ON DYNAMICS OF ELASTIC NETWORKS WITH RIGID JUNCTIONS WITHIN NONLINEAR MICROPOLAR ELASTICITY

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Within the nonlinear micropolar elasticity we discuss effective dynamic (kinetic) properties of elastic networks with rigid joints. The model of a hyperelastic micropolar continuum is based on two constitutive relations, i.e. static and kinetic ones. They introduce a strain energy density and a kinetic energy density, respectively. Here we consider three-dimensional elastic network made of three families of elastic fibers connected through massive rigid joints. So effective elastic properties are inherited from the geometry and material properties of fibers, whereas the kinetic (inertia) properties are determined by the both fibers and joints. Formulae for microinertia tensors are given.

KEY WORDS: *elastic network; rigid junction; micropolar elasticity; dynamics; homogenization; effective properties*

1. INTRODUCTION

The model of micropolar solids was presented in detail in centurial book by Cosserat brothers (Cosserat and Cosserat, 1909). Initially proposed in their theory of elasticity in 1986 (Cosserat and Cosserat, 1896), the model relies on a continuum that could be treated as a set of material particles which possess independent translational and rotational degrees of freedom, as in rigid body dynamics. As a result, we have stresses and couple stresses as static counterparts of translations and rotations. The Cosserat model was discussed by several authors, among others Eringen (1999), Nowacki (1986) and Maugin and Metrikine (2010); in Eremeyev et al. (2013) a complete overview of foundations of the theory and many solutions were presented.

Since several materials used in Civil and Mechanical engineering applications exhibit an internal micro-structure, nowadays, the growing interest in micropolar model relates to the possibility of a proper description of their complex inner micro-structure, when rotational interactions of material particles play an important role. Among such materials, it is worth to mention: granular media - including masonries (Baraldi et al., 2015; de Bellis and Addessi, 2011; Pau and Trovalusci, 2012; Reccia et al., 2018b; Shi et al., 2021); some classes of composites (Addessi et al., 2013, 2016; Leonetti et al., 2018; Pingaro et al., 2019) like random (Reccia et al., 2018a; Trovalusci et al., 2017, 2014, 2015) and regular particles composite; (Colatosti et al., 2021; Fantuzzi et al., 2019, 2020); nanotubes (Izadi et al., 2021a,b); beam-lattice materials (Berkache et al., 2022; Fleck et al., 2010) including foams and porous media(Lakes, 1987, 1986). For example, considering a beam-lattice material as an effective medium it seems quite natural that this medium has to inherit some beam properties, such as sensitivity to applied surface and volumetric couples.

In this work, attention is focused on 3D elastic networks with rigid connections. This typology of material belongs to beam-lattice structures, that find several applications in many engineering areas (Pan et al., 2020; **?**). Periodic networks of interconnected beams or rods, both in two- or three-dimensions, may have interesting mechanical properties related to their micro-structure, such as a higher performance in term of weight/stiffness, in acoustic and thermal responses, as well as in capacity of energy absorption, and greater deformation capacity before fracture/collapse. Moreover, these typology of micro-structured material may be found at all scales, from nano- and micro-scales, up to macro-scale. These aspects make their study a very topical issue, being their application suitable in several engineering areas (Dell'Isola et al., 2015; **?**). In particular, here a three-dimensional network of orthogonal deformable flexible fibres connected together by rigid massive joints, such that they remain orthogonal during deformations, is studied. This kind of material can be found in common applications such as fishnets or metal fences, and it can be considered as a typical example of meta-material exhibiting peculiar mechanical properties related to its internal structure. For such material, the adoption of micropolar model is crucial, thanks to the possibility of properly describe finite deformations of the fibers by means two independent kinematic variables, translations and rotations Eremeyev (2019). A discrete model is adopted where fibers are therefore modelled by the adoption of Cosserat curve Altenbach et al. (2013). At macroscale, the material is modelled as an equivalent micro-polar media (Eremeyev, 2018).

The paper is organized as follows. First, in Section 2 we briefly recall the governing equations of three- and one-dimensional media. Within the micropolar approach we have two kinematical descriptors, that are the fields of translations and rotations. A particular attention is paid to the kinetic constitutive relations, i.e. to the form of a kinetic energy function. We define a kinetic energy density as a positive quadratic form dependent on linear and angular velocities. For comparison, we also consider rigid body motions and the form of kinetic energy for a rigid body. In Section 3 we introduce a beam-lattice network with rigid massive joints. Here we formulate a semi-discrete model of the network considering coupled motion of beams and rigid joints. Using a linear approximation as in Eremeyev (2019), we derive a discrete model of the network. Within this model, we restrict ourselves to translations and rotations given in a finite set of points related to the centres of mass of joints. Comparing discrete model with a similar discrete approximation of three-dimensional (3D) micro-polar continuum, in Section 4 we introduce the notion of equivalent model. We call two models, i.e. of a network and of 3D medium, *equivalent* if their discrete counterparts have the same form. As a result, we can identify the 3D kinetic constitutive relations through inertia properties of beams of joints.

