The editorial version of the paper can be found at IEEE TRANSACTIONS ON AUTOMATIC CONTROL, vol. 61, p. 2708-2713, ISSN: 0018-9286, doi: 10.1109/TAC.2015.2506990

Consensus-based control for a network of diffusion PDEs with boundary local interaction

Alessandro Pilloni, Alessandro Pisano, Yury Orlov and Elio Usai

***Abstract*—In this paper the problem of driving the state of a network of identical agents, modeled by boundary-controlled heat equations, towards a common steady-state profile is addressed. Decentralized consensus protocols are proposed to address two distinct problems. The first problem is that of steering the states of all agents towards the same constant steady-state profile which corresponds to the spatial average of the agents initial condition. The second problem deals with the case where the controlled boundaries of the agents dynamics are corrupted by additive persistent disturbances. To achieve synchronization between agents, while completely rejecting the effect of the boundary disturbances, a nonlinear sliding-mode based consensus protocol is proposed. Simulation results are presented to support the effectiveness of the proposed algorithms.**

***Index Terms*—Average consensus, Synchronization, Heat equation, Boundary control, Sliding mode control.**

I. Introduction

The consensus problem seeks to enforce agreement amongst the states of networked dynamical systems by penalizing their local disagreement with the neighboring nodes in a dynamic manner. The reader should refer, e.g., to [1], [2] for tutorial overviews of consensus-based control in the finite-dimensional setup. The consensus problem for a network of agents modeled as distributed parameter systems (DPSs), however, has not received yet the same level of attention than its finite-dimensional counterpart.

The following papers [3], [4], [5], [6], [7], [8], [9], [10], which have investigated different aspects of consensus and synchronization in the DPSs setting, are worth to mention.

In [10], which appears to be more closely related to the present investigation among the existing references, some noticeable results are attained concerning the synchronization and consensus problems for networks of agents modeled by a class of parabolic PDEs. It should be noted however that in [10] the common steady-state profile of the agents is not established, and furthermore no perturbations are allowed to affect the agents dynamics.

The present work aims to address consensus and synchronization problems for a network of dynamical agents, communicating through an undirected and connected static topology, provided that agents dynamics are governed by a class of diffusion PDEs with Neumann-type boundary actuation. The contribution of the paper is twofold. Firstly, a linear local interaction strategy is proposed. With this strategy, it is shown that the agents states eventually converge pointwise-in-space towards a common constant distribution whose value is given by the spatial average of the agents' initial conditions. Thus, the well-known average consensus algorithm is generalized from a network of integrators to the infinite-dimensional setting of networked heat processes.

Secondly, the more complex scenario where the agents dynamics are perturbed by a class of boundary disturbances, is considered. Based on the second-order sliding-mode control approach [11], a nonlinear protocol is developed to extend the results of [12], [13] from a network of double integrators to the infinite-dimensional framework of networked PDEs. A dynamic input extension, similar to that presented in [14] for stabilizing a unique perturbed diffusion PDE, results in continuous boundary control actions thereby alleviating chattering and yielding another step beyond [12]. It is demonstrated that the proposed nonlinear local interaction protocol enforces the asymptotic synchronization between the agents states while rejecting the persistent matching boundary perturbations.

The motivation to the present investigation comes, e.g., from networked systems of perturbed heat equations that can occur in modeling and controlling industrial furnaces. Heating of certain industrial furnaces (see, e.g., [15, Sect. 1.A]) is made through electrically heated bars aiming to enforce a uniform temperature distribution inside the furnace. Considering these bars as a network of heaters and applying collaborative consensus-based synthesis might be useful in improving the overall performance of the furnaces.

Exploiting the present results in specific application domains certainly requires additional work which is beyond the scope of the present paper.

The paper is organized as follows. In Section II some mathematical preliminaries and useful properties and definitions are recalled. The linear average consensus algorithm is presented in Section III whereas the nonlinear algorithm, providing robust synchronization in the presence of boundary perturbations, is described in Section IV. Simulation results, supporting the proposed designs, are given in Section V, and conclusions and perspectives for next investigations are collected in the final Section VI.

II. Mathematical Preliminaries and Notations

A. Useful definitions and properties

$\mathrm{H}^r(0,1)$, with $r = 0,1,2,\ldots$, denotes the Sobolev space of absolutely continuous scalar functions $z(\varsigma)$ on the domain

Manuscript received September 23, 2015; revised November 19, 2015.

A. Pilloni, A. Pisano and E. Usai are with the Department of Electrical and Electronic Engineering (DIEE), University of Cagliari, Cagliari 09123, Italy. E-mail: alessandro.pilloni@diee.unica.it; pisano@diee.unica.it; eusai@diee.unica.it.

Y. Orlov is with CICESE Research Center, Electronics and Telecommunication Department, Ensenada, Mexico. E-mail: eusai@diee.unica.it

$(0,1)$, with square integrable derivatives $z^{(k)}(\varsigma)$ up to order ℓ and the H^r-norm $\|z(\cdot)\|_{\mathrm{H}^r(0,1)} = \sqrt{\int_0^1 \sum_{k=0}^r \left[z^{(k)}(\xi)\right]^2 \, d\xi}$. Then, the notations

$$\mathrm{H}^{r,N}(0,1) = \underbrace{\mathrm{H}^r(0,1) \times \mathrm{H}^r(0,1) \times \ldots \times \mathrm{H}^r(0,1)}_{N \ \texttt{times}} \tag{1}$$

and

$$\|w(\cdot)\|_{\mathrm{H}^{r,N}(0,1)} = \sqrt{\sum_{i=1}^N \|w_i(\cdot)\|^2_{\mathrm{H}^r(0,1)}} \tag{2}$$

for the corresponding norm of the vector $w(\varsigma) = [w_1(\varsigma), \ldots, w_N(\varsigma)]^T \in H^{r,N}(0,1)$ are utilized. The simplified notation $\|z(\cdot)\|_{\mathrm{H}^r} = \|z(\cdot)\|_{\mathrm{H}^r(0,1)}$, $\|w(\cdot)\|_{\mathrm{H}^{r,N}} = \|w(\cdot)\|_{\mathrm{H}^{r,N}(0,1)}$ will be adopted throughout.

