

Parameter estimation of integrated fractional Brownian motion

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ABSTRACT

Fractional Brownian motion (fBm) is a canonical model for long-memory phenomena. In the presence of large amounts of potentially memory-bearing data, the data are often averaged, which can change the structure of the underlying relationships and affect standard estimation procedures. To address this, we introduce the normalized integrated fractional Brownian motion (nifBm), defined as the average of fBm over a fixed interval. We derive its covariance structure, investigate the stationarity and self-similarity, and extend the framework to linear combinations of independent nifBms and models with deterministic drift. For such linear combinations, we establish the stationarity of increments, investigate the asymptotic behavior of the autocovariance function, and prove an ergodic theorem essential for statistical inference. We consider two statistical models: one driven by a single nifBm and another by a linear combination of two independent nifBms, including cases with deterministic drift. For both models, we propose estimators that are strongly consistent and asymptotically normal for both the drift and the full parameter set. Numerical simulations illustrate the theoretical findings, providing a foundation for modeling averaged fractional dynamics, with potential applications in finance, energy markets, and environmental studies.

1. Introduction

In the past decade, models involving fractional dynamics have become widely used across various scientific fields, including communications, biology, physics, and finance. The growth of interest around these models is primarily due to the ability of fractional dynamics to describe systems with memory effects, where future values are influenced by past values rather than being entirely independent. The most notable example of such processes is fractional Brownian motion (fBm), which is characterized by its Hurst exponent $H \in (0, 1)$: besides characterizing the smoothness of fBm paths, the Hurst exponent is a measure of its long-term memory, capturing the degree of autocorrelation and its decay over time. Fractional Brownian motion has been widely employed to model the stochastic component of observed data in various applied contexts, including soil temperature, telecommunications, signal processing, and hydrology among others [1–4].

As it is very well-known, the Hurst exponent H provides insight into the nature of a time series based on a fBm, allowing classification into three categories: if $H = 0.5$ the time series is a random walk; $H < 0.5$ indicates anti-persistence, which means that

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an up value is more likely to be followed by a down value, and vice versa; if $H > 0.5$, the series is said to be persistent, it means that the direction of the next value is more likely to be the same as the current value.

There are numerous methods to estimate the Hurst exponent and characterize short- or long-range dependence based on the behavior of the autocovariance function (ACVF) and the self-similarity properties. A key issue in the literature, which we aim to highlight, is that in many studies, researchers pre-process the data, often averaging it, before estimating H . This pre-processing step may introduce distortions that affect the accuracy of the exponent's estimation: in fact, nothing guarantees that a linear combination of increments of a fBm, like the typical result of this averaging process, is still a fBm. Common examples of this phenomenon are time series of spot electricity prices in markets having an hourly frequency [5]: in fact, a quite common approach in the literature is to calculate the series using a daily average before performing the estimation of H (see, e.g., Morale and Giordano [6]). Recent contributions in financial econometrics have also emphasized that aggregation or measurement procedures may substantially affect the statistical inference of roughness and memory parameters; see, for instance, Bolko et al. [7], Chong et al. [8]. These works mainly study statistical procedures based on aggregated or integrated volatility-type functionals, typically based on quadratic variation or realized variance, whereas in the present paper, we adopt a complementary approach by introducing an explicit stochastic model obtained by averaging the underlying fractional process itself and by analyzing its probabilistic and statistical properties.

The main motivation of this paper is to make the effect of data aggregation explicit in models whose fine-scale dynamic is compatible with a fractional structure, thereby avoiding the implicit assumption that the aggregated process remains a fractional Brownian motion. To this aim, we analyze the process we call the *normalized integrated fractional Brownian motion* (nifBm), defined as

$$X_t^{h,H} := \frac{1}{h} \int_t^{t+h} W_u^H du.$$

where $h > 0$ is fixed and t varies in the interval $[0, T]$, with $T \in (0, +\infty)$, and some processes related to it. This process is, of course, Gaussian; thus, its distribution is fully characterized by its covariance operator, which in turn will depend on the increment h and on the Hurst exponent H . Extensions of this process that we will consider in this paper are linear combinations of two independent nifBm, i.e., $aX_t^{h,H_1} + bX_t^{h,H_2}$, with $a, b \neq 0$ and $H_1 \neq H_2$, and drifted versions of these processes, where the drift is of the form $\mu G(t)$, with $G: [0, T] \rightarrow \mathbb{R}$ being a known (possibly nonlinear) function and $\mu \neq 0$ being a parameter. We feel that this new class of processes could be easily applied to modeling phenomena where one does not directly observe something happening instantaneously (which could have the dynamics represented as a fBm), but rather the continuous accumulation of it in a given time interval of length h (which could be represented as the increment of a nifBm). For example, two common ways to measure insolation in a given location are solar radiance, which is an instantaneous measurement of solar radiation, and solar insolation, which is the solar radiance averaged over a given time period [9, Chapter 2.5]. Thus, if one assumes that solar radiance follows a fBm, as a logical consequence, solar insolation should follow a nifBm. We emphasize that the value of h is typically determined by the time scale at which data is aggregated in practice. For example, one common choice is to consider hourly or daily averages, as in electricity markets, where one can consider choices such as $h = 1$ h (hourly data) or $h = 24$ h or 1 day (daily averages of hourly data), or photovoltaic production, where Typical Meteorological Year (TMY) data collect hourly solar insolation at a given location for each hour of a given year. The parameter h determines the aggregation scale and therefore influences both the dependence structure of the process and the statistical behavior of the estimators. A more detailed mathematical analysis of the behavior of the model and the estimators as h varies, for instance through asymptotic regimes or scale-invariant approaches, represents a natural direction for future research.

Without claiming to provide a complete presentation of all works devoted to integrated fBm, let us nevertheless mention that it is widely studied from both a theoretical point of view and for applications. For example, in [10] small ball estimates for integrated fBm are considered; in [11] such functionals of integrated fBm as the maximum, the position of the maximum, the occupation time above zero, and some others are studied. In [12] the problem of the log-asymptotic behavior of the probability that integrated fBm does not exceed a fixed level over a long time is solved. The authors of [13] consider the persistence exponent of integrated fBm, a problem that originates in theoretical physics, where the persistence exponent serves as a simple measure of how fast a complicated physical system returns from a disordered initial condition to its stationary state. In [14] the persistence exponent is studied numerically, while in [15] the mean and covariance of ifBm and of some other integrated Gaussian processes are investigated. In [16] biomarker measurement modeling with a longitudinal submodel is analyzed, and to overcome longitudinal submodel challenges, the authors replace random slopes with scaled integrated fBm.

Linear combinations of two fractional processes have also been the subject of extensive research. The motivation for this approach is that one process typically captures short-term variations and exhibits rougher behavior, whereas the other represents long-term dynamics and is smoother. However, most existing studies concentrate on linear combinations of two fractional Brownian motions (fBms) or, as a special case, on the so-called mixed fBm, which is defined as a linear combination of a Wiener process and an fBm. However, we define mixed fBm as a linear combination of any fBms. For instance, the linear combination of two fBms was examined in [17], where estimators were proposed for the coefficients a and b , as well as for the two Hurst exponents H_1 and H_2 . The problem of maximum likelihood estimation of the drift parameter in a model with linear drift driven by two independent fBms was studied in [18,19]. More recently, multivariate extensions of fractional Gaussian models allowing for different Hurst exponents across components and nontrivial cross-dependence structures have been investigated in [20], where a multivariate fractional Ornstein–Uhlenbeck process is introduced and statistical inference for cross-correlation parameters is developed. Moreover, Mishura et al. [21, 22] introduced estimators for the parameter μ in a nonlinear drift term $\mu G(t)$, where the driving process is any Gaussian process with a known and nonsingular covariance operator.