2. GOVERNING EQUATIONS OF MICROPOLAR MEDIA

Let us briefly introduce the basic equations of the micro-polar mechanics considering both three-dimensional (3D) and one-dimensional (1D) solids as well as rigid body dynamics.

2.1 Cosserat (micropolar) continuum

Let $\mathcal{B}$ be an elastic micropolar solid body. A deformation of $\mathcal{B}$ can be considered as an invertible mapping from a reference placement κ into a current placement $\chi(t)$, where t is time. For any point x of $\mathcal{B}$ we introduce its position vectors $\mathbf{X}$ and $\mathbf{x}$ and triples of unit orthogonal vectors called directors $\{\mathbf{D}_k\}$ and $\{\mathbf{d}_k\}$, $k = 1, 2, 3$, defined in κ and χ, respectively. In other words, the position vector and directors play a role of kinematical descriptors in the micropolar elasticity, see Eringen (1999); **?**. As a result, a deformation of $\mathcal{B}$ is given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \quad \mathbf{Q} = \mathbf{Q}(\mathbf{X}, t), \tag{1}$$

where $\mathbf{Q} = \mathbf{D}_k \otimes \mathbf{d}_k$ is a orthogonal tensor of micro-rotation and $\otimes$ is the dyadic product.

Considering hyperelastic materials we introduce a strain energy density W as a function of $\mathbf{x}$ and $\mathbf{Q}$ and their first gradients

$$W = W(\mathbf{x}, \mathbf{Q}, \nabla\mathbf{x}, \nabla\mathbf{Q}), \tag{2}$$

where ∇ is the three-dimensional nabla-operator as defined in Eremeyev et al. (2018); Simmonds (1994). Applying to (2) the principle of material frame indifference by Truesdell and Noll (2004), we get W as a function of two strain Lagrangian strain measures $\mathbf{E}$ and $\mathbf{K}$

$$W = W(\mathbf{E}, \mathbf{K}), \tag{3}$$

where

$$\mathbf{E} = \mathbf{F} \cdot \mathbf{Q}^T, \quad \mathbf{F} = \nabla\mathbf{x}, \quad \mathbf{K} \times \mathbf{I} = -(\nabla\mathbf{Q}) \cdot \mathbf{Q}^T,$$

see Pietraszkiewicz and Eremeyev (2009) for more details. Hereinafter "$\cdot$" and "$\times$" denote the dot and cross products, respectively, $\mathbf{F}$ is the deformation gradient, $\mathbf{I}$ is the 3D unit tensor, and superscript T stands for the transpose of a second-order tensor.

In order to complete the constitutive description of the micro-polar medium we introduce a kinetic energy density as a positive quadratic form of linear $\mathbf{v}$ and angular $\boldsymbol{\omega}$ velocities

$$K = \frac{1}{2}\rho\mathbf{v} \cdot \mathbf{v} + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{j} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathbf{j}_1 \cdot \mathbf{v}, \tag{4}$$

$$\mathbf{v} = \dot{\mathbf{x}}, \quad \boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T)_\times, \tag{5}$$

where ρ is a referential mass density, the overdot denotes the derivative with respect to t, $\mathbf{v}$ and $\boldsymbol{\omega}$ are the linear and angular velocities, respectively, $\mathbf{j}$ and $\mathbf{j}_1$ are tensors of micro-inertia. In addition we introduce the Gibbsian cross or the vectorial invariant of a second-order tensor as an operation which maps a tensor into a vector. For a dyad of two vectors it is defined as follows

$$(\mathbf{a} \otimes \mathbf{b})_\times = \mathbf{a} \times \mathbf{b},$$

and can be extended to any second-order tensor.

Let us note that the form of kinetic energy, i.e. the form of so-called kinetic constitutive relations, plays an essential role in micropolar dynamics Eringen (1999); Eringen and Kafadar (1976). It is worth to mention here similar situation in the case of thin-walled structures, where rotatory inertia may significantly change oscillations and wave propagation, see, e.g., Mindlin (1951); Pietraszkiewicz (2011).

The Lagrangian equation of motion take the form

$$\nabla \cdot \mathbf{T} + \rho\mathbf{f} = \rho\dot{\mathbf{v}} + (\boldsymbol{\omega} \cdot \mathbf{j}_1)^{\cdot}, \quad \mathbf{T} = \frac{\partial W}{\partial \mathbf{E}} \cdot \mathbf{Q}^T, \tag{6}$$

$$\nabla \cdot \mathbf{M} + (\mathbf{F}^T \cdot \mathbf{T})_\times + \rho\mathbf{c} = \mathbf{v} \times \mathbf{j}_1 \cdot \boldsymbol{\omega} + (\mathbf{j}_1 \cdot \mathbf{v})^{\cdot} + (\mathbf{j} \cdot \boldsymbol{\omega})^{\cdot}, \quad \mathbf{M} = \frac{\partial W}{\partial \mathbf{K}} \cdot \mathbf{Q}^T, \tag{7}$$

where $\mathbf{T}$ and $\mathbf{M}$ are the first Piola–Kirchhoff stress and couple stress tensors, respectively, $\mathbf{f}$ and $\mathbf{c}$ are the mass force and couple vectors.

