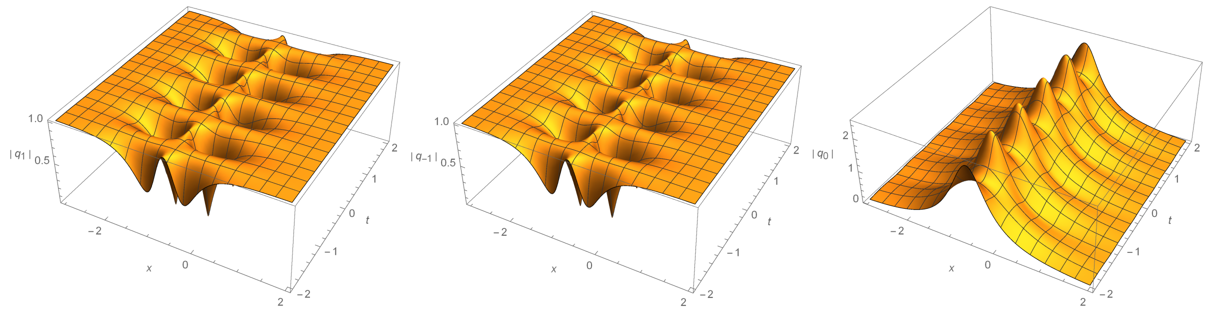


Figure 4: Three components (q_1, q_{-1} and q_0 from left to right) with $Q_+ = I_2$, $\zeta_1 = 2i$ and the entries of C_1 in (4.3) are chosen to be $\gamma_1 = \gamma_{-1} = 0$ and $\gamma_0 = 1$ ($\det C_1 \neq 0$). Note the solution is homoclinic in x and periodic in t, as the Kuznetsov-Ma breather of the focusing NLS equation.

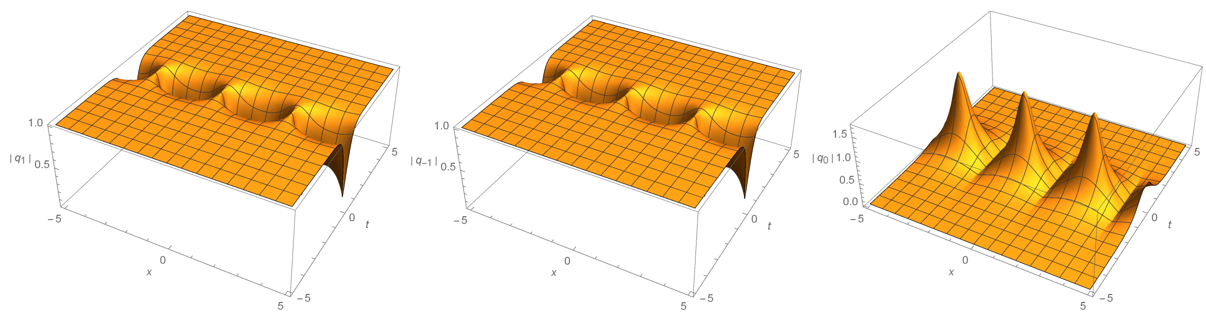


Figure 5: Three components (q_1, q_{-1} and q_0 from left to right) with $Q_+ = I_2$, $\zeta_1 = 1/2 + \sqrt{3}/2 \in C_o$ and and the entries of C_1 in (4.3) are chosen to be $\gamma_1 = \gamma_{-1} = 0$, $\gamma_0 = 1$ ($\det C_1 \neq 0$). Note the solution is homoclinic in t and periodic in x, as the Akhmediev breather of the focusing NLS equation.

homoclinic in t, and hence, as such, they are outside of the class considered for the IST, in which $Q(x,t)$ is assumed to have constant limits as $\to \pm\infty$.

5 Concluding remarks

The analysis of systems modeling spinor BECs has become a subject of increasing attention. The development of far-off-resonant optical techniques for trapping of ultracold atomic gases has opened new directions in the studies of BECs, allowing one to confine atoms regardless of their spin hyperfine state. Accordingly, a number of theoretical works have been dealing with multi-component vector solitons in $F = 1$ spinor BECs and a plethora of soliton pairs has also been found. In general, these systems appear to be non-integrable and as such researchers have often relied on perturbation based techniques of related integrable systems to study solitons and their evolution.