The particular case of mixed fBm with one Wiener component has been investigated in greater detail from theoretical, practical, and statistical perspectives. Several methods for parameter estimation in such processes have been proposed, including the maximum

likelihood method [23], Powell's optimization algorithm [24], and approaches based on power variations [25]. Simultaneous parameter estimation for mixed fBm with drift has also been addressed using different methodologies: a modification of the maximum likelihood approach in [26], and two further approaches – power variation-based estimators and ergodic-type estimators – in [27].

Since the processes considered in this paper are Gaussian, one could in principle apply likelihood-based procedures such as maximum likelihood or Whittle-type estimators. However, in fractional and mixed fractional models these approaches typically require repeated inversion of large covariance matrices and lead to estimators defined only implicitly as maximizers of likelihood functionals; see, e.g., Xiao et al. [23], Zhang et al. [24], Dufitinema et al. [26]. This substantially complicates both their theoretical analysis and numerical implementation, especially in the present setting where integration of the underlying fractional process results in a more involved covariance structure. Although likelihood-based estimators are asymptotically efficient under ideal conditions, their practical use in long-memory models is often limited, while frequency-domain approaches such as Whittle estimation rely on spectral approximations whose accuracy may deteriorate under aggregation or integration, as these operations significantly modify the low-frequency behavior of the process. In contrast, the increments of the models considered here form stationary Gaussian sequences. This makes estimation based on ergodic-type statistics natural and allows one to obtain explicit and computationally tractable estimators together with rigorous asymptotic results via standard limit theorems for stationary Gaussian processes; see, for instance, Mishura et al. [19], Dozzi et al. [25], Kukush et al. [27].

Combining these two important points, namely, the averaging of data with memory and statistical estimation in mixed fractional models (linear combinations of fBms), we arrive at a new and logical continuation: statistical estimation in averaged (integrated and normalized) fractional models and their linear combinations.

The paper is organized as follows. In Section 2, we compute the covariance function of nifBm and of its increments, and investigate the properties of stationarity and self-similarity. Section 3 is devoted to the study of linear combinations of two nifBms. In particular, we establish the stationarity of increments, analyze the asymptotic behavior of the corresponding autocovariance function, and derive an ergodic theorem, which plays a key role in parameter estimation. The subsequent sections address statistical problems related to the processes introduced above. Specifically, Section 4 presents a statistical model involving two nifBms with drift, examines two estimators of the drift parameter, and establishes their strong consistency and normality. As a special case, we also discuss the corresponding model with a single nifBm. The simultaneous estimation of all parameters of a nifBm and of a linear combination of two nifBms is studied in Sections 5–6. Section 5 focuses on the construction of strongly consistent estimators, while Section 6 establishes their joint asymptotic normality. Section 7 presents numerical simulations that support our theoretical results. Finally, Appendix A contains proofs of several auxiliary lemmas, and Appendix B summarizes useful properties of Gaussian random vectors together with a limit theorem employed in the proof of asymptotic normality.

2. Normalized integrated fractional Brownian motion

In this section, we aim to compute covariances and investigate the properties of stationarity and self-similarity of the normalized integral over time of fractional Brownian motion and of its increments.

2.1. Basics of fractional Brownian motion

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which all the processes considered below are correctly defined.

We recall that fractional Brownian motion $W^H = \{W_t^H, t \geq 0\}$ with Hurst index $H \in (0, 1)$ is a centered Gaussian process having the following covariance function:

$$\mathbb{E}[W_s^H W_t^H] = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}).$$

For $H = \frac{1}{2}$, the fBm is a classical Brownian motion. If $H = 1$, then $W_t^H = tW_1^H$ a.s. for all $t \geq 0$. For $H > 1/2$, the increments of the process are positively correlated, while for $H < 1/2$ they are negatively correlated. Moreover, this process has the self-similarity property, with parameter of self-similarity (or scaling exponent) H and possesses stationary increments. Since we will be interested in the incremental covariances, we recall here the result that can be obtained by direct calculations and is well-known.

Lemma 1. For any $H \in (0, 1)$ and any $0 \leq s \leq t \leq u \leq v$

$$\mathbb{E}(W_t^H - W_s^H)(W_v^H - W_u^H) = \frac{1}{2}((v - s)^{2H} + (u - t)^{2H} - (v - t)^{2H} - (u - s)^{2H}).$$

Recall also that, according to [28], fBm W^H is a Gaussian-Volterra process admitting a representation

$$W_t^H := \int_0^t K_H(t, s) dW_s \quad \text{for all } t \geq 0, \quad (1)$$

where W is a Wiener process (so called underlying Wiener process), kernel function K_H has a form

$$K_H(t, s) = c_H \left[\left(\frac{t}{s} \right)^{H-\frac{1}{2}} (t-s)^{H-\frac{1}{2}} + \left(\frac{1}{2} - H \right) \int_s^t u^{H-\frac{3}{2}} \left(\frac{u}{s} - 1 \right)^{H-\frac{1}{2}} du \right],$$

for $H > \frac{1}{2}$ it is simplified to

$$K_H(t, s) = \left(H - \frac{1}{2} \right) c_H s^{1/2-H} \int_s^t u^{H-\frac{1}{2}} (u-s)^{H-\frac{3}{2}} du,$$

and $c_H := \left(\frac{2H\Gamma(H+\frac{1}{2})\Gamma(\frac{3}{2}-H)}{\Gamma(2-2H)} \right)^{1/2}$. It is also well known, see e.g. Mishura [29], that the filtrations generated by W^H and W coincide. Denote this common filtration as $\mathcal{F} = \{\mathcal{F}_t, t \geq 0\}$.

2.2. Covariances and incremental covariances of the normalized integrated fractional Brownian motion. Stationarity of increments

Now, taking into account that the trajectories of fBm are continuous a.s., for any $t \geq 0$ we can consider the normalized Riemann integral of the form

$$X_t^{h,H} := \frac{1}{h} \int_t^{t+h} W_u^H du. \quad (2)$$

In the rest of the paper, we refer to this process as the normalized integrated fractional Brownian motion (nifBm). If the parameter H is clear from the context, we will write simply X_t^h . Of course, under any consideration, even being considered as a function of both parameters, t and h , nifBm is Gaussian and zero-mean process. Therefore, its main characteristics is a covariance function. So, let us calculate covariance and variance of this process and of its increments. In these calculations the length h of the interval of integration will be fixed while t will be shifted. Note that some related calculations for variance and covariance of the integrated fBm were obtained in [15]. However, in order to get the statistical estimators, we need to produce much more detailed analogs.

First, let us compute the covariance of two values of nifBm. For the reader's convenience, we defer the proofs of the auxiliary lemmas in this section to the Appendix.

Lemma 2. For any $s \geq t$

$$\begin{aligned} \mathbb{E}[X_t^h X_s^h] &= \frac{1}{2h(2H+1)} ((s+h)^{2H+1} - s^{2H+1} + (t+h)^{2H+1} - t^{2H+1}) \\ &+ \frac{1}{2h^2(2H+1)(2H+2)} (2(s-t)^{2H+2} - (s-t+h)^{2H+2} - |s-t-h|^{2H+2}). \end{aligned} \quad (3)$$

In particular, for any $t \geq 0$

$$\mathbb{E}[(X_t^h)^2] = \frac{1}{h(2H+1)} ((t+h)^{2H+1} - t^{2H+1}) - \frac{h^{2H}}{(2H+1)(2H+2)}. \quad (4)$$

Remark 3. It is clear from the formula (3), more precisely, from the first term in its right-hand side, as well as from (4), that the process X^h is not stationary in argument t . If we analyze the value $\mathbb{E}[(X_t^h)^2]$ from (4) as $t \rightarrow \infty$, then

$$(t+h)^{2H+1} - t^{2H+1} = t^{2H+1} \left(\left(1 + \frac{h}{t}\right)^{2H+1} - 1 \right) \sim t^{2H+1} (2H+1) \frac{h}{t},$$

whence $\mathbb{E}[(X_t^h)^2] \sim t^{2H}$ as $t \rightarrow \infty$, the same asymptotics as the fBm has.