2.2 Cosserat curve

Cosserat curve model constitutes a particular case of micro-polar media, see Antman (2005); Eremeyev et al. (2013); Rubin (2000). Indeed, this model could be treated as 1D micro-polar continuum embedded into the 3D Euclidean space. We again consider deformations of a Cosserat curve $\mathcal{C}$ as a mapping from a reference placement κ_C into a current placement $\chi_C(t)$. The position and orientation of a material particle z of $\mathcal{C}$ are determined through its position vector and directors defined in both placements. In particular, in κ_C we define a position vector $\mathbf{X}_C(s)$ and directors $\mathbf{D}_k(s)$ given as vector-valued functions of the referential arc-length parameter s. For χ_C, z has a position vector $\mathbf{x}_C(s,t)$ and directors $\mathbf{d}_k(s,t)$ given as a functions of s and t. So the kinematics of $\mathcal{C}$ is defined through the position vector $\mathbf{x}_C(s,t)$ and the micro-rotation tensor $\mathbf{Q}_C(s,t)$.

We introduce a strain energy density W_C defined per unit length in the reference placement as a function of $\mathbf{x}_C$ and $\mathbf{Q}_C$ and their derivatives with respect to s

$$W_C = W_C(\mathbf{x}_C, \mathbf{Q}_C, \mathbf{x}'_C, \mathbf{Q}'_C) \tag{8}$$

where the prime sands for the derivative with respect to s. Using the material frame-indifference principle we transform (8) into the form Altenbach et al. (2013); Bîrsan et al. (2012)

$$W_C = W_C(\mathbf{e}, \mathbf{k}), \qquad \mathbf{e} = \mathbf{x}'_C \cdot \mathbf{Q}_C^T, \quad \mathbf{k} = -\frac{1}{2}(\mathbf{Q}'_C \cdot \mathbf{Q}_C^T)_\times \tag{9}$$

with two vector-valued Lagrangian strain measures.

Within the Cosserat curve approach we introduce a linear $\mathbf{v}_C$ and angular $\boldsymbol{\omega}_C$ velocities given by

$$\mathbf{v}_C = \dot{\mathbf{x}}_C, \quad \boldsymbol{\omega}_C = -\frac{1}{2}(\dot{\mathbf{Q}}_C \cdot \mathbf{Q}_C^T)_\times, \tag{10}$$

so the kinetic energy density defined per unit length in κ_C is given by

$$K_C = \frac{1}{2}\rho_C \mathbf{v}_C \cdot \mathbf{v}_C + \frac{1}{2}\boldsymbol{\omega}_C \cdot \mathbf{j}_C \cdot \boldsymbol{\omega}_C + \boldsymbol{\omega}_C \cdot \mathbf{j}_{C1} \cdot \mathbf{v}_C, \tag{11}$$

where ρ_C is a referential linear mass density, $\mathbf{j}_C$ and $\mathbf{j}_{C1}$ are tensors of inertia.

Lagrangian equations of motion have the form

$$\mathbf{T}_C' + \rho_C \mathbf{f}_C = \rho_C \dot{\mathbf{v}_C} + (\boldsymbol{\omega}_C \cdot \mathbf{j}_{C1})^{\cdot}, \quad \mathbf{T}_C = \frac{\partial W_C}{\partial \mathbf{e}} \cdot \mathbf{Q}_C^T, \tag{12}$$

$$\mathbf{M}_C' + \mathbf{x}_C' \times \mathbf{T}_C + \rho_C \mathbf{c}_C = \mathbf{v}_C \times \mathbf{j}_{C1} \cdot \boldsymbol{\omega}_C + (\mathbf{v}_C \cdot \mathbf{j}_{C1})^{\cdot} + (\mathbf{j}_C \cdot \boldsymbol{\omega}_C)^{\cdot}, \quad \mathbf{M}_C = \frac{\partial W_C}{\partial \mathbf{k}} \cdot \mathbf{Q}_C^T, \tag{13}$$

where $\mathbf{T}_C$ and $\mathbf{M}_C$ are the first Piola–Kirchhoff stress and couple stress vectors, respectively, $\mathbf{f}_C$ and $\mathbf{c}_C$ are the mass force and couple vectors introduced per unit mass in the reference placement. One can easily find similarities between these equations and (6) and (7). In what follows we assume that the center of mass of a cross-section is chosen as a position of the Cosserat curve, so we have $\mathbf{j}_{C1} = 0$.

