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Coherence resource power of isocoherent states

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We address the problem of comparing quantum states with the same amount of coherence in terms of their coherence resource power given by the preorder of incoherent operations. For any coherence measure, two states with null or maximum value of coherence are equivalent with respect to that preorder. This is no longer true for intermediate values of coherence when pure states of quantum systems with dimension greater than two are considered. In particular, we show that, for any value of coherence (except the extreme values, zero and the maximum), there are infinite incomparable pure states with that value of coherence. These results are not peculiarities of a given coherence measure, but common properties of every well-behaved coherence measure. Furthermore, we show that for qubit mixed states there exist coherence measures, such as the relative entropy of coherence, that admit incomparable isocoherent states.

Quantum coherence, which is a consequence of the superposition principle, is one of the fundamental aspects of the quantum theory. It has practical relevance in numerous fields of quantum physics, particularly in quantum information processing[1]. Furthermore, within the paradigm of quantum resource theories[2], quantum coherence is considered as a quantum resource that can be converted, consumed and quantified[3,4].

As any resource theory, the *resource theory of coherence* is built from three basic concepts: free states, resources and free operations. Since coherence is a basis-dependent notion, these three elements are defined in terms of a fixed basis, called *incoherent basis*. The free states of the theory, called *incoherent states*, are quantum states with diagonal density matrix in the incoherent basis. The rest of the states are resources and they are called *coherent states*. Regarding the free operations of the theory, there is no single definition and each proposal leads to a different resource theory for coherence, see *e.g.*[1] and references therein. In this work, we follow the definition of an *incoherent operation* (IO) introduced in[3], which has the property that coherence can not be created from an incoherent state, not even in a probabilistic way.

The resource-theoretic formulation allows us to introduce a *preorder* between quantum states induced by the incoherent operations: one state is more or equally coherent than other if the former can be converted into the later by means of incoherent operations. This preorder is useful for studying coherence transformations and classifying the set of quantum states according to its *coherence resource power*. Given any pair of quantum states, they can be classified as: (i) *IO-comparable*, when one state can be transformed into the other by means of IO, (ii) *IO-equivalent*, when both states can be transformed into the other, and (iii) *IO-incomparable*, when neither state can be transformed into the other.

Another way to capture operational aspects of coherence is by means of coherence quantifiers. There are several coherence quantifiers and most of them can be studied from an axiomatic point of view. More precisely, any *bonafide* coherence measure has to vanish only for incoherent states, to be strong monotone and convex[3] and to be maximal for maximal coherent states as discussed in[5]. Examples of coherence measures are the relative entropy of coherence[3], the ℓ_1-norm of coherence[3] and the coherence of formation[6], among others[7–13].

Clearly, each coherence measure induces a total order on the quantum states. In general, these total orders are different. For instance, the total order induced by the relative entropy of coherence and by ℓ_1-norm of coherence do not coincide[14]. This result motivates the comparison among other total orders induced by different coherence measures (see e.g.[15–20]).

In this work, we address a related but different problem concerning the ordering of quantum states with respect to coherence. In particular, we focus on quantum states with a fixed value of coherence and we ask for their coherence resource power in terms of the preorder induced by the incoherent operations. First, we observe that all quantum states with null coherence are IO-equivalent. Similarly, all quantum states with maximum coherence are also IO-equivalent. However, this is not true in general. In particular, in the case of pure states, we show that, for any value of coherence (except the extreme values, zero and the maximum), there are infinite

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IO-incomparable pure states with that value of coherence, provided that the dimension of the quantum systems is greater than two. These results are not peculiarities of a given coherence measure, but common properties of every well-behaved coherence measure. Furthermore, for qubit mixed states, we show that there are coherence measures, such as the relative entropy of coherence, that admit incomparable isocoherent states. In this way, our work complements other works[14–20] about ordering of quantum states with coherence.

The paper is organized as follows. First, we review the main aspects of the resource theory of coherence, including the definition of incoherent states, incoherent operations and coherence measures, among other relevant results. Then, we provide our main results regarding the comparison of states with the same amount of coherence in terms of their coherence resource power. Finally, we summarize our results and discuss their roots and links with related problems. For the sake of readability all proofs and auxiliary results are given in the "Methods".

Preliminaries: resource theory of coherence, IO preorder and coherence measures

We will focus on quantum systems with finite dimension. The Hilbert space of the system is denoted by $\mathscr{H}$ and its dimension by $d = \dim \mathscr{H}$. The set of all quantum states of $\mathscr{H}$ is denoted by $\mathscr{S}(\mathscr{H})$ and the subset of pure states is denoted by $\mathscr{P}(\mathscr{H})$.

Without loss of generality we choose the computational basis $\mathscr{B} = \{|i\rangle\}_{i=0}^{d-1}$ as the incoherent basis. In this way, incoherent states, are quantum states with diagonal density matrix in the computational basis. More precisely, ρ is an incoherent state if and only if $\rho = \sum_{i=0}^{d-1} \lambda_i |i\rangle\langle i|$, with $\lambda_i \geq 0$ and $\sum_{i=0}^{d-1} \lambda_i = 1$. We denote the set of incoherent states as $\mathscr{I}$.

The incoherent operations introduced in[3], which have the property that coherence can not be created from an incoherent state, not even in a probabilistic way, are defined as follows. A completely positive trace-preserving map $\Lambda : \mathscr{S}(\mathscr{H}) \to \mathscr{S}(\mathscr{H})$ is an incoherent operation (IO) if it admits a representation in terms of Kraus operators $\{K_n\}_{n=1}^{N}$ such that $K_n \rho K_n^\dagger / \mathrm{Tr}(K_n \rho K_n^\dagger) \in \mathscr{I}$ for all $1 \leq n \leq N$ and $\rho \in \mathscr{I}$.

