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The Snyder-de Sitter scalar $\varphi^4_\star$ quantum field theory in $D = 2$

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Abstract

We study the two-dimensional version of a quartic self-interacting quantum scalar field on a curved and noncommutative space (Snyder-de Sitter). We show that the model is renormalizable at the one-loop level and compute the beta functions of the related couplings. The renormalization group flow is then studied numerically, arriving at the conclusion that noncommutative-curved deformations can yield both relevant and irrelevant contributions to the one-loop effective action.

1. Introduction

Originally introduced by Snyder in his search for a natural cutoff in Quantum Field Theories (QFT) [1], noncommutative geometry has become an important branch in mathematics [2]. More important to us is the fact that noncommutative QFT has become one of the main tools in the attempt to gather information and experience on possible effects of Quantum Gravity. In few words, it carries out the conjecture that spacetime should show a granular behaviour at high

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enough energies [3]. Since the literature on the field is really vast, we refer the interested reader to the reports [4,5] and references therein.

Recently, there has been an increasing interest in the consideration of noncommutative theories in curved spaces. This is mainly driven by astronomical observations, which indicate that we live in a universe that is not flat [6]; see [7] for a recent review on quantum gravity phenomenology. The proposals include a curved κ-Minkowski spacetime built from Poisson-Lie algebras structures [8,9], a de Sitter fuzzy space group-theoretically constructed [10–12], deformations of QFTs in de Sitter using embeddings in higher-dimensional spaces [13], and Snyder curved spaces [14–16].

Following the last approach, in [17] we have proposed an action for a scalar self-interacting QFT in Snyder-de Sitter space (SdS), employing techniques which were previously developed for QFT in Snyder space [18–21]. SdS was first introduced in [14] under the name of triply special relativity. One major motivation for considering a QFT on it, is the fact that the modified kinetic term develops a harmonic term. Such a term is well-known for its consequences in the renormalization properties of the Grosse-Wulkenhaar model [22–24], curing the so-called UV/IR mixing problem [25].

The main differences between our model and the Grosse-Wulkenhaar one are given by the structure of the noncommutativity, that in our case is of Snyder type, rather than Moyal, and by the action, that contains more terms due to the symmetries of our model. A duality invariance is present also in our model, but is spoiled by our simplified choice of the action and by the approximations introduced in the calculations.

In spite of this similarity, the renormalization flow in SdS is much more complicated. At the one-loop level and at first order in the noncommutative and curvature parameters, the effective action develops divergences and requires the introduction of new counterterms in the original action. Even more relevant is the fact that the theory apparently has no fixed points. Instead, similarly to the Grosse-Wulkenhaar model and to the nonlocal, albeit covariant theory in [26,27], it admits an asymptotically free regime.

Nevertheless, the proposal remains interesting and deserves further study for three main reasons. First, it provides a natural explanation for the harmonic term, which was introduced by hand in the Grosse-Wulkenhaar model and could have a clear geometric/astronomical meaning, given its link to the cosmological constant in our model. Second, depending on the values of the coupling constants at low energies, the curvature parameter could display a change of sign as a consequence of its renormalization flow. Such a change of sign is compatible with observational data according to [28,29]. Third, it provides a way to circumvent both the Swampland conjecture as proposed in [30], and (even if not mentioned in [17]) the Gross-Coleman theorem, which establishes that there is no scalar asymptotic free theory in $D = 4$ if Poincaré invariance is taken for granted [31].

A toy model that can be useful to disclose some features of the quartic self-interacting scalar QFT in SdS is given by its two-dimensional version. We expect, in fact, that the two-dimensional model should preserve the main properties of the four-dimensional one while rendering the computations easier, in analogy with the Grosse-Wulkenhaar model. In the present article we will show that, in the small-deformations regime (meaning both small noncommutativity and curvature) the $D = 2$ model is one-loop renormalizable, i.e. no additional term should be added to those already present in the action. Moreover, the computed one-loop beta functions show that the presence of deformations allows some flows that differ from the usual commutative and flat ones.

The exposition of the article runs as follows. In Sec. 2 we rederive the main properties of the scalar self-interacting QFT in Euclidean SdS in arbitrary dimensions. Afterwards, in Sec. 4 we write down the divergent contributions and the beta functions for the two-dimensional case, at the order of one-loop. The study of the renormalization flow is explored in Sec. 5 and final remarks are made in Sec. 6. App. A is devoted to the expansion of the quartic potential for a small noncommutative parameter, while in App. B we obtain a closed form expression for some useful integrals involving generalized Laguerre functions.

2. A self-interacting scalar field in curved Snyder space

In this section we will review the main features of the model of a self-interacting scalar field in Euclidean SdS, which was originally derived in [17] and used in [32]. Let us recall that in the curved Snyder scenario in which we are interested, the momentum and position operators do not satisfy the canonical commutation relations, but close instead to a quadratic deformation of the algebra [14],

$$\begin{aligned}&[\hat{x}_i,\hat{x}_j]=\mathrm{i}\beta^2 J_{ij}, \qquad [\hat{p}_i,\hat{p}_j]=\mathrm{i}\alpha^2 J_{ij},\\&[\hat{x}_i,\hat{p}_j]=\mathrm{i}[\delta_{ij}+\alpha^2\hat{x}_i\hat{x}_j+\beta^2\hat{p}_j\hat{p}_i+\alpha\beta(\hat{x}_j\hat{p}_i+\hat{p}_i\hat{x}_j)],\end{aligned} \tag{1}$$

where $\hat{x}_0$ denotes the Euclidean time, $\hat{x}_1$ the only spatial component in $D=2$ (we will work with arbitrary D for convenience) and we have defined

$$J_{ij}=\frac{1}{2}(\hat{x}_i\hat{p}_j-\hat{x}_j\hat{p}_i+\hat{p}_j\hat{x}_i-\hat{p}_i\hat{x}_j). \tag{2}$$

At this point the reader may conjecture that the form of these commutators is rather arbitrary. However, there are two significant symmetries behind them.

First of all, the operators J_{ij} introduced in (2) satisfy the algebra of the Lorentz symmetries, even if the momentum and position operators are deformed. Indeed, a direct computation shows that the commutators among them are those of the usual Lorentz algebra, and their action on position and momentum operators is the one expected for vectors:

$$\begin{aligned}&[J_{ij},J_{kl}]=i(\delta_{ik}J_{jl}-\delta_{il}J_{jk}-\delta_{jk}J_{il}+\delta_{jl}J_{ik}),\\&[J_{ij},\hat{p}_k]=i(\delta_{ik}\hat{p}_j-\delta_{kj}\hat{p}_i),\\&[J_{ij},\hat{x}_k]=i(\delta_{ik}\hat{x}_j-\delta_{kj}\hat{x}_i).\end{aligned} \tag{3}$$

Second, there exists a duality symmetry under the exchange of position with momentum operators, which can be written as

$$\alpha\hat{x}_i\leftrightarrow\beta\hat{p}_i. \tag{4}$$

This symmetry, originally proposed by Born as the reciprocity principle [33], has been more recently studied by Langman and Szabo [34], and exploited by Grosse and Wulkenhaar to obtain an all-order renormalizable and constructable noncommutative QFT [23].

Turning back to (1), one can readily notice that this algebra has two obvious limits: the limit $\alpha\to 0$ corresponds to Snyder space (a noncommutative space or, equivalently, a curved momentum space), while $\beta\to 0$ corresponds to de Sitter space (a curved space). By combining noncommutativity and curvature, we are led to deform also the commutation between positions and momenta as in (1), in order to satisfy the Jacobi identities.

To construct our theory, we will define an action principle in SdS. We will therefore consider, as customary, an action for the real scalar field,

$$S = S_K + S_I, \tag{5}$$

given by the sum of a kinetic term, S_K, and a term of self-interaction, S_I, which we describe in the following.

2.1. The kinetic term

Let us begin by considering the kinetic term. A natural choice is to employ the square of the momentum operator acting on the scalar; in D dimensions we write

$$S_K = \frac{1}{2}\int \mathrm{d}^D x\, \varphi\left[(\hat{p}(\hat{p}\varphi)\right] + m^2\varphi^2 . \tag{6}$$

The key point in the following is that, as can be guessed from (1), $\hat{p}$ will not simply act as a derivative. Notably, the operator $\hat{p}^2$ is Lorentz invariant, which follows from the algebraic properties in Eq. (3). Moreover, we will see that $\hat{p}$ will be Hermitian, meaning that $\hat{p}^2$ will be positive. A further discussion on this choice (and a clarification of the meaning of the brackets) will be left to Sec. 2.3, after we have discussed the interaction term.

