Semiorthomodular BZ*–lattices

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Abstract

Paraorthomodular BZ*-lattices, for short PBZ*-lattices, were introduced in [8] as an abstraction from the lattice of effects of a complex separable Hilbert space, endowed with the spectral ordering. These structures were meant to be a first approximation to a complete description of the equational theory of such lattices of effects. A better approximation is introduced here, together with a preliminary investigation of its properties.

Keywords: Brouwer-Zadeh lattice; PBZ*-lattice; orthomodular lattice; semiorthomodular BZ*-lattice; left-residuated groupoid; quantum structures; unsharp quantum logic.

1 Introduction

The variety $\mathbb{PBZL}^*$ of *paraorthomodular BZ*-lattices*, for short *PBZ*-lattices*, was introduced in the paper [8] and further studied in [9, 10, 11, 18, 12]. The key motivation for this enquiry comes from the foundations of quantum mechanics. Consider the structure

$$\mathbf{E}(\mathbf{H}) = \langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', ^{\sim}, \mathbb{O}, \mathbb{I} \rangle,$$

where:

- $\mathcal{E}(\mathbf{H})$ is the set of all effects of a given complex separable Hilbert space $\mathbf{H}$, i.e., positive linear operators of $\mathbf{H}$ that are bounded by the identity operator $\mathbb{I}$;

- $\wedge_s$ and $\vee_s$ are the meet and the join, respectively, of the *spectral ordering* $\leq_s$ so defined for all $E, F \in \mathcal{E}(\mathbf{H})$:

  $$E \leq_s F \text{ iff } \forall \lambda \in \mathbb{R} : M^F(\lambda) \leq M^E(\lambda),$$

  where for any effect E, M^E is the unique spectral family [15, Ch. 7] such that $E = \int_{-\infty}^{\infty} \lambda \, dM^E(\lambda)$ (the integral is here meant in the sense of norm-converging Riemann-Stieltjes sums [19, Ch. 1]);

- $\mathbb{O}$ and $\mathbb{I}$ are the null and identity operators, respectively;
- $E' = \mathbb{I} - E$ and $E^{\sim} = P_{\ker(E)}$ (the projection onto the kernel of E).

The operations in $\mathbf{E}(\mathbf{H})$ are well-defined. The spectral ordering is indeed a lattice ordering [4] that coincides with the usual ordering of effects induced via the trace functional when both orderings are restricted to the set of projection operators of the same Hilbert space.

A PBZ*-lattice can be viewed as an abstraction from this concrete physical model, much in the same way as an orthomodular lattice can be viewed as an abstraction from a certain structure of projection operators in a complex separable Hilbert space. Indeed, the analogy can be pressed further: the equational theories of both classes of algebras are *incomplete* with respect to the concrete structures they intend to capture. It is well-known (see e.g. [3]) that there are identities — like, e.g., the *orthoarguesian identity* —that fail in certain orthomodular lattices, but hold in all orthomodular lattices of projection operators of Hilbert spaces. A similar phenomenon can be detected in the case of PBZ*-lattices.

In this paper, we enrich the equational theory of PBZ*-lattices so as to obtain a better approximation to the equational properties of PBZ*-lattices of effects. The additional property we consider, called *semiorthomodularity*, corresponds to one of the several possible expressions of the orthomodular property in the language of PBZ*-lattices, which expands the language of orthomodular lattices by an additional unary operator $^{\sim}$. The preliminary investigation of semiorthomodularity offered in this paper barely scratches the surface of its potential uses. To mention only one possible application, we expect, in future work, to be in a position to reconstruct for semiorthomodular PBZ*-lattices at least part of the theory of the *commuting relation* in orthomodular lattices, including a reasonable version of the Foulis-Holland theorem [1].

The paper is structured as follows. We start with some preliminaries. Section 2 contains some background notions and notational conventions on universal algebra and lattice theory, while Section 3 is devoted to a presentation of PBZ*-lattices that is as self-contained as possible — although the reader is still advised to look at the above-cited papers for additional information on these structures. In Section 4 we introduce semiorthomodular PBZ*-lattices and make a note of some of their elementary properties, including a proof that the PBZ*-lattices of the form $\mathbf{E}(\mathbf{H})$, for $\mathbf{H}$ a complex separable Hilbert space, are semiorthomodular. In Section 5 we provide a different characterisation of PBZ*-lattices, with a more order-theoretical flavour. In Section 6 we determine the whereabouts of semiorthomodular PBZ*-lattices in the lattice of subvarieties of PBZ*-lattices. Finally, in Section 7, we prove that the variety of semiorthomodular PBZ*-lattices is term equivalent to a variety of expanded left-residuated groupoids.

2 Preliminaries from Universal Algebra and Lattice Theory

We denote by $\mathbb{N}$ the set of the natural numbers, and by $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. All algebras in this paper will be nonempty, and by *trivial algebra* we mean a singleton algebra. For any signature, the *trivial variety*, consisting of the singleton algebras of that signature, will be denoted by $\mathbb{T}$, and its members will be considered subdirectly irreducible.

Let $\mathbf{A}$ and $\mathbf{B}$ be algebras with reducts in a variety $\mathbb{V}$ of type τ and $\mathbb{C}$ be a class of algebras with reducts in $\mathbb{V}$. For any first order formula φ over τ, we say that $\mathbf{A} \vDash \varphi$ iff the τ–reduct of $\mathbf{A}$ satisfies φ. Now let $n \in \mathbb{N}^*$ and $A_1, \ldots, A_n$ be nonempty subsets of A; if φ contains n variables $x_1, \ldots, x_n$ enumerated here in their order of appearance in φ, then we denote by $\mathbf{A} \vDash_{A_1,\ldots,A_n} \varphi(x_1, \ldots, x_n)$ the property that $\varphi^{\mathbf{A}}(a_1, \ldots, a_n)$ holds for any $a_1 \in A_1, \ldots, a_n \in A_n$; if any of the sets $A_1, \ldots, A_n$ is a singleton $\{a\}$, then $\{a\}$ can be replaced with a in the previous notation.

We denote by $\mathbf{A} \cong_\tau \mathbf{B}$ the fact that the τ–reducts of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic. $\mathrm{H}_{\mathbb{V}}(\mathbb{C})$, $\mathrm{S}_{\mathbb{V}}(\mathbb{C})$ and $\mathrm{P}_{\mathbb{V}}(\mathbb{C})$ will be the classes of the homomorphic images, subalgebras and direct products of the τ–reducts of the members of $\mathbb{C}$, respectively, and we abbreviate by $V_{\mathbb{V}}(\mathbb{C}) = \mathrm{H}_{\mathbb{V}}\mathrm{S}_{\mathbb{V}}\mathrm{P}_{\mathbb{V}}(\mathbb{C})$ the subvariety of $\mathbb{V}$ generated by the τ–reducts of the members of $\mathbb{C}$.

$\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ will denote the lattice of the congruences of the τ–reduct of $\mathbf{A}$. As an immediate consequence of [13, Corollary 2, p. 51], if $\mathbf{A}$ belongs to a variety $\mathbb{W}$, then $\mathrm{Con}_{\mathbb{W}}(\mathbf{A})$ is a complete bounded sublattice of $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$. For any $n \in \mathbb{N}^*$ and any constants $\kappa_1, \ldots, \kappa_n$ in τ, we denote by $\mathrm{Con}_{\mathbb{V}\kappa_1,\ldots,\kappa_n}(\mathbf{A}) = \{\theta \in \mathrm{Con}_{\mathbb{V}}(\mathbf{A}) : (\forall\, i \in \overline{1,n})\,(\kappa_i^{\mathbf{A}}/\theta = \{\kappa_i^{\mathbf{A}}\})\}$: the set of the congruences of the τ–reduct of $\mathbf{A}$ with singleton classes of $\kappa_1^{\mathbf{A}}, \ldots, \kappa_n^{\mathbf{A}}$. Recall from [11] that $\mathrm{Con}_{\mathbb{V}\kappa_1,\ldots,\kappa_n}(\mathbf{A})$ is a complete sublattice of $\mathrm{Con}_{\mathbb{V}}(\mathbf{A})$ and thus a bounded lattice. If $\mathbb{V}$ is the variety of lattices or that of bounded lattices, then the index $\mathbb{V}$ will be eliminated from all previous notations.

For any lattice $\mathbf{L}$, we denote by $\mathbf{L}^d$ the dual of $\mathbf{L}$, and, given any $a \in L$, $[a)$ and $(a]$ will denote the principal filter and the principal ideal generated by a, respectively. For any $n \in \mathbb{N}^*$, $\mathbf{D}_n$ will designate the n–element chain, as well as any algebra having this lattice reduct. The lattice reduct of an arbitrary lattice–ordered algebra $\mathbf{M}$ will be denoted by $\mathbf{M}_l$, and, for any class $\mathbb{K}$ of algebras with lattice reducts, we will denote by $\mathbb{K}_l = \{\mathbf{L}_l : \mathbf{L} \in \mathbb{K}\}$.

Recall that the *ordinal sum* (also called *glued sum*) of a lattice $\mathbf{L}$ with top element $1^{\mathbf{L}}$ with a lattice $\mathbf{M}$ with bottom element $0^{\mathbf{M}}$ is the lattice $\mathbf{L} \oplus \mathbf{M}$ obtained by glueing $1^{\mathbf{L}}$ and $0^{\mathbf{M}}$ together and thus stacking $\mathbf{M}$ on top of $\mathbf{L}$; $L \oplus M$ will denote its universe, obtained by factoring the disjoint union of L with M through the equivalence having $\{1^{\mathbf{L}}, 0^{\mathbf{M}}\}$ as unique nonsingleton class. See the exact definition in [11].

If $(\mathbf{L}_i)_{i \in I}$ is a nonempty family of nontrivial bounded lattices, then the *horizontal sum* of the family $(\mathbf{L}_i)_{i \in I}$ is the nontrivial bounded lattice $\boxplus_{i \in I}\mathbf{L}_i$ obtained by glueing the bottom elements $0^{\mathbf{L}_i}$ of these lattices together, glueing their top elements $1^{\mathbf{L}_i}$ together, and, for every $i, j \in I$ with $i \neq j$, letting every element from $\mathbf{L}_i \setminus \{0^{\mathbf{L}_i}, 1^{\mathbf{L}_i}\}$ be incomparable with every element from $\mathbf{L}_j \setminus \{0^{\mathbf{L}_j}, 1^{\mathbf{L}_j}\}$, so that the union of the orders of the lattices from this family becomes the order of the lattice $\boxplus_{i\in I}\mathbf{L}_i$; we denote by $\boxplus_{i\in I} L_i$ its universe, obtained by factoring the disjoint union of the family $(\mathbf{L}_i)_{i\in I}$ through the equivalence with the classes $\{0^{\mathbf{L}_i} : i \in I\}$, $\{1^{\mathbf{L}_i} : i \in I\}$ and every other class a singleton. See the exact definition in [11]. If α_i is an equivalence on L_i for each $i \in I$, then we denote by $\boxplus_{i\in I}\alpha_i$ the equivalence on $\boxplus_{i\in I} L_i$ generated by $\bigcup_{i\in I} \alpha_i$. Note that $\boxplus_{i\in I}\alpha_i \in \mathrm{Con}_{01}(\boxplus_{i\in I}\mathbf{L}_i)$ if $\alpha_i \in \mathrm{Con}_{01}(\mathbf{L}_i)$ for each $i \in I$.

Note that the ordinal sum of bounded lattices is associative, while the horizontal sum for the case when $|I| \leq 2$ is associative, commutative and has $\mathbf{D}_2$ as a neutral element.

3 Preliminaries on PBZ*-lattices

We recap in this section some preliminary notions on PBZ*-lattices needed to make this paper self-contained. For additional information, we refer the reader to [8, 9, 10, 11, 12, 18].

Recall that a *bounded involution lattice* (in brief, *BI–lattice*) is an algebra $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a bounded lattice and $' : L \to L$ is an order–reversing map that satisfies $a'' = a$ for all $a \in L$. This makes $'$ a dual lattice automorphism of $\mathbf{L}$, called *involution*. If $\mathbf{A}$ is an algebra having a BI–lattice reduct, then this reduct will be denoted by $\mathbf{A}_{bi}$. If $\mathbb{C}$ is a class of algebras with BI–lattice reducts, then we denote by $\mathbb{C}_{BI} = \{\mathbf{M}_{bi} : \mathbf{M} \in \mathbb{C}\}$.

Until mentioned otherwise, let $\mathbf{L}$ be a BI–lattice. We let $S(\mathbf{L}) = \{a \in L : a \wedge a' = 0\}$ and we call the elements of $S(\mathbf{L})$ the *sharp elements* of $\mathbf{L}$. $\mathbf{L}$ is called an *ortholattice* iff $S(\mathbf{L}) = L$. $\mathbf{L}$ is said to be *paraorthomodular* iff, for all $a, b \in L$, if $a \leq b$ and $a' \wedge b = 0$, then $a = b$. $\mathbf{L}$ is called an *orthomodular lattice* iff, for all $a, b \in L$, $a \leq b$ implies $b = (b \wedge a') \vee a$.