Now our goal, taking into account the subsequent construction of statistical estimators that will be based on the ergodic theorem, is to construct a stationary process, starting from nifBm. In this regard, let us consider its increments obtained by shifting time t by an amount nh proportional to h . It is clear that the form of covariance will be different for $n = 0$ (variance, intervals of integration are completely overlapping), $n = 1$ (covariance, intervals are partially overlapping) and $n \geq 2$ (covariance, intervals are not overlapping). Taking this into account, introduce the notation

$$\gamma(H, n) := \begin{cases} \frac{2(2^{2H}-1)}{(2H+1)(H+1)}, & n = 0, \\ (2(2H+1)(2H+2))^{-1} (7 - 4 \cdot 2^{2H+2} + 3^{2H+2}), & n = 1, \\ (2(2H+1)(2H+2))^{-1} ((n-2)^{2H+2} - 4(n-1)^{2H+2} \\ + 6n^{2H+2} - 4(n+1)^{2H+2} + (n+2)^{2H+2}), & n \geq 2, \end{cases}$$

that can be expressed in a more concise form as

$$\gamma(H, n) = \frac{(|n-2|^{2H+2} - 4|n-1|^{2H+2} + 6|n|^{2H+2} - 4|n+1|^{2H+2} + |n+2|^{2H+2})}{4(2H+1)(H+1)}, \quad (5)$$

and proceed with the covariance of the increments of the process X^h .

Theorem 4. For any $t \geq 0$ and $n \geq 0$

$$\mathbb{E}[(X_{t+h}^h - X_t^h)(X_{t+(n+1)h}^h - X_{t+nh}^h)] = h^{2H} \gamma(H, n). \quad (6)$$

Proof. The proof is completely based on Lemma 2, and it is just necessary to calculate, according to the formulas (3) and (4), all the covariances and then add them up. Therefore, we will consider only the case $n = 0$ in detail to show how the terms containing t , mutually cancel each other. The covariances that correspond to the cases $n = 1$ and $n \geq 2$ are calculated in the same spirit, with the help of (3) and (4), therefore we shall omit these very similar but rather tedious calculations. So, let $n = 0$. Then, according to (4),

$$\mathbb{E}[(X_{t+h}^h)^2] = \frac{1}{h(2H+1)} ((t+2h)^{2H+1} - (t+h)^{2H+1}) - \frac{h^{2H}}{(2H+1)(2H+2)},$$

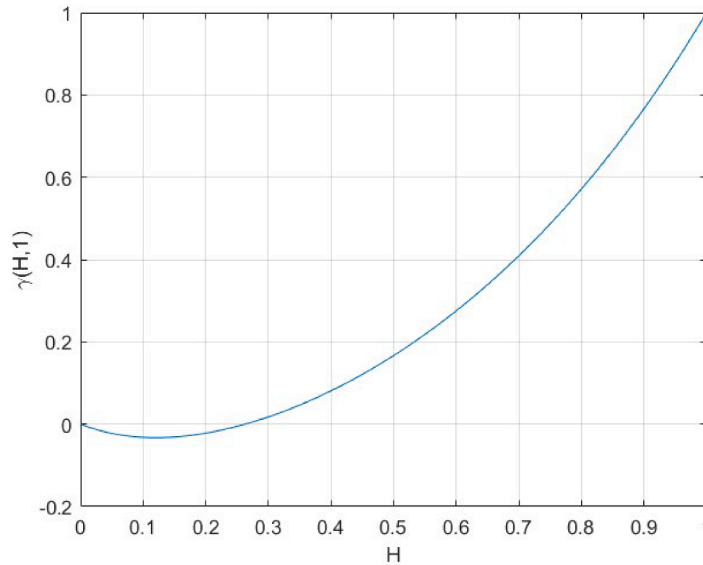


Fig. 1. Graph of the function $\gamma(H, 1)$ for $H \in (0, 1)$. The function becomes zero at $H_0 \approx 0.2626229$.

$\mathbb{E}[(X_t^h)^2]$ is calculated in (4), and the value $\mathbb{E}[X_t^h X_{t+h}^h]$ corresponds to (2) with $s = t + h$, and equals

$$\mathbb{E}[X_t^h X_{t+h}^h] = \frac{1}{2h(2H+1)}((t+2h)^{2H+1} - t^{2H+1}) + \frac{1}{2h^2(2H+1)(2H+2)}(2h^{2H+2} - (2h)^{2H+2}).$$

Therefore

$$\begin{aligned} \mathbb{E}(X_{t+h}^h - X_t^h)^2 &= \frac{1}{h(2H+1)}[(t+2h)^{2H+1} - (t+h)^{2H+1} + (t+h)^{2H+1} - t^{2H+1} - (t+2h)^{2H+1} + t^{2H+1}] \\ &\quad - \frac{2h^{2H}}{(2H+1)(2H+2)} - \frac{2h^{2H+2} - (2h)^{2H+2}}{h^2(2H+1)(2H+2)} \\ &= \frac{h^{2H}}{(2H+1)(2H+2)}(2^{2H+2} - 4) = \frac{2h^{2H}}{(2H+1)(H+1)}(2^{2H} - 1). \end{aligned}$$

□

Now it is interesting to study for which values of $H \in (0, 1)$ the quantities $\gamma(H, n)$ are positive or vice versa, and compare this to the increments of fBm that are positively or negatively correlated for $H > 1/2$ or $H < 1/2$, respectively.

Lemma 5.

- (i) Let $H \in (0, 1)$. The function $\gamma(H, 1)$, is zero for $H = 0$ and admits a unique $H_0 \in (0, 1)$ such that $\gamma(H_0, 1) = 0$, $\gamma(H, 1) < 0$ for $H \in (0, H_0)$ and $\gamma(H, 1) > 0$ for $H \in (H_0, 1]$. Numerical computations give $H_0 \approx 0.2626229$. The graph of the function $\gamma(H, 1)$ over the interval $(0, 1)$ is shown in Fig. 1.
- (ii) For any $n \geq 2$ function $\gamma(H, n)$, $H \in (0, 1)$ is strictly negative for $H \in (0, 1/2)$, zero for $H = 1/2$ and strictly positive for $H \in (1/2, 1)$.

Remark 6. Note that the covariance in (6) does not depend on t . Also, X_t^h is a Gaussian process in both arguments t and h . It means that choosing t proportional to h (as done in the main part of the paper), we get a stationary Gaussian sequence. Additionally, it is interesting to observe that in the brackets on the right-hand side of (6), we have the subsequent binomial coefficients of the binomial expansion for $(a-b)^4$. This tendency is partially observed even in the numerator of $\gamma(H, 1)$, which can be rewritten as $(3^{2H+2} - 4 \cdot 2^{2H+2} + 6) + 1$, where 1 is forced to replace the remaining terms.

Remark 7. Consider briefly smoothness of trajectories of nifBm X^h . Obviously, for any $0 \leq s < t$,

$$X_t^h - X_s^h = \frac{1}{h} \left(\int_t^{t+h} W_u^H du - \int_s^{s+h} W_u^H du \right) = \frac{1}{h} \left(\int_{s+h}^{t+h} W_u^H du \pm \int_s^t W_u^H du \right), \quad (7)$$

where “+” corresponds to the case $t \leq s+h$, and “-” corresponds to the case $t > s+h$.