To analyze Eq. (6), we will employ a fundamental fact of SdS: by using a nonunitary, linear and evidently noncanonical transformation, we can transform the algebra to that of Snyder space. In [35,16] it has been shown that defining new position X and momentum P operators as

$$\hat{x}_i =: X_i + t\,\frac{\beta}{\alpha} P_i, \qquad \hat{p}_i =: (1-t)P_i - \frac{\alpha}{\beta} X_i, \tag{7}$$

one obtains the usual Snyder algebra in the form

$$[X_i, X_j] = \mathrm{i}\beta^2 J_{ij}, \qquad [P_i, P_j] = 0, \qquad [X_i, P_j] = \mathrm{i}(\delta_{ij} + \beta^2 P_i P_j), \tag{8}$$

for any arbitrary value of the parameter t. There is a subtle point with this step: as we will see later, the fact that this transformation is singular for small β (and α if t is nonvanishing), prevents the commutative and flat limit. Choosing $t = 0$, we can conserve a smooth flat limit; the price that we have to pay is the fact that we won't be able to recover the curved commutative limit, but only the commutative flat one. An interesting question is whether these singularities are related to the existence of quantum phases, i.e. related to singular transitions in the properties of the associated group, or are just a consequence of the chosen change of variables [36,37].

Another negative point is the fact that the Born principle does not hold for the variables P and X, and the duality gets hidden. Even if these two disadvantages are not minor ones, working in Snyder space greatly simplifies the subsequent computations. Given that (7) is the only linear transformation that accomplishes this job without introducing further dimensional parameters, we will use it. Alternatively, as a consequence of the duality (4), one could consider a transformation that goes from SdS to de Sitter space, corresponding to $t = 1$.

Starting from (7), we can define a non-linear realization for the operators in Snyder space, in terms of canonical operators x and[1] p

$$P_i =: p_i = -\mathrm{i}\partial_i, \quad X_i =: x_i + \beta^2 x_j p^j p_i = x_i - \beta^2 x^j \partial_j \partial_i . \tag{9}$$

[1] We use Einstein's convention of sum.

Since the operators X_i defined in this way are non-hermitian, we need to apply a symmetrization as described in [18],

$$X_i \to X_i = \hat{x}_i = x_i + \frac{\beta^2}{2}(x_j p^j p_i + p_i p^j x_j). \tag{10}$$

In this way we obtain a simple expression for the momentum:

$$\hat{p}_i = p_i - \frac{\alpha}{\beta} x_i - \frac{\alpha\beta}{2}(x_j p^j p_i + p_i p^j x_j). \tag{11}$$

On physical grounds, we expect both α and β to be small, at least at small energies scales, since they are to be associated with the cosmological constant and the noncommutativity. However, their quotient may be of order unity. Taking this into account, we insert eq. (11) into (6); up to order α^2 and β^2 we get

$$S_K \approx \frac{1}{2}\int \mathrm{d}^D x\, \varphi \left(p^2 + \frac{\alpha^2}{\beta^2} x^2 + 2\alpha^2 x_j p^j p^i x_i + m_{\text{eff}}^2 \right) \varphi, \tag{12}$$

with an effective mass given by[2]

$$m_{\text{eff}}^2 := m^2 - \frac{\alpha^2}{2} D(D+1). \tag{13}$$

Some remarks are now in order. At the end of this procedure, the kinetic term has developed a harmonic contribution which is proportional to the quotient of the curvature and the noncommutativity. This observation, which is similar to the one made in [38], could be a crucial concept in the understanding of the Grosse-Wulkenhaar model [23]. Indeed, such a term was originally introduced by hand in [23] in the so-called vulcanization of the $\lambda\phi_\star^4$ model in Moyal plane, which has been proved to be all-order perturbatively renormalizable.

Additionally, the action (12) displays many common features that arise in commutative curved spaces. First, it is not strange to encounter infrared divergences like the ones that could produce a negative mass as in expression (13). In the commutative de Sitter space, one encounters similar problems arising from the representations of the isometry group $SO(D,1)$ [39]; radiative problems for small masses are believed to be caused by a breaking of the perturbative expansion [40]. Moreover, terms involving both momentum and coordinates, such as the dilation-type operator $x \cdot p$, arise directly from the Laplacian in curved space.

2.2. *The self-interaction term*

As interaction term, we choose a quartic potential for the scalar field φ:

$$S_I = \frac{\lambda}{4!}\int \mathrm{d}^D x\, \varphi(\hat{x})\big[\varphi(\hat{x})\big(\varphi^2(\hat{x})\big)\big]. \tag{14}$$

As a consequence of the coordinate operators' noncommutativity, this expression is not easy to handle. In order to simplify the computations, we can make use of the noncommutative as well as nonassociative star product derived in [18] for the realization in Eqs. (10) and (11),

[2] Notice that in the published version of [17] there are two mistakes: the effective mass appears in the action S and not in its second variation, and the relevant factor is different by a factor five. In spite of these modifications, the qualitative conclusions derived in [17] are still valid.

$$e^{\mathrm{i}k\cdot x}\star e^{\mathrm{i}q\cdot x}:=\frac{e^{\mathrm{i}D(k,q)\cdot x}}{(1-\beta^2 k\cdot q)^{(D+1)/2}}, \tag{15}$$

where the vector D_μ is given by

$$D_\mu(k,q):=\frac{1}{1-\beta^2 k\cdot q}\left[\left(1-\frac{\beta^2 k\cdot q}{1+\sqrt{1+\beta^2k^2}}\right)k_\mu+\sqrt{1+\beta^2k^2}q_\mu\right]. \tag{16}$$

Although the nonassociativity of the product means that the gathering of the different fields in (14) is not unique, at first order in β^2 all the different possible associations lead to the same result. This affirmation is valid at the level of the action, so it does not depend on the choice of the kinetic term.

One other relevant property of the star product defined in expression (15) is that the product of two functions, under the integral sign, is given by the usual (commutative) product [18], i.e.

$$\int \mathrm{d}^D x f(x)\star g(x)=\int \mathrm{d}^D x f(x)g(x). \tag{17}$$

We can now replace the position operators in (14) by star products to finally obtain the equivalent expression

$$\begin{aligned} S_I&=\frac{\lambda}{4!}\int \mathrm{d}^D x\,\varphi(x)\star\left[\varphi(x)\star\left(\varphi(x)\star\varphi(x)\right)\right]\\ &=\frac{\lambda}{4!}\int \mathrm{d}^D x\,\left[\varphi^4+\beta^2\,\varphi^4_{(1)}+\mathcal{O}(\beta^4)\right], \end{aligned} \tag{18}$$

in terms of the first noncommutative correction

$$\varphi^4_{(1)}:=\frac{2}{3}\varphi^3\Big((D+2)+2x^\mu\partial_\mu\Big)\partial^2\varphi. \tag{19}$$

Notice that this expansion involves the Laplacian as well as a dilation operator acting on φ. On one side, the Laplacian is a semi-negative defined operator; on the other side, the involved dilation operator is not Hermitian. Actually, the operator $2x^\mu\partial_\mu+D$ is anti-Hermitian when acting on a complex field, and could thus prompt a unitarity problem. However, acting on a real field gives a real result. This fact entails a possible loss of positivity in the Lagrangian once the noncommutative corrections become big enough or, in other words, opens the door to possible instabilities and phases. Apparently, this issue has passed unnoticed in the literature [41,17,19,20]; instead, the positivity of the effective potential can be checked [20]. We will not deal with it in the present article, leaving it for a future presentation.

2.3. More on the kinetic term

In the last part of this section, we revisit some aspects of the kinetic term that are worth discussing.

The first one is related to the $\star$-product introduced for the interaction term. One could wonder if such product may have a role in the construction of the kinetic term. The answer is that Eq. (11) already takes it implicitly into account. Indeed, remembering the property (17), one can show that

$$S_K=\frac{1}{2}\int \mathrm{d}^D x\,\varphi\star\left[(\hat{p}\star(\hat{p}\star\varphi)\right]+m^2\varphi\star\varphi+\mathcal{O}(\gamma^4), \tag{20}$$

where we denote with γ the scale of either β and α. Additionally, given that we are explicitly including the star product, the replacement $\hat{p}_i \to p_i - \frac{\alpha}{\beta} x_i$ should be understood (cf. Eq. (7) with $t \equiv 0$).

This is of course not the only possibility, since one could gather the terms differently and obtain distinct kinetic operators. However, they won't satisfy the properties of positivity and Lorentz invariance. In order to prove this, notice first of all that if we want to reproduce the commutative results (via the $\alpha \to 0$ and subsequent $\beta \to 0$ limits), the ordering with a $\hat{p}$ on the right should be discarded (the derivative will act trivially on a constant factor). Additionally, from all the $\star$-products inside the action, the last (the more external one) can be removed because of property (17); for this reason the $\star$-product in the mass term is irrelevant so in the rest of this section we will set $m = 0$. We are thus left with a small number of possibilities that can be analyzed case by case.