For later use, an instrumental lemma is further presented:

Lemma 1: *Let $b(\varsigma) \in \mathrm{H}^{1,N}(0,1)$. Then, the following inequality holds:*

$$\|b(\cdot)\|^2_{\mathrm{H}^{0,N}} \leq 2\left(\|b(i)\|_2^2 + \|b_\varsigma(\cdot)\|^2_{\mathrm{H}^{0,N}}\right), \ i = 0,1$$

Proof of Lemma 1: It was proven [14, Lemma 1] that, with reference to a generic scalar function $z(\varsigma) \in H^1(0,1)$, the next estimate holds:

$$\|z(\cdot)\|^2_{\mathrm{H}^0} \leq 2(z(i)^2 + \|z_\varsigma(\cdot)\|^2_{\mathrm{H}^0}), \ i = 0,1. \tag{3}$$

Now let $b(\varsigma) = [b_1(\varsigma), b_2(\varsigma), \ldots, b_N(\varsigma)]^T$ and $b_\varsigma(\varsigma) = [b_{\varsigma 1}(\varsigma), b_{\varsigma 2}(\varsigma), \ldots, b_{\varsigma N}(\varsigma)]^T$ where $b_k(\varsigma) \in \mathrm{H}^1(0,1)$ $\forall k = 1,2,\ldots,N$. By applying definition (2), the following chain of relations is derived by virtue of (3) specified with $z(\cdot) = b_k(\cdot)$:

$$\begin{aligned}\|b(\cdot)\|^2_{\mathrm{H}^{0,N}} &= \sum_{j=1}^N \|b_j(\cdot)\|^2_{\mathrm{H}^0} \leq 2\sum_{j=1}^N \left(b_j(i)^2 + \|b_{\varsigma j}(\cdot)\|^2_{\mathrm{H}^0}\right) \\ &= 2\left(\|b(i)\|_2^2 + \|b_\varsigma(\cdot)\|^2_{\mathrm{H}^{0,N}}\right), \ i = 0,1. \end{aligned} \tag{4}$$

Lemma 1 is proved. □

The identity matrix of dimension N is denoted as $\mathcal{I}_{N\times N} \in \mathbb{R}^{N\times N}$, whereas $1_N = [1,1,\ldots,1]^T \in \mathbb{R}^N$ and $0_N = [0,0,\ldots,0]^T \in \mathbb{R}^N$ stand for the all-ones and all-zeros vectors. Finally, operator $\mathrm{Sign}(x)$ stands for the vector $\mathrm{Sign}(x) = [\mathrm{sign}(x_1), \mathrm{sign}(x_2), \ldots, \mathrm{sign}(x_N)]^T$.

B. Algebraic Graph Theory definitions and properties

We consider a set of N dynamical agents along with an undirected static communication topology represented by the graph $\mathcal{G}(\mathcal{V},\mathcal{E})$, where $\mathcal{V} = \{1,\ldots,N\}$ is the set of vertices representing agents and $\mathcal{E} \subseteq \{\mathcal{V} \times \mathcal{V}\}$ is the set of edges representing the information flow among the agents. The topological structure of $\mathcal{G}$ is encoded in the so-called *Laplacian Matrix* $\mathcal{L} \in \mathbb{R}^{N\times N}$ [2]. For undirected connected graphs, the matrix $\mathcal{L}$ is symmetric and positive semi-definite [2], the properties

$$\mathcal{L}1_N = \mathcal{L}^T 1_N = 0_N \tag{5}$$

hold by construction, and the corresponding eigenvalues λ_i, $i \in \mathcal{V}$, are such that $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$. The smallest nonzero eigenvalue λ_2 is known as *algebraic connectivity* of $\mathcal{G}$.

Next lemma presents useful properties of vector norms involving the Laplacian matrix of the graph.

Lemma 2: *For an undirected connected graph with Laplacian matrix $\mathcal{L}$, and with reference to any vector $x \in \mathbb{R}^N$ such that $1_N^T x = 0$, the next relations are in force*

$$\lambda_N ||x||_2^2 \geq x^T \mathcal{L} x \geq \lambda_2 ||x||_2^2 \tag{6}$$

$$\lambda_N^2 ||x||_2^2 \geq \|\mathcal{L}x\|_2^2 \geq \lambda_2^2 ||x||_2^2 \tag{7}$$

$$\|\mathcal{L}x\|_1 \geq \lambda_2 \|x\|_2 \tag{8}$$

Proof of Lemma 2: The left inequality in (6) comes from well-known properties of quadratic norms. The right inequality in (6) was proven in [2, Th. 3]. To derive (7), observe that $\|\mathcal{L}x\|_2 = \sqrt{x^T \mathcal{L}^2 x}$. The eigenvalues $\{0, \lambda_2^2, \lambda_3^2, \ldots, \lambda_N^2\}$ of $\mathcal{L}^2$ are straightforwardly derived by squaring those of $\mathcal{L}$. Thus, the left inequality of (7) follows from well-known properties of quadratic norms. Additionally, $\mathcal{L}^2$ is symmetric and such that $\mathcal{L}^2 1_N = 0_N$, thus 1_N is the eigenvector associated to the zero eigenvalue of $\mathcal{L}^2$. Therefore, the right inequality of (7) follows from the Courant-Fisher Theorem that can be found, e.g., in [16]. To reproduce (8), it suffices to note that $\|\mathcal{L}\boldsymbol{x}\|_1 \geq \|\mathcal{L}\boldsymbol{x}\|_2$ (see [17, Appendix A]) and then, by applying (7), to derive that $\|\mathcal{L}\boldsymbol{x}\|_2 \geq \lambda_2 ||\boldsymbol{x}||_2$. Lemma 2 is proved. □