A *pseudo–Kleene algebra* is a BI–lattice $\mathbf{L}$ which satisfies the following equational condition that we will refer to as the *Kleene condition*: $a \wedge a' \leq b \vee b'$ for all $a, b \in L$. The involution of a pseudo–Kleene algebra is called *Kleene complement*. Distributive pseudo–Kleene algebras are called *Kleene algebras* or *Kleene lattices*. Clearly, any ortholattice is a pseudo–Kleene algebra, orthomodular lattices are exactly the paraorthomodular ortholattices, distributive orthomodular lattices are exactly Boolean algebras (with their Boolean complements as involutions), and any modular ortholattice is an orthomodular lattice. The bounded lattice $\mathbf{D}_4 \boxplus \mathbf{D}_4$ can be organised as a non–orthomodular, and thus non–paraorthomodular, ortholattice, called the *Benzene ring* and denoted by $\mathbf{B}_6$. Following general usage, for any non–empty set I, if $\kappa = |I|$, we denote by $\mathbf{MO}_\kappa = \boxplus_{i\in I}\mathbf{D}_2^2$ the modular ortholattice of length 3 having exactly $2 \cdot \kappa$ atoms.

For any bounded lattice $\mathbf{M}$ and any BI–lattice $\mathbf{K}$, if $'^{\mathbf{K}}$ is the involution of $\mathbf{K}$ and f is a dual lattice automorphism of $\mathbf{M}$, then the bounded lattice

$\mathbf{M} \oplus \mathbf{K} \oplus \mathbf{M}^d$, can be canonically endowed with the involution $' : M \oplus K \oplus M^d \to M \oplus K \oplus M^d$ defined by: $' \mid_M = f : M \to M^d$, $' \mid_K = '^{\mathbf{K}} : K \to K$ and $' \mid_{M^d} = f : M^d \to M$, thus becoming a BI–lattice that we will also denote by $\mathbf{M} \oplus \mathbf{K} \oplus \mathbf{M}^d$. Clearly, $\mathbf{M} \oplus \mathbf{K} \oplus \mathbf{M}^d$ is a pseudo–Kleene algebra iff $\mathbf{K}$ is a pseudo–Kleene algebra.

For any non–empty family of BI–lattices $(\mathbf{L}_i)_{i\in I}$, the horizontal sum of bounded lattices $\boxplus_{i\in I}(\mathbf{L}_i)_l$ can be organised as a BI–lattice $\boxplus_{i\in I}\mathbf{L}_i$ with the involution that restricts to the involution of $\mathbf{L}_i$ for every $i \in I$. Note that the BI–lattice $\boxplus_{i\in I}\mathbf{L}_i$ is a pseudo–Kleene algebra iff every member of the family $(\mathbf{L}_i)_{i\in I}$ is a pseudo–Kleene algebra and at most one member of this family is not an ortholattice.

A *Brouwer–Zadeh lattice* (in brief, *BZ-lattice*) is an algebra

$$\mathbf{L} = \langle L, \wedge, \vee, ', ^{\sim}, 0, 1\rangle$$

of type $\langle 2, 2, 1, 1, 0, 0\rangle$, such that $\langle L, \wedge, \vee, ', 0, 1\rangle$ is a pseudo–Kleene algebra and the unary operation $^{\sim}$, called *Brouwer complement*, reverses the lattice order and satisfies $a \wedge a^{\sim} = 0$ and $a \leq a^{\sim\sim} = a^{\sim\prime}$ for all $a \in L$. In any BZ-lattice $\mathbf{L}$, we denote by $\Diamond a = a^{\sim\sim}$ and $\Box a = a'^{\sim}$ for any $a \in L$.

It is easy to notice that, in any BZ-lattice $\mathbf{L}$:

- $\{a^{\sim} : a \in L\} = \{a \in L : a = \Diamond a\}$;
- for all $a \in L$, $a^{\sim} \leq a'$, thus $\Box a \leq a'' = a$;
- for all $a, b \in L$, $\Diamond(a \wedge b) \leq \Diamond a \wedge \Diamond b$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

A *BZ*-lattice* is a BZ-lattice $\mathbf{L}$ that satisfies the following equation, which we call *condition* $(*)$: $(a \wedge a')^{\sim} = a^{\sim} \vee \Box a$ for all $a \in L$. We call paraorthomodular BZ*-lattices, in brief, *PBZ*-lattices*.

We denote by $\mathbb{BI}$, $\mathbb{PKA}$, $\mathbb{KA}$, $\mathbb{OL}$, $\mathbb{OML}$, $\mathbb{MOL}$ and $\mathbb{BA}$ the varieties of BI–lattices, pseudo–Kleene algebras, Kleene algebras, ortholattices, orthomodular lattices, modular ortholattices and Boolean algebras, respectively. Note that $\mathbb{BA} \subsetneq \mathbb{MOL} \subsetneq \mathbb{OML} \subsetneq \mathbb{OL} \subsetneq \mathbb{PKA} \subsetneq \mathbb{BI}$ and $\mathbb{BA} \subsetneq \mathbb{KA} \subsetneq \mathbb{PKA}$.

For any BZ-lattice $\mathbf{L}$, we denote by $S(\mathbf{L})$ the set $S(\mathbf{L}_{bi})$ of the sharp elements of the BI–lattice reduct of $\mathbf{L}$ and we call these elements *sharp* or *Kleene sharp elements* of $\mathbf{L}$. If $\mathbf{L}$ is a PBZ*-lattice, then

$$S(\mathbf{L}) = \{a^{\sim} : a \in L\} = \{a \in L : a = \Diamond a\} = \{a \in L : a' = a^{\sim}\}$$

and $S(\mathbf{L})$ is the universe of the largest orthomodular subalgebra of $\mathbf{L}$, that we denote by $\mathbf{S}(\mathbf{L})$; consequently, $\mathbf{L}$ is orthomodular iff all its elements are sharp iff its Kleene complement coincides to its Brouwer complement.

We denote by $\mathbb{BZL}$ and $\mathbb{BZL}^*$ the varieties of BZ-lattices and BZ*-lattices, respectively. PBZ*-lattices form a variety, as well, that we denote by $\mathbb{PBZL}^*$. By the above, the variety $\mathbb{OML}$ can be identified with the subvariety of $\mathbb{PBZL}^*$ consisting of its orthomodular members, each of which is endowed with a Brouwer complement equal to its Kleene complement. A staple result in the theory of orthomodular lattices will be useful in what follows:

Theorem 1 (Foulis–Holland)**.** *[1] If $\mathbf{L} \in \mathbb{OML}$ and $a, b, c \in L$ are such that $(a \wedge b) \vee (a' \wedge b) = b$ and $(a \wedge c) \vee (a' \wedge c) = c$, then the sublattice of $\mathbf{L}_l$ generated by $\{a, b, c\}$ is distributive.*

For any PBZ*-lattice $\mathbf{L}$, we set $D(\mathbf{L}) = \{x \in L : x^\sim = 0\}$; we call the elements of $D(\mathbf{L})$ *dense elements* of $\mathbf{L}$. It is easy to see that any element a of a PBZ*-lattice with $a \geq a'$ is dense.

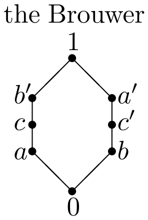
The PBZ*-lattices with no sharp elements besides 0 and 1 are called *antiortholattices.* They form a positive proper universal class that we denote by $\mathbb{AOL}$. Antiortholattices are exactly the PBZ*-lattices $\mathbf{L}$ endowed with the *trivial Brouwer complement*, defined by $0^\sim = 1$ and $a^\sim = 0$ for all $a \in L \setminus \{0\}$. Thus antiortholattices coincide with the PBZ*-lattices in which all nonzero elements are dense.

Any BZ-lattice with the 0 meet–irreducible, in particular any BZ–chain, is an antiortholattice. Moreover, antiortholattices are exactly the pseudo–Kleene algebras with no nontrivial sharp elements, endowed with the trivial Brouwer complement (conditions which imply paraorthomodularity). Hence, if $\mathbf{M}$ is a nontrivial bounded lattice and $\mathbf{K}$ is a pseudo–Kleene algebra, then the canonical pseudo–Kleene algebra $\mathbf{M} \oplus \mathbf{K} \oplus \mathbf{M}^d$, endowed with the trivial Brouwer complement, becomes an antiortholattice, that we denote by $\mathbf{M} \oplus \mathbf{K} \oplus \mathbf{M}^d$, as well.

Proposition 2. *Any pseudo–Kleene algebra $\mathbf{L} = \langle L, \wedge, \vee, ', 0, 1\rangle$ can be endowed with at most one Brouwer complement $^\sim : L \to L$ such that $\langle L, \wedge, \vee, ', ^\sim, 0, 1\rangle$ is a PBZ*-lattice.*

Proof. Let $\mathbf{L} \in \mathbb{PKA}$, and let $^\sim : L \to L$ and $^\circ : L \to L$ be such that both $\mathbf{L}_\sim = \langle L, \wedge, \vee, ', ^\sim, 0, 1\rangle$ and $\mathbf{L}_\circ = \langle L, \wedge, \vee, ', ^\circ, 0, 1\rangle$ are PBZ*-lattices. Then, by the above, $S(\mathbf{L}_\sim) = S(\mathbf{L}_\circ) = S(\mathbf{L}) = \{a \in L : a \vee a' = 1\}$, hence, for any $x \in L$, we have: $x^\sim \in S(\mathbf{L}_\sim) = S(\mathbf{L}_\circ)$, thus $x \leq x^{\sim\sim} = x^{\sim\prime} = x^{\sim\circ}$, hence $x^\circ \geq x^{\sim\circ\circ} \geq x^\sim$, thus $x^\sim \leq x^\circ$. Analogously, $x^\circ \leq x^\sim$, hence $x^\sim = x^\circ$. Therefore the Brouwer complements $^\sim$ and $^\circ$ coincide. $\square$

Example 3. For BZ*-lattices or paraorthomodular BZ-lattices instead of PBZ*–lattices, the previous result does not hold. Indeed, for instance, the 4–element Boolean algebra can be organised as a paraorthomodular BZ-lattice both as an orthomodular lattice, that is with its Brouwer complement equalling its Kleene (Boolean) complement, and with the trivial Brouwer complement, while the following non-paraorthomodular ortholattice can be organised as a BZ*–lattice with any of the Brouwer complements from the table next to its Hasse diagram:



x	0	a	a'	b	b'	c	c'	1
$x^\sim$	1	b	b'	a	a'	c	c'	0
x°	1	b	b'	a	a'	b	b'	0

For any nonempty family of BZ-lattices $(\mathbf{K}_i)_{i \in I}$ such that the BI–lattice reduct of at most one member of this family is not an ortholattice, the horizontal sum of pseudo–Kleene algebras $\boxplus_{i\in I}(\mathbf{K}_i)_{bi}$ becomes a BZ-lattice $\boxplus_{i\in I}\mathbf{K}_i$ when endowed with the union of the Brouwer complements of the members of this family. Clearly, the BZ-lattice $\boxplus_{i\in I}\mathbf{K}_i$ satisfies condition $(*)$ iff each member of the family $(\mathbf{K}_i)_{i\in I}$ satisfies condition $(*)$. By the above, the BZ-lattice $\boxplus_{i\in I}\mathbf{K}_i$ is paraorthomodular iff all members of this family are paraorthomodular and at most one of them is not orthomodular.

Now we recall some notations from [11].

Let $\mathbb{C}$ and $\mathbb{D}$ be nonempty classes of lattice–ordered algebras such that: either $\mathbb{C}$ and $\mathbb{D}$ are subclasses of the variety of bounded lattices, or they are both subclasses of $\mathbb{BI}$, or $\mathbb{C}\subseteq\mathbb{OL}$ and $\mathbb{D}\subseteq\mathbb{PKA}$, or $\mathbb{C}\subseteq\mathbb{OML}$ and $\mathbb{D}\subseteq\mathbb{PBZL}^*$. Then we denote by $\mathbb{C}\boxplus\mathbb{D}=\mathbb{T}\cup\{\mathbf{L}\boxplus\mathbf{M}:\mathbf{L}\in\mathbb{C}\setminus\mathbb{T},\mathbf{M}\in\mathbb{D}\setminus\mathbb{T}\}$, which is a class of bounded lattices, or a subclass of $\mathbb{BI}$, or of $\mathbb{PKA}$, or of $\mathbb{PBZL}^*$, in the cases above, respectively. Since $\mathbf{D}_2$ is a neutral element w.r.t. the horizontal sum, if $\mathbf{D}_2\in\mathbb{D}$, then $\mathbb{C}\subseteq\mathbb{C}\boxplus\mathbb{D}$.

We consider the following equations over $\mathbb{BZL}$, out of which we may note that the equation OML is equivalent with orthomodularity expressed by the implication above:

SDM (*Strong De Morgan*)	$(x\wedge y)^{\sim}\approx x^{\sim}\vee y^{\sim}$
WSDM (*weak SDM*)	$(x\wedge y^{\sim})^{\sim}\approx x^{\sim}\vee\Diamond y$
SK	$x\wedge\Diamond y\leq\Box x\vee y$
J0	$(x\wedge y^{\sim})\vee(x\wedge\Diamond y)\approx x$
J1	$(x\wedge(x\wedge y)^{\sim})\vee(x\wedge\Diamond(x\wedge y))\approx x$
J2	$(x\wedge(y\wedge y')^{\sim})\vee(x\wedge\Diamond(y\wedge y'))\approx x$
OML	$((x\vee y')\wedge y)\vee y'\approx x\vee y'$
MOD	$x\vee(y\wedge(x\vee z))\approx(x\vee y)\wedge(x\vee z)$
DIST	$x\vee(y\wedge z)\approx(x\vee y)\wedge(x\vee z)$.

All subdirectly irreducible members of the variety $V_{\mathbb{BZL}}(\mathbb{AOL})$ generated by antiortholattices belong to $\mathbb{AOL}$, which shows that, for any subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$, the intersection $\mathbb{V}\cap V_{\mathbb{BZL}}(\mathbb{AOL})$ is generated by $\mathbb{V}\cap\mathbb{AOL}$: the antiortholattices that belong to $\mathbb{V}$.