A straightforward computation shows that, following the steps of Sec. 2.1, we have

$$\begin{aligned} S_K^{(2)} &:= \frac{1}{2} \int \mathrm{d}^D x\, \varphi \left[(\hat{p} \star \hat{p}) \star \varphi \right] \\ &= S_K - \mathrm{i}\alpha\beta \int \mathrm{d}^D x\, \varphi p^2 \varphi + \mathcal{O}(\gamma^4). \end{aligned} \tag{21}$$

This kinetic term is complex at first order in the deformation parameters and therefore we discard it. Another possibility, trying to mimic the commutative case, is given by

$$\begin{aligned} S_K^{(3)} &:= -\frac{1}{2} \int \mathrm{d}^D x\, \left(\hat{p} \star \varphi\right) \left(\hat{p} \star \varphi\right) \\ &= -\frac{1}{2} \int \mathrm{d}^D x \left[\left(p^i - \frac{\alpha}{\beta} x^i - \frac{\alpha\beta}{2} (x_k p^k p^i + p^i p_k x^k) \right) \varphi \right] \\ &\qquad \times \left[\left(p_i - \frac{\alpha}{\beta} x_i - \frac{\alpha\beta}{2} (x_j p^j p_i + p_i p^j x_j) \right) \varphi \right]. \end{aligned} \tag{22}$$

In this case, the problem can be seen integrating by parts to obtain an operator acting on just one field: there is a change of sign in the p_i term but not in the others, so that this choice is different from S_K. The difference can be proved to be

$$\Delta S_K \propto \int \mathrm{d}^D x\, \varphi \left(-\frac{\alpha^2}{\beta^2} x^2 - 2\alpha^2 x^i p_i p^j x_j + \alpha^2 \tfrac{D(D+1)}{2} + \mathrm{i}\frac{\alpha\beta}{2} p^2 + \mathrm{i}\frac{\alpha}{\beta}\frac{D}{2} \right) \varphi + \cdots, \tag{23}$$

where the dots denote higher orders in α and β. This term is not real, and is thus also discarded.

A second important comment, is that the action (6) is Lorentz invariant but not fully (A)dS-invariant. The construction of an (A)dS-invariant action is rather involved and will be studied in subsequent publications.

The third remark regards the t parameter that was introduced in Eq. (7) and afterwards set to zero. If this choice is not made, the fields become formal functions of X_i and P_i, with an (at least apparent) all-order dependence in t, given that β/α is not necessarily small. Although it would be desirable to show that the t dependence drops out since it is not a parameter of our algebra, we currently have no proof at our disposal.

Lastly, in this article we consider an Euclidean action, for which the kinetic operator is positive. Working directly with the Minkowskian version seems not feasible, since in that case the kinetic operator has no definite sign. Formally, one can try to perform a Wick rotation at the end of the computations; one should keep in mind the validity of this process would probably be not devoid of subtleties, as is the case in Minkowski Moyal QFT [42] and in usual QFT in curved spaces [43] (see also [44] for recent results on analytic continuation of QFT in dS spacetime).

3. General aspects of the free theory

Before studying the interacting theory, we will consider in this section the free theory with a modified kinetic term defined as in Eq. (12).

3.1. On the operator defining the kinetic term

One way to better understand the kinetic term is by computing the eigenfunctions and eigenvalues of the operator involved in its definition:

$$K := \left(- \partial^2 + \omega^2 x^2 + 2\alpha^2 x_j p^j p^i x_i \right). \tag{24}$$

Since we are interested in a small α, we can proceed with a perturbative computation. We will set $D = 2$ for convenience. The unperturbed operator corresponds thus to a harmonic oscillator in a plane. In polar coordinates,[3] its normalized eigenfunctions can be shown to be proportional to the generalized Laguerre polynomials $L_n^{(k)}(x)$ [45],

$$\phi_{n,l}(\rho,\varphi) := \sqrt{\frac{n!}{\pi(n+|l|)!}\omega^{|l|+1}} e^{-\omega\rho^2/2} \rho^{|l|} L_n^{(|l|)}(\omega\rho^2) e^{\mathrm{i}l\varphi}, \quad n = 0,\, 1\cdots,\, l \in \mathbb{Z}, \tag{25}$$

while the corresponding eigenvalues are

$$\lambda_{n,l} = (2n + |l| + 1)\,\omega. \tag{26}$$

The spectrum of K differs from the one in the undeformed case, which was continuous. In the present case the eigenvalues are integral multiples of the frequency $\omega = \alpha/\beta$, so that the limit of continuous spectrum is approached by taking $\alpha \to 0$ with β fixed.

To proceed to first order in perturbations, notice that

$$\begin{aligned} (p\cdot x) f(\rho,\varphi) &= -\frac{\mathrm{i}}{\rho}\partial_\rho(\rho^2 f(\rho,\varphi)), \\ (x\cdot p) f(\rho,\varphi) &= -\mathrm{i}\rho\partial_\rho f(\rho,\varphi)), \end{aligned} \tag{27}$$

so that the perturbing operator $x_j p^j p^i x_i$ commutes with the angular momentum operator. This means that we can safely compute the shifts in the eigenvalues with standard non-degenerate perturbation theory, i.e.

$$\begin{aligned} \Delta\lambda_{nl}^{(\alpha^2)} :&= 2\alpha^2 \int \mathrm{d}^2 x\, \phi_{n,l}(x\cdot p)(p\cdot x)\phi_{n,l} \\ &= 2\alpha^2 \int \mathrm{d}^2 x\, |(p\cdot x)\phi_{n,l}|^2. \end{aligned} \tag{28}$$

Employing the well-known formula for the derivatives of generalized Laguerre polynomials,

[3] The radial distance and angle respectively correspond to ρ and φ.

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} L_n^{(l)} = (-1)^k L_{n-k}^{(l+k)}, \quad \text{if } k \leq n, \tag{29}$$

we obtain the following formula for the integrand in Eq. (28) (valid for $n > 0$; the $n = 0$ case should be considered separately):

$$(p \cdot x)\phi_{n,l} = -\mathrm{i}\sqrt{\frac{n!}{\pi(n+|l|)!}}\,\omega^{|l|+1} e^{\mathrm{i}l\phi} e^{-\frac{1}{2}(\rho^2\omega)} \rho^{|l|} \\ \times \left(\left(|l| - \rho^2\omega + 2\right) L_n^{|l|}\left(\rho^2\omega\right) - 2\rho^2\omega L_{n-1}^{|l|+1}\left(\rho^2\omega\right)\right). \tag{30}$$

Performing an appropriate change of variables, the calculation of the first perturbative correction has been thus reduced to the computation of integrals involving a product of two generalized Laguerre polynomials, the corresponding exponential measure and powers of its argument. In spite of the existence of some general results [46], they are not directly applicable to our computation. We leave to App. B the explanation of how to compute those integrals; employing Eq. (64), we can express the first-order contribution to the eigenvalues as

$$\Delta\lambda_{nl}^{(\alpha^2)} = 2\alpha^2\Big[(2n+1)\,|l| + 2\left(n^2+n+1\right)\Big]. \tag{31}$$

This formula deserves some comments. First, notice that Eq. (31) provides an explicit quantitative proof that the kinetic operator is positive, since $\Delta\lambda_{nl}^{(\alpha^2)} \geq 4\alpha^2$ and therefore always compensates the term in the effective mass proportional to α^2, cf. Eq. (13).

Secondly, the corrections increase with n and $|l|$; hence, the contribution of the potential can be considered small only for a finite number of eigenfunctions. Heuristically, this can be understood from the functional dependence of the potential: if the expectation values of p or x become large (which is what happens when n or $|l|$ increase), then the potential will naturally get larger. Coming back to the field theory, the term containing the operator $x_j p^j p^i x_i$ will intuitively become large for configurations that have sizeable momentum components or space extensions. However, in our Euclidean setup they are suppressed by the kinetic term, so the small-α expansion of the quantum theory is expected to be well-defined.

3.2. The propagator at tree level

As we have seen in the previous section, the peculiarities of the model are already displayed at the level of the kinetic term. This implies that also the propagator is intrinsically modified at the tree level with respect to the standard case. We can define it as

$$\begin{aligned} G_\alpha &:= (-\partial^2 + \omega^2 x^2 + m^2 + 2\alpha^2 (x\cdot p)(p\cdot x))^{-1} \\ &= G_0 + \alpha^2 G_\alpha^{(1)} + \cdots, \\ G_0 &:= (-\partial^2 + \omega^2 x^2 + m^2)^{-1}, \\ G_\alpha^{(1)} &:= -2G_0 (x\cdot p)(p\cdot x) G_0, \end{aligned} \tag{32}$$

where G_0 denotes the unperturbed propagator, i.e. the one corresponding to $\alpha = 0$, and, as previously done with other physical quantities, we have performed a perturbative expansion in α.

A direct computation of the propagator in terms of its eigenfunctions using the results of the previous section seems rather involved. Instead, one simple way to compute it is to recall the relation between the inverse A^{-1} of an operator and its heat-kernel K_A,

$$A^{-1}(x,x') = \int_0^\infty \mathrm{d}\beta K_A(x,x';\beta), \tag{33}$$

which can be easily proved. In Cartesian coordinates, the oscillator-like operator G_0 can be divided into a sum of operators involving one single coordinate; this translates at the heat-kernel level into a product of heat-kernels. Recalling the well-known result by Mehler [47] for the heat-kernel with harmonic terms, we find in D dimensions

$$G_0(x,x') = \int_0^\infty \mathrm{d}\beta \left(\frac{\omega}{2\pi \sinh 2\omega\beta}\right)^{D/2} e^{-\beta m^2} e^{-\frac{\omega}{2\sinh 2\omega\beta}\left\{(x^2+x'^2)\cosh 2\omega\beta - 2x\cdot x'\right\}}. \tag{34}$$

Notice that one can also write Mehler's kernel as a function of $x_\pm := (x \pm x')/2$,

$$G_0(x,x') = \int_0^\infty \mathrm{d}\beta \left(\frac{\omega}{2\pi \sinh 2\omega\beta}\right)^{D/2} e^{-\beta m^2} e^{-\frac{1}{4}x_+^2\omega\tanh(\beta\omega) - \frac{1}{4}x_-^2\omega\coth(\beta\omega)}, \tag{35}$$

so that one can readily see the divergence of the propagator arising at coinciding points $x = x'$.