2.3 Rigid body dynamics

Finally, in order to describe a rigid joint motion let us briefly consider elements of rigid body dynamics. Let $\mathcal{B}$ be a rigid body loaded by a net force $\mathbf{N}$ and total torque $\mathbf{L}$. Following Eremeyev et al. (2013); Lurie (2001) the kinematics of $\mathcal{B}$ could be described as a translation of an arbitrary point O of $\mathcal{B}$ called the pole and a rotation about O. Using this description we introduce position vectors of another point P of $\mathcal{B}$ in reference κ_B and current χ_B placements as follows

$$\mathbf{X} = \mathbf{X}_0 + \boldsymbol{\xi}, \quad \mathbf{x}(t) = \mathbf{x}_0(t) + \boldsymbol{\eta}(t), \tag{14}$$

where $\mathbf{X}_0$ and $\mathbf{x}_0$ are position vectors of O, whereas $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are vectors $\overrightarrow{OP}$ directed from O to P in κ_B and χ_B, respectively. The latter vectors are related to each other through a rotation tensor $\mathbf{Q}$, so

$$\boldsymbol{\eta}(t) = \mathbf{Q}(t) \cdot \boldsymbol{\xi}. \tag{15}$$

As a result, the displacement vector of P is given by

$$\mathbf{u}(t) \equiv \mathbf{x}(t) - \mathbf{X} = \mathbf{x}_0(t) - \mathbf{X}_0 + \mathbf{Q}(t) \cdot \boldsymbol{\xi} - \boldsymbol{\xi}. \tag{16}$$

From (16) we get the formulae for linear $\mathbf{v}$ and angular $\boldsymbol{\omega}$ velocities

$$\mathbf{v} \equiv \dot{\mathbf{u}} = \mathbf{v}_0 + \boldsymbol{\omega} \times \boldsymbol{\eta}, \quad \mathbf{v}_0 = \dot{\mathbf{x}}_0, \quad \boldsymbol{\omega} = -\frac{1}{2}(\dot{\mathbf{Q}} \cdot \mathbf{Q}^T)_\times, \tag{17}$$

where $\mathbf{v}_0$ is a velocity of the pole.

The kinetic energy of $\mathcal{B}$ is given by

$$K_B = \frac{1}{2} \iiint\limits_{v_B} \rho_B \mathbf{v} \cdot \mathbf{v} \, dv, \tag{18}$$

where ρ_B is a mass density of $\mathcal{B}$ and v_B is a volume which $\mathcal{B}$ occupies in χ_B. With (17) we have

$$\begin{aligned} K_B =& \frac{1}{2} \iiint\limits_{v_B} \rho_B (\mathbf{v}_0 + \boldsymbol{\omega} \times \boldsymbol{\eta}) \cdot (\mathbf{v}_0 + \boldsymbol{\omega} \times \boldsymbol{\eta}) \, dv \\ =& \frac{1}{2} \iiint\limits_{v_B} \rho_B \, dv \mathbf{v}_0 \cdot \mathbf{v}_0 - \frac{1}{2} \boldsymbol{\omega} \cdot \iiint\limits_{v_B} \rho_B \boldsymbol{\eta} \times \mathbf{I} \times \boldsymbol{\eta} \, dv \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \iiint\limits_{v_B} \rho_B \mathbf{I} \times \boldsymbol{\eta} \, dv \cdot \mathbf{v}_0. \end{aligned} \tag{19}$$

Introducing the mass of $\mathcal{B}$ and the tensors of inertia by the formulae

$$M_B = \iiint\limits_{v_B} \rho_B \, dv, \quad \mathbf{J} = -\iiint\limits_{v_B} \rho_B \boldsymbol{\eta} \times \mathbf{I} \times \boldsymbol{\eta} \, dv, \quad \mathbf{J}_1 = \iiint\limits_{v_B} \rho_B \mathbf{I} \times \boldsymbol{\eta} \, dv, \tag{20}$$

we transform (19) into

$$K_B = \frac{1}{2} M_B \mathbf{v}_0 \cdot \mathbf{v}_0 + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{J} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathbf{J}_1 \cdot \mathbf{v}_0. \tag{21}$$

In what follows we use the center of mass of $\mathcal{B}$ as a pole, so $\mathbf{J}_1 = 0$. Using (15) we have

$$\mathbf{J} = \mathbf{q} \cdot \mathbf{J}_0 \cdot \mathbf{Q}^T, \quad \mathbf{J}_0 = \iiint\limits_{V_B} \rho_B \boldsymbol{\xi} \times \mathbf{I} \times \boldsymbol{\xi} \, dV, \tag{22}$$

where $\mathbf{J}_0$ is the referential tensor of inertia and V_B is a domain of $\mathcal{B}$ in κ_B.

Finally, the equations of motion of $\mathcal{B}$ have the form

$$M_B \dot{\mathbf{v}}_0 = \mathbf{N}, \quad (\mathbf{J} \cdot \boldsymbol{\omega})^{\cdot} = \mathbf{L}. \tag{23}$$

If we assume that considered previously 3D and 1D solids are rigid, we immediately come from (6) and (7) or (12) and

(13) to (23). In fact, for presented above model we face forces and couples as primary dynamic measures. Moreover, one can see, that similar to rigid body dynamics, tensors of inertia are presented in all micro-polar media, in general.