III. Average consensus for networked heat processes

A network of N dynamical agents whose communication topology is described by an undirected connected static graph $\mathcal{G}(\mathcal{V},\mathcal{E})$ is under study. The i-th agent has state $Q_i(\varsigma,t)$, $i \in \mathcal{V}$, with the spatial variable $\varsigma \in (0,1)$ and time variable $t \geq 0$. Let $Q(\varsigma,t) = \left[Q_1(\varsigma,t), Q_2(\varsigma,t), \ldots, Q_N(\varsigma,t)\right]^T$ be the vector collecting the states of all agents, and let the dynamics of $Q(\varsigma,t)$ be governed by the vector heat equation

$$Q_t(\varsigma,t) = \theta \cdot Q_{\varsigma\varsigma}(\varsigma,t), \tag{9}$$

The scalar parameter $\theta \in \mathbb{R}^+$ is a positive unknown coefficient, called "diffusivity parameter", which is supposed to be identical for all agents. Throughout, Neumann-type Boundary Conditions (BCs) of the form

$$Q_\varsigma(0,t) = 0, \qquad Q_\varsigma(1,t) = U(t), \tag{10}$$

are considered, where $U(t) = \left[u_1(t), u_2(t), \ldots, u_N(t)\right]^T \in \mathbb{R}^N$ is a modifiable source term (boundary vector control input).

The Initial Conditions (ICs) are

$$Q(\varsigma,0) = Q_0(\varsigma) \tag{11}$$

To deal with classical solutions of class $\mathrm{H}^{2,N}(0,1)$, the admissible initial functions are specified by the next assumption.

Assumption 1: *The initial function $Q_0(\varsigma)$ in the ICs* (11) *is assumed to be of class* $\mathrm{H}^{2,N}(0,1)$ *and compatible to the BCs* $Q_{0\varsigma}(0) = 0$ *and* $Q_{0\varsigma}(1) = U(0)$.

The objective of the present section is to introduce a linear local interaction strategy providing closed-loop stability and the point-wise consensus condition

$$\lim_{t\to\infty} Q(\varsigma,t) = Q^* \cdot 1_N, \quad \forall \varsigma \in (0,1), \tag{12}$$

where the constant

$$Q^* = \frac{1}{N}\int_0^1 1_N^T Q_0(\varsigma)d\varsigma \tag{13}$$

corresponds to the spatial averaging of the agents initial conditions.

To achieve the control goal, the local interaction protocol

$$U(t) = -\mathcal{L}Q(1,t) \tag{14}$$

is proposed. Under the assumptions, imposed on the ICs and BCs, the well-posedness of the system in question is straightforwardly verified by applying [18, Theorem 2.1.10] to the classical solutions of the homogeneous linear Boundary-Value Problem (BVP) (9)-(11), (14).

We are now in a position to state the first main result of this paper.

Theorem 1: Consider the multi-agent system (9)-(11), with Assumption 1, communicating through an undirected connected static graph with Laplacian matrix $\mathcal{L}$. Let it be subject to the boundary local interaction control strategy (14). Then, the closed-loop system is stable in the space $H^2(0,1)$ and the average consensus condition (12)-(13) is achieved. □

Proof of Theorem 1: Let us note that the eigenspace of the closed-loop BVP (9)-(11), (14), associated with the zero eigenvalue, is one dimensional and it is spanned by the uniform distribution $Q(\varsigma) = 1_N$, whereas the remaining eigenvalues are strictly negative (see [19] for details).

Furthermore, one observes that the projection

$$\frac{1}{N}\int_0^1 1_N^T Q(\xi,t)d\xi\, 1_N \tag{15}$$

of the solution of the closed-loop system to the eigenspace, associated to the zero eigenvalue, remains constant, whereas all the remaining modes tend to zero because the other eigenvalues are strictly negative. It follows that the state $Q(\varsigma,t)$ eventually converges point-wise to the constant spatial distribution

$$\frac{1}{N}\int_0^1 1_N^T Q_0(\xi)d\xi\, 1_N = Q^* \cdot 1_N \tag{16}$$

thereby establishing relations (12)-(13). This completes the proof of Theorem 1. □

IV. Robust synchronization for networked heat processes with perturbations

A perturbed version of the BVP (9)-(11), with the only difference in the BCs (10) which now take the perturbed form

$$Q_\varsigma(0,t) = 0, \quad Q_\varsigma(1,t) = U(t) + \Psi(t), \tag{17}$$

is under investigation, where $\Psi(t) = \left[\psi_1(t), \psi_2(t), \ldots, \psi_N(t)\right]^T \in \mathbb{R}^N$ represents an uncertain, sufficiently smooth, persistent disturbance.

The class of admissible ICs and disturbances is specified by the next assumption.

Assumption 2: *The initial function $Q_0(\varsigma)$ is assumed to be of class $\mathrm{H}^{4,N}(0,1)$ and compatible to the perturbed BCs $Q_{0\varsigma}(0) = 0$, $Q_{0\varsigma}(1) = \Psi(0)$, whereas the disturbance $\Psi(t)$ is supposed to be twice continuously differentiable, and there exists an* a-priori *known constant $\Pi > 0$ such that*

$$\left\|\dot{\Psi}(t)\right\|_\infty \leq \Pi. \tag{18}$$

Note that for technical reasons higher degree of smoothness of the ICs is required in the present perturbed scenario. The objective of the present section is to develop a local interaction strategy providing the attainment of the synchronization condition

$$\lim_{t\to\infty}\left|Q_i(\varsigma,t) - Q_j(\varsigma,t)\right| = 0, \quad \forall\, i,j \in \mathcal{V}, \forall \varsigma \in (0,1), \tag{19}$$

despite the presence of the uncertain boundary disturbance $\Psi(t)$ of arbitrary shape and possibly unbounded in magnitude.