According to Eq. (32), the first perturbative correction to the propagator is given by the contribution

$$\begin{aligned} G_\alpha^{(1)}(x,x') = 2\int \mathrm{d}^D x'' \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_2 \left(\frac{b_1 b_2}{\pi^2}\right)^{D/2} e^{-(\beta_1+\beta_2)m^2} \\ \times e^{-a_1(x^2+x''^2)+2b_1 x\cdot x''}(x''\cdot\partial_{x''})(\partial_{x''}\cdot x'')e^{-a_2(x''^2+x'^2)+2b_2 x''\cdot x'}, \end{aligned} \tag{36}$$

where we have abbreviated

$$a_i := \frac{\omega}{2}\coth 2\omega\beta_i, \tag{37}$$

$$b_i := \frac{\omega}{2\sinh 2\omega\beta_i}. \tag{38}$$

Notice that the integral in x'' is Gaussian and thus computable; a straightforward calculation gives:

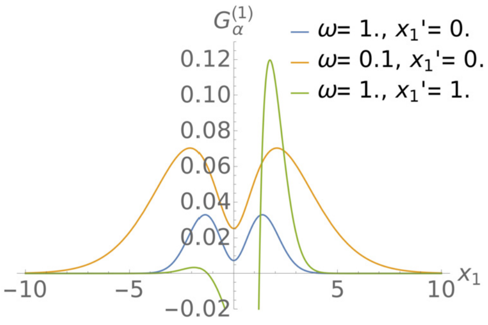

Fig. 1. The first correction to the tree propagator, $G_\alpha^{(1)}(x,x')$, computed numerically in $D=2$ as a function of $x=(0,x_1)$. We set all quantities in units of the mass and $x'_0=0$. We choose ($\omega=1$, $x'_1=0$), ($\omega=0.1$, $x'_1=0$) and ($\omega=1$, $x'_1=1.$) for the blue, orange and green line respectively.

$$
\begin{aligned}
G_\alpha^{(1)}(x,x') = \int_0^\infty & \mathrm{d}\beta_1 \mathrm{d}\beta_2 \left(\frac{b_1 b_2}{\pi(a_1+a_2)}\right)^{D/2} e^{-(\beta_1+\beta_2)m^2} e^{-a_1x^2-a_2x'^2} \frac{e^{\frac{(b_2x'+b_1x)^2}{a_1+a_2}}}{(a_1+a_2)^4} \\
\times \Bigg[& Da_2(a_1+a_2)^2\big(-(3+D)a_1+(1+D)a_2\big) \\
& + 2{b_1}^2x^2a_2\Big(4{b_1}^2x^2a_2-(a_1+a_2)\big((3+D)a_1-(5+3D)a_2\big)\Big) \\
& + 2{b_2}^2x'^2a_1\big(4{b_2}^2x'^2a_1+3a_1^2+Da_1^2-8{b_1}^2x^2a_2 \\
& \qquad -2a_1a_2-2Da_1a_2-5a_2^2-3Da_2^2+8b_1b_2(a_1-a_2)x'\cdot x\big) \\
& - 2b_1b_2x'\cdot x\Big(8{b_1}^2x^2(a_1-a_2)a_2 \\
& \qquad -(a_1+a_2)\big((1+D)a_1^2-4(3+D)a_1a_2+3(1+D)a_2^2\big) \\
& \qquad -4b_1b_2(a_1-a_2)^2x'\cdot x\Big)\Bigg].
\end{aligned}
\tag{39}
$$

This expression can be studied numerically in $D=2$. In Fig. 1 we have plotted the propagator as a function of x for different values of ω and x', employing the mass as unit. To further simplify, we have set $x_0 \equiv x'_0 \equiv 0$. On the one hand, if $x'_1=0$, the function is symmetric with respect to the $x_1=0$ line, as happens for the blue ($\omega=1$) and orange ($\omega=0.1$) cases. As ω decreases, the peaks of $G_\alpha^{(1)}$ get larger since we approach the usual commutative situation where the internal propagators should be UV-regularized.

On the other hand, the $x'_1 \neq 0$ case is more involved; see for example the green line, which corresponds to ($\omega=1$, $x'_1=1$). In this case the plot is not symmetric: the bump on the left (right) gets larger for x'_1 smaller (bigger) than zero. Moreover, the function develops a logarithmic divergence at $x_1=x'_1$. This fact can be analytically derived from Eq. (39). Indeed, notice that the divergence for small propertimes β_i is tamed only by an exponential term that depends on x_-^2, as happens in the commutative case; expanding carefully for small x_- we then have

$$
\begin{aligned}
G^{(1)}_\alpha(x,x')\big|_{x_-\to 0} \to \int_0^\infty \mathrm{d}\beta_1 \mathrm{d}\beta_2\, e^{-\frac{x_-^2}{4(\beta_1+\beta_2)}} \Bigg[-\frac{x_+^2}{32\pi(\beta_1+\beta_2)^2} \\
+ \frac{32\beta_1(3\beta_1-5\beta_2)+x_+^2(4m^2+\omega^2 x_+^2)(\beta_1+\beta_2)^2}{128\pi(\beta_1+\beta_2)^3}\Bigg] + \cdots \\
= \frac{x_+^2 \log(x_-^2)}{32\pi} + \cdots ,
\end{aligned}
\tag{40}
$$

where the dots denote regular terms in the coinciding point limit. This is the same divergence present in the commutative situation, meaning that, at the perturbative level, the UV region is not strongly modified.

4. One-loop contribution to the effective action

4.1. The worldline formalism

Our focus will be centred on the computation of the divergent one-loop contributions arising for the SdS model, cf. eq. (6), from which the running of the coupling constants in the Modified Minimal Subtraction scheme can be read. One economic way to do so is by employing the Worldline Formalism, which is closely related to heat-kernel techniques. Some recent applications include the computation of transition amplitudes in curved spacetimes [48], the calculation of fermion propagators in electromagnetic backgrounds [49] and the consideration of vacuum energies with generalized boundary conditions [50]. The noncommutative version of this technique [51] has been extended to consider all-order harmonic contributions in the study of the Grosse-Wulkenhaar model [21]; it has also proven helpful even in presence of terms with higher momentum powers in the action [41].

In the present case, the computation goes at follows. One introduces first the classical (or mean) field ϕ; for example, in a path integral approach we define

$$
\phi(x) := \frac{\int \mathcal{D}\varphi\, e^{-S[\varphi]+\int dx\, J(x)\varphi(x)}\varphi(x)}{\int \mathcal{D}\varphi\, e^{-S[\varphi]+\int dx\, J(x)\varphi(x)}}. \tag{41}
$$

The one-loop contribution to the effective action can be written in terms of it as

$$
\Gamma_{1-\text{loop}}[\phi] = S[\phi] - \frac{1}{2}\int_0^\infty \frac{\mathrm{d}T}{T}\mathrm{Tr}\left(e^{-T\,\delta^2 S}\right), \tag{42}
$$

where the kernel of the operator $\delta^2 S$ is the second variation of the action evaluated at the classical field ϕ:

$$
\delta^2 S f(x) := \int dy \frac{\delta^2 S}{\delta\varphi(x)\delta\varphi(y)}[\phi] f(y). \tag{43}
$$

Because of the nonlocality encoded in the star product of the interaction term, this operator will involve an infinite number of derivatives, rendering the computation highly nontrivial.

The situation becomes more tractable once we expand to first order in the deforming parameter β^2. In this case the operator can be written as a local differential operator of second order, whose coefficients depend on the classical field ϕ. The explicit expressions read

$$
\begin{aligned}
\delta^2 S_I f(x) = &-\frac{\lambda}{4!}\frac{1}{2}\int \frac{\mathrm{d}q_1}{(2\pi)^D}\frac{\mathrm{d}q_2}{(2\pi)^D}\tilde{\phi}_1\tilde{\phi}_2 e^{\mathrm{i}x(q_1+q_2)} \\
&\times\left[4! + \beta\left(a_{ij}(x)(-i\partial^i)(-\mathrm{i}\partial^j) + b_j(x)(-\mathrm{i}\partial^j) + c(x)\right)\right] f(x), \\
\delta^2 S_K f(x) = &\left(-\partial^2 + \frac{\alpha^2}{\beta^2}x^2 - 2\alpha^2 x_j\partial^j\partial^i x_i + m_{\text{eff}}^2\right) f(x),
\end{aligned}
\tag{44}
$$

where $\delta^2 S_I$ and $\delta^2 S_K$ correspond respectively to the contributions of the interaction and kinetic term. In this expression we have defined the Fourier transform $\tilde{\phi}_i := \tilde{\phi}(q_i)$ as

$$
\phi(x) =: \int \frac{\mathrm{d}q_i}{(2\pi)^D}\, e^{\mathrm{i}xq_i}\, \tilde{\phi}(q_i), \tag{45}
$$

and the tensorial coefficients are given by

$$
\begin{aligned}
a_{ij}(x) :&= 8\mathrm{i}(s_1+s_2)\left(2x_i(q_1+q_2)_j + (q_1+q_2)\cdot x\delta_{ij}\right), \\
b_j(x) :&= 8\mathrm{i}(s_1+s_2)\left(x_i(q_1+q_2)^2 + 2(q_1+q_2)\cdot x\,(q_1+q_2)_j\right) \\
&\qquad + 8(2+D)(s_1+s_2)(q_1+q_2)_j, \\
c(x) :&= 8\mathrm{i}(s_1+s_2)\left((q_1\cdot x)(2q_1\cdot q_2 + q_2^2) + (q_2\cdot x)(2q_1\cdot q_2 + q_1^2)\right) \\
&\qquad + 8(2+D)(s_1+s_2)q_1\cdot q_2.
\end{aligned}
\tag{46}
$$

Notice that the derivatives are intended to act on every function on the right.