3. ELASTIC NETWORKS

Let us consider a regular elastic network made of three families of flexible fibres connected to each other through rigid joints as shown in Fig. 1. For simplicity we assume that all fibres (links) have the same mechanical and geometrical properties. This includes forms of strain and kinetic energies W_C, K_C, links length ℓ, ρ_C, $\mathbf{j}_C$, $\mathbf{J}_0$, M_B, etc. We mark each joint through indices i, j, and k, $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $k = 1, \ldots, l$. For example, a center of mass of the i, j, k-th joint we denote as $O_{i,jk}$, see Fig. 2.

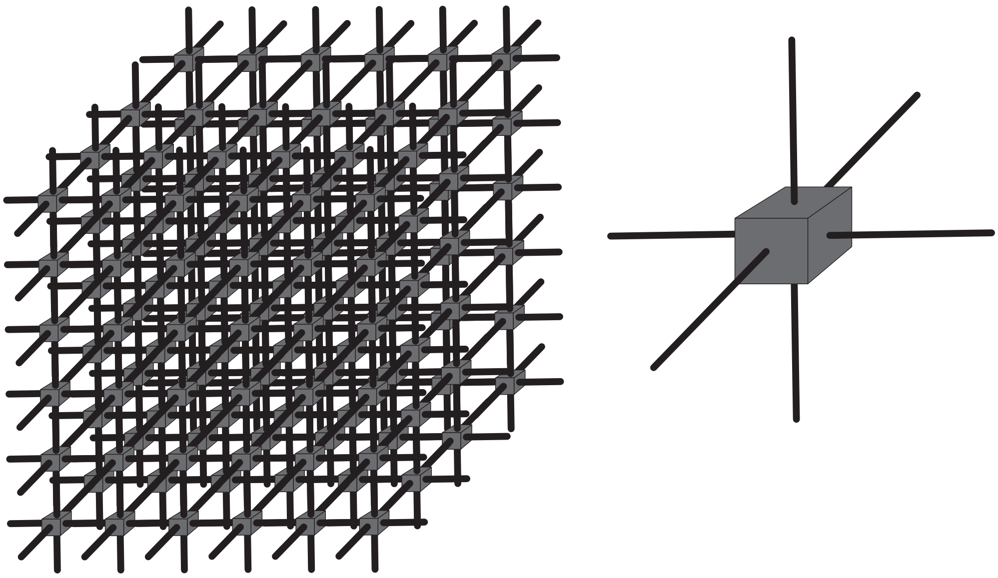

FIG. 1: Elastic network with rigid joints and one "elementary cell" of it.

Lagrangian equations of motion of the considered network consist of partial differential equations (PFEs) for elastic links and ordinary differential equations (ODEs) for joints. Following Eremeyev (2019) the latter system of equations takes the form

$$\mathbf{T}'_{C1,1} + \rho_C \mathbf{f}_{C1} = \rho_C \dot{\mathbf{v}}_{C1}, \quad s_1 \in (s_1^i, s_1^{i+1}), \quad i = 1, \dots, m-1, \tag{24}$$

$$\mathbf{M}'_{C1,1} + \mathbf{x}'_{C1,1} \times \mathbf{T}_{C1} + \rho_C \mathbf{c}_{C1} = (\mathbf{j}_{C1} \cdot \boldsymbol{\omega}_{C1})^{\cdot}, \tag{25}$$

$$\mathbf{T}'_{C2,2} + \rho_C \mathbf{f}_{C2} = \rho_C \dot{\mathbf{v}}_{C2}, \quad s_2 \in (s_2^j, s_2^{j+1}), \quad j = 1, \dots, n-1, \tag{26}$$

$$\mathbf{M}'_{C2,2} + \mathbf{x}'_{C2,2} \times \mathbf{T}_{C2} + \rho_C \mathbf{c}_{C2} = (\mathbf{j}_{C2} \cdot \boldsymbol{\omega}_{C2})^{\cdot}, \tag{27}$$

$$\mathbf{T}'_{C3,3} + \rho_C \mathbf{f}_{C3} = \rho_C \dot{\mathbf{v}}_{C3}, \quad s_3 \in (s_3^k, s_3^{k+1}), \quad k = 1, \dots, l-1, \tag{28}$$

$$\mathbf{M}'_{C3,3} + \mathbf{x}'_{C3,3} \times \mathbf{T}_{C3} + \rho_C \mathbf{c}_{C3} = (\mathbf{j}_{C3} \cdot \boldsymbol{\omega}_{C3})^{\cdot}, \tag{29}$$

$$M_B \dot{\mathbf{v}}_{i,j,k} = \mathbf{N}_{i,j,k}, \quad (\mathbf{J}_{i,j,k} \cdot \boldsymbol{\omega}_{i,j,k})^{\cdot} = \mathbf{L}_{i,j,k}. \tag{30}$$