Interesting enough, incoherent operations induce a preorder among quantum states in terms of their coherence resource power:

Definition 1 We say that the state ρ *is more or equally coherent than* σ if it is possible to transform the former into the latter by means of an incoherent operation. We denote this by $\rho \underset{\mathrm{IO}}{\to} \sigma$.

In other words, $\rho \underset{\mathrm{IO}}{\to} \sigma$ if there is an incoherent operation Λ such that $\sigma = \Lambda(\rho)$. This defines a preorder on the set $\mathscr{S}(\mathscr{H})$, since it satisfies the two conditions:

1. Reflexivity: $\rho \underset{\mathrm{IO}}{\to} \rho$ for all $\rho \in \mathscr{S}(\mathscr{H})$.
2. Transitivity: If $\rho \underset{\mathrm{IO}}{\to} \omega$ and $\omega \underset{\mathrm{IO}}{\to} \sigma$, then $\rho \underset{\mathrm{IO}}{\to} \sigma$ for all $\rho, \omega, \sigma \in \mathscr{S}(\mathscr{H})$.

Property 1 follows from the fact that doing nothing, i.e., applying the identity operator, is an IO, whereas property 2 follows from the fact that the composition of IOs is an IO.

Furthermore, we classify the states as follows. We say that two states ρ and σ are *IO-comparable* when $\rho \underset{\mathrm{IO}}{\to} \sigma$ or $\sigma \underset{\mathrm{IO}}{\to} \rho$. In particular, if both of these transformations are possible, we say that they are *IO-equivalent*, and we denote that as $\rho \underset{\mathrm{IO}}{\leftrightarrow} \sigma$. On the contrary, if neither of these transformations are possible, we say that the states are *IO-incomparable*, and we denote that as $\rho \underset{\mathrm{IO}}{\nleftrightarrow} \sigma$.

We recall that, in this resource theory of coherence, there exist *maximally coherent states* (MCSs), which are states that can be transformed into any other state by means of IO. The canonical MCS is a pure state of the form $\rho^{\mathrm{mcs}} = |\psi_d^{\mathrm{mcs}}\rangle\langle\psi_d^{\mathrm{mcs}}|$[3], with

$$|\psi_d^{\mathrm{mcs}}\rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |i\rangle. \tag{1}$$

Any MCSs can be obtained from the state ρ^{mcs} by applying a unitary incoherent operations of the form $U_{\mathrm{IO}} = \sum_{i=0}^{d-1} e^{i\theta_i} |\pi(i)\rangle\langle i|$, where π is a permutation acting on the set $\{0, 1, \ldots, d-1\}$ and $\theta_i \in \mathbb{R}$[5].

In addition, we recall that for pure states the preorder induced by IO is equivalent to the majorization preorder of the corresponding coherence vectors (see[7,8,21–23]). More precisely, given $|\psi\rangle \in \mathscr{H}$, its *coherence vector* is defined as the vector

$$\psi = \left(|\langle 0|\psi\rangle|^2, \ldots, |\langle d-1|\psi\rangle|^2 \right). \tag{2}$$

Notice that $\psi \in \Delta_d$, where Δ_d is the set of d-dimensional probability vectors, i.e., $\Delta_d = \{\psi = (\psi_0, \ldots, \psi_{d-1}) \in \mathbb{R}^d : \psi_i \geq 0, \sum_{i=0}^{d-1} \psi_i = 1\}$.

Given two probability vectors $\psi = (\psi_0, \ldots, \psi_{d-1})$ and $\phi = (\phi_0, \ldots, \phi_{d-1})$, we say that ψ is majorized by ϕ, and we denote it by $\psi \preceq \phi$, if[24]

$$\sum_{i=0}^{k} \psi_i^\downarrow \leq \sum_{i=0}^{k} \phi_i^\downarrow \quad \forall\, 0 \leq k \leq d-2, \tag{3}$$

where $\downarrow$ indicates that the entries of ψ and ϕ are sorted in non-increasing order, i.e., $\psi_i^\downarrow \geq \psi_{i+1}^\downarrow$ and $\phi_i^\downarrow \geq \phi_{i+1}^\downarrow$ for all $0 \leq i \leq d - 2$.

As we say before, the majorization relation defines a preorder on the set of probability vectors Δ_d. If $\psi \preceq \phi$ or $\phi \preceq \psi$, we say that the probability vectors are *comparable*. If both relation are satisfied, ψ and ϕ are equal up to a permutation. In this case we say that the probability vectors are *equivalent*. For dimensions greater than two[24], there are cases in which neither $\psi \preceq \phi$ nor $\phi \preceq \psi$ are possible. When this is the case, we say that the probability vectors are *incomparable*.

Taking into account these definitions, we can state the following result that connects both preorders (see[7,8,21–23]).

Theorem 1 *Let $|\psi\rangle$ and $|\phi\rangle$ be two pure states. Then,*

$$|\psi\rangle \underset{\text{IO}}{\rightarrow} |\phi\rangle \iff \psi \preceq \phi. \tag{4}$$

According to this theorem, two pure states $|\psi\rangle$ and $|\phi\rangle$ are IO-comparable (or IO-incomparable) if and only if their corresponding coherence vectors are comparable (or incomparable).

In general, for mixed states, a finite number of conditions are not sufficient to fully characterize coherent transformations[25]. From a generalized notion of coherence vector, it can be obtained a necessary condition in terms of a majorization relation[13]. However, qubit transformations under incoherent operations are completely characterized[26,27].