Here, the complete IST with non-zero boundary conditions at infinity is developed for a matrix nonlinear Schrödinger-type equation which has been proposed as a model to describe these hy-

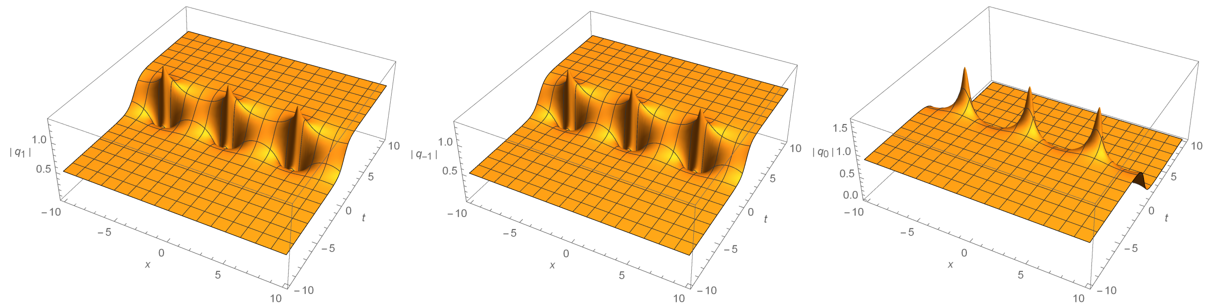


Figure 6: Three components (q_1, q_{-1} and q_0 from left to right) with $Q_+ = I_2$, $\zeta_1 = 1/2 + \sqrt{3}/2 \in C_o$ and the entries of C_1 in (4.3) are chosen to be $\gamma_1 = \gamma_{-1} = \gamma_0 = 1$ ($\det C_1 = 0$). As in Fig. 5, the solution is homoclinic in t and periodic in x.

perfine spin $F = 1$ spinor Bose-Einstein condensates with either repulsive interatomic interactions and anti-ferromagnetic spin-exchange interactions (self-defocusing case), or attractive interatomic interactions and ferromagnetic spin-exchange interactions (self-focusing case). The general expression of one-soliton solutions for the self-focusing equation is also presented, both in the case when the norming constant has rank 1 (ferromagnetic states) and when it is full rank (polar states). Finally, reductions to some special cases not previously discussed in the literature are derived. These correspond to soliton solutions obtained when the discrete eigenvalue is purely imaginary (analog of the Kuznetsov-Ma breather in the scalar case), yielding a solution that is periodic in t and homoclinic in x, as well as the limiting value of the soliton solution when the discrete eigenvalue is taken on the cut. In this latter case, which is the analog in the spinor system of the Akhmediev breather, the solution is homoclinic in t and periodic in x.

Acknowledgements

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A On the boundary condition Q_+

In this Appendix we will show that one can indeed choose without loss of generality the boundary value for $Q(x,t)$ as $x \to +\infty$ to be $Q_+ = k_o I_2$ up to a unitary transformation, and also provide an explicit construction for such transformation.

The matrix equation (1.1) is invariant if Q is multiplied from the left and from the right by arbitrary constant (i.e., independent of x and t) unitary matrices A and B. Therefore, starting from $Q(x,t)$ with arbitrary boundary condition Q_+ as $x \to +\infty$, one can define a new potential matrix $\hat{Q}(x,t) = A\,Q(x,t)\,B$ satisfying the same nonlinear PDE and such that $\hat{Q}_+ = k_o I_2$. This obviously implies that $A\,Q_+ B = k_o I_2$, i.e., $B = Q_+^\dagger A^\dagger / k_o$, and consequently $\hat{Q}(x,t) = A\,Q(x,t)\,(Q_+^\dagger/k_o)A^\dagger$. Note, however, that one needs to also impose require that $\hat{Q}(x,t)$ be symmetric (as $Q(x,t)$ is), which then implies that A must be such that

$$A^*(Q_+^\dagger/k_o)Q(x,t)\,A^T = A\,Q(x,t)\,(Q_+^\dagger/k_o)\,A^\dagger\,,$$

or equivalently

$$Q(x,t) = A^\dagger A^*(Q_+^\dagger/k_o)Q(x,t)\,A^T A\,(Q_+/k_o)\,. \tag{A.1}$$

Now note that since A is unitary $A^T A$ is unitary and symmetric, and the same is true for both Q_+/k_o and $Q_+^\dagger/k_o$. Clearly, (A.1) is satisfied if

$$A^T A = Q_+^\dagger/k_o\,, \tag{A.2}$$

and the existence of such A is guaranteed by Takagi's factorization algorithm.