In any case, considering some interval $[0, T]$, we have that

$$\begin{aligned} |X_t^h - X_s^h| &\leq \frac{1}{h} \left(\int_{s+h}^{t+h} |W_u^H| du + \int_s^t |W_u^H| du \right) \\ &\leq \frac{2}{h} \sup_{0 \leq u \leq T+h} |W_u^H| (t-s). \end{aligned}$$

Thus, X^h is a Lipschitz function. Moreover, $X^h \in C^1(\mathbb{R})$, and

$$(X^h)'_t = \frac{1}{h} (W_{t+h}^H - W_t^H).$$

Of course, $(X^h)'_t$ is a.s. unbounded as $h \downarrow 0$. However, if in (7) we have $h < t - s$, then

$$X_t^h - X_s^h = \frac{1}{h} \left(\int_{s+h}^{t+h} W_u^H du - \int_s^t W_u^H du \right) = \frac{1}{h} (\psi_h - \psi_0),$$

where $\psi_h = \int_{s+h}^{t+h} W_u^H du$, and

$$X_t^h - X_s^h = \psi'_h(\theta_h(\omega)) = W_{t+\theta_h(\omega)}^H - W_{s+\theta_h(\omega)}^H,$$

where $\theta_h(\omega) \in (0, h)$. Therefore, uniformly in h , for any $0 < \varepsilon < H$,

$$|X_t^h - X_s^h| \leq C_\varepsilon(\omega) |t - s|^{H-\varepsilon}.$$

2.3. Self-similarity of the normalized integral

Recall that fBm is a self-similar process with self-similarity index H . This property may be crucial in many estimation procedures. Let us establish self-similarity of the process X_t^h , $t \geq 0$. The standard definition of self-similarity is the following one.

Definition 8. A stochastic process $\{Y_t, t \geq 0\}$ is called self-similar with self-similarity parameter $\alpha \in (0, 1)$ if for all $c > 0$

$$\{Y_{ct}, t \geq 0\} \stackrel{d}{=} \{c^\alpha Y_t, t \geq 0\}.$$

Given that nifBm defined in (2) depends on t and h , we consider the following modified property of self-similarity.

Lemma 9. For any $h > 0$ a nifBm $X^h = \{X_t^h, t \geq 0\}$ is self-similar with self-similarity parameter H in the sense that for all $c > 0$

$$\{X_{ct}^{ch}, t \geq 0\} \stackrel{d}{=} \{c^H X_t^h, t \geq 0\}.$$

2.4. Nonsingularity of the incremental covariance matrix

Fix some $h > 0$ and define the discrete-time increments $\Delta X_k^h := X_{(k+1)h}^h - X_{kh}^h$, $k \in \mathbb{N}_0$. This of course defines a Gaussian sequence $\Delta X^h = \{X_{(k+1)h}^h - X_{kh}^h, k \in \mathbb{N}_0\}$. From Theorem 4, it is immediate to notice that

$$\mathbb{E}[\Delta X_k^h \Delta X_{k+n}^h] = h^{2H} \gamma(H, n), \quad \text{for all } k \in \mathbb{N}_0. \quad (8)$$

In the next lemma, we show that any covariance matrix of the increments ΔX_k^h is non-singular.

Proposition 10. If $0 \leq k_1 < \dots < k_n$, then $\Sigma_{k_1, \dots, k_n} := (\text{Cov}(\Delta X_{k_i}^h, \Delta X_{k_j}^h))_{i,j=1}^n$ has rank n .

Proof. Let us prove this by contradiction. Assume that for some $0 \leq k_1 < \dots < k_n$

$$\text{rank } \Sigma_{k_1, \dots, k_{n-1}} = \text{rank } \Sigma_{k_1, \dots, k_n} = n - 1$$

(if $\text{rank } \Sigma_{k_1, \dots, k_{n-1}} < n - 1$, a similar argument could be applied). Then it would be possible to express $\Delta X_{k_n}^h$ as a linear combination of the other components, i.e. there would exist $a_1, \dots, a_{n-1} \in \mathbb{R}$ such that

$$\Delta X_{k_n}^h = \sum_{i=1}^{n-1} a_i \Delta X_{k_i}^h. \quad (9)$$

Note that

$$\Delta X_{k_n}^h = \frac{1}{h} (I_n + \zeta_n), \quad (10)$$

where $I_n = \int_{(k_n+1)h}^{(k_n+2)h} W_s^H ds$ and ζ_n is $\mathcal{F}_{(k_n+1)h}$ -adapted. Also, $\sum_{i=1}^{n-1} a_i \Delta X_{k_i}^h$ is $\mathcal{F}_{(k_{n-1}+2)h} \subset \mathcal{F}_{(k_n+1)h}$ -adapted. Taking into account this adaptedness and (9)–(10), we conclude that I_n is $\mathcal{F}_{(k_n+1)h}$ -adapted, and therefore $\mathbb{E}(I_n | \mathcal{F}_{(k_n+1)h}) = I_n$. However, according to representation (1),

$$\mathbb{E}(I_n | \mathcal{F}_{(k_n+1)h}) = \int_{(k_n+1)h}^{(k_n+2)h} \int_0^{(k_n+1)h} K_H(s, u) dW_u ds \neq I_n = \int_{(k_n+1)h}^{(k_n+2)h} \int_0^s K_H(s, u) dW_u ds,$$

whence the proof follows. $\square$

Remark 11. In fact, we did not use in the proof stationarity of increments ΔX_k^h . However, taking into account stationarity and assuming that equality (9) holds, we get that for any $m \geq 0$

$$\Delta X_{k_n+m}^h = \sum_{i=1}^{n-1} a_i \Delta X_{k_i+m}^h.$$

Now, by the same reason, we can present all $\Delta X_{k_i+m}^h$ with $k_i + m \geq k_n$ via linear combination of the increments with smaller indices. By induction, it means that assumption (9) leads to the possibility to predict without error all the future increments ΔX_k^h for $k \geq k_n$ via increments until k_{n-1} . This observation is useful in studying stationary Gaussian sequences.

3. Linear combination of two nifBms and its main properties

In this section we study the properties of linear combination of two independent incremental processes constructed from two nifBms.

3.1. Stationarity of the increments of the linear combination of nifBms

Starting with the process (2), put in it $t = kh$ and define the following stochastic sequence:

$$\tilde{X}_k^h = \frac{a}{h} \int_{kh}^{(k+1)h} W_s^{H_1} ds + \frac{b}{h} \int_{kh}^{(k+1)h} W_s^{H_2} ds = aX_{kh}^{h,H_1} + bX_{kh}^{h,H_2}, \quad k \in \mathbb{N}_0, h > 0, \quad (11)$$

where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Here coefficients a and b are two real numbers, and W^{H_1}, W^{H_2} are two independent fractional Brownian motions with Hurst parameters $H_i \in (0, 1)$, $i = 1, 2$. So, we are working with equidistant observations, because one of our goals is to estimate the parameters $\{H_1, H_2, a, b\}$ of the model using these observations.

Remark 12. As usual, taking into account the symmetry of the Gaussian processes W^{H_1} and W^{H_2} , we can assume that $a > 0$, $b > 0$. Otherwise, in what follows we can just estimate $|a|$ and $|b|$.