We denote by $\mathbb{SK}$, $\mathbb{SDM}$, $\mathbb{WSDM}$, $\mathbb{MOD}$ and $\mathbb{DIST}$ the subvarieties of $\mathbb{PBZL}^*$ relatively axiomatised by SK, SDM, WSDM, MOD and DIST w.r.t. $\mathbb{PBZL}^*$, respectively. We also denote by $\mathbb{SAOL}=\mathbb{SDM}\cap V_{\mathbb{BZL}}(\mathbb{AOL})$, which, by the above, is generated by the antiortholattices with the Strong De Morgan property, which are exactly the PBZ*–lattices with the 0 meet–irreducible. We have $\mathbb{OML}\subsetneq\mathbb{SDM}\subsetneq\mathbb{WSDM}\subsetneq\mathbb{PBZL}^*$.

By the above, in $\mathbb{PBZL}^*$, OML is equivalent to $x'\approx x^{\sim}$. Clearly, OML implies SDM, SDM implies WSDM, J0 implies J1 and J2, and, by [10], J0 implies WSDM. By [10], J0 axiomatises the variety $V_{\mathbb{BZL}}(\mathbb{AOL})$ over $\mathbb{PBZL}^*$. $V_{\mathbb{BZL}}(\mathbb{AOL})$ is incomparable to $\mathbb{OML}$, it intersects it at the single atom $\mathbb{OML}\cap V_{\mathbb{BZL}}(\mathbb{AOL})=\mathbb{OML}\cap\mathbb{DIST}=\mathbb{BA}$ of the lattice of subvarieties of $\mathbb{PBZL}^*$ and it joins it strictly below $\mathbb{PBZL}^*$, at the variety $\mathbb{OML}\vee V_{\mathbb{BZL}}(\mathbb{AOL})$ axiomatised by $\{\text{WSDM},\text{J1}\}$ or, equivalently, by $\{\text{WSDM},\text{J2}\}$ w.r.t. $\mathbb{PBZL}^*$, according to [10].

By [11], J1 and J2 are equivalent under WSDM. Also by [11], we have the following:

- $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \subsetneq V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL}) \subsetneq V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL}))$;
- the varieties $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})$ and $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL}))$ are incomparable to each of $\mathbb{SDM}$ and $\mathbb{WSDM}$;
- $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL}) \cap \mathbb{WSDM} = V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL})) \cap \mathbb{WSDM} = \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$.

In the same paper, an equational basis for $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})$ relative to $\mathbb{PBZL}^*$ is given, and it is shown that J1 is preserved (and reflected) by horizontal sums with orthomodular lattices, while J2 is not; in fact, every member of the class $\mathbb{OML} \boxplus (V_{\mathbb{BZL}}(\mathbb{AOL}) \setminus \mathbb{AOL})$ fails J2.

4 Semiorthomodular BZ*–Lattices

In [8] there is a short discussion concerning the determination of the equational properties valid in all PBZ*–lattices of effects of some Hilbert space, as opposed to the equational properties characterising PBZ*–lattices in general. As a contribution towards shedding more light on this problem, we set out to investigate one of the possible ways to express the orthomodular property in the language of PBZ*-lattices, which we call *semiorthomodularity* since it does not imply, although it is implied by, the identity $x' \approx x^\sim$. The semiorthomodular identity holds in all PBZ*–lattices of effects of some Hilbert space, but not throughout $\mathbb{PBZL}^*$.

Definition 4. We call *semiorthomodular BZ*–lattice* a BZ*-lattice that satisfies the following identity, hereafter referred to as *semiorthomodularity*:

$$\text{SOML} \quad ((x \vee y^\sim) \wedge \Diamond y) \vee y^\sim \approx x \vee y^\sim.$$

We denote by $\mathbb{SOML}$ the variety of semiorthomodular BZ*–lattices. Next, we provide some examples of such structures. First, observe that all orthomodular lattices, as well as all members of $V_{\mathbb{BZL}}(\mathbb{AOL})$, satisfy SOML. The next proposition shows, as already claimed, that SOML holds in PBZ*–lattices of effects — where, as recalled in the introduction, $\wedge_s$ and $\vee_s$ denote the lattice operations derived from the spectral ordering.

Proposition 5. *If* $\mathbf{H}$ *is a complex separable Hilbert space, the PBZ*–lattice*

$$\mathbf{E}(\mathbf{H}) = \langle \mathcal{E}(\mathbf{H}), \wedge_s, \vee_s, ', ^\sim, \mathbb{O}, \mathbb{I} \rangle$$

of effects of $\mathbf{H}$ *is semiorthomodular.*

Proof. Let $E, F \in \mathcal{E}(\mathbf{H})$. By [5, p. 109], if E and F commute, the set $\{E, F\}$ generates a finite von Neumann algebra. However, it is shown in [4] that the PBZ*-lattice of effects of a finite von Neumann algebra is modular. Thus, in order to show our claim it suffices to prove that $E \vee_s F^\sim$ commutes with $F^\sim$. Now, since $F^\sim \leq_s E \vee_s F^\sim$ and $F^\sim$ is a projection operator, our conclusion follows. □

We now provide some equivalent characterisations of semiorthomodularity. For the next theorem, note that SOML clearly implies the quasiequation in (1), and vice versa, substituting x with $x \vee y^{\sim}$ in this quasiequation gives us SOML. Similarly for the equivalences between the quasiequations and the equations stated directly in conditions (2) through (8) from this theorem.

Theorem 6. *In the variety of BZ-lattices, the following equational conditions are equivalent:*

(1) semiorthomodularity, or, equivalently: $y^{\sim} \leq x \Rightarrow x \approx y^{\sim} \vee (x \wedge \Diamond y)$*;*

(2) $\Diamond y \leq x \Rightarrow x \approx \Diamond y \vee (x \wedge y^{\sim})$*, otherwise written:* $x \vee \Diamond y \approx ((x \vee \Diamond y) \wedge y^{\sim}) \vee \Diamond y$*;*

(3) $y \leq x \Rightarrow x \approx y \vee (x \wedge (\Box y)^{\sim})$*, otherwise written:* $x \vee y \approx ((x \vee y) \wedge (\Box y)^{\sim}) \vee y$*, or, equivalently:* $x \approx (x \wedge y) \vee (x \wedge (\Box(x \wedge y))^{\sim})$*;*

(4) $x \leq y \Rightarrow x \approx (x \vee y^{\sim}) \wedge y$*, otherwise written:* $x \wedge y \approx ((x \wedge y) \vee y^{\sim}) \wedge y$*, or, equivalently:* $x \approx (x \vee (x \vee y)^{\sim}) \wedge (x \vee y)$*;*

(5) $x \leq y \Rightarrow x \approx (x \vee y^{\sim}) \wedge \Diamond y$*, otherwise written:* $x \wedge y \approx ((x \wedge y) \vee y^{\sim}) \wedge \Diamond y$*, or, equivalently:* $x \approx (x \vee (x \vee y)^{\sim}) \wedge \Diamond (x \vee y)$*;*

(6) $x \leq y^{\sim} \Rightarrow x \approx (x \vee y) \wedge y^{\sim}$*, otherwise written:* $x \wedge y^{\sim} \approx ((x \wedge y^{\sim}) \vee y) \wedge y^{\sim}$*;*

(7) $x \leq y^{\sim} \Rightarrow x \approx (x \vee \Diamond y) \wedge y^{\sim}$*, otherwise written:* $x \wedge y^{\sim} \approx ((x \wedge y^{\sim}) \vee \Diamond y) \wedge y^{\sim}$*;*

(8) $x \leq \Diamond y \Rightarrow x \approx (x \vee y^{\sim}) \wedge \Diamond y$*, otherwise written:* $x \wedge \Diamond y \approx ((x \wedge \Diamond y) \vee y^{\sim}) \wedge \Diamond y$*;*

(9) $x \wedge y \approx ((x \wedge \Diamond y) \vee y^{\sim}) \wedge y$*.*

Proof. Both implications in each of the equivalences (1) $\Leftrightarrow$ (2) and (7) $\Leftrightarrow$ (8), as well as the implication (6) $\Rightarrow$ (8) can be obtained by replacing y with $y^{\sim}$. The implication (4) $\Rightarrow$ (8) can be obtained by replacing y by $\Diamond y$. Now let $\mathbf{L}$ be a BZ-lattice and let $a, b \in L$. (1) $\Leftrightarrow$ (8). If $\mathbf{L}$ satisfies (1), then:

$$\begin{aligned} a \wedge \Diamond b &= a'' \wedge \Diamond b \\ &= (a' \vee b^{\sim})' \\ &= (((a' \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim})' \\ &= ((a' \vee b^{\sim}) \wedge \Diamond b)' \wedge \Diamond b \\ &= ((a \wedge \Diamond b) \vee b^{\sim}) \wedge \Diamond b. \end{aligned}$$

The converse implication is handled similarly. Analogously, (2) $\Leftrightarrow$ (7). (8) $\Rightarrow$ (9). If $\mathbf{L}$ satisfies (8), then:

$$\begin{aligned} a \wedge b &= a \wedge b \wedge \Diamond b \\ &= b \wedge ((a \wedge \Diamond b) \vee b^{\sim}) \wedge \Diamond b && \text{(by (8))} \\ &= b \wedge ((a \wedge \Diamond b) \vee b^{\sim}) && \text{(by } b \leq \Diamond b\text{)} \\ &= ((a \wedge \Diamond b) \vee b^{\sim}) \wedge b. \end{aligned}$$

(9) $\Rightarrow$ (4). If $\mathbf{L}$ satisfies (9) and $a \leq b$, then $a = a \wedge b = ((a \wedge \Diamond b) \vee b^{\sim}) \wedge b = (a \vee b^{\sim}) \wedge b$. (8) $\Rightarrow$ (5). Our conclusion follows since $a \leq a \vee b \leq \Diamond (a \vee b)$. (5) $\Rightarrow$ (6). If $\mathbf{L}$ satisfies (5) and $a \leq b^{\sim}$, then:

$$\begin{array}{lll} a & = (a \vee (a \vee b^{\sim})^{\sim}) \wedge \Diamond (a \vee b^{\sim}) & \text{(by (5))} \\ & = (a \vee \Diamond b) \wedge b^{\sim} & \text{(by assumption)} \\ & \geq (a \vee b) \wedge b^{\sim} & \\ & \geq a. & \end{array}$$

(2) $\Rightarrow$ (3). If $b \leq a$, then $\Diamond(\Box b) = b'^{\sim\sim\sim} = b'^{\sim} = \Box b \leq b \leq a$, hence, by substituting x with a and y with $\Box b$ in (2), we get that $a = \Diamond(\Box b) \vee (a \wedge (\Box b)^{\sim}) = \Box b \vee (a \wedge (\Box b)^{\sim}) \leq b \vee (a \wedge (\Box b)^{\sim}) \leq a$ since $\Box b \leq b$, $a \wedge (\Box b)^{\sim} \leq a$ and, by the hypothesis, $b \leq a$, as well, therefore $a = b \vee (a \wedge (\Box b)^{\sim})$. (3) $\Rightarrow$ (2). If $\Diamond b \leq a$, then, by substituting x with a and y with $\Diamond b$ in (3), we get that $a = \Diamond b \vee (a \wedge (\Box(\Diamond b))^{\sim}) = \Diamond b \vee (a \wedge b^{\sim})$ since $(\Box(\Diamond b))^{\sim} = b^{\sim\sim\prime\sim\sim} = b^{\sim\sim\sim\sim\sim} = b^{\sim}$. $\Box$

Notice that the equation in (7) from Theorem 6 is the lattice dual of semiorthomodularity. Observe that the identities corresponding to the lattice duals of the first equation in (4):

$$\text{(4d) } x \vee y \approx ((x \vee y) \wedge y^{\sim}) \vee y$$
$$\text{(4d') } x \vee y \approx ((x \vee y) \wedge x^{\sim}) \vee x,$$

fail in $\mathbb{SOML}$. For a counterexample e.g. to (4d), consider $\mathbf{D}_3$ and denote by a its single unsharp element. Then:

$$(x \vee y)^{\mathbf{D}_3} (1, a) = 1 \neq a = (((x \vee y) \wedge y^{\sim}) \vee y)^{\mathbf{D}_3} (1, a) .$$

In an axiomatisation of $\mathbb{SOML}$ relative to $\mathbb{BZL}^*$, paraorthomodularity doesn't need to be postulated. Indeed, upon recalling that

Lemma 7. *[10] In the variety of BZ*-lattices, paraorthomodularity is equivalent to the following equational condition, called $\Diamond$–*orthomodularity*:*

$$(x^{\sim} \vee (\Diamond x \wedge \Diamond y)) \wedge \Diamond x \leq \Diamond y,$$

we obtain that

Corollary 8. SOML *implies paraorthomodularity in $\mathbb{BZL}^*$, but not in in $\mathbb{BZL}$.*

Proof. By the equivalence of (1) with (7) in Theorem 6, SOML is equivalent to $((x \wedge y^{\sim}) \vee \Diamond y) \wedge y^{\sim} \approx x \wedge y^{\sim}$, in which, by replacing x with $\Diamond y$ and y with $x^{\sim}$ and switching the joinands in the lhs, we obtain:

$$(x^{\sim} \vee (\Diamond x \wedge \Diamond y)) \wedge \Diamond x \approx \Diamond x \wedge \Diamond y \leq \Diamond y,$$

which thus implies $\Diamond$–orthomodularity, which in turn is equivalent to paraorthomodularity in $\mathbb{BZL}^*$, according to Lemma 7. The benzene ring $\mathbf{B}_6$, endowed with the trivial Brouwer complement, is a BZ-lattice that satisfies SOML, but fails condition $(*)$ and also fails paraorthomodularity. $\Box$

Corollary 9. *Any semiorthomodular BZ*–lattice is a PBZ*–lattice.*

5 An Order–Theoretic Characterisation of $\mathbb{SOML}$

In this section, we aim at a better description of $\mathbb{SOML}$ from the order–theoretic viewpoint.