Theorem 2 *Let ρ and σ be two qubit states with Bloch vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$, respectively. Then, $\rho \underset{\text{IO}}{\rightarrow} \sigma$ if and only if*

$$r_1 \geq r_2, \tag{5}$$

$$\frac{1 - z_1^2}{r_1^2} \leq \frac{1 - z_2^2}{r_2^2}, \tag{6}$$

with $r_1 = \sqrt{x_1^2 + y_1^2}$ and $r_2 = \sqrt{x_2^2 + y_2^2}$.

In addition to the comparability notions between states, it is relevant to quantify the coherence amount of quantum states. In this paper, we mainly follow the axiomatic formulation for coherence measures.

Definition 2 A coherence measure C is a real function defined on $\mathscr{S}(\mathscr{H})$, satisfying the following conditions:

1. Vanishing only on incoherent states: $C(\rho) = 0$ if and only if $\rho \in \mathscr{I}$.
2. Strong monotonicity under IO: $C(\rho) \geq \sum_i p_i C(\sigma_i)$, where $\{p_i, \sigma_i\}$ is an ensemble obtained from the state ρ by means of IO.
3. Convexity: $C\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i C(\rho_i)$.
4. Maximum coherence: $\arg\max_{\rho \in \mathscr{S}(\mathscr{H})} C(\rho)$ coincides with the set of maximally coherent states.

It can be shown that conditions 2 and 3 imply monotonicity under IO, that is, $C(\rho) \geq C(\Lambda(\rho))$ for any incoherent operation Λ and any state ρ. The relevance of condition 4 is discussed in[5]. In particular, this condition excludes some problematic cases as the one given in Ex. 4 of[28].

An interesting result is that any coherence measure restricted to pure states can be expressed in terms of a real, symmetric and concave functions defined on Δ_d. More precisely, given the set

$$\mathscr{F} = \left\{ f : \Delta_d \to [0, 1] : f \text{ is symmetric and concave}, f(1, 0, \ldots, 0) = 0, \text{ and } \arg\max_{\psi \in \Delta_d} f(\psi) = (1/d, \ldots, 1/d) \right\}, \tag{7}$$

we have the following result[7,8].

Theorem 3 *Let C be a coherence measure. Then, there exists a function $f_C \in \mathscr{F}$, such that the restriction of C to the set of pure states, denoted by $C|_{\mathscr{P}(\mathscr{H})}$, satisfies*

$$C|_{\mathscr{P}(\mathscr{H})}(|\psi\rangle\langle\psi|) = f_C(\psi), \tag{8}$$

where ψ is the coherence vector of $|\psi\rangle$.

By abuse of notation, we will use $C(|\psi\rangle)$ when evaluating the restriction of the measure C on a pure state, instead of $C|_{\mathscr{P}(\mathscr{H})}(|\psi\rangle\langle\psi|)$. In particular, we will focus on functions of $\mathscr{F}$ that are also strictly Schur-concave. Namely, a real function f defined on Δ_d is said to be *Schur-concave*, if $f(\psi) \geq f(\phi)$ whenever $\psi \preceq \phi$. If, in addition, $f(\psi) > f(\phi)$ whenever $\psi \preceq \phi$ and $\psi \neq \Pi\phi$, with Π a permutation matrix, then f is said to be *strictly Schur-concave* (see Def. A.1 in[24]). We note that any function $f \in \mathscr{F}$ is Schur-concave, since they are symmetric and concave (see Prop. C.2 in[24]), but they are not necessarily strictly Schur-concave. The condition that f_C be

strictly Schur-concave is not so restrictive, since the most used coherence measures satisfy it, including the relative entropy of coherence, the ℓ_1-norm of coherence, and the coherence of formation.

Now, we review some relevant coherence measures. The first one is the *relative entropy of coherence C_{re}*, defined as

$$C_{re}(\rho) = \min_{\sigma \in \mathscr{I}} S(\rho||\sigma), \qquad (9)$$

where $S(\rho||\sigma) = \text{Tr}(\rho(\ln \rho - \ln \sigma))$. Alternatively, the relative of coherence can be expressed as $C_{re}(\rho) = S(\rho_{diag}) - S(\rho)$, where $\rho_{diag} = \sum_{i=0}^{d-1} \langle i|\rho|i\rangle|i\rangle\langle i|$ and $S(\rho) = -\text{Tr}(\rho \ln \rho)$ is the von Neumann entropy. Accordingly, the associated function $f_{C_{re}} \in \mathscr{F}$ of the relative of coherence is the Shannon entropy, i.e., $f_{C_{re}}(\psi) = H(\psi) = -\sum_{i=0}^{d-1} \psi_i \ln \psi_i$, which is also strictly Schur-concave. The relative entropy of coherence has a particular operational interpretation. It coincides with the distillable coherence, that is, the maximal number of maximally coherent single-qubit states $|\psi_2^{mcs}\rangle$ which can be obtained per copy of a given state ρ by means of incoherent operations in the asymptotic limit[6].

Another relevant coherence measure is given in terms of the off-diagonal elements of ρ. More precisely, the *ℓ_1-norm of coherence C_{ℓ_1}* is defined as

$$C_{\ell_1}(\rho) = \sum_{\substack{i, i' = 0 \\ i \neq i'}}^{d-1} |\langle i|\rho|i'\rangle|. \qquad (10)$$

In this case, the associated function $f_{C_{\ell_1}} \in \mathscr{F}$ is given by $f_{C_{\ell_1}}(\psi) = \left(\sum_{i=0}^{d-1} \sqrt{\psi_i}\right)^2 - 1$, which is also strictly Schur-concave. We remark that the ℓ_1-norm of coherence is useful for characterizing quantum interference and obtaining complementarity relations between coherence and path information in multipath interferometers[29–31].