Of course, the sequence $\{\tilde{X}_k^h, k \in \mathbb{N}_0\}$ is Gaussian but not stationary. Again, as in Section 2, in order to construct statistical estimators of all parameters basing on the ergodic theorem, we want to transform it into a stationary Gaussian stochastic sequence. This can be done in several ways, but the easiest way is to create an incremental process of the form $\Delta \tilde{X}_k^h := \tilde{X}_{k+1}^h - \tilde{X}_k^h$. Thus, we have

$$\begin{aligned} \Delta \tilde{X}_k^h &= a(X_{(k+1)h}^{h,H_1} - X_{kh}^{h,H_1}) + b(X_{(k+1)h}^{h,H_2} - X_{kh}^{h,H_2}) = \frac{a}{h} \int_{kh}^{(k+1)h} (W_{s+h}^{H_1} - W_s^{H_1}) ds + \frac{b}{h} \int_{kh}^{(k+1)h} (W_{s+h}^{H_2} - W_s^{H_2}) ds \\ &=: a\Delta X_k^{h,H_1} + b\Delta X_k^{h,H_2}. \end{aligned}$$

Proposition 13. The Gaussian sequence $\{\Delta \tilde{X}_k^h, k \in \mathbb{N}_0\}$ is stationary, with $\mathbb{E}[\Delta \tilde{X}_k^h] = 0$ for all $k \in \mathbb{N}_0$ and

$$\mathbb{E}[\Delta \tilde{X}_k^h \Delta \tilde{X}_{k+n}^h] = a^2 h^{2H_1} \gamma(H_1, n) + b^2 h^{2H_2} \gamma(H_2, n), \quad (12)$$

for all $k, n \in \mathbb{N}_0$.

Proof. Since $\{\Delta \tilde{X}_k^h, k \in \mathbb{N}_0\}$ is Gaussian, and $\mathbb{E}[\Delta \tilde{X}_k^h] = 0$ for all $k \in \mathbb{N}_0$, it is sufficient to prove that its covariance $\mathbb{E}[\Delta \tilde{X}_k^h \Delta \tilde{X}_{k+n}^h]$ does not depend on k . To prove this, we take into account independence and zero mean of $\Delta X_k^{h,H_1}$ and $\Delta X_k^{h,H_2}$, and state that

$$\mathbb{E}[\Delta \tilde{X}_k^h \Delta \tilde{X}_{k+n}^h] = a^2 \mathbb{E}[\Delta X_k^{h,H_1} \Delta X_{k+n}^{h,H_1}] + b^2 \mathbb{E}[\Delta X_k^{h,H_2} \Delta X_{k+n}^{h,H_2}].$$

Now, in order to finish the proof, it is sufficient to note, that, in terms of (6),

$$\mathbb{E}[\Delta X_k^{h,H} \Delta X_{k+n}^{h,H}] = \mathbb{E}[(X_{t+h}^h - X_t^h)(X_{t+(n+1)h}^h - X_{t+nh}^h)],$$

with $t = kh$, and apply Theorem 4. $\square$

3.2. Asymptotic behavior of the covariance

To assess whether we can use the ergodic theorems [30, Chapter 2] for Gaussian sequences, we need the following property: the covariance function (12) tends to zero as $n \rightarrow \infty$. It is sufficient to establish that for any $H \in (0, 1)$, $\gamma(H, n)$ tends to zero as $n \rightarrow \infty$. It is the subject of the next lemma.

Lemma 14. For any $H \in (0, 1)$ the sequence $\gamma(H, n), n \in \mathbb{N}_0$ defined by equality (5), is asymptotically negligible, i.e., tends to zero as $n \rightarrow \infty$. More precisely,

$$\frac{n^{2-2H} \gamma(H, n)}{2H(2H-1)} \rightarrow 1,$$

as $n \rightarrow \infty$.

Proof. Let us rewrite $\gamma(H, n)$ for $n \geq 2$ by taking out the multiplier n^{2H+2} :

$$\gamma(H, n) = \frac{n^{2H+2}}{2(2H+1)(2H+2)} g(n^{-1}),$$

where

$$g(x) = ((1-2x)^{2H+2} - 4(1-x)^{2H+2} + 6 - 4(1+x)^{2H+2} + (1+2x)^{2H+2}).$$

The asymptotic behavior of $\gamma(H, n)$ can be deduced from the Taylor expansion of $g(x)$ at the origin. Indeed, let us compute the subsequent derivatives of $g(x)$:

$$\begin{aligned} g'(x) &= (2H+2)(-2(1-2x)^{2H+1} + 4(1-x)^{2H+1} - 4(1+x)^{2H+1} + 2(1+2x)^{2H+1}), \\ g''(x) &= (2H+2)(2H+1)(4(1-2x)^{2H} - 4(1-x)^{2H} - 4(1+x)^{2H} + 4(1+2x)^{2H}), \\ g'''(x) &= (2H+2)(2H+1)(2H)(-8(1-2x)^{2H-1} + 4(1-x)^{2H-1} - 4(1+x)^{2H-1} + 8(1+2x)^{2H-1}), \\ g^{(iv)}(x) &= (2H+2)(2H+1)(2H)(2H-1)(16(1-2x)^{2H-2} - 4(1-x)^{2H-2} - 4(1+x)^{2H-2} + 16(1+2x)^{2H-2}). \end{aligned}$$

Observe that $g(0) = g'(0) = g''(0) = g'''(0) = 0$, and the first non-zero term in the Taylor expansion of g at zero is the fourth derivative.

$$g^{(iv)}(0) = 24(2H+2)(2H+1)(2H)(2H-1)$$

Then we get from the Taylor expansion that

$$g(x) = (2H+2)(2H+1)(2H)(2H-1)x^4 + o(x^4).$$

Hence, as $n \rightarrow \infty$, $\gamma(H, n)$ behaves as $2H(2H-1)n^{2H-2}$, which implies that

$$\frac{n^{2-2H}\gamma(H, n)}{2H(2H-1)} \rightarrow 1, n \rightarrow \infty.$$

□

Remark 15. It is interesting to note that the covariance of the increments of the process $\{X_{kh}^h, k \geq 1\}$ exhibits the same asymptotic behavior as that of a fractional Brownian motion. Specifically, when $H > \frac{1}{2}$, the function $\gamma^H(n)$ is strictly positive and decays slowly, whereas for $H < \frac{1}{2}$, $\gamma^H(n)$ is strictly negative and exhibits a faster decay. This can be easily explained if for $n \geq 2$ we transform the incremental covariance as follows:

$$\frac{1}{h^2} \mathbb{E} \int_0^h (W_{s+h}^H - W_s^H) ds \int_{nh}^{(n+1)h} (W_{t+h}^H - W_t^H) dt = \frac{1}{h^2} \int_0^h \int_0^h \mathbb{E}(W_{s+h}^H - W_s^H)(W_{t+(n+1)h}^H - W_{t+nh}^H) ds dt,$$

and recall that the increments of fBm are positively (negatively) correlated if $H > 1/2$ ($H < 1/2$). Moreover, applying the expansion of $(1+x)^{2H}$, we get that

$$\begin{aligned} \mathbb{E}(W_{s+h}^H - W_s^H)(W_{t+(n+1)h}^H - W_{t+nh}^H) &= \frac{1}{2} \left[(t-s+(n+1)h)^{2H} + (t-s+(n-1)h)^{2H} - 2(t-s+nh)^{2H} \right] \\ &= \frac{(nh)^{2H}}{2} \left[\left(1 + \frac{t-s+h}{nh}\right)^{2H} + \left(1 + \frac{t-s-h}{nh}\right)^{2H} - 2\left(1 + \frac{t-s}{nh}\right)^{2H} \right] \\ &= \frac{H(2H-1)(nh)^{2H}}{2} \left[\left(\frac{t-s+h}{nh}\right)^2 + \left(\frac{t-s-h}{nh}\right)^2 - 2\left(\frac{t-s}{nh}\right)^2 + o(n^{-2}) \right] \\ &\sim H(2H-1)h^{2H}n^{2H-2}, \end{aligned}$$

and we can apply the Lebesgue dominated convergence theorem in order to get the same asymptotic behavior of standard and integrated fBm. Figs. 2 and 3 illustrate the comparison of the incremental covariance of the integrated and standard fractional Brownian motions for $H = 0.3$ and $H = 0.7$, respectively. One more observation: it follows from Lemma 5, (ii), that for $k \geq j+2$ and for $H = \frac{1}{2}$ the respective increments are independent. Therefore, in this case $g_{1/2}(\frac{1}{n}) = 0, n \geq 2$, where

$$g_{1/2}(x) = (1-2x)^3 - 4(1-x)^3 + 6 - 4(1+x)^3 + (1+2x)^3.$$

Moreover, equality $g_{1/2}(x) = 0$ can be easily checked for any $x \leq \frac{1}{2}$.