Lemma 10. *Let* $\mathbf{L}$ *be a BZ–lattice. Then:*

(1) $\mathbf{L} \vDash \mathrm{SOML}$ *iff, for all* $a, b \in L$, $\mathbf{L} \vDash_{a,b^\sim} \mathrm{SOML}$*;*

(2) for any $a, b \in L$*:* $\mathbf{L} \vDash_{a,b^\sim} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a,b^\sim} \mathrm{OML}$*, and* $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a,\Diamond b} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a,\Diamond b} \mathrm{OML}$*;*

(3) if $\mathbf{L}$ *is a PBZ***–lattice, then:* $\mathbf{L} \vDash \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{L,S(\mathbf{L})} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{L,S(\mathbf{L})} \mathrm{OML}$.

Proof. (1) $\mathbf{L} \vDash \mathrm{SOML}$ iff, for all $a, b \in L$, $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$. Both implications follow by changing b with $b^\sim$ in the current equation.
(2) By the fact that $b^{\sim\prime} = b^{\sim\sim}$ and $\Diamond b^\sim = b^\sim$ for any $b \in L$.
(3) By (2) and the fact that, if $\mathbf{L}$ is a PBZ*–lattice, then $S(\mathbf{L}) = \{b^\sim : b \in L\}$. □

Lemma 11. *For any BZ–lattice* $\mathbf{L}$ *and any* $a, b \in L$*, we have:*

(1) $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a\vee b^\sim,b} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a\vee b^\sim,\Diamond b} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a\vee b^\sim,\Diamond b} \mathrm{OML}$*;* $\mathbf{L} \vDash_{a,b^\sim} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a,b^\sim} \mathrm{OML}$ *iff* $\mathbf{L} \vDash_{a\vee\Diamond b,b^\sim} \mathrm{SOML}$ *iff* $\mathbf{L} \vDash_{a\vee\Diamond b,b^\sim} \mathrm{OML}$*;*

(2) if $a \leq b^\sim$*, then* $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *and* $\mathbf{L} \vDash_{a,\Diamond b} \mathrm{SOML}$*;*

(3) $\mathbf{L} \vDash_{(b^\sim],b} \mathrm{SOML}$ *and* $\mathbf{L} \vDash_{(b^\sim],\Diamond b} \mathrm{SOML}$.

Proof. (1) Both the lhs and the rhs of the equation SOML evaluated in (a, b) coincide with the same terms evaluated in $(a \vee b^\sim, b)$, hence the first equivalence, from which, along with Lemma 10.(2), we get the rest.
(2) If $a \leq b^\sim$, then $((a \vee b^\sim) \wedge \Diamond b) \vee b^\sim = (b^\sim \wedge \Diamond b) \vee b^\sim = 0 \vee b^\sim = b^\sim = a \vee b^\sim$, thus $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$, hence $\mathbf{L} \vDash_{a,\Diamond b} \mathrm{SOML}$, as well, according to Lemma 10.(2).
(3) By (2). □

Lemma 12. *Let* $\mathbf{L}$ *be a BZ–lattice and* $a, b \in L$*.*

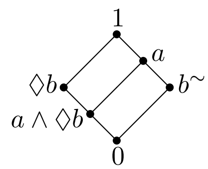
(1) if $a \geq b^\sim$*, then:* $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *iff* $(a \wedge \Diamond b) \vee b^\sim = a$ *iff* $(a \wedge \Diamond b) \vee b^\sim \geq a$ *iff* $(a \wedge \Diamond b) \vee b^\sim \not< a$ *iff* $\{0, b^\sim, a \wedge \Diamond b, a\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *iff* $\{0, b^\sim, a \wedge \Diamond b, \Diamond b, a, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$*;*

(2) if $a > b^\sim$*, then:* $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *iff* $\{0, b^\sim, a \wedge \Diamond b, a\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2^2$*;*

(3) if $1 \neq a > b^\sim$*, then:* $\mathbf{L} \vDash_{a,b} \mathrm{SOML}$ *iff* $\{0, b^\sim, a\wedge\Diamond b, \Diamond b, a, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2 \times \mathbf{D}_3$.

Proof. (1) Let us recall that $\mathbf{L} \vDash_{a,b}$ SOML iff $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} = a \vee b^{\sim}$, which in this case when $a \geq b^{\sim}$ is equivalent to $(a \wedge \Diamond b) \vee b^{\sim} = a$.

Since $(a \vee b^{\sim}) \wedge \Diamond b \leq a \vee b^{\sim}$ and thus $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \leq a \vee b^{\sim}$, it follows that $\mathbf{L} \vDash_{a,b}$ SOML iff $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \geq a \vee b^{\sim}$ iff $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \not< a \vee b^{\sim}$.



$a \wedge \Diamond b \wedge b^{\sim} = a \wedge 0 = 0$, thus $\mathbf{L} \vDash_{a,b}$ SOML iff $\{0, b^{\sim}, a \wedge \Diamond b, a\}$ is the set reduct of a sublattice of $\mathbf{L}_l$.

Since $a \geq b^{\sim}$, we have $a \vee \Diamond b = 1$, thus $\{a \wedge \Diamond b, \Diamond b, a, 1\}$ is the set reduct of a sublattice of $\mathbf{L}_l$, hence $\{0, b^{\sim}, a \wedge \Diamond b, a\}$ is the universe of a sublattice of $\mathbf{L}_l$ iff $\{0, b^{\sim}, a \wedge \Diamond b, \Diamond b, a, 1\}$ is the universe of a bounded sublattice of $\mathbf{L}_l$.
(2),(3) By (1). ☐

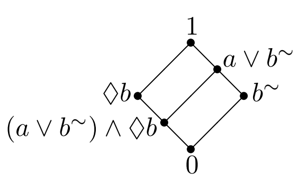
Proposition 13. *Let $\mathbf{L}$ be a BZ–lattice and $a, b \in L$. Then:*

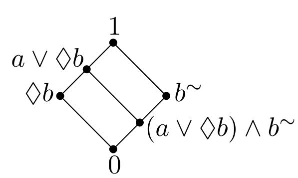
(1) $\mathbf{L} \vDash_{a,b}$ SOML *iff* $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \geq a \vee b^{\sim}$ *iff* $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \not< a \vee b^{\sim}$ *iff* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, a \vee b^{\sim}\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *iff* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, \Diamond b, a \vee b^{\sim}, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$*;* $\mathbf{L} \vDash_{a,b^{\sim}}$ SOML *iff* $((a \vee \Diamond b) \wedge b^{\sim}) \vee \Diamond b \geq a \vee \Diamond b$ *iff* $((a \vee \Diamond b) \wedge b^{\sim}) \vee \Diamond b \not< a \vee \Diamond b$ *iff* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, a \vee \Diamond b\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *iff* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, b^{\sim}, a \vee \Diamond b, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$*;*

(2) if $a \nleq b^{\sim}$*, then:* $\mathbf{L} \vDash_{a,b}$ SOML *iff* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, a \vee b^{\sim}\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2^2$*; if* $a \nleq b^{\sim}$ *and* $a \vee b^{\sim} \neq 1$*, then:* $\mathbf{L} \vDash_{a,b}$ SOML *iff* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, \Diamond b, a \vee b^{\sim}, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2 \times \mathbf{D}_3$*;*

(3) if $a \nleq \Diamond b$*, then:* $\mathbf{L} \vDash_{a,b^{\sim}}$ SOML *iff* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, a \vee \Diamond b\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2^2$*; if* $a \nleq \Diamond b$ *and* $a \vee \Diamond b \neq 1$*, then:* $\mathbf{L} \vDash_{a,b^{\sim}}$ SOML *iff* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, b^{\sim}, a \vee \Diamond b, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2 \times \mathbf{D}_3$*.*

Proof. By Lemma 11.(1) and Lemma 12 in which we replace a by $a \vee b^{\sim}$, then by $a \vee \Diamond b$.





☐

Theorem 14. *Let $\mathbf{L}$ be a BZ–lattice. Then the following are equivalent:*

- $\mathbf{L} \vDash$ SOML*;*
- *for all* $a, b \in L$*,* $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \geq a \vee b^{\sim}$*;*
- *for all* $a, b \in L$*,* $((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \not< a \vee b^{\sim}$*;*
- *for all* $a, b \in L$*,* $((a \vee \Diamond b) \wedge b^{\sim}) \vee \Diamond b \geq a \vee \Diamond b$*;*
- *for all* $a, b \in L$*,* $((a \vee \Diamond b) \wedge b^{\sim}) \vee \Diamond b \not< a \vee \Diamond b$*;*
- *for all* $a, b \in L$*,* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, a \vee b^{\sim}\}$ *is the universe of a sublattice of* $\mathbf{L}_l$*;*
- *for all* $a, b \in L$ *such that* $a \nleq b^{\sim}$*,* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, a \vee b^{\sim}\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2^2$*;*
- *for all* $a, b \in L$*,* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, \Diamond b, a \vee b^{\sim}, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$*;*
- *for all* $a, b \in L$ *such that* $a \nleq b^{\sim}$ *and* $a \vee b^{\sim} \neq 1$*,* $\{0, b^{\sim}, (a \vee b^{\sim}) \wedge \Diamond b, \Diamond b, a \vee b^{\sim}, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2 \times \mathbf{D}_3$*;*
- *for all* $a, b \in L$*,* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, a \vee \Diamond b\}$ *is the universe of a sublattice of* $\mathbf{L}_l$*;*
- *for all* $a, b \in L$ *such that* $a \nleq \Diamond b$*,* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, a \vee \Diamond b\}$ *is the universe of a sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2^2$*;*
- *for all* $a, b \in L$*,* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, b^{\sim}, a \vee \Diamond b, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$*;*
- *for all* $a, b \in L$ *such that* $a \nleq \Diamond b$ *and* $a \vee \Diamond b \neq 1$*,* $\{0, \Diamond b, (a \vee \Diamond b) \wedge b^{\sim}, b^{\sim}, a \vee \Diamond b, 1\}$ *is the universe of a bounded sublattice of* $\mathbf{L}_l$ *isomorphic to* $\mathbf{D}_2 \times \mathbf{D}_3$.

Proof. By Proposition 13, Lemma 11.(2) and Lemma 10.(1). □

Corollary 15. *A BZ–lattice* $\mathbf{L}$ *fails* SOML *iff there exist elements* $a, b \in L$ *with* $a > b^{\sim}$ *that fail the equivalent statements from Lemma 12.(1).*

Proof. $\mathbf{L} \nvDash$ SOML iff there exist $c, b \in L$ such that $\mathbf{L} \nvDash_{c,b}$ SOML. If we denote by $a = c \vee b^{\sim} \geq b^{\sim}$ and we take into account the fact that, if $a = b^{\sim}$, that is if $c \leq b^{\sim}$, then $\mathbf{L} \vDash_{c,b}$ SOML by Lemma 11.(2), then we get the characterisation in the enunciation. □

In any BZ–lattice $\mathbf{L}$, the elements of $\{b^{\sim} : b \in L\}$ are called $\Diamond$*–sharp elements* of $\mathbf{L}$, while the elements whose Kleene and Brouwer complements coincide are called *Brouwer–sharp elements*.

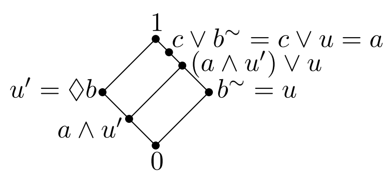
Proposition 16. *If* $\mathbf{L}$ *is a PBZ*–lattice, then, for any* $u \in S(\mathbf{L})$*, we have* $\mathbf{L} \vDash_{(u],u'}$ SOML.

Proof. If $\mathbf{L}$ is a PBZ*–lattice, then any $u \in S(\mathbf{L})$ is of the form $u = b^\sim$ for some $b \in L$, hence, according to Lemma 11.(3), we have $\mathbf{L} \vDash_{(b^\sim],\Diamond b}$ SOML, that is $\mathbf{L} \vDash_{(u],u'}$ SOML since u is sharp. □

Theorem 17. *A PBZ*–lattice* $\mathbf{L}$ *fails* SOML *iff it contains a sharp element u and an element $a > u$ such that one of the following equivalent properties holds:*

- $(a \wedge u') \vee u \neq a$;
- $(a \wedge u') \vee u \ngeq a$;
- $(a \wedge u') \vee u < a$;
- $\{0, a \wedge u', u, a\}$ *is not the set reduct of a sublattice of* $\mathbf{L}_l$;
- $\{0, a \wedge u', u, u', a, 1\}$ *is not the set reduct of a bounded sublattice of* $\mathbf{L}_l$.

Proof. By taking $u = b^\sim$ in Corollary 15 and using the fact that, in PBZ*–lattices, the $\Diamond$–sharp elements coincide to the (Kleene–)sharp elements and also to the Brouwer–sharp ones.

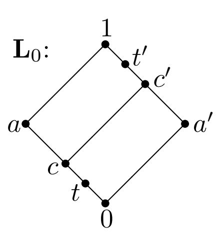


□

6 Placing $\mathbb{SOML}$ in the lattice of subvarieties of $\mathbb{PBZL}^*$

In Section 4 we already gave some examples of semiorthomodular PBZ*-lattices, remarking that $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \subseteq \mathbb{SOML}$. In this section, we investigate the place occupied by $\mathbb{SOML}$ in the lattice of subvarieties of $\mathbb{PBZL}^*$.