Hereinafter we introduce Cartesian coordinate system $(x = s_1, y = s_2, z = s_3)$ and corresponding unit base vectors $\mathbf{i}_1$, $\mathbf{i}_2$, $\mathbf{i}_3$ in such a way that s_1 is the arc-length parameter of fibres aligned in ith direction and $\mathbf{i}_1$ is the tangent vector to this fibre in the reference placement, respectively. Coordinates s_2 and s_3 are chosen similarly for fibres which constitutes second and third families of the network, respectively. In addition we use notations

$$(\dots)'_{,1} = \frac{\partial}{\partial s_1}, \quad (\dots)'_{,2} = \frac{\partial}{\partial s_2}, \quad (\dots)'_{,3} = \frac{\partial}{\partial s_3}.$$

In Eqs. (24)–(30) $\mathbf{f}_{C1}$, $\mathbf{c}_{C1}$, $\mathbf{f}_{C2}$, $\mathbf{c}_{C2}$, $\mathbf{f}_{C3}$, $\mathbf{c}_{C3}$, $\mathbf{N}_{i,j,k}$, and $\mathbf{L}_{i,j,k}$ are corresponding forces and couples.

The corner stone of the further description of network motions is kinematic compatibility conditions, which describes mutual deformations of fibres connected via joints. Let us consider a contact point P of a fibre perfectly connected to a joint. One can find that the linear velocity of P is given by the formula

$$\mathbf{v}_C = \mathbf{v}_O + \boldsymbol{\xi} \times \boldsymbol{\omega}_O, \tag{31}$$

whereas the angular velocity of P and O are equal

$$\boldsymbol{\omega}_C = \boldsymbol{\omega}_O. \tag{32}$$

In (31) $\boldsymbol{\xi}$ is a vector $\overrightarrow{OP}$ from the center of mass O to P. As i, j, kth joint is connected to six fibres, we have six

$\boldsymbol{\xi}$=vectors which are denoted as

$$\boldsymbol{\xi}_{i_-,j,k}, \quad \boldsymbol{\xi}_{i_+,j,k}, \quad \boldsymbol{\xi}_{i,j_-,k}, \quad \boldsymbol{\xi}_{i,j_+,k}, \quad \boldsymbol{\xi}_{i,j,k_-}, \quad \boldsymbol{\xi}_{i,j,k_+},$$

see Fig. 2.

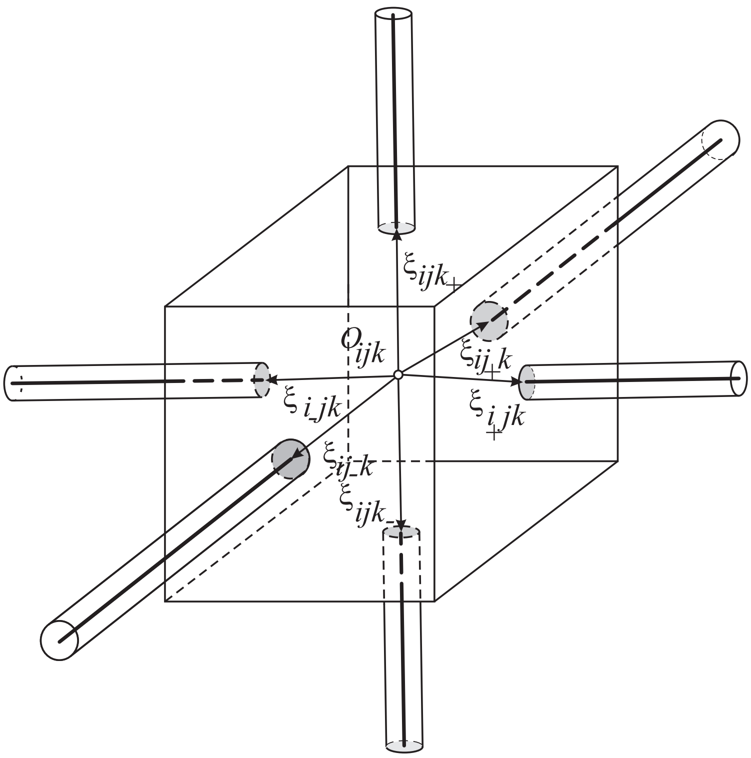

FIG. 2: Geometry in the vicinity of a i, j, k-joint.

Dynamic compatibility conditions could be derived using the least action principle

$$\delta\mathcal{H} = 0, \tag{33}$$

where $\mathcal{H}$ is the action functional. It could be written in a standard way

$$\mathcal{H} = \int_{t_1}^{t_2} (\mathcal{K}_N - \mathcal{W}_N)\, dt, \tag{34}$$

where $\mathcal{K}_N$ and $\mathcal{W}_N$ are kinetic and potential energies of the network given by the relations

$$\begin{aligned}\mathcal{K}_N =& \sum_{k=1}^{l}\sum_{j=1}^{n}\sum_{i=1}^{m-1}\int_{s_1^i}^{s_1^{i+1}} K_C(s_1)\,ds_1 + \sum_{k=1}^{l}\sum_{j=1}^{n-1}\sum_{i=1}^{m}\int_{s_2^j}^{s_2^{j+1}} K_C(s_2)\,ds_2\\ &+ \sum_{k=1}^{l-1}\sum_{j=1}^{n}\sum_{i=1}^{m}\int_{s_3^k}^{s_3^{k+1}} K_C(s_3)\,ds_3 + \sum_{k=1}^{l}\sum_{j=1}^{n}\sum_{i=1}^{m} K_B, \end{aligned} \tag{35}$$