Finally, we recall that there are different ways to extend a coherence measure defined on pure states to mixed states[7,8,12,13]. The most common way is the convex roof method. For any $f \in \mathscr{F}$ the *convex roof measure of coherence C_f^{cr}* is given by[7,8]

$$C_f^{cr}(\rho) = \min_{\{q_k, |\psi_k\rangle\}_{k=1}^M \in \mathscr{D}(\rho)} \sum_{k=1}^M q_k f(\psi_k), \qquad (11)$$

where $\mathscr{D}(\rho) = \left\{ \{q_k, |\psi_k\rangle\}_{k=1}^M : \rho = \sum_{k=1}^M q_k |\psi_k\rangle\langle\psi_k| \right\}$ is set of all pure state decompositions of ρ. For instance, choosing the function $f \in \mathscr{F}$ as the q-Tsallis entropy[32], i.e., $f_q^T(\psi) = \left(1 - \sum_{i=0}^{d-1} \psi_i^q\right)/(q-1)$ for $q \in (0, 1) \cup (1, +\infty)$, leads to the q-Tsallis coherence of formation C_q^T. Notice that f_q^T is strictly Schur-concave[33] and $C_1^T(\rho) = \lim_{q \to 1} C_q^T(\rho) = C_{CoF}(\rho)$, recovering the coherence of formation. This measure coincides with the coherence cost, that is, the minimal number of maximally coherent single-qubit states $|\psi_2^{mcs}\rangle$ required to produce a given state ρ by means of incoherent operations, in the asymptotic limit[6].

Results

We are interesting in comparing states with the same amount of coherence. For a given coherence measure C and a non-negative number α, we introduce the set of isocoherence states $\mathscr{E}_{C,\alpha}$ as follows

$$\mathscr{E}_{C,\alpha} = \{\rho \in \mathscr{S}(\mathscr{H}) : C(\rho) = \alpha\}. \qquad (12)$$

On the one hand, from the condition 1 of definition 2, it follows $\mathscr{E}_{C,0} = \mathscr{I}$. Thus, as a consequence of that all incoherent states are IO-equivalent (see Observation 1), the states belonging to $\mathscr{E}_{C,0}$ are all IO-equivalent. On the other hand, from condition 4 of definition 2, it follows that the set $\mathscr{E}_{C,M_C}$, with $M_C = C(\rho^{mcs})$, is formed by maximally coherent states. Thus, the states belonging to $\mathscr{E}_{C,M_C}$ are all IO-equivalent. Therefore, as one might expect, all isocoherent states with an extreme value of coherence have the same coherence resource power. Moreover, this is also true for pure isocoherent states of qubit systems for any value of coherence.

Proposition 1 *For any function $f \in \mathscr{F}$ strictly Schur-concave, pure isocoherent states of qubit systems are IO-equivalent.*

A natural question that arises from the previous observations is: In higher dimensional systems, do isocoherent pure states with an intermediate value of coherence have the same coherence resource power? In other words, for systems with $d > 2$, we are asking if states of $\mathscr{E}_{C,\alpha}$, with $\alpha \in (0, M_C)$, are IO-equivalent.

We will show that this is not the case. More precisely, we will prove that for each value of coherence α in the interval (I_C, M_C), there are infinite IO-incomparable pure states with that amount of coherence, where $I_C = \inf_{\psi \in \text{ri}(\Delta_d)} f_C(\psi)$ with $\text{ri}(\Delta_d)$ the relative interior of the set Δ_d, i.e., $\text{ri}(\Delta_d) = \{\psi \in \mathbb{R}^d : \psi_i > 0, \sum_{i=0}^{d-1} \psi_i = 1\}$.

Proposition 2 *Let $d > 2$ and let $C : \mathscr{S}(\mathscr{H}) \to \mathbb{R}$ be a coherence measure such that its restriction to pure states has an associated function $f_C \in \mathscr{F}$ strictly Schur-concave. For any $\alpha \in (I_C, M_C)$, there are infinite pure states $\{|\psi^{(i)}\rangle\langle\psi^{(i)}|\}_{i \in I} \subseteq \mathscr{E}_{C,\alpha}$ (with I a set of index), such that $|\psi^{(i)}\rangle \underset{IO}{\not\leftrightarrow} |\psi^{(i')}\rangle$ for all $i \neq i' \in I$.*
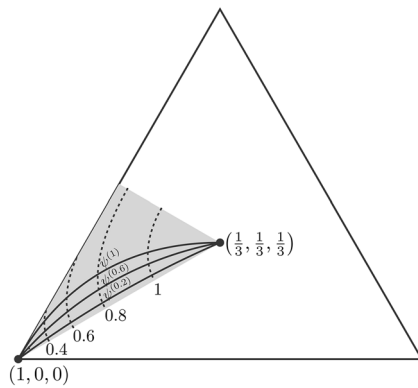
Figure 1. The big triangle is the set of three-dimensional probability vectors Δ_3. The gray triangle is the subset of ordered probability vectors $\Delta_3^{\downarrow}$. We depict the curves $\psi^{(a)}$, defined in Eq. (13), for $a \in \{0.2, 0.6, 1\}$ (solid lines) and the contours plot of $-\sum_{i=0}^{2} \psi_i \ln \psi_i = \alpha$ for $\alpha \in \{0.2, 0.4, 0.8, 1\}$ (dashed lines). The intersection of a given contour plot with the curves $\psi^{(a)}$ gives a family of incomparable probability vectors. From this family and Eq. (14), we obtain a family of mutually IO-incomparable pure states with relative entropy of coherence equal to α.