For a start, observe that $\mathbb{SOML}$ is a proper subvariety of $\mathbb{PBZL}^*$. The PBZ*-lattice $\mathbf{L}_0$ displayed below:



x	0	a	a'	c	c'	t	t'	1
$x^\sim$	1	a'	a	a'	0	a'	0	0

fails SOML. Note, also, that $\mathbf{L}_0$ satisfies SDM, hence $\mathbb{SDM} \nsubseteq \mathbb{SOML}$.

It is easy to prove that the variety $\mathbb{MOD}$ of the modular PBZ*-lattices is strictly included in $\mathbb{SOML}$, the strictness of the inclusion being proven by the fact that, for instance, the non–modular antiortholattice $\mathbf{D}_2 \oplus (\mathbf{D}_2^2 \boxplus \mathbf{D}_4) \oplus \mathbf{D}_2$ satisfies SOML according to the above.

Another important subvariety of $\mathbb{PBZL}^*$ that fails to be included in $\mathbb{SOML}$ is $\mathbb{SK}$, although a proof of this claim calls for a little more work.

Lemma 18. *Let $\mathbf{L} \in \mathbb{SK}$. Then, for all $a, b \in L$:*

(1) $a \wedge (\Box a \vee b) = a \wedge (\Box a \vee \Diamond b)$;

(2) $a = \Box a \vee (a \wedge a')$;

(3) if $\Diamond b \leq a$, *then* $a = \Box a \vee (a \wedge b^\sim)$.

Proof. (1) Since $\mathbf{L} \models$ SK, we have $a \wedge \Diamond(\Box a \vee b) \leq \Box a \vee \Box a \vee b = \Box a \vee b$, thus

$$\begin{aligned} a \wedge (\Box a \vee b) &= a \wedge \Diamond(\Box a \vee b) \wedge (\Box a \vee b) = a \wedge \Diamond(\Box a \vee b) \\ &= a \wedge (\Diamond(\Box a) \vee \Diamond b) = a \wedge (\Box a \vee \Diamond b). \end{aligned}$$

(2) By (1), condition $(*)$ and Theorem 1 applied to $\mathbf{S}(\mathbf{L})$,

$$\begin{aligned} a \wedge (\Box a \vee (a \wedge a')) &= a \wedge (\Box a \vee \Diamond(a \wedge a')) = a \wedge (\Box a \vee (\Diamond a \wedge \Diamond a')) \\ &= a \wedge (\Box a \vee \Diamond a) \wedge (\Box a \vee \Diamond a') = a \wedge \Diamond a \wedge 1 = a, \end{aligned}$$

hence $a \leq \Box a \vee (a \wedge a') \leq a$, thus $a = \Box a \vee (a \wedge a')$.
(3) Assume that $\Diamond b \leq a$, so that $\Box b \leq a$ and thus $\Box b \vee (\Diamond b \wedge a) \leq a$, and also $a' \leq (\Diamond b)' = (\Diamond b)^\sim = b^\sim$. By (2), it follows that $a = \Box a \vee (a \wedge a') \leq \Box a \vee (a \wedge b^\sim) \leq a$, thus $a = \Box a \vee (a \wedge b^\sim)$. $\Box$

Theorem 19. $\mathbb{SK} \subsetneq \mathbb{SOML}$.

Proof. Let $\mathbf{L} \in \mathbb{SK}$ and let $a, b \in L$ be such that $\Diamond b \leq a$. Then $a = \Box a \vee (a \wedge b^\sim)$ by Lemma 18.(3) and $a' \leq b^\sim$ so that $a' \leq b^\sim \wedge (\Diamond b \vee a')$, so $(b^\sim \wedge (\Diamond b \vee a'))^\sim \leq \Box a$. We have that:

$$\begin{aligned} &(b^\sim \wedge (\Diamond b \vee a'))^{\sim\prime} \wedge \Box a \\ &= (b^\sim \wedge (\Diamond b \vee a'))^{\sim\sim} \wedge \Box a \\ &\leq b^\sim \wedge (\Diamond b \vee a')^{\sim\sim} \wedge \Box a \\ &= b^\sim \wedge \Box a \wedge ((\Diamond b)^\sim \wedge \Box a)^\sim \\ &= b^\sim \wedge \Box a \wedge (b^\sim \wedge \Box a)^\sim = 0. \end{aligned}$$

Since $\mathbf{L}$ is paraorthomodular, it follows that $\Box a = (b^\sim \wedge (\Diamond b \vee a'))^\sim$. By the above,

$$\begin{aligned}
a &= \Box a \vee (a \wedge b^{\sim}) \\
&= (b^{\sim} \wedge (\Diamond b \vee a'))^{\sim} \vee (a \wedge b^{\sim}) \\
&\leq (b^{\sim} \wedge (\Diamond b \vee a'))' \vee (a \wedge b^{\sim}) \\
&= b^{\sim\prime} \vee (\Diamond b \vee a')' \vee (a \wedge b^{\sim}) \\
&= \Diamond b \vee (a \wedge b^{\sim\sim\prime}) \vee (a \wedge b^{\sim}) \\
&= \Diamond b \vee (a \wedge b^{\sim}) \vee (a \wedge b^{\sim}) \\
&= \Diamond b \vee (a \wedge b^{\sim}).
\end{aligned}$$

By Theorem 6.(1)⇔(2), it follows that $\mathbf{L} \in \mathbb{SOML}$. Therefore $\mathbb{SK} \subseteq \mathbb{SOML}$. Since $\mathbb{AOL} \subseteq \mathbb{SOML}$, for instance, the antiortholattice $\mathbf{D}_4 \in \mathbb{SOML} \setminus \mathbb{SK}$, thus $\mathbb{SOML} \nsubseteq \mathbb{SK}$. Hence $\mathbb{SK} \subsetneq \mathbb{SOML}$. □

Next, we determine the inclusion relationships between $\mathbb{SOML}$ and some notable varieties of PBZ*–lattices generated by horizontal sums. Let $\mathbf{A}$ and $\mathbf{B}$ be nontrivial bounded lattices (or lattice–ordered algebras whose horizontal sum can be defined). We say that $\mathbf{A} \boxplus \mathbf{B}$ is a *nontrivial horizontal sum* of bounded lattices iff $|A| > 2$ and $|B| > 2$, that is iff $A \boxplus B \notin \{A, B\}$.

Clearly, if $\mathbf{A} \boxplus \mathbf{B}$ is nontrivial, then $|A \boxplus B| \geq 4$ and: $\mathbf{A} \boxplus \mathbf{B}$ is distributive iff $|A \boxplus B| = 4$ iff (the underlying lattice of) $\mathbf{A} \boxplus \mathbf{B}$ is isomorphic to the 4-element Boolean algebra. If $\mathbf{A}$ is an orthomodular lattice and $\mathbf{B}$ is a PBZ*–lattice, then $\mathbf{A}$ is not the 3-element chain, thus, if $\mathbf{A} \boxplus \mathbf{B}$ is nontrivial, then $|A \boxplus B| > 4$.

Lemma 20. *(1) Any non–distributive nontrivial horizontal sum of bounded lattices is directly indecomposable. Equivalently, no direct product of a bounded lattice of cardinality at least* 2 *with a bounded lattice of cardinality at least* 3 *is a nontrivial horizontal sum of bounded lattices.*

(2) The underlying lattice $\mathbf{L}_l \boxplus \mathbf{A}_l$ of any nontrivial horizontal sum of an orthomodular lattice $\mathbf{L}$ and a PBZ*–lattice $\mathbf{A}$ is directly indecomposable, thus the PBZ*–lattice $\mathbf{L} \boxplus \mathbf{A}$ is directly indecomposable.

Equivalently, no direct product of two nontrivial PBZ*–lattices is a nontrivial horizontal sum of an orthomodular lattice with a PBZ*–lattice.

(3) If $\mathbf{A}$ and $\mathbf{B}$ are nontrivial PBZ*–lattices and $\mathbb{C}$ is a class of PBZ*–lattices such that $\mathbf{A} \times \mathbf{B} \in \mathbb{OML} \boxplus \mathbb{C}$, then $\mathbf{A} \times \mathbf{B} \in \mathbb{OML} \cup \mathbb{C}$.

Proof. (1) Let $\mathbf{L}$ and $\mathbf{A}$ be bounded lattices each of cardinality at least 3. If $|L \boxplus A| = 5$, then $\mathbf{L} \boxplus \mathbf{A} \cong \mathbf{N}_5$, which is, of course, even subdirectly irreducible. If $|L| > 3$ and $|A| > 3$, then assume by absurdum that $\mathbf{L} \boxplus \mathbf{A} = \mathbf{X} \times \mathbf{Y}$ for some nontrivial bounded lattices $\mathbf{X}$ and $\mathbf{Y}$, so that $0^{\mathbf{X}} \neq 1^{\mathbf{X}}$ and $0^{\mathbf{Y}} \neq 1^{\mathbf{Y}}$. Say, for instance, that $\langle 0^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle \in L$. Then

$$\langle 0^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle \in L \setminus \{\langle 0^{\mathbf{X}}, 0^{\mathbf{Y}} \rangle, \langle 1^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle\} = L \setminus \{0, 1\},$$

hence every element $a \in A \setminus \{0,1\}$ satisfies $a \vee \langle 0^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle = 1 = \langle 1^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle$ and $a \wedge \langle 0^{\mathbf{X}}, 1^{\mathbf{Y}} \rangle = 0 = \langle 0^{\mathbf{X}}, 0^{\mathbf{Y}} \rangle$, therefore $a = \langle 1^{\mathbf{X}}, 0^{\mathbf{Y}} \rangle$, thus $A = \{0, \langle 1^{\mathbf{X}}, 0^{\mathbf{Y}} \rangle, 1\}$, which contradicts the fact that $|A| > 3$. Therefore $\mathbf{L} \boxplus \mathbf{A}$ is directly indecomposable. By noticing that $\mathbf{D}_2 \times \mathbf{D}_2 = \mathbf{D}_2^2 \in \mathbb{BA} \subset \mathbb{OML}$, we get the equivalent statement.

(2) By (1).

(3) By (2), if $\mathbf{L} \in \mathbb{OML} \setminus \mathbb{T}$ and $\mathbf{M} \in \mathbb{C} \setminus \mathbb{T}$ are such that $\mathbf{A} \times \mathbf{B} = \mathbf{L} \boxplus \mathbf{M}$, then $\mathbf{L} \cong_{\mathbb{BZL}} \mathbf{D}_2$ or $\mathbf{M} \cong_{\mathbb{BZL}} \mathbf{D}_2$, thus $\mathbf{A} \times \mathbf{B} = \mathbf{L} \boxplus \mathbf{M} = \mathbf{M} \in \mathbb{C}$ or $\mathbf{A} \times \mathbf{B} = \mathbf{L} \boxplus \mathbf{M} = \mathbf{L} \in \mathbb{OML}$, so $\mathbf{A} \times \mathbf{B} \in \mathbb{OML} \cup \mathbb{C}$. □

Next, let us define, for any subclass $\mathbb{C}$ of $\mathbb{PBZL}^*$: $\Xi(\mathbb{C}) = \mathbb{OML} \boxplus \mathbb{C} = \mathbb{T} \cup \{\mathbf{L} \boxplus \mathbf{A} : \mathbf{L} \in \mathbb{OML} \setminus \mathbb{T}, \mathbf{A} \in \mathbb{C} \setminus \mathbb{T}\}$. Clearly, the operator Ξ preserves arbitrary unions and arbitrary intersections and, for all $\mathbb{C} \subseteq \mathbb{PBZL}^*$ such that $\mathbf{D}_2 \in \mathbb{C}$: $\mathbb{C} \subseteq \mathbb{OML} \cup \mathbb{C} \subseteq \Xi(\mathbb{C})$, $\Xi\Xi(\mathbb{C}) = \Xi(\mathbb{C})$, in particular Ξ is a closure operator on the class of the subclasses of $\mathbb{PBZL}^*$ which contain the two–element chain. Let us investigate the commutation properties of Ξ with the usual class operators $\mathrm{H}, \mathrm{S}, \mathrm{P}$ (see also [11, Section 6, Proposition 9]).

For any nontrivial subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$, we have $\mathbb{OML} \cup \mathbb{V} \subseteq \Xi(\mathbb{V})$. If $\mathbf{D}_2 \in \mathbb{C} \subseteq \mathbb{OML}$, then $\Xi(\mathbb{C}) = \mathbb{OML}$. On the other hand, if $\mathbf{D}_2 \in \mathbb{C} \subseteq \mathbb{PBZL}^*$ and $\mathbb{C}$ is closed w.r.t. Ξ, that is $\Xi(\mathbb{C}) = \mathbb{C}$, then $\mathbb{OML} \subseteq \mathbb{C}$. In particular, all nontrivial subvarieties of $\mathbb{PBZL}^*$ which are closed w.r.t. Ξ include $\mathbb{OML}$.