$$\begin{aligned}\mathcal{W}_N =& \sum_{k=1}^{l}\sum_{j=1}^{n}\sum_{i=1}^{m-1}\int_{s_1^i}^{s_1^{i+1}} W_C(s_1)\,ds_1 + \sum_{k=1}^{l}\sum_{j=1}^{n-1}\sum_{i=1}^{m}\int_{s_2^j}^{s_2^{j+1}} W_C(s_2)\,ds_2\\ &+ \sum_{k=1}^{l-1}\sum_{j=1}^{n}\sum_{i=1}^{m}\int_{s_3^k}^{s_3^{k+1}} W_C(s_3)\,ds_3. \end{aligned} \tag{36}$$

Obviously, joints do not contribute in the potential energy of the network whereas their contribution to the kinetic energy could be significant.

In (33) variations of kinematic descriptors are also satisfy to (31) and (32):

$$\delta\mathbf{u}_C = \delta\mathbf{u}_O + \boldsymbol{\xi}\times\delta\boldsymbol{\psi}_O, \quad \delta\boldsymbol{\psi}_C = \delta\boldsymbol{\psi}_O, \tag{37}$$

where $\delta\mathbf{u}_C$, $\delta\mathbf{u}_O$, and $\delta\boldsymbol{\psi}_C$, $\delta\boldsymbol{\psi}_O$ are the virtual translations and vectors of virtual rotations, see Eremeyev et al. (2013).

Equations (24)–(30) constitute a semidiscrete model of a network. In order to introduce an effective homogenized medium we extend the approach by Eremeyev (2019) to the case of dynamics.

4. EQUIVALENT CONTINUUM MODEL OF A NETWORK AND ITS EFFECTIVE PROPERTIES

Considering statics of an elastic network with rigid joints Eremeyev (2019) introduced an equivalent micro-polar medium which strain energy density inherited elastic properties of network fibres. By equivalent model we mean a continuum medium which discretization coincides with discretization of the semi-discrete model. As a result, a strain energy density of the equivalent micro-polar model has the form

$$W_E = \tilde{W}_C(\mathbf{i}_1\cdot\mathbf{E}\cdot\mathbf{i}_1, \mathbf{i}_1\cdot\mathbf{K}\cdot\mathbf{i}_1) + \tilde{W}_C(\mathbf{i}_2\cdot\mathbf{E}\cdot\mathbf{i}_2, \mathbf{i}_2\cdot\mathbf{K}\cdot\mathbf{i}_2) + \tilde{W}_C(\mathbf{i}_3\cdot\mathbf{E}\cdot\mathbf{i}_3, \mathbf{i}_3\cdot\mathbf{K}\cdot\mathbf{i}_3), \tag{38}$$

where $\tilde{W}_C$ is a normalized strain energy of Cosserat curve, see Eremeyev (2019) for more details.

Here we extend the same approach for derivation of an equivalent kinetic energy K_E. First, let us introduce the effective mass density ρ_E by the formula

$$\rho_E V = 3\rho_C \ell + M_B, \tag{39}$$

where V is the volume of the minimal rectangular cuboid which includes the elementary cell. It could be calculated as follows $V = 3S_C\ell + V_B$, where S_C and V_B are an area of a fiber cross-section and volume of joint, respectively. Then, we replace integrals in (35) using the trapezoidal rule as follows

$$\begin{aligned}
\int_{s_1^i}^{s_1^{i+1}} K_C(s_1)\, ds_1 &= \frac{\ell}{2}\left[K_C(s_1^i) + K_C(s_1^{i+1})\right],\\
\int_{s_2^j}^{s_2^{j+1}} K_C(s_2)\, ds_2 &= \frac{\ell}{2}\left[K_C(s_2^j) + K_C(s_2^{j+1})\right],\\
\int_{s_3^k}^{s_3^{k+1}} K_C(s_3)\, ds_3 &= \frac{\ell}{2}\left[K_C(s_3^k) + K_C(s_3^{k+1})\right].
\end{aligned}$$

As a result, $\mathcal{K}_N$ became a function given at ends of fibres.