In the Worldline Formalism one rewrites the heat-kernel in Eq. (42) as a path integral in a first quantization of an auxiliary particle. The action of this auxiliary particle can be immediately read from the Weyl-ordered expression of the operator $\delta^2 S$, which we will denote using a subscript[4] W. After the Weyl-ordering, the positions x_i and derivatives ∂_i in the symmetrized expressions are replaced respectively by position (x_i) and momentum (p_i) variables, which are the variables of the corresponding path integral [52]:

$$
\operatorname{Tr}\left(e^{-T\,\delta^2 S}\right) = \int_{\text{PBC}} \mathcal{D}x\mathcal{D}p\, e^{-\int_0^T \mathrm{d}t\,\delta^2 S_W(p(t),x(t))}. \tag{47}
$$

In this expression, the subscript PBC denotes periodic boundary conditions in the position variables, which is a consequence of considering the trace of the heat-kernel. If we were interested in the local elements of the heat-kernel, we would instead impose local boundary conditions.

For a quartic potential in SdS, a master equation has been obtained in [17], considering a first-order expansion in the noncommutative parameter β^2 and the curvature α^2. Let us recall that the Weyl-ordered expression for the second variation of S is given by

$$
\delta^2 S_W = p^2 + \omega^2 x^2 + \alpha^2 (x_i x^j p_j p^i)_S + m^2 + V_W^I. \tag{48}
$$

Notice that we have defined $\omega^2 := \alpha^2/\beta^2$ and the Weyl-ordered potential V_W^I reads

[4] The Weyl-ordering of a polynomial expression consists in obtaining its totally symmetric expression by performing the necessary commutations. As a simple example, if we have $A = px$ with $p \equiv -\mathrm{i}\partial$, its Weyl-ordered expression is given by $A_W = \frac{px+xp}{2} - \mathrm{i}/2 = (xp)_S - \mathrm{i}/2$, where the subscript S means symmetrized. For further information see [52].

$$V_W^I := -\frac{1}{2}\frac{\lambda}{4!}\int \frac{\mathrm{d}^D q_1 \mathrm{d}^D q_2}{(2\pi)^{2D}} e^{\mathrm{i}x(q_1+q_2)}\left[4! + \beta^2\left(\alpha'_{ij} p^i p^j + \beta'_j p^j + \gamma'\right)\right]\tilde{\phi}_1\tilde{\phi}_2, \tag{49}$$

in terms of the tensorial coefficients α_{ij}, β_j and γ reported in App. A. These coefficients are functions[5] of the momenta q_1, q_2 and derivatives acting on $\tilde{\phi}_{i=1,2}$. Importantly, although the full potential term is nonlocal in the classical field ϕ, it becomes a local operator at the order in which we are working.

4.2. The divergent terms of the effective action

Recall that we are interested in the first contributions in the deformation parameters α and β to the effective action. According to the discussion before Eq. (12), the frequency of the oscillator is not necessarily small; therefore, we can perform a perturbative expansion around a quadratic action which includes the oscillator potential. Performing an appropriate rescaling in the internal times and the phase space coordinates, we may take thus

$$S_0[k, j] := \int_0^1 \mathrm{d}t\left[p^2(t) + \omega^2 x^2(t) - \mathrm{i}p(t)\dot{q}(t) + k(t)p(t) + j(t)q(t)\right] \tag{50}$$

as the undeformed action of the worldline particle. We have included sources $j(t)$ and $k(t)$ for positions and momenta anticipating the fact that we will be interested in the computation of the particle's partition function, from which mean values of quantities depending on the phase space coordinates may be obtained as variations in the sources. A straightforward computation [21] shows that the partition function is given by

$$\begin{aligned} Z_{\mathrm{per}}[k, j] &:= \frac{\int_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}p\, e^{-S_0[k,j]}}{\int_{\mathrm{PBC}} \mathcal{D}x\mathcal{D}p\, e^{-S_0[0,0]}} \\ &= \exp\left\{\int_0^1\int_0^1 dt dt' \begin{pmatrix} k(t) & j(t)\end{pmatrix} G^{(\mathrm{per})}(t-t') \begin{pmatrix} k(t') \\ j(t')\end{pmatrix}\right\}, \end{aligned} \tag{51}$$

in terms of the matrix-valued Green function

$$G^{(\mathrm{per})}(\Delta) := \frac{1}{4\sinh \omega T}\begin{pmatrix} G_{pp}^{(\mathrm{per})} & G_{px}^{(\mathrm{per})} \\ G_{xp}^{(\mathrm{per})} & G_{xx}^{(\mathrm{per})}\end{pmatrix}, \tag{52}$$

whose components are

$$\begin{aligned} G_{pp}^{(\mathrm{per})} &:= \omega T\, \cosh[\omega T(2|\Delta| - 1)], \\ G_{px}^{(\mathrm{per})} &:= i\,\epsilon(\Delta)\, \sinh[\omega T(2|\Delta| - 1)], \\ G_{xp}^{(\mathrm{per})} &:= -i\,\epsilon(\Delta)\, \sinh[\omega T(2|\Delta| - 1)], \\ G_{xx}^{(\mathrm{per})} &:= \frac{1}{\omega T}\, \cosh[\omega T(2|\Delta| - 1)]. \end{aligned} \tag{53}$$

5 The tensorial coefficients, α_{ij} and β_j, should not be confused with the curvature and noncommutative parameters, α and β.

Turning back to our model, a perturbative expansion of the one-loop effective action in terms of the deformation parameters can be written as

$$\Gamma_{1-\text{loop}}[\phi] = -\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n!}\int_0^1 \cdots \int_0^1 \prod_{i=1}^{n} \mathrm{d}t_i \, V_W(\delta_{j_i}, \delta_{k_i}) \times \\ \times \int_0^{\infty} \mathrm{d}T \, \frac{T^{n-1}}{(2\sinh \omega T)^D} e^{-Tm^2} \, Z_{\text{per}}[k(t)/\sqrt{T}, \sqrt{T}\, j(t)] , \quad (54)$$

where we have employed the abbreviation $\delta_{j_i} := \frac{\delta}{\delta j(t_i)}$ for the variation with respect to sources at a given time t_i (analogous definition is valid for k). In that expression, V_W includes the terms coming from the interaction V_W^I, as well as the one proportional to α^2 in Eq. (48), i.e.

$$V_W(x, p) := V_W^I + V_W^{\alpha}, \quad (55)$$

$$V_W^{\alpha} := -2\alpha^2 x_i x^j p_j p^i . \quad (56)$$

To isolate the UV divergences in (54), one should notice that they come from the small-propertime T behaviour of the integrand; these divergences are adequately regulated in dimensional regularization. Additionally, as happens in the commutative case, each potential term contributes a power of T, rendering the integral more convergent at higher orders in the potential. However, in this case also inverse powers of T are generated because of the dependence of the potential on the momenta; this can be readily seen from expression (54): the partition function is evaluated at $k(t)/\sqrt{T}$.

In our noncommutative case, the number of divergent terms turns out to be finite. Defining $D =: 2 - \varepsilon$ and retaining only the divergent contributions for $\varepsilon \to 0$, a direct albeit lengthy computation gives

$$\Gamma^{\text{div}}_{1-\text{loop}} = \frac{\lambda}{24\pi\varepsilon}\int \mathrm{d}^2x \left[\frac{6\alpha^2 m^2}{\omega^2} + 4\beta^2 m^2 + 3 + x^2\left(9\alpha^2 + 8\beta^2\omega^2\right)\right]\phi^2(x) \\ + \frac{\lambda^2}{48\pi\varepsilon}\int \mathrm{d}^2x \left[\frac{3\alpha^2}{\omega^2} + 2\beta^2\right]\phi^4(x), \quad (57)$$

telling us that in order to have a well-defined theory we should renormalize just the mass m, the frequency ω and the coupling constant λ by introducing appropriate counterterms.