B Trace formula

By trace formula we mean an expression for the analytic entry/entries of the scattering matrices $a(z)$ and $\bar{a}(z)$ in terms of scattering data (in the scalar case: discrete eigenvalues and reflection coefficient, see for instance [30]). As shown in [12, 4], in the case of NZBCs this trace formula also provides a relationship (usually referred to as θ-condition) between the asymptotic phase difference of the potential and the scattering data. In the vector/matrix case, similar formulas can be obtained for $\det a(z)$ and $\det \bar{a}(z)$ (whose zeros provide the discrete eigenvalues), while in general the problem of reconstructing the full matrices $a(z)$ and $\bar{a}(z)$ is significantly more complicated, as it requires the solution of a matrix factorization problem such as (2.33) or (2.47). Also, unlike their determinants (see below), $a(z)$ and $\bar{a}(z)$ depend not only on discrete eigenvalues and reflection coefficients, but also explicitly on the norming constants (see, for instance, [7] where this problem was solved in the reflectionless case for the scattering problem associated with the Manakov system). In the following we will derive the trace formula for $\det a(z)$, from which $\det \bar{a}(z)$ can be obtained by symmetry, in the focusing case. This will also provide a weak version of the θ-condition, establishing a relationship between the asymptotic phases of $\det Q_+$ and $\det Q_-$ and the spectral data. Analogous results for the defocusing case can be obtained in a similar way, and the construction is simplified by the fact that the eigenvalues/zeros appear in complex conjugate pairs, instead of quartets (see (2.52)).

Simple zeros. Let us first consider the case where all discrete eigenvalues are simple zeros of $\det a(z)$, i.e., $\det a(z_n) = 0$ and $(\det a)'(z_n) \neq 0$. [In this case the corresponding norming constants are of rank one, i.e., $\det C_n = 0$].

$\det a(z)$ is analytic in D^+ with zeros at $z = z_n$ and $z = -k_o^2/z_n^*$; $\det \bar{a}(z)$ is analytic in D^- with zeros at $z = z_n^*$ and $z = -k_o^2/z_n$. Therefore, we can define

$$\tilde{\alpha}^+(z) = \det a(z) \prod_{n=1}^{\mathcal{N}} \frac{(z - z_n^*)(z + k_o^2/z_n)}{(z - z_n)(z + k_o^2/z_n^*)}\,, \qquad z \in D^+\,,$$

$$\tilde{\alpha}^-(z) = \det \bar{a}(z) \prod_{n=1}^{\mathcal{N}} \frac{(z - z_n)(z + k_o^2/z_n^*)}{(z - z_n^*)(z + k_o^2/z_n)}\,, \qquad z \in D^-\,,$$

where $\tilde{\alpha}^\pm(z)$ are analytic in $D^\pm$, respectively, and they don't have zeros. Moreover, from (2.77) it follows that $\alpha^\pm(z) \to 1$ as $z \to \infty$ in the proper region, and from (2.33) and (2.32) for $\sigma = -1$ one has

$$\tilde{\alpha}^+(z)\tilde{\alpha}^-(z) = \det[I_2 + \rho^\dagger(z^*)\rho(z)]^{-1}\,, \qquad z \in \Sigma\,,$$

or equivalently

$$\log \tilde{\alpha}^{+}(z) - \log(1/\tilde{\alpha}^{-}(z)) = -\log\det[I_2 + \rho^\dagger(z^*)\rho(z)]\,, \qquad z \in \Sigma\,.$$

The latter is nothing but a RHP for $\tilde{\alpha}^{\pm}(z)$ across the contour Σ, and applying the Cauchy projectors introduced in Section 3.1 one can write the formal solution of the RHP as follows:

$$\log \tilde{\alpha}^{\pm}(z) = \mp\frac{1}{2\pi i}\int_\Sigma \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta - z}\,, \qquad z \in D^{\pm}\,.$$

So, we obtain a weak form of the trace formula:

$$\det a(z) = \exp\left\{-\frac{1}{2\pi i}\int_\Sigma \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta - z}\right\}\prod_{n=1}^{\mathcal{N}}\frac{(z - z_n)(z + k_o^2/z_n^*)}{(z - z_n^*)(z + k_o^2/z_n)}\,. \tag{B.1}$$

From (2.78) one can compute the behavior of $\det a(z)$ as $z \to 0$:

$$\det a(z) \sim \frac{1}{k_o^4}\det Q_+ \det Q_-^\dagger\,, \qquad z \to 0\,.$$

Combining (B.1) and the above asymptotic behavior, we find

$$\det Q_+ \det Q_-^\dagger = k_o^4 \exp\left\{-\frac{1}{2\pi i}\int_\Sigma \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta}\right\}\prod_{n=1}^{\mathcal{N}} \mathrm{e}^{4i\delta_n}\,,$$

where δ_n denotes the phase of the discrete eigenvalue z_n, i.e., $z_n = |z_n|\mathrm{e}^{i\delta_n}$.