Proposition 21. *For any nonempty subclass $\mathbb{C}$ of $\mathbb{PBZL}^*$ such that $\mathbf{D}_2 \in \mathbb{C}$, the following hold:*

(1) $\Xi\mathrm{S}_{\mathbb{BZL}}(\mathbb{C}) = \mathrm{S}_{\mathbb{BZL}}\Xi(\mathbb{C})$*;*

(2) $\Xi\mathrm{H}_{\mathbb{BZL}}(\mathbb{C}) \supseteq \mathrm{H}_{\mathbb{BZL}}\Xi(\mathbb{C})$*; if $\mathbb{C}$ is closed w.r.t. quotients or the proper congruences in each member of $\mathbb{C}$ have singleton classes of* 0 *(equivalently, of* 1*), then* $\Xi\mathrm{H}_{\mathbb{BZL}}(\mathbb{C}) = \mathrm{H}_{\mathbb{BZL}}\Xi(\mathbb{C})$*;*

(3) if $\mathbb{C}$ is closed w.r.t. direct products, then $\Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C}) \subseteq \mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{C})$*; if* $\Xi(\mathbb{C})$ *is closed w.r.t. direct products, then* $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{C}) \subseteq \mathbb{OML} \cup \mathbb{C} \subseteq \Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C})$*; if $\mathbb{C}$ and $\Xi(\mathbb{C})$ are closed w.r.t. direct products, then* $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{C}) = \mathbb{OML} \cup \mathbb{C} = \Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C})$.

Proof. Let $\mathbf{L} \in \mathbb{OML} \setminus \mathbb{T}$ and $\mathbf{A} \in \mathbb{C} \setminus \mathbb{T}$.
(1) Since the BZ-lattice operations are defined by restriction to the summands, we have $\mathrm{S}_{\mathbb{BZL}}(\mathbf{L} \boxplus \mathbf{A}) = \{\mathbf{M} \boxplus \mathbf{B} : \mathbf{M} \in \mathrm{S}_{\mathbb{BZL}}(\mathbf{L}), \mathbf{B} \in \mathrm{S}_{\mathbb{BZL}}(\mathbf{A})\}$. Thus Ξ commutes with the operator $\mathrm{S}_{\mathbb{BZL}}$.
(2) We have that

$$\begin{aligned}\mathrm{Con}_{\mathbb{BZL}}(\mathbf{L} \boxplus \mathbf{A}) &= \{\lambda \boxplus \alpha : \lambda \in \mathrm{Con}_{\mathbb{BZL}01}(\mathbf{L}), \alpha \in \mathrm{Con}_{\mathbb{BZL}01}(\mathbf{A})\} \cup \{\nabla_{L \boxplus A}\} \\ &= \{\Delta_L \boxplus \alpha : \alpha \in \mathrm{Con}_{\mathbb{BZL}01}(\mathbf{A})\} \cup \{\nabla_{L \boxplus A}\}\end{aligned}$$

and, for any $\alpha \in \mathrm{Con}_{\mathbb{BZL}01}(\mathbf{A})$, $(\mathbf{L} \boxplus \mathbf{A})/(\Delta_L \boxplus \alpha) \cong_{\mathbb{BZL}} \mathbf{L}/\Delta_L \boxplus \mathbf{A}/\alpha \cong_{\mathbb{BZL}} \mathbf{L} \boxplus \mathbf{A}/\alpha$. Hence $\Xi\mathrm{H}_{\mathbb{BZL}}(\mathbb{C}) \supseteq \mathrm{H}_{\mathbb{BZL}}\Xi(\mathbb{C})$, and, in the particular case when

$\mathrm{Con}_{\mathbb{BZL}}(\mathbf{B}) = \mathrm{Con}_{\mathbb{BZL}01}(\mathbf{B}) \cup \{\nabla_B\}$ for all $\mathbf{B} \in \mathbb{C}$, such as the case when $\mathbb{C} \subseteq \mathbb{AOL}$, the converse inclusion holds, as well. Furthermore, in the particular case when $\mathrm{H}_{\mathbb{BZL}}(\mathbb{C}) = \mathbb{C}$, we have $\Xi\mathrm{H}_{\mathbb{BZL}}(\mathbb{C}) = \Xi(\mathbb{C}) \subseteq \mathrm{H}_{\mathbb{BZL}}\Xi(\mathbb{C})$, so this is another situation in which we also have the converse inclusion.
(3) If $\mathbb{C}$ is closed w.r.t. direct products, then $\Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C}) = \Xi(\mathbb{C}) \subseteq \mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{C})$.

Since the operator Ξ is order–preserving, we have $\Xi(\mathbb{C}) \subseteq \Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C})$, so, if $\Xi(\mathbb{C})$ is closed w.r.t. direct products, then $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{C}) = \Xi(\mathbb{C}) \subseteq \Xi\mathrm{P}_{\mathbb{BZL}}(\mathbb{C})$. But, by Lemma 20.(3), if $\Xi(\mathbb{C})$ is closed w.r.t. direct products, then $\Xi(\mathbb{C}) \subseteq \mathbb{OML} \cup \mathbb{C}$.

Hence the equalities in the case when $\mathbb{C}$ and $\Xi(\mathbb{C})$ are closed w.r.t. direct products. □

Corollary 22. *For any nontrivial subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$, the following are equivalent:*

(1) $\Xi(\mathbb{V})$ *is a variety, that is* $V_{\mathbb{BZL}}\Xi(\mathbb{V}) = \Xi(\mathbb{V})$*;*

(2) $\Xi(\mathbb{V})$ *is closed w.r.t. direct products, that is* $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{V}) = \Xi(\mathbb{V})$*;*

(3) $\Xi(\mathbb{V}) = \mathbb{OML} \cup \mathbb{V}$ *and* $\mathbb{OML} \cup \mathbb{V}$ *is closed w.r.t. direct products.*

Proof. (1)⇒(2) and (3)⇒(1): Clear. (2)⇒(1): By Proposition 21.(1)–(2). (2)⇒(3): By Proposition 21.(3). □

Corollary 23. *For any nontrivial subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$, the following are equivalent:*

(1) $\Xi(\mathbb{V}) = \mathbb{V}$*;*

(2) $V_{\mathbb{BZL}}\Xi(\mathbb{V}) = \mathbb{V}$*;*

(3) $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{V}) = \mathbb{V}$.

Proof. (1)⇒(2): Clear. (2)⇒(1): By the fact that $V_{\mathbb{BZL}}\Xi(\mathbb{V}) \supseteq \Xi(\mathbb{V}) \supseteq \mathbb{V}$.
(1)⇒(3): By the fact that $\mathbb{V}$ is closed w.r.t. direct products. (3)⇒(1): By the fact that $\mathrm{P}_{\mathbb{BZL}}\Xi(\mathbb{V}) \supseteq \Xi(\mathbb{V}) \supseteq \mathbb{V}$. □

Corollary 24. *For any nontrivial subvariety $\mathbb{V}$ of $\mathbb{PBZL}^*$:*

(1) if $\Xi(\mathbb{V})$ *is a variety, then* $\mathbb{V} \subseteq \mathbb{OML}$ *or* $\mathbb{OML} \subseteq \mathbb{V}$*, hence* $\Xi(\mathbb{V}) = \mathbb{OML}$ *or* $\Xi(\mathbb{V}) = \mathbb{V}$*;*

(2) $\Xi(\mathbb{V})$ *is a variety iff* $\mathbb{V} \subseteq \mathbb{OML}$ *or* $\Xi(\mathbb{V}) = \mathbb{V}$*;*

(3) $\Xi(\mathbb{V}) = \mathbb{V}$ *iff* $\mathbb{OML} \subseteq \mathbb{V}$ *and* $\Xi(\mathbb{V})$ *is a variety.*

Proof. (1) If $\Xi(\mathbb{V})$ is a variety, but $\mathbb{V} \nsubseteq \mathbb{OML}$ and $\mathbb{OML} \nsubseteq \mathbb{V}$, then there exist an $\mathbf{A} \in \mathbb{V} \setminus \mathbb{OML}$ and a $\mathbf{B} \in \mathbb{OML} \setminus \mathbb{V}$, so that $\mathbf{A} \boxplus \mathbf{B} \in \Xi(\mathbb{V}) \setminus (\mathbb{OML} \cup \mathbb{V})$, contradicting Corollary 22.(3), since $\mathbf{A}, \mathbf{B} \in \mathrm{S}_{\mathbb{BZL}}(\mathbf{A} \boxplus \mathbf{B})$ and $\mathbf{A} \notin \mathbb{OML}$, while $\mathbf{B} \notin \mathbb{V}$.
(2) and (3) follow from (1). □

Not any supervariety $\mathbb{V}$ of $\mathbb{OML}$ satisfies $\Xi(\mathbb{V}) = \mathbb{V}$. For instance, if we let $\mathbb{V} = V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})$, then $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{V}) = V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})) \supseteq V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL})) \supsetneq \mathbb{V}$ [11], thus $\Xi(\mathbb{V}) = \mathbb{OML} \boxplus \mathbb{V} \nsubseteq \mathbb{V}$, so $\Xi(\mathbb{V}) \neq \mathbb{V}$.

An example of a subvariety of $\mathbb{PBZL}^*$ that has the equivalent properties in Corollary 23 is precisely $\mathbb{SOML}$, as we now proceed to show.

Proposition 25. *(1) For any subclass $\mathbb{C} \subseteq \mathbb{PBZL}^*$, we have: $\Xi(\mathbb{C}) \subseteq \mathbb{SOML}$ iff $\mathbb{C} \subseteq \mathbb{SOML}$.*

(2) $\mathbb{SOML} = \Xi(\mathbb{SOML}) = V_{\mathbb{BZL}}(\Xi(\mathbb{SOML}))$.

Proof. (1) We prove that for any $\mathbf{M} \in \mathbb{OML}$ and any $\mathbf{L} \in \mathbb{PBZL}^*$, $\mathbf{M} \boxplus \mathbf{L} \vDash$ SOML iff $\mathbf{L} \vDash$ SOML. The direct implication is trivial. For the converse, let us denote by $\mathbf{A} = \mathbf{M} \boxplus \mathbf{L}$. Then, since $\mathbb{OML} \vDash$ SOML and $\mathbf{L} \vDash$ SOML by the hypothesis of this implication, it follows that $\mathbf{A} \vDash_{M,M}$ SOML and $\mathbf{A} \vDash_{L,L}$ SOML. Now let $x, y \in A \backslash \{0,1\}$. If $x \in L$ and $y \in M$, then $((x \vee y^{\sim}) \wedge \Diamond y) \vee y^{\sim} = ((x \vee y') \wedge y) \vee y' = (1 \wedge y) \vee y' = y \vee y' = 1 = x \vee y' = x \vee y^{\sim}$, so $\mathbf{L}_{x,y} \vDash$ SOML. Now assume that $x \in M$ and $y \in L$. Observe that, for any PBZ*-lattice $\mathbf{L}$, $\mathbf{L} \vDash_{L,D(\mathbf{L})}$ SOML. Thus, if $x \in D(\mathbf{A}) = D(\mathbf{L})$, then $\mathbf{L}_{x,y} \vDash$ SOML. If $x \notin D(\mathbf{A})$, then $((x \vee y^{\sim}) \wedge \Diamond y) \vee y^{\sim} = (1 \wedge \Diamond y) \vee y^{\sim} = \Diamond y \vee y^{\sim} = 1 = x \vee y^{\sim}$, thus $\mathbf{L}_{x,y} \vDash$ SOML. Therefore $\mathbf{M} \boxplus \mathbf{L} \vDash$ SOML.

(2) By (1). $\square$

We may easily notice that Proposition 25 also holds for $\mathbb{SK}$:

Proposition 26.

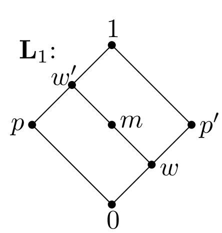
- *For any $\mathbf{M} \in \mathbb{OML}$ and any $\mathbf{L} \in \mathbb{PBZL}^*$, $\mathbf{M} \boxplus \mathbf{L} \vDash$ SK iff $\mathbf{L} \vDash$ SK.*
- *For any subclass $\mathbb{C} \subseteq \mathbb{PBZL}^*$, we have: $\Xi(\mathbb{C}) \subseteq \mathbb{SK}$ iff $\mathbb{C} \subseteq \mathbb{SK}$.*
- *$\mathbb{SK} = \Xi(\mathbb{SK}) = V_{\mathbb{BZL}}(\Xi(\mathbb{SK}))$.*

Since PBZ*–lattices are lattice–ordered and thus congruence–distributive, the lattice of subvarieties of $\mathbb{PBZL}^*$ is distributive, thus, as noted in [18], any subvarieties $\mathbb{V}$, $\mathbb{W}$, $\mathbb{U}$ of $\mathbb{PBZL}^*$ satisfy $\mathbb{V} \vee \mathbb{W} = \mathbb{V} \times_s \mathbb{W}$, thus $Si(\mathbb{V} \vee \mathbb{W}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$, and therefore: $\mathbb{U} = \mathbb{V} \vee \mathbb{W}$ iff $Si(\mathbb{U}) = Si(\mathbb{V}) \cup Si(\mathbb{W})$.

Corollary 27. $(\mathbb{SDM} \cap \mathbb{SOML}) \vee V_{\mathbb{BZL}}(\Xi(V_{\mathbb{BZL}}(\mathbb{AOL}))) \subsetneq \mathbb{SOML}$.