Another condition that coherence measures usually satisfy is the continuity condition, that is

5. Continuity: C is continuous on $\mathscr{S}(\mathscr{H})$.

 If a coherence measure also satisfies the continuity condition (5), we have that for any possible value of coherence (except the extreme cases zero and the maximal value M_C) there are an infinite number of IO-incomparable pure states with that amount of coherence.

Corollary 1 *Let $d > 2$ and let $C : \mathscr{S}(\mathscr{H}) \to \mathbb{R}$ be a coherence measure satisfying condition (5) and such that its restriction to pure states has an associated function $f_C \in \mathscr{F}$ strictly Schur-concave. For any $\alpha \in (0, M_C)$, there are infinite pure states $\{|\psi^{(i)}\rangle\langle\psi^{(i)}|\}_{i \in I} \subseteq \mathscr{E}_{C,\alpha}$ (with I a set of index), such that $|\psi^{(i)}\rangle \underset{\text{IO}}{\leftrightarrow} |\psi^{(i')}\rangle$ for all $i \neq i' \in I$.*

In particular, we remark that, Propositions 1 and 2, and Corollary 1 are valid for the relative entropy of coherence C_{re} and the ℓ_1-norm of coherence C_{ℓ_1}.

Finally, for the three-dimensional case, we provide an example of a family of IO-incomparable pure states with the same value of the relative entropy of coherence. For each $a \in [0, 1]$, we define the curve

$$\begin{cases} \psi^{(a)} : [0, 1] \to \Delta_d^{\downarrow} \\ \psi^{(a)}(t) = u + t(v - u) + at(t - 1)w, \end{cases} \tag{13}$$

where $u = (1/3, 1/3, 1/3)$, $v = (1, 0, 0)$ and $w = (0, -1/3, 1/3)$. For each $a \in [0, 1]$ and $t \in [0, 1]$, $\psi^{(a)}(t)$ is a probability vector sorted in a non-increasing way. Moreover, it can be proved that different curves do not have equivalent probability vectors in common, except the extreme vectors u and v.

For some $\alpha \in (0, M_C)$ (with $M_C = \ln 3$) and for each $a \in [0, 1]$, we consider the intersection of the contour plot $C_r(|\psi\rangle) = -\sum_{i=0}^{2} \psi_i \ln \psi_i = \alpha$ with the curve $\psi^{(a)}$. We denote this intersection by $\psi^{(a)}(t_a^*)$, with $t_a^* \in (0, 1)$. In Fig. 1, we depict the curves $\psi^{(a)}$, for $a \in \{0.2, 0.6, 1\}$ (solid lines), and the contour plots $C_r(|\psi\rangle) = \alpha$, for $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1\}$ (dashed lines).

We consider the family of probability vectors $\{\psi^{(a)}(t_a^*)\}_{a \in [0,1]}$, and the corresponding family of pure states $\{|\psi^{(a)}\rangle\}_{a \in [0,1]}$, where

$$|\psi^{(a)}\rangle = \sum_{i=0}^{d-1} \sqrt{(\psi^{(a)}(t_a^*))_i} |i\rangle. \tag{14}$$

Then, we have $C_r(|\psi^{(a)}\rangle) = f_{C_r}(\psi^{(a)}(t_a^*)) = \alpha$ for all the states of the family. Since Shannon entropy is strictly Schur-concave, then the family of probability vectors only has equivalent or incomparable pairs of vectors. Moreover, since different curves do not have pairs of equivalent vectors (except the extreme vectors), the family of probability vectors does not have pairs of equivalent vectors. This implies that all probability vectors of the family $\{\psi^{(a)}(t_a^*)\}_{a \in [0,1]}$ are mutually incomparable. We conclude that all the states of the family $\{|\psi^{(a)}\rangle\}_{a \in [0,1]}$ are mutually IO-incomparable. In this way, we have found a family of mutually IO-incomparable pure states with relative entropy of coherence equal to α.

Regarding qubit mixed states, we can distinguish two situations depending on the coherence measure. An arbitrary coherence measure for a qubit state ρ, with Bloch vector $\mathbf{r} = (x, y, z)$, can be expressed in terms of $r = \sqrt{x^2 + y^2}$ and z, i.e., $C(\rho) = C(r, z)$. For measures that do not depend on z and are strictly increasing on r, it is easy to see that there are no pair of incomparable isocoherent states. More precisely, let ρ and σ be two
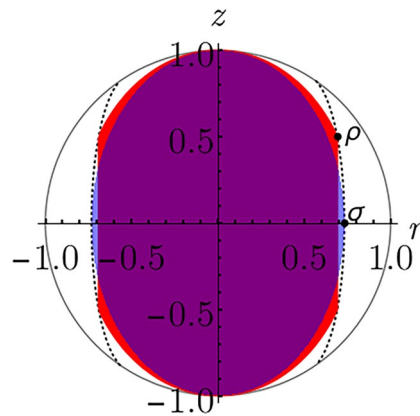
Figure 2. Projection of the Bloch sphere on the $z - r$ plane. The black dots represent two qubit states: ρ (upper dot, with $r_1 = 0.69135$, $z_1 = 0.5$) and σ (lower dot, with $r_2 = 0.732828$, $z_2 = 0$). The red and blue regions represent the projection of the sets $\{\rho' : \rho \xrightarrow{\text{IO}} \rho'\}$ and $\{\rho' : \sigma \xrightarrow{\text{IO}} \rho'\}$ on the $z - r$ plane, respectively. The dotted black curve represents the set $\{\rho' : C_{\text{re}}(\rho') = 0.3\}$ projected on the $z - r$ plane. From the figure, it can be observed that ρ and σ are IO-incomparable, and $C_{\text{re}}(\rho) = C_{\text{re}}(\sigma) = 0.3$.