Let us now consider the kinetic energy density K_C at an end of a fibre. Using (31) and (32) we came to the equation

$$K_C = \frac{1}{2}\rho_C\left[\mathbf{v}_O\cdot\mathbf{v}_O + 2\mathbf{v}_O\cdot(\boldsymbol{\xi}\times\mathbf{I})\cdot\boldsymbol{\omega}_O - \boldsymbol{\omega}_O\cdot(\boldsymbol{\xi}\times\mathbf{I}\times\boldsymbol{\xi})\cdot\boldsymbol{\omega}_O\right] + \frac{1}{2}\boldsymbol{\omega}_O\cdot\mathbf{j}_C\cdot\boldsymbol{\omega}_O. \tag{40}$$

As a result, the kinetic energy of the elementary cell has the form

$$\begin{aligned}
VK_E = &3\ell\rho_C\left[\mathbf{v}_O\cdot\mathbf{v}_O + 2{\sum}'\mathbf{v}_O\cdot(\boldsymbol{\xi}'\times\mathbf{I})\cdot\boldsymbol{\omega}_O - {\sum}'\boldsymbol{\omega}_O\cdot(\boldsymbol{\xi}'\times\mathbf{I}\times\boldsymbol{\xi}')\cdot\boldsymbol{\omega}_O\right] + \boldsymbol{\omega}_O\cdot 3\ell\mathbf{j}_C\cdot\boldsymbol{\omega}_O\\
&+ \frac{1}{2}M_B\mathbf{v}_O\cdot\mathbf{v}_O + \frac{1}{2}\boldsymbol{\omega}_O\cdot\mathbf{j}_C\cdot\boldsymbol{\omega}_O.
\end{aligned} \tag{41}$$

Here we use summation ${\sum}'$ with respect to all connection points of i, j, k-th joint

$${\sum}'(\ldots) = (\ldots)\big|_{i_-,j,k} + (\ldots)\big|_{i_+,j,k} + (\ldots)\big|_{i,j_-,k} + (\ldots)\big|_{i,j_+,k} + (\ldots)\big|_{i,j,k_-} + (\ldots)\big|_{i,j,k_+}.$$

Finally, the effective kinetic energy could be written in a more compact way

$$K_E = \frac{1}{2}\rho_E\mathbf{v}_O\cdot\mathbf{v}_O + \mathbf{v}_O\cdot\mathbf{j}_{1E}\cdot\boldsymbol{\omega}_O + \frac{1}{2}\boldsymbol{\omega}_O\cdot\mathbf{j}_E\cdot\boldsymbol{\omega}_O, \tag{42}$$

where we have introduced two micro-inertia tensors

$$V\mathbf{j}_E = \mathbf{J} + 3\ell\mathbf{j}_C - \sum{}' \frac{\rho_C \ell}{2}(\boldsymbol{\xi}' \times \mathbf{I} \times \boldsymbol{\xi}'), \quad V\mathbf{j}_{1E} = \frac{1}{2}\sum{}'(\boldsymbol{\xi}' \times \mathbf{I}). \tag{43}$$

If we neglect inertia properties of fibres, i.e. consider massive joints and light fibres, these formulae could be simplified

$$V\mathbf{j}_E = \mathbf{J} - \sum{}' \frac{\rho_C \ell}{2}(\boldsymbol{\xi}' \times \mathbf{I} \times \boldsymbol{\xi}'), \quad V\mathbf{j}_{1E} = \frac{1}{2}\sum{}'(\boldsymbol{\xi}' \times \mathbf{I}), \tag{44}$$

or even as follows if we neglect also the mass of fibers

$$V\mathbf{j}_E = \mathbf{J}, \quad V\mathbf{j}_{1E} = 0. \tag{45}$$

One can see that mass and inertia properties of joints essentially affect effective kinetic energy density.

Formulae (43) or their simplified counterparts (44) and (45) could be extended for fibres of different properties and even for less regular networks when rigid joints connect to various number of fibres.

CONCLUSIONS

We have discussed kinetic constitutive equations for an elastic network from the point of view of the micro-polar elasticity. Here we restrict ourselves to elastic networks with rigid massive joints. Considering the network as a homogenized micro-polar continuum we have shown that elastic properties are determined through the properties of network links, whereas dynamic properties, i.e. micro-inertia tensors, depend on both mass distribution along elastic links and joints. In particular, for massive joints micro-inertia tensors are almost entirely determined through inertia properties of joints. Let us note that joints could be non-symmetric with respect to elastic links connections, which results in appearance of two micro-inertia tensors in a kinetic energy density of the homogenized micropolar medium. This will result in dynamic coupling between translational and rotational degrees of freedom, in general. Moreover, this brings in the micro-polar theory two micro-inertia tensors whereas usually they assume $\mathbf{j}_1 = 0$ and $\mathbf{j} = j\mathbf{I}$ with scalar measure of rotational inertia j, see, e.g., Eringen (1999). This case corresponds to symmetric material particles such as spheres. Dynamic properties introduced through two micro-inertia tensors could be taken into account considering material symmetry as in Eremeyev and Konopińska-Zmysłowska (2020); see also Vilchevskaya et al. (2022), where other references on micro-inertia tensors could be found. Derived here formulae for the micro-inertia tensors complete the description by Eremeyev (2019) of a network undergoing large deformations within micro-polar elasticity. Further development of this research will be devoted to improve the assessment of the dynamic behaviour of elastic networks with rigid junctions. Whit this purpose, the characteristics of wave propagation in media may be exploited. In particular, an effective correlation between the micro-structure and the way in which waves propagate in the medium, may be found. This relation may be very useful both to better understand the mechanical behaviour of such materials, both to improve their design and modelling in order to achieve specific required properties.

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