From the constraint (1.4) on the boundary conditions, it follows that $|\det Q_\pm| = k_o^2$; if we then write

$$\det Q_+ = k_o^2 \mathrm{e}^{i\theta_+}\,, \qquad \det Q_- = k_o^2 \mathrm{e}^{i\theta_-}\,,$$

we obtain a weak form of the θ-condition:

$$\theta_+ - \theta_- = \frac{1}{2\pi}\int_\Sigma \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta} + 4\sum_{n=1}^{\mathcal{N}}\delta_n\,. \tag{B.2}$$

Double zeros. Next, we consider the case where all discrete eigenvalues are double zeros, i.e., $\det a(z_n) = (\det a)'(z_n) = 0$ and $\det''(z_n) \neq 0$. In this case, we introduce

$$\tilde{\alpha}^{+}(z) = \det a(z)\prod_{n=1}^{\mathcal{N}}\frac{(z - z_n^*)^2(z + k_o^2/z_n)^2}{(z - z_n)^2(z + k_o^2/z_n^*)^2}\,, \qquad z \in D^{+}\,,$$

$$\tilde{\alpha}^{-}(z) = \det \bar{a}(z)\prod_{n=1}^{\mathcal{N}}\frac{(z - z_n)^2(z + k_o^2/z_n^*)^2}{(z - z_n^*)^2(z + k_o^2/z_n)^2}\,, \qquad z \in D^{-}\,,$$

where again by construction $\tilde{\alpha}^{\pm}$ do not have zeros. Proceeding as before we get:

$$\det a(z) = \exp\left\{-\frac{1}{2\pi i}\int_\Sigma \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta - z}\right\}\prod_{n=1}^{\mathcal{N}}\frac{(z - z_n)^2(z + k_o^2/z_n^*)^2}{(z - z_n^*)^2(z + k_o^2/z_n)^2}\,. \tag{B.3}$$

and

$$\theta_+ - \theta_- = \frac{1}{2\pi}\int_{\Sigma_o} \log\det[I_2 + \rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta} - 8\sum_{n=1}^{\mathcal{N}}\delta_n\,. \tag{B.4}$$

Of course, one can easily combine the two cases to account for both simple and double zeros. Let $\{z_n\}_{n=1}^{\mathcal{N}_1}$ be all the simple zeros of $\det a(z)$, and let $\{\breve{z}_n\}_{n=1}^{\mathcal{N}_2}$ be all the double zeros of $\det a(z)$. Then the trace formula and the θ-condition can be written as:

$$\det a(z) = \exp\left\{ -\frac{1}{2\pi i}\int_\Sigma \log\det[I_2+\rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta - z}\right\} \\ \times \prod_{n=1}^{\mathcal{N}_1}\frac{(z-z_n)(z+k_o^2/z_n^*)}{(z-z_n^*)(z+k_o^2/z_n)}\prod_{n=1}^{\mathcal{N}_2}\frac{(z-\breve{z}_n)^2(z+k_o^2/\breve{z}_n^*)^2}{(z-\breve{z}_n^*)^2(z+k_o^2/\breve{z}_n)^2}\,. \tag{B.5}$$

$$\theta_+ - \theta_- = \frac{1}{2\pi}\int_\Sigma \log\det[I_2+\rho^\dagger(\zeta^*)\rho(\zeta)]\frac{\mathrm{d}\zeta}{\zeta} - 4\sum_{n=1}^{\mathcal{N}_1}\delta_n - 8\sum_{n=1}^{\mathcal{N}_2}\breve{\delta}_n\,, \tag{B.6}$$

where $z_n = |z_n|\,e^{i\delta_n}$ and $\breve{z}_n = |\breve{z}_n|\,e^{i\breve{\delta}_n}$.