To achieve the control goal, the following dynamic local interaction protocol

$$\dot{U}(t) = \dot{U}_1(t) + \dot{U}_2(t) \tag{20}$$

$$\dot{U}_1(t) = -a\mathrm{Sign}(\mathcal{L}Q(1,t)) - b\mathrm{Sign}(\mathcal{L}Q_t(1,t)) \tag{21}$$

$$\dot{U}_2(t) = -W_1 \cdot \mathcal{L}Q(1,t) - W_2 \cdot \mathcal{L}Q_t(1,t) - W_3 \cdot Q_t(1,t) \tag{22}$$

$$U_1(0) = U_2(0) = 0_N. \tag{23}$$

is proposed, where a, b, W_1, W_2 and W_3 are nonnegative tuning constants.

It is worth to note that the discontinuities affect the time derivative of the boundary control vector, whereas the boundary control signal is smoothed by passing these discontinuities through an integrator, thereby alleviating chattering. By virtue of this dynamic extension, the system state is augmented by Q_t being viewed as a component of the augmented state vector $(Q, Q_t) \in \mathrm{H}^{4,N}(0,1) \times \mathrm{H}^{2,N}(0,1)$ which is particularly why the initial function $Q_0(\varsigma)$ was assumed to be of class $\mathrm{H}^{4,N}(0,1)$.

The well-posedness of the underlying closed-loop system, under the assumptions, imposed on the ICs and BCs, is actually verifiable in accordance with [18, Theorem 3.3.3] by taking into account that the dynamic local interaction rule (20)-(22) is twice piece-wise continuously differentiable along the state trajectories. Thus, in the remainder, it is assumed the following:

Assumption 3: *The closed loop networked system* (9)-(11), (17), (20)-(23) *possesses a unique Filippov solution $Q(\cdot,t) \in \mathrm{H}^{4,N}(0,1)$ and its time derivative $Z(\cdot,t) = Q_t(\cdot,t) \in \mathrm{H}^{2,N}(0,1)$ verifies the auxiliary boundary-value problem*

$$Z_t(\varsigma,t) = \theta Z_{\varsigma\varsigma}(\varsigma,t) \tag{24}$$

$$Z_\varsigma(0,t) = 0, \qquad Z_\varsigma(1,t) = \dot{U}(t) + \dot{\Psi}(t), \tag{25}$$

$$Z(\varsigma,0) = \theta Q_{0\varsigma\varsigma}(\varsigma) \in \mathrm{H}^{2,N}(0,1). \tag{26}$$

Extension of the Filippov solution concept towards the infinite dimensional setting can be found, e.g., in [20]. Notice that (24)-(25) are formally obtained by differentiating (9)-(11), (17), in the time variable t, whereas the IC (26) is straightforwardly derived from (9) and (11).

The following distributed disagreement vectors

$$\delta_1(\cdot,t) = \mathcal{L}_\mathcal{C}\, Q(\cdot,t), \quad \delta_2(\cdot,t) = \mathcal{L}_\mathcal{C}\, Q_t(\cdot,t) \tag{27}$$

$$\mathcal{L}_\mathcal{C} = \left(\mathcal{I}_{N\times N} - \frac{\mathbf{1}_N \cdot \mathbf{1}_N^T}{N}\right), \tag{28}$$

will be considered in the present investigation for analysis purposes. The BVP governing the dynamics of the disagreement vectors reads as

$$\begin{aligned}&\delta_{1t}(\varsigma,t)=\delta_2(\varsigma,t),\\&\delta_{2t}(\varsigma,t)=\theta\delta_{2\varsigma\varsigma}(\varsigma,t),\end{aligned}\tag{29}$$

$$\delta_{2\varsigma}(0,t)=0,\quad \delta_{2\varsigma}(1,t)=\boldsymbol{\mathcal{L}}_{\mathcal{C}}\left[\dot{U}(t)+\dot{\Psi}(t)\right],\tag{30}$$

$$\begin{aligned}&\delta_1(\varsigma,0)=\boldsymbol{\mathcal{L}}_{\mathcal{C}}Q_0(\varsigma)\in \mathrm{H}^{4,N}(0,1)\\&\delta_2(\varsigma,0)=\theta\boldsymbol{\mathcal{L}}_{\mathcal{C}}Q_{0\varsigma\varsigma}(\varsigma)\in \mathrm{H}^{2,N}(0,1)\end{aligned}\tag{31}$$

Presenting the second main result of this paper is preceded by the following instrumental lemma.

Lemma 3: The functional

$$\begin{aligned}V(\delta_1,\delta_2)=&\theta a\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1+\frac{1}{2}\theta W_1\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_2^2\\&+\frac{1}{2}\int_0^1\delta_2(\xi,t)^T\boldsymbol{\mathcal{L}}\delta_2(\xi,t)d\xi\end{aligned}\tag{32}$$

being computed on the solutions $(\delta_1(\cdot,t),\delta_2(\cdot,t))$ of the BVP (29)-(31), is equivalent to the $\mathrm{H}^{2,N}(0,1)\times\mathrm{H}^{0,N}(0,1)$ norm of these solutions in the sense that

$$\begin{aligned}&\eta_1\left(\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}^2+\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2\right)\le V(\delta_1,\delta_2)\\&\le\eta_2\left(\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}^2+\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2+\sum_{i=1}^N\|\delta_{1,i}(\cdot,t)\|_{\mathrm{H}^2}\right)\end{aligned}\tag{33}$$

for an arbitrary solution $(\delta_1(\cdot,t),\delta_2(\cdot,t))$ of (29)-(31), for all $t\ge 0$, and for some positive constants η_1 and η_2.