In particular, this means that in two dimensions the theory is renormalizable at the one-loop level. This result is non-trivial, since a power counting argument would say that the theory is non-renormalizable or, analogously, that some operators are classically relevant in the UV. This is in contrast with the four-dimensional case [17], where we had to perform a renormalization of the noncommutative parameter and of the curvature, and even to introduce some additional terms in the original action.

It is now straightforward to introduce the necessary counterterms and compute the beta functions for the dimensionful couplings. Introducing the renormalization scale μ and defining $\beta_x = \frac{\partial x}{\partial \log \mu}$, we are led to[6]

[6] We are still working at $\mathcal{O}(\alpha^2, \beta^2)$.

$$\begin{aligned}
\beta_{\omega^2} &= -\frac{\lambda}{12\pi}\left(9\alpha^2 + 8\beta^2\omega^2\right), \\
\beta_{m^2_{\text{eff}}} &= -\frac{\lambda}{12\pi}\left(\frac{6\alpha^2 m^2}{\omega^2} + 4\beta^2 m^2 + 3\right), \\
\beta_{\lambda} &= -\frac{\lambda^2}{\pi}\left[\frac{3\alpha^2}{\omega^2} + 2\beta^2\right], \\
\beta_{\beta^2} &= 0, \\
\beta_{\alpha^2} &= 0.
\end{aligned} \tag{58}$$

In any case remember that, when we consider the Callan-Symanzik equation [53], we need to add a contribution proportional to the classical dimension of the coupling in each differential equation in (58).

5. Analysis of the beta functions

First of all, let us look for fixed points (FPs) of the system at the one-loop level, which are obtained by equating to zero the system of eqs. (58) with the addition of the classical dimensions. Since α and β have no contribution from anomalous dimensions to compensate their classical dimensions, the only solution for them is the trivial one. Replacing these values in the remaining equations, we get just one solution, namely the trivial Gaussian FP. There is however a subtlety: as a consequence of our exact computation in ω, the beta functions contain inverse powers of the frequency. This means that the FP can only be obtained dynamically, i.e. we can not just set our parameters to the values of the FP, since at that point the expressions are not well-defined. Hence, our one-loop computation should be trusted only if the evolution is given in a certain region of the flow in the (α, ω) plane, where the quotient $\frac{\alpha}{\omega}$ remains bounded.

In order to study the system of coupled differential equations, we will tackle the nontrivial ones numerically. In the following the vector $v_i = (\omega_i^2, m_i^2, \alpha_i^2, \beta_i^2, \lambda_i)$ will label the initial conditions at a scale $\mu = 1$ in arbitrary units.

First of all, let us recall the commutative and flat result in $D = 2$ without harmonic term, remembering that one should take the $\alpha \to 0$ limit before the $\omega \to 0$. In this case, the only nontrivial running corresponds to the mass. It is easy to see that the Gaussian FP is actually an attractor in the UV, so that the theory is asymptotically safe. Instead, if one trusts these one-loop results, the theory would be strongly coupled in the IR. The addition of a harmonic term introduces no new effect.

On the other side, the introduction of the noncommutativity involves an operator which at first glance, given the trivial scaling of β, is relevant in the UV. However, the coupling involved in the noncommutative contribution to the potential is $\lambda\beta^2$; thus, there exists a dispute between the respectively decreasing and increasing behaviour of λ and β. One can numerically see that the product in which we are interested tends to zero for large energies. As an example, choosing $v_1 := (0.5,\ 1,\ 0,\ 10^{-3},\ 1.)$, we have plotted both $\lambda\beta^2$ (in green solid line) and λ (in purple dashed line) in the left panel of Fig. 2. According to this plot, the theory is asymptotically free, even if the noncommutative contribution becomes more important.

The situation becomes unstable if instead of considering Snyder space we analyze a change of sign in β^2, i.e. we consider anti-Snyder space. This happens even in the case of a small noncommutativity. We depict this setting in the right panel of Fig. 2, with the choice $v_2 = (0.5,\ 1,\ 0,\ -10^{-3},\ 1.)$. Under these circumstances the anomalous dimension of λ, even if not big enough to counteract the classical dimension, generates a growth in the absolute value of

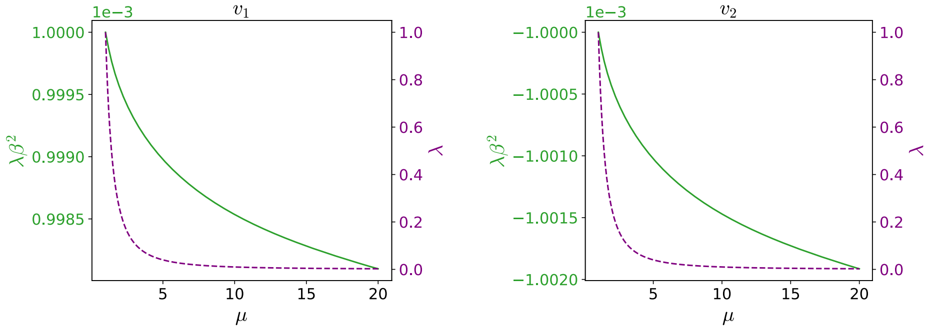

Fig. 2. The product $\lambda\beta^2$ (green continuous line) and the coupling constant λ (purple dashed line) as a function of the energy μ in arbitrary units. The plot on the left corresponds to the choice of parameters $v_1 = (0.5,\ 1,\ 0,\ 10^{-3},\ 1.)$, while the plot on the right belongs to $v_2 = (0.5,\ 1,\ 0,\ -10^{-3},\ 1.)$.

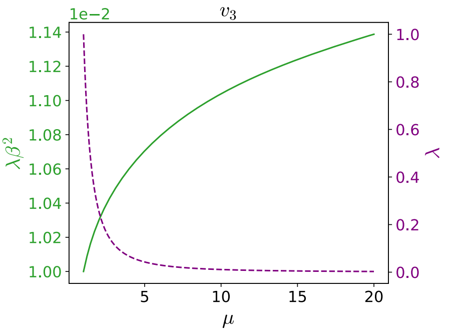

Fig. 3. The product $\lambda\beta^2$ (green continuous line) and the coupling constant λ (purple dashed line) as a function of the energy μ in arbitrary units, corresponding to the parameters $v_3 = (0.1,\ 1,\ -0.005,\ 0.01,\ 1)$.

$\lambda\beta^2$. In other words, the operator associated with the noncommutative sector of the interaction will become relevant in the UV if we are allowed to extrapolate this one-loop computation.

Let us now come to the full noncommutative and curved theory. Although the consideration of a curvature α of the same positive sign[7] as β leads always to a vanishing coupling λ at high energies, once we consider a negative curvature or noncommutativity the situation is more involved. This can be already seen from eq. (58), where the sign of the derivative depends on the relative magnitude of α and β, with an effect of the curvature parameter that could be enhanced by the frequency. To be explicit, consider the Snyder-anti-de Sitter case, where the initial values of α and β are not so small, viz. $v_3 := (0.1,\ 1,\ -0.005,\ 0.01,\ 1)$. This case yields the situation depicted in Fig. 3, with an asymptotically free commutative contribution to the potential, and a noncommutative contribution that is relevant in the UV if the one-loop is trustable enough.

Finally, let us mention an interesting alternative possibility related to the study of the beta functions, cf. (58). Following the results in [23] we could define dimensionless couplings that scale either with the curvature or the noncommutative parameter. This is natural, at least at the

[7] Recall that $\alpha^2 > 0$ should be associated with a de Sitter geometry, while $\alpha^2 < 0$ corresponds to an anti-de Sitter one.

one-loop level, since they have no anomalous dimension. This will not imply a change in the results obtained; it just renders the comparison with [23,24] simpler.

Perhaps the most natural rescaling would be $\omega \to \frac{\alpha}{\beta}\tilde{\omega}$, rendering the frequency parameter dimensionless. Afterwards, some of the resulting beta functions could be made to vanish for some values of the parameters, if the relative sign of ω with respect to α and β is chosen appropriately. In any case, this will allow a vanishing beta function for the mass and for either the frequency or the coupling (but never for both ω and λ). Consequently, the fixed point structure in [24] will not be totally reproduced. Indeed, in [24] both the beta functions for the coupling and the dimensionless frequency vanish at the dual point, while the beta function for the mass remains non-null.

6. Conclusions

We have shown that in two dimensions the SdS QFT is renormalizable at the one-loop level, contrary to what happens in the four-dimensional case. Indeed, we have seen that one needs to renormalize only the frequency, the mass and the coupling constant, without the introduction of new terms in the action.

Additionally, we have computed the beta functions of all the involved couplings. The frequency, the mass, the coupling constant λ and α generically tend to zero as the energy increases. Instead, the operator corresponding to the noncommutative deformation of the potential, i.e. the term proportional to β^2, can be either relevant or irrelevant depending on the initial conditions.