Proof. By Proposition 25.(1) and the fact that $V_{\mathbb{BZL}}(\mathbb{AOL}) \subseteq \mathbb{SOML}$, we have $V_{\mathbb{BZL}}(\Xi(V_{\mathbb{BZL}}(\mathbb{AOL}))) \subseteq \mathbb{SOML}$. Let us consider the following PBZ*-lattice:

$\mathbf{L}_1$:



x	0	p	p'	m	w'	w	1
$x^{\sim}$	1	p'	p	0	0	p	0

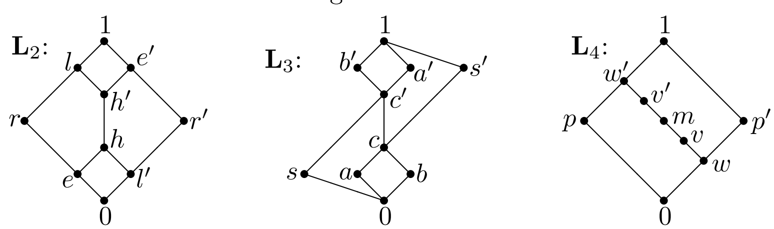
$\mathbf{L}_1$ satisfies SOML and fails SDM and J1, thus $\mathbf{L}_1 \in \mathbb{SOML} \setminus (\mathbb{SDM} \cap \mathbb{SOML}) \cup V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL})))$ since $V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL}))) \vDash$ J1 [11].

Note that $\mathrm{Con}_{\mathbb{BZL}}(\mathbf{L}_1) = \{\Delta_{L_1}, \theta, \nabla_{L_1}\} \cong \mathbf{D}_3$, where $L_1/\theta = \{\{0,p\}, \{w,m,w'\}, \{p',1\}\}$, thus $\mathbf{L}_1$ is subdirectly irreducible, hence $\mathbf{L}_1 \in Si(\mathbb{SOML}) \setminus (Si(\mathbb{SDM} \cap \mathbb{SOML}) \cup Si(V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL}))))) = Si(\mathbb{SOML}) \setminus Si((\mathbb{SDM} \cap \mathbb{SOML}) \vee V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL}))))$ by the above, so $\mathbf{L}_1 \in \mathbb{SOML} \setminus (\mathbb{SDM} \cap \mathbb{SOML}) \vee V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL})))$. $\square$

Proposition 28. *$\mathbb{SOML}$ is incomparable to $\mathbb{SDM}$ and to $\mathbb{WSDM}$ and we have:*

- $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \subsetneq \mathbb{SOML} \cap \mathbb{WSDM}$*;*
- $\mathbb{OML} \vee \mathbb{SAOL} \subsetneq \mathbb{SOML} \cap \mathbb{SDM}$*;*
- $\mathbb{SOML} \cap \mathbb{SDM} \subsetneq \mathbb{SOML} \cap \mathbb{WSDM}$*;*
- $\mathbb{SDM} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \subsetneq \mathbb{WSDM}$*;*
- $\mathbb{WSDM} \vee \mathbb{SOML} \subsetneq \mathbb{PBZL}^*$*.*

Proof. Let us consider the following PBZ*–lattices:



endowed with the following Brouwer complements, respectively:

x	0	r	r'	e	e'	l'	h	h'	l	1
$x^{\sim}$	1	r'	r	r'	0	r	0	0	0	0

x	0	s	s'	a	b	c	a'	b'	c'	1
$x^{\sim}$	1	s'	s	s	s	s	0	0	0	0

x	0	p	p'	m	v'	v	w'	w	1
$x^{\sim}$	1	p'	p	0	0	p	0	p	0

By the above, $\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL}) \subseteq \mathbb{SOML} \cap \mathbb{WSDM}$ and $\mathbb{OML} \vee \mathbb{SAOL} \subseteq \mathbb{SOML} \cap \mathbb{SDM}$. $\mathbf{L}_2$ satisfies SOML and SDM, thus also WSDM, and fails J2, thus also J1, hence, by the results recalled at the end of Section 3,

$$\begin{aligned} \mathbf{L}_2 &\in (\mathbb{SOML} \cap \mathbb{SDM}) \setminus (\mathbb{OML} \vee \mathbb{SAOL}) \\ &= (\mathbb{SOML} \setminus V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL}))) \cap \mathbb{SDM} \\ &\subseteq (\mathbb{SOML} \setminus V_{\mathbb{BZL}}(\Xi\,(V_{\mathbb{BZL}}(\mathbb{AOL}))) \cap \mathbb{WSDM} \\ &= (\mathbb{SOML} \cap \mathbb{WSDM}) \setminus (\mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})). \end{aligned}$$

Since $\mathbb{AOL} \subset \mathbb{SOML} \cap \mathbb{WSDM}$, any antiortholattice that fails SDM, for instance $\mathbf{D}_2^2 \oplus \mathbf{D}_2^2$, belongs to $(\mathbb{SOML} \cap \mathbb{WSDM}) \setminus (\mathbb{SOML} \cap \mathbb{SDM})$. By the above, $\mathbb{SOML}$ is incomparable to $\mathbb{SDM}$ and also to $\mathbb{WSDM}$.

$\mathbf{L}_3$ satisfies WSDM and fails SDM and J0, hence $\mathbf{L}_3 \in \mathbb{WSDM} \setminus (\mathbb{SDM} \cup V_{\mathbb{BZL}}(\mathbb{AOL}))$.

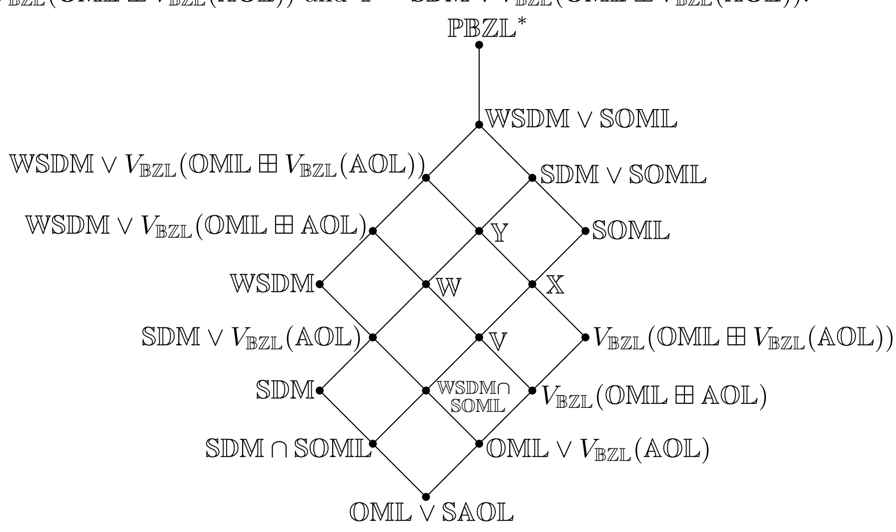
Note that $\mathrm{Con}_{\mathbb{BZL}}(\mathbf{L}_3) = \{\Delta_{L_3}, \alpha, \beta, \nabla_{L_3}\}$, where: $L_3/\alpha = \{\{0,b\},\{a,c\},\{s\},\{s'\},\{c',a'\},\{b',1\}\}$ and $L_3/\beta = \{\{0,a,b,c,s'\},\{s,a',b',c',1\}\}$, so that $\alpha \subset \beta$ and thus $\mathrm{Con}_{\mathbb{BZL}}(\mathbf{L}_3) \cong \mathbf{D}_4$, hence $\mathbf{L}_3$ is subdirectly irreducible. Therefore $\mathbf{L}_3 \in Si(\mathbb{WSDM}) \setminus (Si(\mathbb{SDM}) \cup Si(V_{\mathbb{BZL}}(\mathbb{AOL}))) = Si(\mathbb{WSDM}) \setminus Si(\mathbb{SDM} \vee V_{\mathbb{BZL}}(\mathbb{AOL}))$ by the above, so $\mathbf{L}_3 \in \mathbb{WSDM} \setminus (\mathbb{SDM} \vee V_{\mathbb{BZL}}(\mathbb{AOL}))$.

$\mathbf{L}_4$ fails WSDM and SOML, thus $\mathbf{L}_4 \in \mathbb{PBZL}^* \setminus (\mathbb{WSDM} \cup \mathbb{SOML})$.

It is easy to check that $\mathrm{Con}_{\mathbb{BZL}}(\mathbf{L}_4) = \{\Delta_{L_4}, \zeta, \nabla_{L_4}\} \cong \mathbf{D}_3$, where $L_4/\zeta = \{\{0,p\},\{w,v,m,v',w'\},\{p',1\}\}$, hence $\mathbf{L}_4$ is subdirectly irreducible, thus, as above, $\mathbf{L}_4 \in Si(\mathbb{PBZL}^*) \setminus (Si(\mathbb{WSDM}) \cup Si(\mathbb{SOML})) = Si(\mathbb{PBZL}^*) \setminus Si(\mathbb{WSDM} \vee \mathbb{SOML})$, so $\mathbf{L}_4 \in \mathbb{PBZL}^* \setminus (\mathbb{WSDM} \vee \mathbb{SOML})$. $\square$

As noticed in [18], the distributivity of the lattice of subvarieties of $\mathbb{PBZL}^*$ entails that, for any subvarieties $\mathbb{V}$, $\mathbb{W}$ of $\mathbb{PBZL}^*$, the map $(\mathbb{S}_1, \mathbb{S}_2) \mapsto \mathbb{S}_1 \vee \mathbb{S}_2$ from $[\mathbb{V} \cap \mathbb{W}, \mathbb{V}] \times [\mathbb{V} \cap \mathbb{W}, \mathbb{W}]$ to $[\mathbb{V} \cap \mathbb{W}, \mathbb{V} \vee \mathbb{W}]$ is a lattice isomorphism, whose inverse maps $\mathbb{S} \mapsto (\mathbb{S} \cap \mathbb{V}, \mathbb{S} \cap \mathbb{W})$, where we consider these intervals in the lattice of subvarieties of $\mathbb{PBZL}^*$.

By the above and the fact that $V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL}) \cap \mathbb{WSDM} = V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL})) \cap \mathbb{WSDM} = \mathbb{OML} \vee V_{\mathbb{BZL}}(\mathbb{AOL})$ [11], we get that the following is a sublattice of the lattice of subvarieties of $\mathbb{PBZL}^*$, where $\mathbb{V} = (\mathbb{SDM} \cap \mathbb{SOML}) \vee V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})$, $\mathbb{W} = \mathbb{SDM} \vee V_{\mathbb{BZL}}(\mathbb{OML} \boxplus \mathbb{AOL})$, $\mathbb{X} = (\mathbb{SDM} \cap \mathbb{SOML}) \vee V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL}))$ and $\mathbb{Y} = \mathbb{SDM} \vee V_{\mathbb{BZL}}(\mathbb{OML} \boxplus V_{\mathbb{BZL}}(\mathbb{AOL}))$:



7 A term equivalence result

Some years ago, Chajda and Länger [2] proved that the variety of orthomodular lattices is term equivalent to a certain variety of bounded left-residuated lattice-ordered groupoids. Indeed, the Sasaki projection $x \circ y := (x \vee y') \wedge y$ and the

Sasaki hook $x \rightsquigarrow y := (y \wedge x) \vee x'$ satisfy the *left-residuation law*

$$x \circ y \leq z \text{ iff } x \leq y \rightsquigarrow z.$$

This result is important in so far as it provides a conceptually significant, if partial, bridge between the theories of orthomodular lattices and *residuated structures*, which are ubiquitous and of crucial importance both in algebra and in logic [7, 16, 17]. Since PBZ*-lattices are a generalisation of orthomodular lattices, it is natural to ask whether they also contain some left-residuated pair of operation. In general, this is not true, but if we restrict our attention to the subvariety $\mathbb{SOML}$ the defined term operation $x \cdot y := (x \vee y^{\sim}) \wedge \Diamond y$ is indeed left-residuated.

In this subsection we present a term equivalence between $\mathbb{SOML}$ and an expansion of the variety of left-residuated groupoids in a language containing an additional *pseudo-Kleene implication* $\Rightarrow$. The precise definition of this class follows.

Definition 29. A *SOML left-residuated groupoid* is an algebra

$$\mathbf{A} = \langle A, \cdot, \rightarrow, \Rightarrow, \wedge, \vee, 0, 1\rangle$$

of type $\langle 2, 2, 2, 2, 2, 0, 0\rangle$ such that:

(1) the term reduct $\langle A, \wedge, \vee, ', 0, 1\rangle$, where $a' = a \Rightarrow 0$, is a pseudo-Kleene algebra;

(2) $\langle A, \cdot, 1\rangle$ is a right-unital groupoid, i.e., $\cdot$ is a binary operation and $a \cdot 1 = a$ for all $a \in A$;

(3) for all $a, b, c \in A$, $a \cdot b \leq c$ iff $a \leq b \rightarrow c$ (left residuation);

(4) the following conditions hold for all $a, b, c \in A$:

(Str.idemp.) $a \cdot (a \vee b) = a$;
(Antit) if $a \leq b$, then $b \rightarrow 0 \leq a \rightarrow 0$;
(*) $(a \wedge (a \Rightarrow 0)) \rightarrow 0 = (a \rightarrow 0) \vee ((a \Rightarrow 0) \rightarrow 0)$;
(Sas1) $a \cdot b = (a \vee (b \rightarrow 0)) \wedge ((b \rightarrow 0) \rightarrow 0)$;
(Ksas) $a \Rightarrow b = (a \wedge b) \vee (a \Rightarrow 0)$;
(Str.invol.) $a \rightarrow b = ((b \Rightarrow 0) \cdot a) \Rightarrow 0$.

The class of SOML left-residuated groupoids will be denoted by $\mathbb{SLRG}$. Hereafter, we often denote $\cdot$ by plain juxtaposition. We follow the convention that $\cdot$ binds stronger than the lattice operations, which in turn bind stronger than either $\rightarrow$ or $\Rightarrow$.

Observe that, by the results in [6], the quasi-equational conditions in Definition 29 can be replaced by identities, whence $\mathbb{SLRG}$ is a variety.