Proof of Lemma 3: It is preliminarily demonstrated that the condition

$$\alpha_1\cdot\tilde{V}(\delta_1,\delta_2)\le V(\delta_1,\delta_2)\le\alpha_2\cdot\tilde{V}(\delta_1,\delta_2)\tag{34}$$

holds, where α_1 and α_2 are positive constants and $\tilde{V}(\delta_1,\delta_2)=\theta a\|\delta_1(1,t)\|_1+\frac{1}{2}\theta W_1\|\delta_1(1,t)\|_2^2+\frac{1}{2}\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2$.

By considering (8) and the well-known inequality $\|x\|_1\le\sqrt{N}\,\|x\|_2$, both specialized with $x=\delta_1(1,t)$, one derives

$$\frac{\lambda_2}{\sqrt{N}}\cdot\|\delta_1(1,t)\|_1\le\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1\le\|\boldsymbol{\mathcal{L}}\|_1\|\delta_1(1,t)\|_1\tag{35}$$

Specializing (7) with $x=\delta_1(1,t)$, and (6) with $x=\delta_2(\varsigma,t)$, one obtains

$$\lambda_2^2\|\delta_1(1,t)\|_2^2\le\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_2^2\le\lambda_N^2\|\delta_1(1,t)\|_2^2\tag{36}$$

$$\lambda_2\|\delta_2(\varsigma,t)\|_2^2\le\delta_2(\varsigma,t)^T\boldsymbol{\mathcal{L}}\delta_2(\varsigma,t)\le\lambda_N\|\delta_2(\varsigma,t)\|_2^2\tag{37}$$

Noticing that, by construction, $\int_0^1\|\delta_2(\xi,t)\|_2^2d\xi=\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2$, the next estimate is derived after spatial integration of all terms in (37)

$$\lambda_2\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2\le\int_0^1\delta_2(\xi,t)^T\boldsymbol{\mathcal{L}}\delta_2(\xi,t)d\xi\le\lambda_N\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2.\tag{38}$$

By (35)-(36) and (38), relation (34) is derived with the positive constants $\alpha_1=\min\{\lambda_2/\sqrt{N},\lambda_2^2\}$ and $\alpha_2=\max\{\|\boldsymbol{\mathcal{L}}\|_1,\lambda_N^2,\lambda_N\}$. Furthermore, by (2), functional $\tilde{V}(\delta_1,\delta_2)$ can be rewritten as follows:

$$\tilde{V}(\delta_1,\delta_2)=\sum_{i=1}^N\tilde{V}_i(\delta_{1i},\delta_{2i})\tag{39}$$

where $\tilde{V}_i=\theta a|\delta_{1i}(1,t)|+\frac{1}{2}\theta W_1\delta_{1i}(1,t)^2+\frac{1}{2}\|\delta_{2i}(\cdot,t)\|_{\mathrm{H}^0}^2$. From that, by applying the estimate that was derived in [14, Lemma 2], it straightforwardly results

$$\begin{aligned}&\beta_1\left(\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}^2+\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2\right)\le\tilde{V}(\delta_1,\delta_2)\\&\le\beta_2\left(\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}^2+\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2+\sum_{i=1}^N\|\delta_{1i}(\cdot,t)\|_{\mathrm{H}^2}\right)\end{aligned}\tag{40}$$

Finally, relation (33), being specified with the constants $\eta_1=\alpha_1\beta_1$ and $\eta_2=\alpha_2\beta_2$, is straightforwardly derived by combining (34), (39) and (40). Lemma 3 is proved. □

Next theorem presents the second main result of this work.

Theorem 2: Consider the perturbed multi-agent system (9), (11), (17), with Assumptions 2 and 3, communicating through an undirected connected static graph with Laplacian matrix $\boldsymbol{\mathcal{L}}$. Let the boundary local interaction strategy (20)-(22) be applied, with the tuning parameters selected according to

$$a>b+\Pi,\quad b>\Pi,\quad W_1>0,\quad W_2>0,\quad W_3>0\tag{41}$$

Then, condition (19) is achieved. □

Proof of Theorem 2: A sketched version of the proof is given. An extended version of it, including all detailed mathematical derivations, is given in [19].

Consider the Lyapunov function (32). Its time derivative is

$$\begin{aligned}\dot{V}(t)=&\theta a\delta_2(1,t)^T\boldsymbol{\mathcal{L}}\mathrm{Sign}(\boldsymbol{\mathcal{L}}\delta_1(1,t))+\theta W_1\delta_2(1,t)^T\boldsymbol{\mathcal{L}}^2\delta_1(1,t)\\&+\int_0^1\delta_2(\xi,t)^T\boldsymbol{\mathcal{L}}\delta_{2t}(\xi,t)d\xi\end{aligned}\tag{42}$$

Substituting (29) into the last term of (42), and performing straightforward manipulations taking advantage of (6)-(8) and of relations (30) and (20)-(22), the next inequality

$$\begin{aligned}\dot{V}(t)\le&-\theta\cdot(b-\Pi)\cdot\|\boldsymbol{\mathcal{L}}\delta_2(1,t)\|_1-\theta\lambda_2\cdot\|\delta_{2\varsigma}(\cdot,t)\|_{\mathrm{H}^{0,N}}^2\\&-\theta W_2\lambda_2^2\cdot\|\delta_2(1,t)\|_2^2-\theta W_3\lambda_2\cdot\|\delta_2(1,t)\|_2^2\end{aligned}\tag{43}$$

is concluded. Due to (43) and (41), the time derivative of the Lyapunov functional $V(t)$, being computed along the solutions of the closed-loop system, is negative semi-definite, and $V(t)$ is therefore a non-increasing function of time. Thereby, the set

$$\mathscr{D}_R^V=\left\{(\delta_1,\delta_2)\in\mathrm{H}^{2,N}\times\mathrm{H}^{0,N}:V(\delta_1,\delta_2)\le R\right\}\tag{44}$$

specified for an arbitrary $R\ge V(0)$, is invariant. By exploiting the invariance of the domain $\mathscr{D}_R^V$, the next estimates are derived by straightforward manipulations of the inequality $V(\cdot)\le R$ in light of (32), (36) and (38):

$$\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1\le R/\theta a\tag{45}$$

$$\|\delta_1(1,t)\|_2^2\le 2R/\theta W_1\lambda_2^2\tag{46}$$

$$\|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2\le 2R/\lambda_2\tag{47}$$