The analysis also shows that the system possesses no fixed points. Although one can introduce by hand the relative scale of one of the parameters, for example by replacing $\omega \to \frac{\alpha}{\beta}\tilde{\omega}$, and then study the running of the dimensionless parameter $\tilde{\omega}$, one would need to introduce also an additional relative sign in order to allow the vanishing of some beta functions. In any case, the fixed-point pattern of the Grosse-Wulkenhaar model [24] is never attained. Related to the renormalizability of the model, recently some numerical simulations have pointed out that the absence of a so-called stripe phase could be behind the renormalization properties of the Grosse-Wulkenhaar model [54,55]. It could be thus worth to numerically analyze the nonperturbative phase structure of the SdS model both in $D = 2$ and $D = 4$, where they could be compared with our perturbative analytic computations.

One possibility that has not yet been studied in SdS is a reformulation in terms of a matrix model, which was one of the keys of the success in [23]. This could render the renormalization properties more explicit. Another interesting idea could be to consider an associative realization of the model, following [56]. This would however require an additional compactification of the extra dimensions.

More generally, our model is intended to be an effective field theory in a noncommutative and curved space. One of the main features of any effective field theory regards the identification of its underlying symmetries, in order to include in the action every possible term compatible with it. In our case, more focus should be made in the future on Born duality [33], which is patent in our starting algebra (1) and lost in the building of the theory. Although, as can be seen from (12), the kinetic term in the SdS action develops several terms that treat p and x on equal footing, the potential term does not satisfy such a symmetry, at least in this small-deformations expansion. We believe that this is an important point on which one should focus, since it is a related symmetry, Langmann-Szabo's symmetry [34], that is behind the properties of the celebrated Grosse-Wulkenhaar model [23]. To achieve this end, one should presumably figure out a method that does not rely on the transformations (7). Such issue is currently investigated.

CRediT authorship contribution statement

Sebastián Franchino-Viñas: Conceptualization, Methodology, Writing-Original draft preparation. **Salvatore Mignemi**: Conceptualization, Methodology, Writing- Original draft preparation.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Coefficients in the expansion of the potential V_W

The tensorial coefficients $\alpha'_{\mu\nu}$, β'_μ and γ', which are used in eq. (49), have been derived for the first time in [41]. As a matter of completion, we write them explicitly, omitting the factors of the Fourier-transformed fields $\tilde{\phi}_i$, which were taken out as overall factors in (49):

$$
\begin{aligned}
\alpha'_{ij} &= -8(s_1+s_2)\left(2(q_1+q_2)^j \partial_{q_1^i} + (q_1+q_2)\cdot\partial_{q_1}\delta_{ij} + (D+2)\delta_{ij}\right),\\
\beta'_j &= 0,\\
\gamma' &= -2(s_1+s_2)\Big[4(2q_1\cdot q_2+q_2^2)(q_1\cdot\partial_{q_1}) + 4(2q_1\cdot q_2+q_1^2)(q_2\cdot\partial_{q_1})\\
&\qquad - 3(q_1+q_2)^2(q_1+q_2)\cdot\partial_{q_1} - (2+D)(q_1^2-2q_1q_2-3q_2^2)\Big].
\end{aligned}
\tag{59}
$$

In these expressions, the derivatives are intended to apply solely to the right, what in our case means only on the $\tilde{\phi}$ factors.

Appendix B. Integrals involving a product of generalized Laguerre functions

In Sec. 3.1 we need to compute integrals involving the product of two generalized Laguerre functions, the corresponding exponential measure and powers of its argument,

$$
I_{m,n}(a,\alpha,\beta) := \int_0^\infty \mathrm{d}x\, e^{-x} x^a L_m^{(\alpha)}(x) L_n^{(\beta)}(x). \tag{60}
$$

This type of integrals has already been investigated in the literature; for example, the authors of [46] find the expression

$$I_{m,n}(a,\alpha,\beta)=\frac{\Gamma(a+1)\Gamma(n+\beta+1)\Gamma(m+\alpha-a)}{m!n!\Gamma(\alpha-a)\Gamma(\beta+1)} \times {}_3F_2(-n,a+1,a-\alpha+1;\beta+1,a+1-\alpha-m;1) \tag{61}$$

in terms of the generalized hypergeometric function ${}_3F_2$, which is valid as long as no singularity arises.

Unfortunately, the expressions that we need in Sec. 3.1 correspond to singular limits of this result. To circumvent this inconvenient, notice that, for fixed indices of the Laguerre polynomials, the result should be an analytic function of the power a. We can thus obtain the desired results by using (61) with an arbitrary power a of the variable x (for fixed n and m) and performing an analytic continuation.

Let us be more concrete considering an example. If we want to compute $I_{n-1,n}(|l|+1,|l|+1,|l|)$, we may introduce an auxiliary variable s so that

$$\begin{aligned} I_{n-1,n}(|l|+1,|l|+1,|l|) &= \int_0^\infty \mathrm{d}z\, e^{-z} z^{|l|+s+1} L_{n-1}^{|l|+1}(z) L_n^{|l|}(z)\Big|_{s=0} \\ &= \frac{\Gamma(n-s-1)\Gamma(n+|l|+1)\Gamma(s+|l|+2)}{(n-1)!n!\Gamma(-s)\Gamma(|l|+1)} \\ &\quad \times {}_3F_2(-n,s+1,s+|l|+2;-n+s+2,|l|+1;1)\Big|_{s=0}. \end{aligned} \tag{62}$$

If one naively takes the limit $s\to 0$, then the divergent $\Gamma(-s)$ in front of the expression generates a vanishing result. However, if n is a fixed positive integer, the hypergeometric function has a hidden simple pole as a function of s. Indeed, by definition we have

$${}_3F_2(-n,s+1,s+|l|+2;-n+s+2,|l|+1;1)=\sum_{i=0}^{n}\frac{(-n)_i(s+1)_i(s+|l|+2)_i}{(-n+s+2)_i(|l|+1)_i}\frac{1}{i!}, \tag{63}$$

where $(\cdot)_i$ denotes the Pochhammer symbol (or generalized factorial). The singular behaviour comes from the factor $\frac{(-n)_i}{(-n+s+2)_i}$, which displays a pole in s for $i=(n-1),n$. These are the only terms that will contribute to Eq. (62), so that we can isolate them and obtain the correct result.

In this way we compute the necessary integrals:

$$\begin{aligned} I_{n-1,n}(|l|+1,|l|+1,|l|) &= -\frac{\Gamma(n+|l|+1)}{\Gamma(n)}, \\ I_{n-1,n}(|l|+2,|l|+1,|l|) &= -\frac{(3n+2|l|+1)\Gamma(n+|l|+1)}{\Gamma(n)}, \\ I_{n,n}(|l|+1,|l|,|l|) &= \frac{(2n+|l|+1)\Gamma(n+|l|+1)}{\Gamma(n+1)}, \\ I_{n,n}(|l|+2,|l|,|l|) &= \frac{(6n(n+1)+|l|(6n+|l|+3)+2)\Gamma(n+|l|+1)}{\Gamma(n+1)}. \end{aligned} \tag{64}$$

References

[1] Hartland S. Snyder, Quantized space-time, Phys. Rev. 71 (1947) 38–41.