3.3. Ergodic theorem for the normalized integral process

Taking into account that the normalized integral process is Gaussian, Proposition 13 that guarantees its stationarity, Lemma 14 that guarantees that the increments are asymptotically uncorrelated, and Theorem 5, Section 8 [30], we can formulate the following theorem.

Theorem 16. The Gaussian sequence $\Delta \tilde{X}_k^h := \tilde{X}_{k+1}^h - \tilde{X}_k^h, k \in \mathbb{N}_0$ is ergodic, i.e., for any function $Q : \mathbb{R} \rightarrow \mathbb{R}$ such that $\mathbb{E}[Q(\Delta \tilde{X}_0^h)] < \infty$, we have that

$$\frac{1}{N} \sum_{k=0}^{N-1} Q(\Delta \tilde{X}_k^h) \rightarrow \mathbb{E}[Q(\Delta \tilde{X}_0^h)]$$

a.s., as $N \rightarrow \infty$.

Remark 17. Of course, the same is true for the sequence $\Delta X_k^h := X_{(k+1)h}^h - X_{kh}^h, k \in \mathbb{N}_0$ with any Hurst index H .

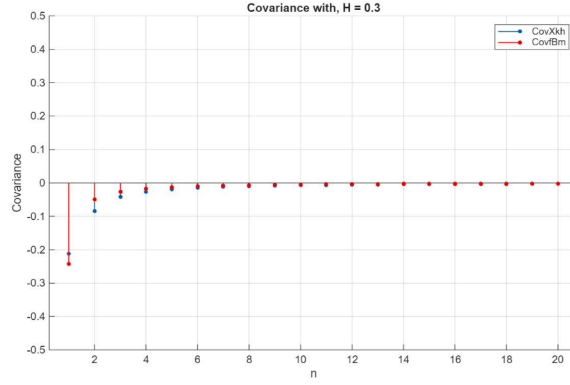


Fig. 2. Comparison of the covariance of the increments in our process ΔX_k^h and the fractional Brownian motion with $H = 0.3$.

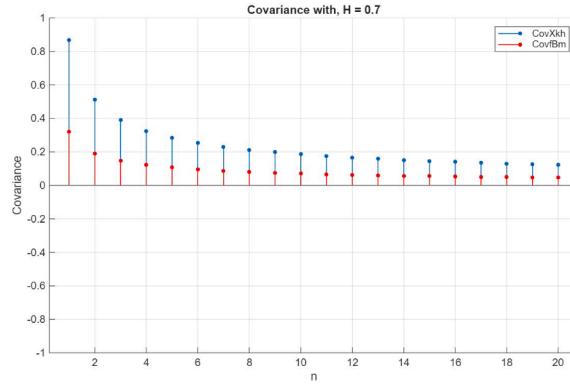


Fig. 3. Comparison between the covariance of the increments in our process ΔX_k^h and the fractional Brownian motion with $H = 0.7$.

4. Mixed nifBm with drift: Model description and drift parameter estimation

Starting from this section, we focus on the statistical inference for a process defined as a linear combination of two independent nifBms with a nonlinear drift component. As a special case, we also consider a related model driven by a single nifBm. This section is devoted to the estimation of the drift parameter, while the estimation of the noise parameters will be addressed in [Sections 5](#) and [6](#).

4.1. Drift parameter estimation in the model with drift and two nifBms

Let us begin by describing the statistical model. Fix $h > 0$, and let $X^{h,H_i} = \{X_t^{h,H_i}, t \geq 0\}$, $i = 1, 2$, denote two independent nifBms with Hurst parameters $H_1, H_2 \in (0, 1)$, as defined in [\(2\)](#). We define the mixed nifBm with drift $Y = \{Y_t, t \geq 0\}$ by

$$Y_t = \mu G(t) + aX_t^{h,H_1} + bX_t^{h,H_2},$$

where $\mu, a, b \in \mathbb{R}$ are constants, and G is a known deterministic function.

Suppose that the process Y is observed at discrete time points $t_k = kh$, for $k = 0, \dots, N$. The corresponding observations $Y_1, \dots, Y_N$ are given by

$$Y_k^h = \mu G(kh) + aX_{kh}^{h,H_1} + bX_{kh}^{h,H_2}, \quad k = 0, \dots, N. \quad (13)$$

Denoting $G_k^h := G(kh)$ and using the notation [\(11\)](#), we can rewrite [\(13\)](#) as follows:

$$Y_k^h = \mu G_k^h + \tilde{X}_k^h, \quad k = 0, \dots, N. \quad (14)$$

In what follows, we assume that $G_0^h = 0$ and $G_k^h \neq 0$ at least for one k . Also, in this section, we assume that $H_1, H_2 \in (0, 1)$ and the coefficients $a, b \in \mathbb{R} \setminus \{0\}$ are known. The goal is to estimate the drift parameter $\mu \in \mathbb{R}$ based on the discrete observations. Estimation of the remaining parameters will be discussed in [Section 5](#).

Define the vector of increments $\Delta Y^{(N)} = (\Delta Y_0, \dots, \Delta Y_{N-1})^\top$, where, for $k = 0, \dots, N-1$,

$$\Delta Y_k = Y_{(k+1)h} - Y_{kh} = \mu \Delta G_k^h + \Delta \tilde{X}_k^h,$$

and $\Delta G_k^h = G_{k+1}^h - G_k^h$, $\Delta \tilde{X}_k^h = \tilde{X}_{k+1}^h - \tilde{X}_k^h$.

Clearly, the vector $\Delta Y^{(N)}$ is Gaussian with distribution $\mathcal{N}(\mu \Delta G^{(N)}, \tilde{\Sigma}^{(N)})$, where $\Delta G^{(N)} = (\Delta G_0^h, \dots, \Delta G_{N-1}^h)^\top$, and $\tilde{\Sigma}^{(N)}$ is the covariance matrix of the Gaussian vector

$$\Delta \tilde{X}^{(N)} := (\Delta \tilde{X}_0^h, \dots, \Delta \tilde{X}_{N-1}^h)^\top.$$

Due to the independence of X^{h,H_1} and X^{h,H_2} , the covariance matrix $\tilde{\Sigma}^{(N)}$ can be decomposed as

$$\tilde{\Sigma}^{(N)} = a^2 \Sigma_{H_1}^{(N)} + b^2 \Sigma_{H_2}^{(N)},$$

where $\Sigma_{H_i}^{(N)}$ is the covariance matrix of $(\Delta X_0^{h,H_i}, \dots, \Delta X_{N-1}^{h,H_i})^\top$, $i = 1, 2$. By [Proposition 10](#), both $\Sigma_{H_1}^{(N)}$ and $\Sigma_{H_2}^{(N)}$ are positive definite, and hence $\Sigma^{(N)}$ is positive definite and non-singular whenever $a^2 + b^2 > 0$.