Now consider the "augmented" functional

$$V_R(t)=V(t)+\kappa_R\cdot\bar{V}(t)\tag{48}$$

$$\bar{V}(t)=\frac{1}{2}\theta W_2\cdot\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_2^2+\int_0^1\delta_1(1,t)^T\boldsymbol{\mathcal{L}}\delta_2(\xi,t)d\xi\tag{49}$$

where κ_R is a positive constant to subsequently be specified. Taking advantage of (45) and exploiting well-known properties

of vector norms (see [17, Appendix A]) one derives that in the invariant domain $\mathscr{D}_R^V$ the augmented functional $V_R(t)$ is lower estimated in terms of $V(t)$:

$$V_R(t) \geq \min\left\{\frac{\theta a}{\theta a - \frac{\kappa_R R}{2\theta a}}, \frac{\lambda_2}{\lambda_2 - \kappa_R}, \frac{W_1}{W_1 + \kappa_R W_2}\right\} V(t). \quad (50)$$

Thus, the positive definitiveness of $V_R(t)$ is guaranteed by selecting the positive constant κ_R small enough.

Differentiating (48) along the solutions of (29)-(31), and exploiting the identity $\boldsymbol{\mathcal{L}}\boldsymbol{\mathcal{L}}_{\boldsymbol{\mathcal{C}}} = \boldsymbol{\mathcal{L}}$, it yields

$$\begin{aligned}\dot{V}_R(t) = \dot{V}(t) &+ \kappa_R \theta W_2 \delta_2(1,t) \boldsymbol{\mathcal{L}}^2 \delta_1(1,t) \\ &+ \kappa_R \int_0^1 \theta \delta_1(1,t)^T \boldsymbol{\mathcal{L}} \delta_{2,\xi\xi}(\xi,t) d\xi \\ &+ \kappa_R \int_0^1 \delta_2(1,t)^T \boldsymbol{\mathcal{L}} \delta_2(\xi,t) d\xi. \end{aligned} \quad (51)$$

Employing (20)-(22), the BCs (30), the estimations (45)-(47), Lemma 1, and the properties (6)-(8), one manipulates the right hand side of (51) to get

$$\begin{aligned}\dot{V}_R(t) \leq &- c_1 \cdot \|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1 - c_2 \cdot \|\boldsymbol{\mathcal{L}}\delta_2(1,t)\|_1 \\ &- c_3 \cdot \|\delta_1(1,t)\|_2^2 - c_4 \cdot \|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2, \end{aligned} \quad (52)$$

with the coefficients $c_1 = \kappa_R \theta (a - b - \Pi)$, $c_2 = \theta\left(b - \Pi - \kappa_R\left(\sqrt{\frac{2R}{\theta^2\lambda_2}} + \frac{W_3}{\lambda_2}\sqrt{\frac{2R}{\theta W_1}}\right)\right)$, $c_3 = \kappa_R \theta W_1 \lambda_2^2$ and $c_4 = \theta\lambda_2 \min\{1, (W_2\lambda_2 + W_3)\}$. It is clear that due to the proposed specifications of constants c_1, c_2, c_3, all terms, appearing in the right-hand side of (52), are nonpositive provided that the tuning conditions (41), imposed on the controller parameters, hold and the condition $\kappa_R \leq \min\left\{\frac{2\theta^2 a^2}{R}, \lambda_2, \frac{b-\Pi}{\sqrt{\frac{2R}{\theta^2\lambda_2}} + \frac{W_3}{\lambda_2}\sqrt{\frac{2R}{\theta W_1}}}\right\}$ is additionally satisfied. It then follows from (52) that

$$\dot{V}_R(t) \leq -\gamma_1(\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1 + \|\delta_1(1,t)\|_2^2 + \|\delta_2(\cdot,t)\|_{\mathrm{H}^{0,N}}^2) \quad (53)$$

with $\gamma_1 = \min\{c_1, c_3, c_4\} > 0$.

By (32) and (48), taking advantage of (6), (7) and (45), and exploiting well-known properties of vector norms (see [17, Appendix A]), one also obtains

$$V_R(t) \leq \gamma_2(\|\boldsymbol{\mathcal{L}}\delta_1(1,t)\|_1 + \|\delta_1(1,t)\|_2^2 + \|\delta_2(\cdot,t)\|_{[\mathrm{H}^0(0,1)]^n}^2) \quad (54)$$

where $\gamma_2 = \min\{\theta a - \frac{\kappa_R R}{2\theta a}, \frac{(\lambda_N - \kappa_R)}{2}, \frac{\theta\lambda_N^2(W_1 + \kappa_R W_2)}{2}\} > 0$. Thus, one derives from (53) and (54) that $\dot{V}_R(t) \leq -\rho_R \cdot V_R(t)$, $\rho_R = \gamma_1/\gamma_2$, thereby concluding the exponential decay of $V_R(t)$, initialized within the invariant set $\mathscr{D}_R^V$ in (44).

To complete the proof, it remains to note that due to the upper estimate (50) the functional $V(t)$ decays, too. By applying Lemma 3, the local asymptotic stability of (29)-(31) is then established in the space $H^{2,N}(0,1) \times H^{0,N}(0,1)$ for the initial set (44). Since (44) can be specified with an arbitrarily large $R > 0$, thus capturing an arbitrarily large initial domain, and the tuning conditions (41) do not depend on R, the global asymptotic stability is then concluded in the space $\mathrm{H}^{2,N}(0,1) \times \mathrm{H}^{0,N}(0,1)$. It follows from (33) that $\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}$ asymptotically vanishes too, which results in the following component-wise relations $\lim_{t\to\infty}\|\delta_{1i}(\cdot,t)\|_{\mathrm{H}^2} = 0$ $\forall i = 1,2,...,N$, where $\delta_{1i}(\varsigma,t)$ is the i-th entry of $\delta_1(\varsigma,t)$.