[2] Alain Connes, Noncommutative Geometry, ISBN 9780121858605, 1994.
[3] Sergio Doplicher, Klaus Fredenhagen, John E. Roberts, The quantum structure of space-time at the Planck scale and quantum fields, Commun. Math. Phys. 172 (1995) 187–220, arXiv:hep-th/0303037 [hep-th].
[4] Michael R. Douglas, Nikita A. Nekrasov, Noncommutative field theory, Rev. Mod. Phys. 73 (2001) 977–1029, arXiv:hep-th/0106048 [hep-th].
[5] Richard J. Szabo, Quantum field theory on noncommutative spaces, Phys. Rep. 378 (2003) 207–299, arXiv:hep-th/0109162 [hep-th].
[6] N. Aghanim, et al., Planck 2018 results. VI. Cosmological parameters, Astron. Astrophys. 641 (2020), arXiv:1807.06209 [astro-ph.CO].
[7] A. Addazi, et al., Quantum gravity phenomenology at the dawn of the multimessenger era - a review, Prog. Part. Nucl. Phys. 125 (2022) 103948, https://doi.org/10.1016/j.ppnp.2022.103948, arXiv:2111.05659 [hep-ph].
[8] Ivan Gutierrez-Sagredo, et al., Quantum groups, non-commutative Lorentzian spacetimes and curved momentum spaces, in: C. Duston, M. Holman (Eds.), Spacetime Physics 1907 - 2017, Minkowski Institute Press, Montreal, ISBN 978-1-927763-48-3, 2019, pp. 261–290, arXiv:1907.07979 [hep-th].
[9] Angel Ballesteros, Ivan Gutierrez-Sagredo, Francisco J. Herranz, The κ-(A)dS noncommutative spacetime, Phys. Lett. B 796 (2019) 93–101, arXiv:1905.12358 [math-ph].
[10] M. Buric, D. Latas, Singularity resolution in fuzzy de Sitter cosmology, in: 10th Mathematical Physics Meeting: School and Conference on Modern Mathematical Physics, 2020, pp. 27–36.
[11] Maja Buric, Dusko Latas, Discrete fuzzy de Sitter cosmology, Phys. Rev. D 100 (2) (2019) 024053, arXiv:1903.08378 [hep-th].
[12] M. Burić, D. Latas, Discreteness of fuzzy de Sitter space, Phys. Part. Nucl. 49 (5) (2018) 918–920.
[13] M.B. Fröb, A. Much, Strict deformations of quantum field theory in de Sitter spacetime, J. Math. Phys. 62 (6) (2021) 062302, https://doi.org/10.1063/5.0046666, arXiv:2012.11313 [hep-th].
[14] J. Kowalski-Glikman, Lee Smolin, Triply special relativity, Phys. Rev. D 70 (2004) 065020, arXiv:hep-th/0406276.
[15] S. Mignemi, Doubly special relativity in de Sitter spacetime, Ann. Phys. 522 (2010) 924–940, arXiv:0802.1129 [gr-qc].
[16] Salvatore Mignemi, Rina Štrajn, Quantum mechanics on a curved Snyder space, Adv. High Energy Phys. 2016 (2016) 1328284, arXiv:1501.01447 [hep-th].
[17] Sebastián A. Franchino-Viñas, Salvatore Mignemi, Asymptotic freedom for $\lambda\phi^4_\star$ QFT in Snyder-de Sitter space, Eur. Phys. J. C 80 (5) (2020) 382, arXiv:1911.08921 [hep-th].
[18] S. Meljanac, et al., Quantum field theory in generalised Snyder spaces, Phys. Lett. B 768 (2017) 321–325, arXiv:1701.05862 [hep-th].
[19] Stjepan Meljanac, et al., Nonassociative Snyder ϕ^4 quantum field theory, Phys. Rev. D 96 (4) (2017) 045021, arXiv:1703.10851 [hep-th].
[20] Stjepan Meljanac, et al., UV-IR mixing in nonassociative Snyder ϕ^4 theory, Phys. Rev. D 97 (5) (2018) 055041, arXiv:1711.09639 [hep-th].
[21] Sebastián Franchino Viñas, Pablo Pisani, Worldline approach to the Grosse- Wulkenhaar model, J. High Energy Phys. 11 (2014) 087, arXiv:1406.7336 [hep-th].
[22] Harald Grosse, Alexander Hock, Raimar Wulkenhaar, Solution of the selfdual Φ^4 QFT-model on four-dimensional Moyal space, J. High Energy Phys. 01 (2020) 081, arXiv:1908.04543 [math-ph].
[23] Harald Grosse, Raimar Wulkenhaar, Renormalization of phi**4 theory on noncommutative R**4 to all orders, Lett. Math. Phys. 71 (2005) 13–26, arXiv:hep-th/0403232.
[24] Harald Grosse, Raimar Wulkenhaar, The beta function in duality covariant noncommutative phi**4 theory, Eur. Phys. J. C 35 (2004) 277–282, arXiv:hep-th/0402093.
[25] Shiraz Minwalla, Mark Van Raamsdonk, Nathan Seiberg, Noncommutative perturbative dynamics, J. High Energy Phys. 0002 (2000) 020, arXiv:hep-th/9912072 [hep-th].
[26] Roberto Trinchero, Critical behavior of a nonlocal ϕ^4 field theory and asymptotic freedom, Phys. Rev. D 98 (5) (2018) 056023, arXiv:1807.03691 [hep-th].
[27] R. Trinchero, Scalar field on non-integer dimensional spaces, Int. J. Geom. Methods Mod. Phys. 09 (2012) 1250070, arXiv:1201.4365 [math-ph].
[28] Özgür Akarsu, et al., Graduated dark energy: observational hints of a spontaneous sign switch in the cosmological constant, Phys. Rev. D 101 (6) (2020) 063528, arXiv:1912.08751 [astro-ph.CO].
[29] Özgür Akarsu, et al., Rastall gravity extension of the standard ΛCDM model: theoretical features and observational constraints, Eur. Phys. J. C 80 (11) (2020) 1050, arXiv:2004.04074 [astro-ph.CO].
[30] Senarath de Alwis, et al., Asymptotic safety, string theory and the weak gravity conjecture, arXiv:1907.07894 [hep-th], 2019.

[31] Sidney R. Coleman, David J. Gross, Price of asymptotic freedom, Phys. Rev. Lett. 31 (1973) 851–854.
[32] S.A. Franchino-Viñas, S. Mignemi, Snyder-de Sitter meets the Grosse-Wulkenhaar model, in: F. Finster, D. Giulini, J. Kleiner, J. Tolksdorf (Eds.), Progress and Visions in Quantum Theory in View of Gravity, Birkhäuser, Cham, 2020, https://doi.org/10.1007/978-3-030-38941-3_6.
[33] Max Born, Elementary particles and the principle of reciprocity, Nature 163 (1949) 207.
[34] Edwin Langmann, Richard J. Szabo, Duality in scalar field theory on noncommutative phase spaces, Phys. Lett. B 533 (2002) 168–177, arXiv:hep-th/0202039.
[35] S. Mignemi, The Snyder-de Sitter model from six dimensions, Class. Quantum Gravity 26 (2009) 245020.
[36] V.P. Nair, A.P. Polychronakos, Quantum mechanics on the noncommutative plane and sphere, Phys. Lett. B 505 (2001) 267–274, arXiv:hep-th/0011172 [hep-th].
[37] Stefano Bellucci, Armen Nersessian, Corneliu Sochichiu, Two phases of the noncommutative quantum mechanics, Phys. Lett. B 522 (2001) 345349, arXiv:hep-th/0106138 [hep-th].
[38] Maja Buric, Michael Wohlgenannt, Geometry of the Grosse-Wulkenhaar model, J. High Energy Phys. 1003 (2010) 053, arXiv:0902.3408 [hep-th].
[39] Donald Marolf, Ian A. Morrison, Mark Srednicki, Perturbative S-matrix for massive scalar fields in global de Sitter space, Class. Quantum Gravity 30 (2013) 155023, arXiv:1209.6039 [hep-th].
[40] Diana López Nacir, Francisco D. Mazzitelli, Leonardo G. Trombetta, To the sphere and back again: de Sitter infrared correlators at NTLO in 1/N, J. High Energy Phys. 08 (2019) 052, arXiv:1905.03665 [hep-th].
[41] S.A. Franchino-Viñas, S. Mignemi, Worldline formalism in Snyder spaces, Phys. Rev. D 98 (6) (2018) 065010, arXiv:1806.11467 [hep-th].
[42] D. Bahns, et al., Field theory on noncommutative spacetimes: quasiplanar Wick products, Phys. Rev. D 71 (2005) 025022, arXiv:hep-th/0408204.
[43] L.E. Parker, D.J. Toms, Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity, Cambridge University Press, ISBN 978-0521-87787-9, 2009.
[44] Atsushi Higuchi, Donald Marolf, Ian A. Morrison, On the equivalence between Euclidean and In-In formalisms in de Sitter QFT, Phys. Rev. D 83 (2011) 084029, arXiv:1012.3415 [gr-qc].
[45] W. Pauli, Wave Mechanics, Dover Publications Inc., Mineola, New York, 2000.
[46] M.L. Glasser, Norman J. Morgenstern Horing, Ground-state energy of a two-dimensional electron gas in a magnetic field: Hartree-Fock approximation, Phys. Rev. B 31 (7 Apr. 1985) 4603–4611.
[47] F.G. Mehler, Ueber die Entwicklung einer Function von beliebig vielen Variablen nach Laplaceschen Functionen hoeherer Ordnung, J. Reine Angew. Math. 66 (1866) 161.
[48] Olindo Corradini, Luigi Crispo, Maurizio Muratori, Dimensional regularization for the particle transition amplitude in curved space, arXiv:2109.11804 [hep-th], Sept. 2021.
[49] Naser Ahmadiniaz, et al., Worldline master formulas for the dressed electron propagator, part 2: on-shell amplitudes, arXiv:2107.00199 [hep-th], June 2021.
[50] S.A. Franchino-Viñas, F.D. Mazzitelli, Effective action for delta potentials: spacetime-dependent inhomogeneities and Casimir self-energy, Phys. Rev. D 103 (6) (2021) 065006, arXiv:2010.11144 [hep-th].
[51] R. Bonezzi, et al., Worldline approach to noncommutative field theory, J. Phys. A 45 (2012) 405401, arXiv:1204.1013 [hep-th].
[52] Fiorenzo Bastianelli, Peter van Nieuwenhuizen, Trace anomalies from quantum mechanics, Nucl. Phys. B 389 (1993) 53–80, arXiv:hep-th/9208059 [hep-th].
[53] Michael E. Peshkin, Daniel V. Schroeder, Introduction to Quantum Field Theory, Addison-Wesley, Reading, Massachusetts, 1995.
[54] Dragan Prekrat, Renormalization footprints in the phase diagram of the Grosse-Wulkenhaar model, arXiv:2104.00657 [hep-th], Apr. 2021.
[55] Dragan Prekrat, Kristina Neli Todorović-Vasović, Dragana Ranković, Detecting scaling in phase transitions on the truncated Heisenberg algebra, J. High Energy Phys. 03 (2021) 197, arXiv:2002.05704 [hep-th].
[56] S. Meljanac, S. Mignemi, Associative realizations of the extended Snyder model, Phys. Rev. D 102 (2020) 126011, arXiv:2007.13498 [physics.gen-ph].