C Asymptotics of the one-soliton solutions as $x \to -\infty$

In this Appendix we will analyze the large-x behavior of the one soliton solutions derived in Section 4. Recall that $\theta(\zeta_j) = \lambda(\zeta_j)(x + 2k(\zeta_j)t)$, and since

$$\mathrm{Im}(\lambda(\zeta_j)) = -\,\mathrm{Im}(\lambda(\zeta_j^*)) = \frac{1}{2}(\mathrm{Im}\,\zeta_j)\left(1 - \frac{k_o^2}{|\zeta_j|^2}\right)$$

and for $\zeta_j \in D^+$ the coefficient is positive, it follows that $e^{2i\theta(\zeta_j)}$ and $e^{-2i\theta(\zeta_j^*)}$ decay exponentially as $x \to +\infty$, and grow exponentially as $\to -\infty$.

Then as $x \to +\infty$ from (4.7a) it follows that $\Delta_1 \to 1$, and all of $A_1, A_2, B_1, B_2, X_1, X_2 \to I_2$, so that $Q(x,t) \to Q_+$.

On the other hand, the behavior as $x \to -\infty$ depends on whether $\det C_1 = 0$ or $\det C_1 \neq 0$. Specifically, as $x \to -\infty$ one has:

$$\Delta_1 \sim \begin{cases} \dfrac{ik_o\,\mathrm{trace}(C_1^\dagger)}{(\zeta_1^*)^2+k_o^2}\,e^{-2i\theta(\zeta_1^*)} & \text{if } \det C_1 = 0 \\[2ex] -\dfrac{k_o^2\,\det(C_1^\dagger)}{((\zeta_1^*)^2+k_o^2)^2}\,e^{-4i\theta(\zeta_1^*)} & \text{if } \det C_1 \neq 0 \end{cases}$$

$$A_1 \sim \begin{cases} I_2 + \dfrac{\zeta_1^2+k_o^2}{\zeta_1(\zeta_1^*-\zeta_1)\mathrm{trace}(C_1)}\,C_1 & \text{if } \det C_1 = 0 \\[2ex] I_2 + \dfrac{\zeta_1^2+k_o^2}{\zeta_1(\zeta_1^*-\zeta_1)}\,I_2 \equiv \dfrac{|\zeta_1|^2+k_o^2}{\zeta_1(\zeta_1^*-\zeta_1)}I_2 & \text{if } \det C_1 \neq 0 \end{cases}$$

$$A_2 \sim \begin{cases} I_2 + \dfrac{\zeta_1((\zeta_1^*)^2+k_o^2)}{k_o^2(\zeta_1^*-\zeta_1)\mathrm{trace}(C_1^\dagger)}\,C_1^\dagger & \text{if } \det C_1 = 0 \\[2ex] I_2 + \dfrac{\zeta_1((\zeta_1^*)^2+k_o^2)}{k_o^2(\zeta_1^*-\zeta_1)}\,I_2 \equiv \dfrac{\zeta_1^*(|\zeta_1|^2+k_o^2)}{k_o^2(\zeta_1^*-\zeta_1)}I_2 & \text{if } \det C_1 \neq 0 \end{cases}$$

$$B_1 \sim \begin{cases} \dfrac{ik_o}{(\zeta_1^*)^2+k_o^2}\, C_1^\dagger \left[I_2 - \dfrac{|\zeta_1^2+k_o^2|^2}{k_o^2(\zeta_1^*-\zeta_1)^2 \mathrm{trace}(C_1)} C_1 \right] e^{-2i\theta(\zeta_1^*)} & \text{if } \det C_1 = 0 \\[2ex] -i\dfrac{(|\zeta_1|^2+k_o^2)^2}{k_o(\zeta_1^*-\zeta_1)^2((\zeta_1^*)^2+k_o^2)}\, C_1^\dagger e^{-2i\theta(\zeta_1^*)} & \text{if } \det C_1 \neq 0 \end{cases}$$

$$B_2 \sim \begin{cases} \dfrac{-ik_o}{\zeta_1^2+k_o^2} C_1 \left[I_2 - \dfrac{|\zeta_1^2+k_o^2|^2}{k_o^2(\zeta_1^*-\zeta_1)^2 \mathrm{trace}(C_1^\dagger)} C_1^\dagger \right] e^{2i\theta(\zeta_1)} & \text{if } \det C_1 = 0 \\[2ex] i\dfrac{(|\zeta_1|^2+k_o^2)^2}{k_o(\zeta_1^*-\zeta_1)^2(\zeta_1^2+k_o^2)}\, C_1 e^{2i\theta(\zeta_1)} & \text{if } \det C_1 \neq 0 \end{cases}$$