Following the methodology of [\[21\]](#), the maximum likelihood estimator (MLE) of the drift parameter μ is given by

$$\hat{\mu}_N = \frac{(\Delta G^{(N)})^\top (\tilde{\Sigma}^{(N)})^{-1} \Delta Y^{(N)}}{(\Delta G^{(N)})^\top (\tilde{\Sigma}^{(N)})^{-1} \Delta G^{(N)}}. \quad (15)$$

Proposition 18. Let $a \neq 0$, $b \neq 0$, and $H_1, H_2 \in (0, 1)$.

1. The MLE $\hat{\mu}_N$ is unbiased and normally distributed with variance

$$\text{Var}(\hat{\mu}_N) = \frac{1}{(\Delta G^{(N)})^\top (\tilde{\Sigma}^{(N)})^{-1} \Delta G^{(N)}}. \quad (16)$$

2. If

$$\frac{N^{H_1 \vee H_2}}{G_N^h} \rightarrow 0 \quad \text{as } N \rightarrow \infty, \quad (17)$$

then the estimator $\hat{\mu}_N$ is consistent both in L^2 and almost surely.

Proof. 1. Since the process Y is Gaussian and the estimator $\hat{\mu}_N$ is a linear functional of the observations, it is normally distributed. From the model representation

$$\Delta Y^{(N)} = \mu \Delta G^{(N)} + \Delta \tilde{X}^{(N)},$$

it follows that

$$\hat{\mu}_N = \mu + \frac{(\Delta G^{(N)})^\top (\tilde{\Sigma}^{(N)})^{-1} \Delta \tilde{X}^{(N)}}{(\Delta G^{(N)})^\top (\tilde{\Sigma}^{(N)})^{-1} \Delta G^{(N)}},$$

which shows that $\hat{\mu}_N$ is unbiased and has variance given by [\(16\)](#).

2. According to [\[22, Theorem 1\]](#), a sufficient condition for both L^2 -consistency and strong consistency of $\hat{\mu}_N$ is that

$$\frac{\text{Var}(\tilde{X}_N^h)}{(G_N^h)^2} \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

This condition holds under assumption [\(17\)](#), since

$$\text{Var}(\tilde{X}_N^h) = \text{Var}(aX_{Nh}^{h,H_1} + bX_{Nh}^{h,H_2}) \sim a^2(Nh)^{2H_1} + b^2(Nh)^{2H_2}, \quad \text{as } N \rightarrow \infty,$$

see [Remark 3](#). $\square$

Remark 19. In this section, we have assumed that the process Y is observed on an equidistant grid $\{h, 2h, \dots, Nh\}$. However, the same results for the MLE of the drift parameter μ remain valid for any grid of the form $\{k_1 h, k_2 h, \dots, k_N h\}$, where $k_1 < k_2 < \dots < k_N$. The L^2 - and strong consistency of the corresponding MLE holds under assumption [\(17\)](#), provided that $k_N \rightarrow \infty$ as $N \rightarrow \infty$.

In contrast, the ergodic-type estimators of H_1 , H_2 , a , and b rely on the stationarity of the increment sequence and therefore require an equidistant observation scheme. For irregular sampling, this stationarity property is generally lost, and one would need to work directly with the covariance structure of the non-equidistant increments, which requires a different analytical approach.

Let us consider a simpler alternative estimator based on only two observations:

$$\tilde{\mu}_N = \frac{Y_N^h - Y_0^h}{G_N^h}. \quad (18)$$

Observe that this estimator can be interpreted as the MLE of μ based on two observations Y_0^h and Y_N^h . Therefore, taking into account [Remark 19](#) and [Proposition 18](#), we immediately obtain the following result.

Proposition 20. Let $a \neq 0$, $b \neq 0$, and $H_1, H_2 \in (0, 1)$.

1. The estimator $\tilde{\mu}_N$ is unbiased and normal with

$$\text{Var}(\tilde{\mu}_N) = \frac{\text{Var}(\tilde{X}_N^h - \tilde{X}_0^h)}{(G_N^h)^2}.$$

2. Under assumption (17), the estimator $\tilde{\mu}_N$ is consistent both in L^2 and almost surely.

Remark 21. 1. The MLE $\hat{\mu}_N$ is more efficient than the estimator $\tilde{\mu}_N$, in the sense that $\text{Var}(\hat{\mu}_N) \leq \text{Var}(\tilde{\mu}_N)$. This result follows from [21, Corollary 1], where the variances of MLEs based on samples of different sizes are compared.

2. A key advantage of the estimator $\tilde{\mu}_N$ is its independence from the noise parameters H_1, H_2, a , and b , which may be unknown in practice.

3. Under the stronger assumption that $\frac{G_N^h}{N} \geq c > 0$, $N \geq 1$, the strong consistency of $\tilde{\mu}_N$ can be derived from the ergodic Theorem 16. Indeed, this theorem implies that

$$\frac{\tilde{X}_N^h - \tilde{X}_0^h}{N} = \frac{1}{N} \sum_{k=0}^{N-1} \Delta \tilde{X}_k^h \rightarrow \mathbb{E}[\Delta \tilde{X}_0^h] = 0 \quad \text{a.s., as } N \rightarrow \infty,$$

which yields, under the stated assumption, that $\tilde{\mu}_N = \mu + \frac{N}{G_N^h} \cdot \frac{\tilde{X}_N^h - \tilde{X}_0^h}{N} \rightarrow \mu$, a.s., as $N \rightarrow \infty$.

4.2. Drift parameter estimation in the model with drift and one nifBm

Let us consider a particular case of the model (13) with $b = 0$. In this setting, the statistical model takes the form

$$Y_k^h = \mu G_k^h + a X_{kh}^{h,H}, \quad (19)$$

where $X^{h,H}$ is a nifBm with Hurst parameter $H \in (0, 1)$, as defined in (2).

For the process (19), the MLE of the drift parameter μ is given by

$$\hat{\mu}_N = \frac{(\Delta G^{(N)})^\top (\Sigma_H^{(N)})^{-1} \Delta Y^{(N)}}{(\Delta G^{(N)})^\top (\Sigma_H^{(N)})^{-1} \Delta G^{(N)}}, \quad (20)$$

where $\Sigma_H^{(N)}$ denotes the covariance matrix of the vector $(\Delta X_0^{h,H}, \dots, \Delta X_{N-1}^{h,H})^\top$.

The following result summarizes the properties of the estimator (20). It follows directly from the results of [21,22], using the same approach as in Proposition 18. For this reason, we omit the proof.

Proposition 22. Let $a \neq 0$ and $H \in (0, 1)$.

1. The MLE $\hat{\mu}_N$, defined in (20), is unbiased and normally distributed with variance

$$\text{Var}(\hat{\mu}_N) = \frac{a^2}{(\Delta G^{(N)})^\top (\Sigma_H^{(N)})^{-1} \Delta G^{(N)}}. \quad (21)$$

2. Suppose that

$$\frac{N^H}{G_N^h} \rightarrow 0 \quad \text{as } N \rightarrow \infty. \quad (22)$$

Then the MLE $\hat{\mu}_N$ is consistent both in the L^2 -sense and almost surely.

The following result gives the properties of an alternative estimator in the model with drift and one nifBm.

Proposition 23. Let $a \neq 0$ and $H \in (0, 1)$.

1. The estimator

$$\tilde{\mu}_N = (Y_N^h - Y_0^h)/G_N^h \quad (23)$$

is unbiased and normally distributed with variance $\text{Var}(\tilde{\mu}_N) = a^2 \mathbb{E}[(X_{N^h}^{h,H} - X_0^{h,H})^2]/(G_N^h)^2$.