isocoherent states, with Bloch vectors $\mathbf{r}_1 = (x_1, y_1, z_1)$ and $\mathbf{r}_2 = (x_2, y_2, z_2)$ respectively. Then, since the coherence measure only depends on r, and it is strictly increasing, we have that $r_1 = r_2$. Finally, from Theorem 2, we conclude that, if $|z_1| \leq |z_2|$, then $\sigma \xrightarrow{\text{IO}} \rho$, and if $|z_2| \leq |z_1|$, then $\rho \xrightarrow{\text{IO}} \sigma$. In other words, the isocoherent states ρ and σ are IO-comparable . In particular, the ℓ_1-norm of coherence is an example of this kind of coherence measures: $C_{\ell_1}(\rho) = r$.

For coherence measures that also depend on z, there are examples of isocoherent states which are incomparable. For instance, for the relative entropy of coherence, which can be expressed as $C_{\text{re}}(\rho) = h\left(\frac{1+z}{2}\right) - h\left(\frac{1+\sqrt{r^2+z^2}}{2}\right)$ with $h(x) = -x \ln x - (1 - x) \ln(1 - x)$, we show in Fig. 2 an example of two incomparable and isocoherent states.

Discussions

In this work, we have considered the subset of quantum states formed by those states with a fixed value of coherence for a given coherence measure. We have analyzed its coherence resource power in terms of the preorder induced by the incoherent operations.

First, we have observed that, as one might expect, isocoherent states with an extreme value of coherence have the same coherence resource power in terms of the incoherent preorder. Second, we have shown that pure isocoherent states of qubit systems with arbitrary value of coherence are IO-equivalent (Proposition 1).

Third, we have proved that, in higher dimensional systems ($d > 2$), pure isocoherent states are not necessarily IO-equivalent. In particular, for any value of coherence, we have shown that there are infinite IO-incomparable pure states with that value of coherence (Proposition 2 and Corollary 1). The essence of these results is that, in general, the coherence measures do not fully preserve the preorder structure of the quantum states induced by incoherent operations. Indeed, coherence measures map the set of quantum states to the positive real numbers, which is a total order set. In this way, the quantum states go from being pre-ordered by IO to being totally ordered by the coherence measure. Our results Proposition 2 and Corollary 1 arise as a consequence of this discrepancy. Another related consequence is the fact that different coherence measures induce different total orders on the set of quantum states, as it was observed in[14,16–18]. In this way, our work complements these studies about ordering quantum states with coherence.

Regarding qubit mixed states, we have distinguished two situations depending on the coherence measure. For measures that do not depend on z and are strictly increasing on r, we have shown that there are no pair of incomparable isocoherent states. In particular, the ℓ_1-norm of coherence is an example of this kind of coherence measures. For coherence measures that also depend on z, we have shown that there are examples of isocoherent mixed states which are incomparable. In particular, we have considered the case of the relative entropy of coherence.

Finally, we remark that we have focused on the resource theory of coherence to illustrate these observations due to its topicality and practical relevance[1]. However, as our proofs are posed in a wider and simpler context based on majorization theory[24], the results can be easily extended to any majorization-based quantum resource theory[34], such as entanglement theory[35] and nonuniformity[36,37]. In fact, the observations made for the entanglement entropy in[38], namely there are infinite incomparable bipartite pure states with a fixed value of entanglement entropy, can be extend to any entanglement monotone[39] by exploiting similar majorization arguments to those given in our proofs. The reason behind this generality is again that the preorder induced by the free operations of the resource theory and the total order induced by the measures are not isomorphic.

Methods

First, we provide a proof of the following intuitive result: all incoherent states have the same coherence resource power. Indeed, for any quantum resource theory that admits a tensor product structure, like the quantum coherence resource theory considered here, it is valid that the free states are interconvertible by means of the free operations of the theory[2]. For the sake of completeness, we provide a directly proof of the Observation 1, giving an explicit incoherent operation that allows to transform an arbitrary incoherent state into another arbitrary one.

Observation 1 All incoherent states are IO-equivalent.

Proof Let us prove that any incoherent state ρ is IO-equivalent to the state $|0\rangle\langle 0|$. By definition, $\rho = \sum_{i=0}^{d-1} \lambda_i |i\rangle\langle i|$, with $\lambda_i \geq 0$ and $\sum_{i=0}^{d-1} \lambda_i = 1$.

Firstly, let us consider the quantum operation Λ_1 with Kraus operators $K_i = |0\rangle\langle i|$, for $i = 0, \ldots, d-1$. It is easy to see that Λ_1 is an incoherent operation. Moreover, $\Lambda_1(\rho) = |0\rangle\langle 0|$. Therefore, for all $\rho \in \mathscr{I}$, $\rho \xrightarrow[\text{IO}]{} |0\rangle\langle 0|$.

Secondly, let define the quantum operation Λ_2 with Kraus operators $K_i = \sqrt{\lambda_i}|i\rangle\langle 0|$, for $i = 0, \ldots, d-1$ and $G = I - |0\rangle\langle 0|$. Again, it is easy to see that Λ_2 is an incoherent operation, and $\Lambda_2(|0\rangle\langle 0|) = \rho$. Then, for all $\rho \in \mathscr{I}$, $|0\rangle\langle 0| \xrightarrow[\text{IO}]{} \rho$.