It is well known [21] that the Sobolev space $H^2(0,1)$ is continuously embedded in the Banach space $C(0,1)$ equipped with the supremum norm. In other words, there exists a constant $M > 0$ such that

$$sup_{\xi\in[0,1]}|\delta_{1i}(\xi,t)| \leq M\|\delta_{1i}(\cdot,t)\|_{\mathrm{H}^2}, \qquad \forall\ i \in \boldsymbol{\mathcal{V}}, \quad (55)$$

Thus, one concludes the spatially point-wise decay of all entries of $\delta_1(\cdot,t)$. This property, coupled to the identity $Q_i(\varsigma,t) - Q_j(\varsigma,t) = \delta_{1i}(\varsigma,t) - \delta_{1j}(\varsigma,t)$, yields (19). The proof of Theorem 2 is completed. □

V. Simulation Results

The connected network of $N = 10$ agents displayed in Figure 1 is considered, with the diffusivity parameter $\theta = 1$.

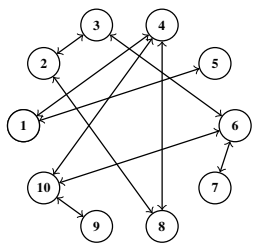

Fig. 1. The considered network topology.

A. Average consensus

System (9)-(10), coupled with the local interaction protocol (14), is under investigation. In the first simulation run (TEST 1), spatially varying ICs have been selected as follows: $Q_1(\varsigma,0) = 10 + \omega_1\cos(3\pi\varsigma)$, $Q_2(\varsigma,0) = 10 + \omega_2\cos(3\pi\varsigma)$, $Q_3(\varsigma,0) = 8 + \omega_3\cos(3\pi\varsigma)$, $Q_4(\varsigma,0) = 10 + \omega_4\cos(3\pi\varsigma)$, $Q_5(\varsigma,0) = 6 + \omega_5\cos(3\pi\varsigma)$, $Q_6(\varsigma,0) = 10 + \omega_6\cos(3\pi\varsigma)$, $Q_7(\varsigma,0) = 10 + \omega_7\cos(3\pi\varsigma)$, $Q_8(\varsigma,0) = -5 + \omega_8\cos(3\pi\varsigma)$, $Q^9(\varsigma,0) = 10 + \omega_9\cos(3\pi\varsigma)$, $Q_{10}(\varsigma,0) = 10 + \omega_{10}\cos(2.5\pi\varsigma)$, with $\omega_i = 1 + 4(i-1)/9$, $i = 1,2\ldots,10$. The corresponding spatial average Q^*, evaluated according to (13), is $Q^* = 7.9637$. Figure 2 shows the spatiotemporal evolutions of the states $Q_6(\varsigma,t)$ and $Q_{10}(\varsigma,t)$. The steady-state profile of both agents is constant and takes the expected value Q^*.

B. Robust Synchronization

The performance of the robust synchronization protocol (20)-(22), presented in Section IV, is now verified with reference to the perturbed PDEs (9), (11), (17). The entries $\psi_i(t)$ of the disturbance vector $\Psi(t)$ are selected as $\psi_i(t) = 4k_i \cdot t + \sin(k_i\pi t)$ $(i = 1,2,...,10)$, where the coefficients k_i are randomly chosen in the interval [0,2]. The considered disturbance, which is unbounded in magnitude as time grows, meets the restriction (18) with a constant upper-bound constant $\Pi = 2\pi + 8$. The chosen ICs are

$$Q_i(\varsigma,0) = 10 + (i - 4.5)\cos(4\pi\varsigma) \qquad i = 1,2,...,10.$$

The tuning parameters $a = 40, b = 20, W_1 = W_2 = W_3 = 5$, were chosen according to Theorem 2. This simulation run is referred to as TEST 2. Figure 3 shows the spatiotemporal

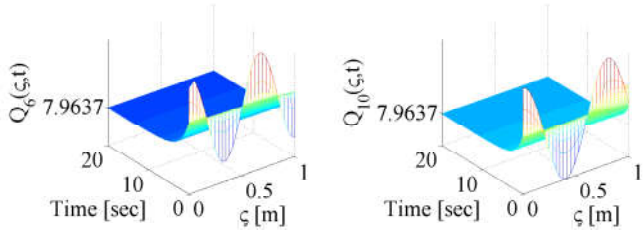

Fig. 2. TEST 1: Spatiotemporal profiles $Q_6(\varsigma,t)$ (left) and $Q_{10}(\varsigma,t)$ (right).

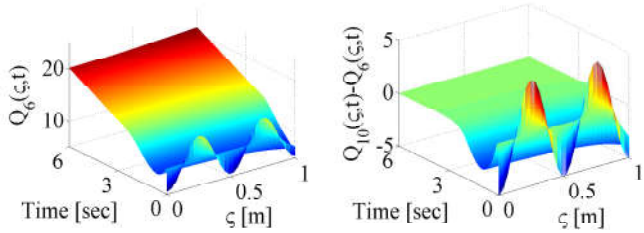

Fig. 3. TEST 2: Spatiotemporal profiles $Q_6(\varsigma,t)$ (left) and $Q_6(\varsigma,t)-Q_{10}(\varsigma,t)$ (right).