Lemma 30. *Let $\mathbf{A} \in \mathbb{SLRG}$. Then the following holds for all $a, b, c \in A$:*

(1) $ab \leq (b \to 0) \to 0$;

(2) The map $\varphi(x) = (x \to 0) \to 0$ *is a closure operator on* $\langle A, \leq \rangle$*;*

(3) $a \to 0 \leq a \Rightarrow 0$*;*

(4) $((a \to 0) \Rightarrow 0)(a \to 0) = 0$*;*

(5) $(a \wedge b) b = a \wedge b$*;*

(6) if $a \leq b$*, then* $ac \leq bc$*;*

(7) $a \wedge b \leq ab$*;*

(8) $a(a \to 0) = 0$*;*

(9) $(a \to b) a \leq ((a \to 0) \to 0) \wedge b$*;*

(10) if $a \leq b$*, then* $c \to a \leq c \to b$*.*

Proof. (1) By (Sas1), $ab = (a \vee (b \to 0)) \wedge ((b \to 0) \to 0) \leq (b \to 0) \to 0$.

(2) We show that φ is reflexive. Using (Str.idemp) and (Sas1), we get:

$$a = a(a \vee a) = aa = (a \vee (a \to 0)) \wedge ((a \to 0) \to 0) \leq ((a \to 0) \to 0).$$

Using this fact and (Antit), we obtain that φ is order-preserving and idempotent.

(3) By (2), (Str.invol.), (Sas1) and pseudo-Kleene algebra properties, we get:

$$a \leq ((a \to 0) \to 0) = 1a = (((0 \Rightarrow 0) a) \Rightarrow 0) \Rightarrow 0 = (a \to 0) \Rightarrow 0,$$

whence, by pseudo-Kleene algebra properties, $a \to 0 \leq a \Rightarrow 0$.

(4) By (Sas1), $1(a \to 0) = (1 \vee ((a \to 0) \to 0)) \wedge (a \to 0) \leq a \to 0$. This implies $1 = (a \to 0) \to (a \to 0)$. Using (Str.invol.) and pseudo-Kleene algebra properties, $((a \to 0) \Rightarrow 0)(a \to 0) = ((a \to 0) \to (a \to 0)) \Rightarrow 0 = 0$.

(5) By (Str.idemp.) $a \wedge b = (a \wedge b)((a \wedge b) \vee b) = (a \wedge b) b$.

(6) By left residuation, $b \leq c \to bc$. Thus, if $a \leq b$, then $a \leq c \to bc$, whence $ac \leq bc$.

(7) Since $a \wedge b \leq a$, by (5) and (6) $a \wedge b = (a \wedge b) b \leq ab$.

(8) By (2) and left-residuation.

(9) By (1), left-residuation and lattice properties.

(10) Let $a \leq b$. By (9), $(c \to a) c \leq ((c \to 0) \to 0) \wedge a \leq ((c \to 0) \to 0) \wedge b \leq b$, whence $c \to a \leq c \to b$. □

The next results highlight the fact that $\mathbb{SLRG}$, as a variety of expanded left-residuated groupoids, is fairly well-behaved. It satisfies a form of Modus Ponens that is also found in Heyting algebras, and its lattice order is a *divisibility* order.

Lemma 31. *Let* $\mathbf{A} \in \mathbb{SLRG}$*. Then the following holds for all* $a, b \in A$*:*

MP $a \wedge b = (b \to a) \wedge b$*.*

Proof. By Lemma 30.(5), $(a \wedge b)\, b = a \wedge b \leq a$, so $a \wedge b \leq b \to a$, whence $a \wedge b \leq (b \to a) \wedge b$. Conversely, by Lemma 30.(7) and left residuation, $(b \to a) \wedge b \leq (b \to a)\, b \leq a$, and since also $(b \to a) \wedge b \leq b$, we obtain $(b \to a) \wedge b \leq a \wedge b$. $\square$

Lemma 32. *In any* $\mathbf{A} \in \mathbb{SLRG}$ *the lattice order is a divisibility order, i.e. the next two equivalent conditions hold for all* $a, b \in A$*:*

Div1 if $a \leq b$, then $(b \to a)\, b = a$;

Div2 if $a \leq b$, then there is $c \in A$ s.t. $a = cb$.

Proof. We first show that (Div1) and (Div2) are equivalent. For the nontrivial implication, suppose that (Div2) holds and that $a \leq b$. Then $a \wedge b = a = cb$, for some $c \in A$. In particular, $cb \leq a \wedge b$, i.e., $c \leq b \to a \wedge b$. Hence, using Lemma 30.(6):

$$(b \to a)\, b \leq a = a \wedge b = cb \leq (b \to a \wedge b)\, b = (b \to a)\, b,$$

and the consequent of (Div1) follows.

Now we show that (Div1) holds. Suppose that $a \leq b$. Then by Lemmas 30.(7) and 31,

$$a = a \wedge b = (b \to a) \wedge b \leq (b \to a)\, b.$$

Since the converse inequality always holds, we obtain our conclusion. $\square$

Next, we present our term equivalence. We provide mutually inverse functions mapping members of $\mathbb{SLRG}$ to members of $\mathbb{SOML}$, and back.

Definition 33. (1) Let $\mathbf{A} = \langle A, \cdot, \to, \Rightarrow, \wedge, \vee, 0, 1\rangle$ be a member of $\mathbb{SLRG}$. We define

$$f(\mathbf{A}) = \langle A, \wedge, \vee, ', ^{\sim}, 0, 1\rangle,$$

where for any $a \in A$, $a^{\sim} = a \to 0$ and $a' = a \Rightarrow 0$.

(2) Let $\mathbf{L} = \langle L, \wedge, \vee, ', ^{\sim}, 0, 1\rangle$ be a semiorthomodular BZ*-lattice. We define

$$g(\mathbf{L}) = \langle L, \cdot, \to, \Rightarrow, \wedge, \vee, 0, 1\rangle,$$

where for any $a, b \in L$, $ab = (a \vee b^{\sim}) \wedge \Diamond b$, $a \to b = (b \wedge \Diamond a) \vee a^{\sim}$ and $a \Rightarrow b = (b \wedge a) \vee a'$.

For a start, we show that the maps in Definition 33 are well-defined.

Lemma 34. $f(\mathbf{A})$ *is a semiorthomodular BZ*-lattice.*

Proof. $\langle A, \wedge, \vee, ', 0, 1\rangle$ is a pseudo-Kleene algebra by Definition 29. We now check the BZ*-axioms.

$[x \wedge x^{\sim} \approx 0]$ By Lemma 30.(7)-(8), $a \wedge (a \to 0) \leq a\,(a \to 0) = 0$.

$[x \leq x^{\sim\sim}]$ By Lemma 30.(2).

$[x \leq y$ implies $y^{\sim} \leq x^{\sim}]$ By (Antit).

$[x^{\sim\prime} \approx x^{\sim\sim}]$ By Lemma 30.(3) $(a \to 0) \to 0 \leq (a \to 0) \Rightarrow 0$. By Lemma 30.(4), $((a \to 0) \Rightarrow 0)\,(a \to 0) = 0$, whence $(a \to 0) \Rightarrow 0 \leq (a \to 0) \to 0$.

$[(x \wedge x')^{\sim} = x^{\sim} \vee \Box x]$ By (*).

We are left with semiorthomodularity. By (Sas1) and Theorem 6, all we have to prove is that for all a, b such that $a \leq b \to 0$, we have that $a = a\,(b \to 0)$. However, if $a \leq b \to 0$, then by (Str.idemp) $a = a\,(a \vee (b \to 0)) = a\,(b \to 0)$. $\Box$

Lemma 35. *$g(\mathbf{L})$ is a SOML left-residuated groupoid.*

Proof. Clearly, $\langle L, \wedge, \vee, ', 0, 1\rangle$ is a pseudo-Kleene algebra, and for all $a \in L$, $a1 = (a \vee 1^{\sim}) \wedge \Diamond 1 = a$. We check left residuation.

$$\begin{array}{rll} ab \leq c & \text{implies} & (a \vee b^{\sim}) \wedge \Diamond b \leq c \\ & \text{implies} & (a \vee b^{\sim}) \wedge \Diamond b \leq c \wedge \Diamond b \\ & \text{implies} & a \leq a \vee b^{\sim} \\ & & = ((a \vee b^{\sim}) \wedge \Diamond b) \vee b^{\sim} \\ & & \leq (c \wedge \Diamond b) \vee b^{\sim} \\ & & = b \to c. \end{array}$$

Conversely,

$$\begin{array}{rll} a \leq b \to c & \text{implies} & a \leq (c \wedge \Diamond b) \vee b^{\sim} \\ & \text{implies} & a \vee b^{\sim} \leq (c \wedge \Diamond b) \vee b^{\sim} \\ & \text{implies} & ab = (a \vee b^{\sim}) \wedge \Diamond b \\ & & \leq ((c \wedge \Diamond b) \vee b^{\sim}) \wedge \Diamond b \\ & & = c \wedge \Diamond b \leq c. \end{array}$$

As to the other axioms, (Antit) and (*) are clear. Moreover, for all $a, b \in L$, (Str.idemp.) $a\,(a \vee b) = (a \vee (a \vee b)^{\sim}) \wedge \Diamond\,(a \vee b) = a$, by Theorem 6.(5)

(Sas1) This holds true by the definition of product in $g(\mathbf{L})$.

(Ksas) $a \Rightarrow b = (b \wedge a) \vee a' = (b \wedge a) \vee (a \Rightarrow 0)$.

(Str.Invol.) $a \to b = (b \wedge \Diamond a) \vee a^{\sim} = ((b' \vee a^{\sim}) \wedge \Diamond a)' = ((b \Rightarrow 0)\,a) \Rightarrow 0$. $\Box$

Theorem 36. *There exists a term equivalence between $\mathbb{SOML}$ and $\mathbb{SLRG}$, induced by the correspondences f and g in Definition 33.*

Proof. By Lemmas 34 and 35, all we have to prove is that f and g are mutually inverse. Let $\mathbf{L} = \langle L, \wedge, \vee, ', ^{\sim}, 0, 1\rangle$ be a semiorthomodular PBZ*-lattice, and let $a \in L$. Then:

- $a^{\sim f(g(\mathbf{L}))} = a \to^{g(\mathbf{L})} 0 = \left(0 \wedge^{\mathbf{L}} \Diamond a\right) \vee^{\mathbf{L}} a^{\sim \mathbf{L}} = a^{\sim \mathbf{L}}$.
- $a^{\prime f(g(\mathbf{L}))} = a \Rightarrow^{g(\mathbf{L})} 0 = \left(0 \wedge^{\mathbf{L}} a\right) \vee^{\mathbf{L}} a^{\prime \mathbf{L}} = a^{\prime \mathbf{L}}$.

Let now $\mathbf{A} = \langle A, \cdot, \to, \Rightarrow, \wedge, \vee, 0, 1\rangle$ be a SOML left-residuated groupoid, and let $a, b \in A$. Then:

- $a \cdot^{g(f(\mathbf{A}))} b = \left(a \vee b^{\sim f(\mathbf{A})}\right) \wedge \Diamond^{f(\mathbf{A})} b = (a \vee (b \to 0)) \wedge ((b \to 0) \to 0) = a \cdot^{\mathbf{A}} b$ by (Sas1).
- $a \Rightarrow^{g(f(\mathbf{A}))} b = a \Rightarrow^{\mathbf{A}} b$, by (Ksas).

- Let $c = \big(a \to^{\mathbf{A}} 0\big) \to^{\mathbf{A}} 0$. Then:

$$\begin{array}{lll} a \to^{g(f(\mathbf{A}))} b & = \big(b \wedge \Diamond^{f(\mathbf{A})} a\big) \vee a^{\sim f(\mathbf{A})} & \\ & = \Big(\big(\big(b \Rightarrow^{\mathbf{A}} 0\big) \vee \big(a \to^{\mathbf{A}} 0\big)\big) \wedge c\Big) \Rightarrow^{\mathbf{A}} 0 & \text{(pseudo-Kleene)} \\ & = \big(\big(b \Rightarrow^{\mathbf{A}} 0\big)\, a\big) \Rightarrow^{\mathbf{A}} 0 & \text{(Sas1)} \\ & = a \to^{\mathbf{A}} b. & \text{(Str. invol.)} \end{array}$$

□

Note from the definition of the map f and the fact that orthomodular lattices are exactly the PBZ*–lattices whose complements coincide that this term equivalence takes $\mathbb{OML}$ into the subvariety $\mathbb{V} = \{\mathbf{A} \in \mathbb{SLRG} : \to^{\mathbf{A}} = \Rightarrow^{\mathbf{A}}\}$ of $\mathbb{SLRG}$.

One can define the horizontal sum $\mathbf{A} \boxplus \mathbf{B}$ of any nontrivial member $\mathbf{A}$ of this subvariety $\mathbb{V}$ with any nontrivial SOML left–residuated groupoid $\mathbf{B}$ by letting its bounded lattice reduct be the horizontal sum of the bounded lattice reducts of $\mathbf{A}$ and $\mathbf{B}$ and defining its multiplication and two implications by restriction to the summands. A handy way to see that the result of this horizontal sum is indeed a SOML left–residuated groupoid is denoting $\mathbb{V} \boxplus \mathbb{SLRG} = \mathbb{T} \cup \{\mathbf{A} \boxplus \mathbf{B} : \mathbf{A} \in \mathbb{V} \setminus \mathbb{T}, \mathbf{B} \in \mathbb{SLRG} \setminus \mathbb{T}\}$ and using Proposition 25.(2) and the definition of the map f to get that $\mathbb{V} \boxplus \mathbb{SLRG} = \mathbb{SLRG}$ in the variety of all algebras of the same signature as SOML left–residuated groupoids.

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