Both results imply that, for all $\rho \in \mathscr{I}$, $|0\rangle\langle 0| \underset{\text{IO}}{\leftrightarrow} \rho$. Then, we can conclude that all incoherent states are IO-equivalent. □

Proof of Proposition 1

Proof Let $|\psi\rangle$ and $|\phi\rangle$ be two pure isocoherent states of a qubit system, and let $\psi^\downarrow = (\psi_0^\downarrow, 1 - \psi_0^\downarrow)$ and $\phi^\downarrow = (\phi_0^\downarrow, 1 - \phi_0^\downarrow)$ be their ordered coherence vectors.

In this case, Theorem 1 reduces to $|\psi\rangle \xrightarrow[\text{IO}]{} |\phi\rangle \iff \psi_0^\downarrow \leq \phi_0^\downarrow$. Therefore, ψ and ϕ are comparable or, equivalently, $|\psi\rangle$ and $|\phi\rangle$ are IO-comparable.

Moreover, since $f_C(\psi) = f_C(\phi)$ and f_C is strictly Schur-concave, we have that ψ and ϕ are equivalent. Therefore, $|\psi\rangle$ and $|\phi\rangle$ are IO-equivalent. □

For the proof of Proposition 2 we need the following lemma.

Lemma 1 Let $C : \mathscr{S}(\mathscr{H}) \to \mathbb{R}$ be an coherence measure and f_C its associated function.

   (i) f_C is continuous on $\text{ri}(\Delta_d)$, where $\text{ri}(\Delta_d)$ is the relative interior of Δ_d, i.e., $\text{ri}(\Delta_d) = \{\psi \in \mathbb{R}^d : \psi_i > 0, \sum_{i=0}^{d-1} \psi_i = 1\}$,
   (ii) If C satisfies condition (5), then f_C is continuous on Δ_d.
   (iii) $I_C \equiv \inf_{\psi \in \text{ri}(\Delta_d)} f_C(\psi) = \lim_{n\to\infty} f_C(v_n)$, with $v_n = (1 - 1/n, 1/[n(d-1)], \ldots, 1/[n(d-1)])$. In particular, if f_C is continuous on Δ_d, $I_C = 0$.

Proof

   (i) Since f_C is a concave function on Δ_d, $-f_C$ is a convex function on the same domain. We consider the following extension of f_C over all $\mathbb{R}^d$:

$$g_C(\psi) = \begin{cases} -f_C(\psi) & \text{if } \psi \in \Delta_d \\ +\infty & \text{if } \psi \notin \Delta_d \end{cases}$$

   The function g_C is convex on $\mathbb{R}^d$, therefore it is continuous on $\text{ri}(\text{dom } g)$ (see Th. 10.1 in[40]). Since $\text{ri}(\text{dom } g_C) = \text{ri}(\Delta_d)$, g_C is continuous on $\text{ri}(\Delta_d)$. Finally, we conclude that f_C is continuous on $\text{ri}(\Delta_d)$.
   (ii) We consider the continuous map $h : \Delta_d \mapsto \mathscr{H}$, given by

$$h(\psi) = \sum_{i=0}^{d-1} \sqrt{\psi_i}|i\rangle. \tag{15}$$

   We can express the function f_C in terms of C and h as follows,

$$f_C(\psi) = C(h(\psi)).$$

   Therefore, if C is continuous, then f_C is also continuous.
   (iii) The probability vectors $v_n = (1 - 1/n, 1/[n(d-1)], \ldots, 1/[n(d-1)])$ satisfy $v_n \prec v_{n+1}$ for all $n > 1$. Then, since f_C is a bounded Schur-concave function, $\{f_C(v_n)\}_{n \in \mathbb{N}_{>1}}$ is a bounded and monotonic decreasing sequence. Therefore, there exists $L = \lim_{n\to\infty} f_C(v_n)$, and $L \leq f_C(v_n)$ for all $n > 1$.

Let $\psi \in \text{ri}(\Delta_d)$. Since $\lim_{n\to\infty} v_n = (1, 0, \ldots, 0)$, we have that there is some $n_\psi \in \mathbb{N}$, such that $\psi \preceq v_{n_\psi}$. Then, $f_C(\psi) \geq f_C(v_{n_\psi}) \geq L$. Therefore, for all $\psi \in \text{ri}(\Delta_d)$, $f_C(\psi) \geq L$, which implies $I_C = \inf_{\psi \in \text{ri}(\Delta_d)} f_C(\psi) \geq L$.

On the other hand, for all $\psi \in \text{ri}(\Delta_d)$, $I_C \leq f_C(\psi)$. In particular, $I_C \leq f_C(v_n)$ for all $n \in \mathbb{N}$. This implies, $I_C \leq \lim_{n\to\infty} f_C(v_n)$.

Summing up, $I_C = \lim_{n\to\infty} f_C(v_n)$. In particular, if f_C is continuous on Δ_d, we have $I_C = \lim_{n\to\infty} f_C(v_n) = f_C(1, 0, \ldots, 0) = 0$.

$\square$

Proof of Proposition 2.

Proof On the one hand, $M_C = C(\rho^{\text{mcs}}) = f_C(u)$, with $u = (1/d, \ldots, 1/d)$. On the other hand, due to Lemma 1, $I_C = \lim_{n\to\infty} f_C(v_n)$. Since, for all $n \in \mathbb{N}$, $u \preceq v_n$, but $v_n \not\preceq u$, and f_C is strictly Schur-concave, we have $f_C(v_n) < f_C(u)$. Therefore, $I_C < M_C$.