If $\det C_1 \neq 0$, the above asymptotics allow one to compute the asymptotics of $X_1 = A_1 B_1^{-1}$ and $X_2 = A_2 B_2^{-1}$ as $x \to -\infty$, and verify that they are indeed exponentially decaying. On the other hand, the potential is given by

$$Q(x,t) = k_o I_2 - iA_1 B_1^{-1} C_1^\dagger e^{-2i\theta(\zeta_1^*)} + \frac{ik_o^2}{\zeta_1^2} A_2 B_2^{-1} C_1 e^{2i\theta(\zeta_1)}$$

and so if $\det C_1 \neq 0$ in the limit $x \to -\infty$ one has:

$$Q(x,t) \sim Q_- \equiv e^{-4i\delta} k_o\, I_2 \tag{C.1}$$

where δ is the phase of the eigenvalues ζ_1, i.e., $\zeta_1 = |\zeta_1| e^{i\delta}$. This shows that for any polar state one always has Q_- and Q_+ differing by an overall phase.

On the other hand, if $\det C_1 = 0$ one cannot use the above asymptotics for B_1 and B_2 to compute B_1^{-1} and B_2^{-1} because C_1 and $C_1^\dagger$ are not invertible. From the expressions of B_1 and B_2 it follows

$$B_1^{-1} = \frac{1}{\beta}\left[I_2 + e^{-2i\theta(\zeta_1^*)} \mathrm{cof}\left[\frac{ik_o}{(\zeta_1^*)^2+k_o^2} I_2 - \frac{1}{(\zeta_1^*-\zeta_1)^2} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1 \right] \mathrm{cof}(C_1^\dagger) \right],$$

$$B_2^{-1} = \frac{1}{\beta^*}\left[I_2 + e^{2i\theta(\zeta_1)} \mathrm{cof}\left[-\frac{ik_o}{\zeta_1^2+k_o^2} I_2 - \frac{1}{(\zeta_1^*-\zeta_1)^2} \frac{e^{-2i\theta(\zeta_1^*)}}{\Delta_1} C_1^\dagger \right] \mathrm{cof}(C_1) \right],$$

$$\beta = \det B_1 = 1 + e^{-2i\theta(\zeta_1^*)} \mathrm{trace}\left[\frac{ik_o}{(\zeta_1^*)^2+k_o^2} C_1^\dagger - \frac{1}{(\zeta_1^*-\zeta_1)^2} \frac{e^{2i\theta(\zeta_1)}}{\Delta_1^*} C_1^\dagger C_1 \right].$$

It is important to note that in this case B_1^{-1} and B_2^{-1} are $O(1)$ as $x \to -\infty$, and so are X_1 and X_2. However, in spite of this their contribution to the potential is still finite because

$$B_1^{-1} C_1^\dagger = C_1^\dagger/\beta, \qquad B_2^{-1} C_1 = C_1/\beta^*,$$

as a consequence of the fact that $\mathrm{cof}(C_1)C_1 = \mathrm{cof}(C_1^\dagger)C_1^\dagger = 0$ when $\det C_1 = 0$. Then as $x \to -\infty$:

$$Q_- = Q_+ + \beta_o (C_1 C_1^\dagger + C_1^\dagger C_1) - iC_1^\dagger/\beta_- + i(k_o^2/\zeta_1^2) C_1/\beta_-^*, \tag{C.2}$$

with

$$\beta_o = -i \frac{\zeta_1^2+k_o^2}{\zeta_1(\zeta_1^*-\zeta_1)\beta_- \mathrm{trace}(C_1)} \equiv \frac{i}{\zeta_1} \frac{(\zeta_1^*)^2+k_o^2}{(\zeta_1^*-\zeta_1)\beta_-^* \mathrm{trace}(C_1^\dagger)}, \tag{C.3}$$

$$\beta_- = \mathrm{trace}\left[\frac{i}{(\zeta_1^*)^2+1}C_1^\dagger - i\frac{\zeta_1^2+1}{(\zeta_1^*-\zeta_1)^2\mathrm{trace}(C_1)}C_1^\dagger C_1\right]. \tag{C.4}$$

So for ferromagnetic states Q_- in general is not diagonal, and the individual densities will have different amplitudes as $x \to -\infty$ and as $x \to +\infty$, obviously still in accordance with (1.5).

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