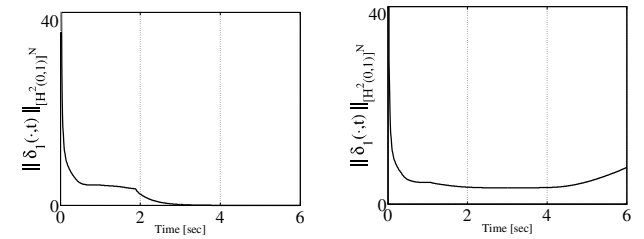

Fig. 4. Time evolution of the norm $\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,N}}$ in TEST 2 (left) and TEST 3 (right).

evolution of the state $Q_6(\varsigma,t)$ and of the state mismatch $Q_{10}(\varsigma,t)-Q_6(\varsigma,t)$. It is clear that both states converge towards the same steady-state profile. The time evolution of the disagreement vector norm $\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,10}}$, shown in Figure 4-left, tends to zero as shown in the Theorem 2. To verify the conservativeness of the approach, another simulation run, called TEST 3, was made where the terms $\alpha_i t^2$, $i=1,2,\ldots,10$ (α_i being randomly chosen constants in the interval $[0,20]$) has been added to the disturbance entries $\psi_i(t)$ used in TEST 2. Due to the insertion of these additional terms, not only the magnitudes of the disturbances are increasing with time but also those of their derivatives are. Thus, the tuning conditions (41) are deliberately violated. Figure4-right depicts the resulting divergence trend of the disagreement vector norm $\|\delta_1(\cdot,t)\|_{\mathrm{H}^{2,10}}$, thereby supporting the theoretical results.

VI. Conclusion

In this work, an infinite-dimensional counterpart of the well-known finite-dimensional average consensus algorithm has been derived with reference to an unperturbed network of diffusion processes under a linear decentralized local interaction policy. Along with this, the problem of guaranteeing the asymptotic agreement among the agents' states while rejecting a class of persistent and possibly unbounded disturbances has been tackled by devising a nonlinear local interaction policy based on the second-order sliding-mode control approach.

Future activities will be targeted to relaxing the topological restrictions on the network structure by covering, e.g., directed and possibly switching communication graphs. Further investigation is called for the extension of these results to more general classes of (possibly non identical) distributed parameters agents' dynamics and additionally to more general consensus-based problems in the infinite dimensional setting such as e.g. leader following.

References

[1] W. Ren, R. W. Beard, and E. M. Atkins, "Information consensus in multivehicle cooperative control: Collective group behavior through local interaction," *IEEE Control Syst. Mag.*, vol. 27, no. 2, pp. 71–82, Sep. 2007.

[2] R. Olfati-Saber, J. Fax, and R. Murray, "Consensus and cooperation in networked multi-agent systems," *Proc. IEEE*, vol. 95, no. 1, pp. 215–233, Jan. 2007.

[3] H. Chao, Y. Chen, and W. Ren, "Consensus of information in distributed control of a diffusion process using centroidal voronoi tessellations," in *IEEE Conf. on Decision and Control*, 2007, pp. 1441–1446.

[4] C. Tricaud and Y. Chen, "Optimal mobile actuator/sensor network motion strategy for parameter estimation in a class of cyber physical systems," in *IEEE American Control Conf.*, 2009, pp. 367–372.

[5] M. A. Demetriou, "Design of consensus and adaptive consensus filters for distributed parameter systems," *Automatica*, vol. 46, no. 2, pp. 300–311, Feb. 2010.

[6] ——, "Natural consensus filters for second order infinite dimensional systems," *Systems & Control Letters*, vol. 58, no. 12, pp. 826–833, Dec. 2009.

[7] ——, "Guidance of mobile actuator-plus-sensor networks for improved control and estimation of distributed parameter systems," *IEEE Trans. Automat. Contr.*, vol. 55, no. 7, pp. 1570–154, Jul. 2010.

[8] ——, "Enforcing consensus on adaptive parameter estimation of structurally perturbed infinite dimensional systems," *IEEE Trans. Automat. Contr.*, vol. 57, no. 12, pp. 3147–3152, Dec. 2012.

[9] J. Qi, R. Vazquez, and M. Krstic, "Multi-agent deployment in 3-d via pde control," *IEEE Trans. Automat. Contr.*, vol. 60, no. 4, pp. 891–906, Apr. 2015.

[10] M. A. Demetriou, "Synchronization and consensus controllers for a class of parabolic distributed parameter systems," *Systems & Control Letters*, vol. 62, no. 1, pp. 70–76, Jan. 2013.

[11] A. Pisano and E. Usai, "Sliding mode control: a survey with applications in math," *Mathematics and Computers in Simulation*, vol. 81, no. 5, pp. 954–979, Jan. 2011.

[12] A. Pilloni, A. Pisano, M. Franceschelli, and E. Usai, "Finite-time consensus for a network of perturbed double integrators by second-order sliding mode technique," in *IEEE Conf. on Decision and Control*, 2013, pp. 2145–2150.

[13] ——, "Recent advances in sliding-mode based consensus strategies," in *IEEE Int. Workshop on Variable Structure Systems*, 2014, pp. 1–6.

[14] A. Pisano and Y. Orlov, "Boundary second-order sliding-mode control of an uncertain heat process with unbounded matched perturbation," *Automatica*, vol. 48, no. 8, pp. 1768–1775, Aug. 2012.

[15] R. Caponetto, A. Pisano, and E. Usai, "Second order sliding mode approaches to fault detection and control of infinite dimensional systems," in *IEEE European Control Conf.*, 2014, pp. 2297–2303.

[16] R. A. Horn and C. R. Johnson, *Matrix analysis*. Cambridge University Press, 1990.

[17] H. Khalil, *Nonlinear Systems*. Prentice Hall, 2002.

[18] R. F. Curtain and H. Zwart, *An introduction to infinite-dimensional linear systems theory*. Springer New York, 1995, vol. 21. New York, NY.

[19] A. Pilloni, A. Pisano, Y. Orlov, and U. E, "Consensus-based control for a network of diffusion pdes with boundary local interaction," *ArXiv preprint arXiv:1509.06755*, 2015.

[20] Y. Orlov, *Discontinuous systems: Lyapunov analysis and robust synthesis under uncertainty conditions*. Springer, 2008.

[21] D. Henry, *Geometric theory of semilinear parabolic equations*. Springer, 1981.