Let $\alpha \in (I_C, M_C)$. Since $\alpha > I_C = \lim_{n\to\infty} f_C(v_n)$ and $\{f_C(v_n)\}_{n\in\mathbb{N}_{>1}}$ is monotonic decreasing, there is some $n_\alpha > 1$, such that $f_C(v_{n_\alpha}) < c$. Therefore, $f_C(v_{n_\alpha}) < \alpha < f_C(u)$.

Now, we construct a family of probability vectors, such that the value of f_C on these vectors is equal to α. For each $a \in [0, 1/d]$, we define the curve

$$\psi^{(a)}(t) = u + t(v_{n_\alpha} - u) + at(t-1)w \quad \text{for} \quad t \in [0, 1],$$

with $v_{n_\alpha} = (1 - 1/n_\alpha, 1/[n_\alpha(d-1)], \ldots, 1/[n_\alpha(d-1)])$ and $w = (0, -1/d, \ldots, -1/d, (d-2)/d)$.

Notice that the entries of $\psi^{(a)}(t)$ are

$$\psi_1^{(a)}(t) = \frac{(1-t)}{d} + \frac{t(n_\alpha - 1)}{n_\alpha},$$

$$\psi_i^{(a)}(t) = \frac{(1-t)}{d} + \frac{t}{n_\alpha(d-1)} + \frac{at(1-t)}{d}, \quad \text{for } 2 \leq i < d,$$

$$\psi_d^{(a)}(t) = \frac{(1-t)}{d} + \frac{t}{n_\alpha(d-1)} - \frac{at(1-t)(d-2)}{d}.$$

Then, for $t \in [0, 1]$ and $a \in [0, 1/d]$, all the entries are greater than zero, and $\sum_{i=0}^{d-1} \psi_i^{(a)}(t) = 1$. In other words, for each $a \in [0, 1/d]$, the curve $\psi^{(a)}(t)$ is formed by probability vectors. In addition, $\psi^{(a)}(t) \in \text{ri}(\Delta_d)$ for $t \in [0, 1]$ and $a \in [0, 1/d]$.

Next, we show that different curves do not have equivalent probability vectors in common, except u and v_{n_α}. First, we observe that their entries are decreasingly ordered:

$$\psi_1^{(a)}(t) - \psi_2^{(a)}(t) = t - \frac{td}{n_\alpha(d-1)} - \frac{at(1-t)}{d} \geq 0,$$

$$\psi_i^{(a)}(t) - \psi_{i+1}^{(a)}(t) = 0, \quad \text{for } 2 \leq i < d-1,$$

$$\psi_{d-1}^{(a)}(t) - \psi_d^{(a)}(t) = \frac{at(1-t)(d-1)}{d} \geq 0.$$

This implies that two probability vectors $\psi^{(a)}(t)$ and $\psi^{(a')}(t')$ are equivalent if, and only if, their entries are the same. There are three cases: (i) $t = t' = 0$, (ii) $t = t' = 1$ or (iii) $t = t'$ and $a = a'$. Therefore, different curves do not have equivalent probability vectors in common, except u and v_{n_c}.

Finally, we construct a family of IO-incomparable pure states with amount of coherence equal to c. From Lemma 1, we have that f_C is continuous on $\text{ri}(\Delta_d)$. Moreover, $\psi^{(a)}(t) \in \text{ri}(\Delta_d)$ for $t \in [0, 1]$ and $a \in [0, 1/d]$. Therefore, for each $a \in [0, 1/d]$, $f_C(\psi^{(a)}(t))$ is a continuous function of the variable $t \in [0, 1]$. Since $f_C(\psi^{(a)}(0)) = f_C(u)$, $f_C(\psi^{(a)}(1)) = f_C(v_{n_c})$ and $f_C(v_{n_c}) < c < f_C(u)$, there is a $t_a^* \in (0, 1)$ such that $f_C(\psi^{(a)}(t_a^*)) = c$. From the probability vectors $\psi^{(a)}(t_a^*)$, we define the pure states $|\psi^{(a)}\rangle = h(\psi^{(a)}(t_a^*))$, with map h as in Eq. (15).

All pure states of the family $\{|\psi^{(a)}\rangle\}_{a\in[0,1/d]}$ satisfy $C(|\psi^{(a)}\rangle) = \alpha$. Moreover, since the probability vectors of the family $\{\psi^{(a)}(t_a^*)\}_{a\in[0,1/d]}$ are not mutually equivalent, then their respective states are not mutually IO-equivalent. Now, let us prove that they are IO-incomparable. Given $a, a' \in [0, 1/d]$, suppose that $|\psi^{(a)}\rangle \underset{\text{IO}}{\to} |\psi^{(a')}\rangle$. Then, $|\psi^{(a')}\rangle \underset{\text{IO}}{\not\to} |\psi^{(a)}\rangle$. In terms of the probability vectors this implies that $\psi^{(a')} \preceq \psi^{(a)}$, but $\psi^{(a)} \not\preceq \psi^{(a')}$. Then, $\alpha = f_C(\psi^{(a)}) < f_C(\psi^{(a')}) = \alpha$, which is absurd. Therefore, the pure states of the family $\{|\psi^{(a)}\rangle\}_{a\in[0,1/d]}$ are mutually IO-incomparable. $\square$

Proof of Corollary 1.

Proof This result follows from Lemma 1 and Proposition 2. $\square$

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Author contributions

G.M.B. conceived the idea of the work and wrote the first version of the manuscript. M.L. contributed to the developments of the proofs, the formal analysis, and the revision and edition of the manuscript. H.F. and G.S. contributed to discussions and validation of the proofs. All authors discussed the results and contributed to the writing of the final version of the manuscript.

Competing interests

The authors declare no competing interests.

Additional information

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