2. Under assumption (22), the estimator $\tilde{\mu}_N$ is consistent both in the L^2 -sense and almost surely.

5. Construction of strongly consistent estimators for the parameters of (mixed) nifBm

In the previous section, we focused on estimating the drift parameter μ in statistical models (14) and (19) driven by two or one nifBms, respectively. In the present section, we address the simultaneous estimation of the parameters of the noise term. Specifically, we construct strongly consistent estimators for these parameters, relying on the ergodic Theorem 16.

For technical convenience, we first consider the case in which the drift parameter μ is known and equals zero. However, our methodology can be extended to the general case of an unknown drift. In that situation, μ should first be estimated using the estimator $\tilde{\mu}_N$ introduced in (18). Further details of this extension for the models with two and one nifBms will be provided in Remarks 25 and 32, respectively.

5.1. Estimation of the parameters H_1 , H_2 , a^2 , and b^2 in the model with two nifBms

Now, we aim to estimate the vector parameter $\theta = (H_1, H_2, a^2, b^2)$ of the process

$$\tilde{X}_k^h = aX_{kh}^{h,H_1} + bX_{kh}^{h,H_2}.$$

That is, we assume that the drift parameter μ in model (14) is equal to zero. If this assumption does not hold, the parameter μ should first be estimated using the estimator $\tilde{\mu}$ from (18). Subsequently, one may apply the estimator $\hat{\theta}_N = (\hat{H}_{1,N}, \hat{H}_{2,N}, \hat{a}_N^2, \hat{b}_N^2)$ defined in Theorem 26, where the observations $\tilde{X}_k^h$ are replaced by

$$\tilde{Y}_k^h := Y_k^h - \tilde{\mu}_N G_k^h = \tilde{X}_k^h + (\mu - \tilde{\mu}_N) G_k^h. \quad (24)$$

This approach requires additional assumptions on the sequence $\{G_k^h, k \geq 1\}$, which are detailed in Remark 25.

Having four parameters to estimate, we apply Theorem 16 to construct four corresponding statistics. The first statistic, ξ_N^1 , is based directly on the stationary increment process $\Delta \tilde{X}_k^h = \tilde{X}_{k+1}^h - \tilde{X}_k^h$, which was considered in Theorem 16. The remaining three statistics are constructed analogously from the increments $\Delta \tilde{X}_k^{jh} = \tilde{X}_{k+1}^{jh} - \tilde{X}_k^{jh}$, $j = 2, 4, 8$, whose stationarity can be established in the same way as for $\Delta \tilde{X}_k^h$. Thus, we define the following family of statistics:

$$\xi_N^j := \frac{1}{N} \sum_{k=0}^{N-1} \left(\tilde{X}_{k+1}^{jh} - \tilde{X}_k^{jh} \right)^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left(\Delta \tilde{X}_k^{jh} \right)^2, \quad j = 1, 2, 4, 8.$$

All four statistics are constructed in the same manner, differing only in the time step used to compute the increment of the process (11). This design ensures that the exponents appearing in the estimating equations are powers of two, which simplifies the solution for the estimator $\hat{\theta}$ of the parameter vector θ .

Lemma 24. *The following a.s. convergences hold for $N \rightarrow \infty$:*

$$\xi_N^j \rightarrow \mathbb{E}(\xi_1^j) = A_1 a^2 (2^{2H_1} - 1) j^{2H_1} + A_2 b^2 (2^{2H_2} - 1) j^{2H_2}, \quad (25)$$

where $j = 1, 2, 4, 8$ and

$$A_i = \frac{2h^{2H_i}}{(2H_i + 1)(H_i + 1)}, \quad i = 1, 2.$$

Proof. Since the random sequence $\{\Delta \tilde{X}_k^h, k \geq 0\}$ is stationary and ergodic for any $h > 0$, the proof of all a.s. convergence follows directly from the ergodic Theorem 16 with $Q(x) = x^2$, the independence of the processes W_i^H , $i = 1, 2$ and the values of $\mathbb{E}(\xi_1^j)$ that are calculated in accordance with formula (12) from Proposition 13, substituting subsequently $h, 2h, 4h, 8h$ in it. $\square$

Remark 25. For $\mu \neq 0$, the random sequence $\{\Delta Y_k^h, k \geq 0\}$, where Y_k^h is defined by (14), is not ergodic. However, in this case an analogue of Lemma 24 can be obtained. For the modified observations (24), define

$$\tilde{\xi}_N^j := \frac{1}{N} \sum_{k=0}^{N-1} \left(\tilde{Y}_{k+1}^{jh} - \tilde{Y}_k^{jh} \right)^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left(\Delta \tilde{Y}_k^{jh} \right)^2, \quad j \in \{1, 2, 4, 8\}.$$

Then

$$\begin{aligned} \tilde{\xi}_N^j &= \frac{1}{N} \sum_{k=0}^{N-1} \left(\Delta \tilde{X}_k^{jh} + (\mu - \tilde{\mu}_N) \Delta G_k^{jh} \right)^2 \\ &= \xi_N^j + (\mu - \tilde{\mu}_N)^2 \frac{1}{N} \sum_{k=0}^{N-1} \left(\Delta G_k^{jh} \right)^2 + (\mu - \tilde{\mu}_N) \frac{2}{N} \sum_{k=0}^{N-1} \Delta \tilde{X}_k^{jh} \Delta G_k^{jh}. \end{aligned} \quad (26)$$

If assumption (17) holds, then $\tilde{\mu}_N \rightarrow \mu$ a.s. as $N \rightarrow \infty$, by Proposition 20. Therefore, under the additional assumption

$$\sum_{k=0}^{N-1} \left(\Delta G_k^{jh} \right)^2 = O(N), \quad N \rightarrow \infty, \quad (27)$$

we obtain the almost sure convergence

$$\tilde{\xi}_N^j \rightarrow A_1 a^2 (2^{2H_1} - 1) j^{2H_1} + A_2 b^2 (2^{2H_2} - 1) j^{2H_2}, \quad N \rightarrow \infty. \quad (28)$$

Indeed, the first term on the right-hand side of (26) converges to the stated limit by (25); the second term tends to zero due to (27), and the third term also vanishes, since it can be bounded via the first two terms using the Cauchy–Schwarz inequality.

The convergence in (28) guarantees the strong consistency, in the model with drift, of the estimator $\hat{\theta}_N = (\hat{H}_{1,N}, \hat{H}_{2,N}, \hat{a}_N^2, \hat{b}_N^2)$ defined in Theorem 26, where $\tilde{\xi}_N^j$ are used instead of ξ_N^j .

Assumption (27) is satisfied, for instance, if the sequence $\{\Delta G_k^{jh}, k \geq 0\}$ is bounded; in particular, for a linear drift function $G(t) = t$, we have $\Delta G_k^{jh} = jh$ for all k (in this case, condition (17) also holds). More generally, (27) is valid whenever G satisfies a Lipschitz or Hölder condition. Moreover, under (27) assumed to hold a.s., it is not necessary to suppose that the drift is deterministic. This can extend the class of the models under consideration; however, it could happen that the standard methods for the drift parameter estimation can possibly not be valid anymore.

Now we are in a position to construct strongly consistent estimators of the parameter $\theta = (H_1, H_2, a^2, b^2)^\top$.

The subsequent notation will be used: $\log x = \log_e x$,

$$\log_+ x = \begin{cases} \log x, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0, \end{cases} \quad \sqrt[+]{x} = \begin{cases} \sqrt{x}, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

Also, for any $N \geq 1$ denote

$$D_N = (\xi_N^4 \xi_N^2 - \xi_N^8 \xi_N^1)^2 - 4(\xi_N^4 \xi_N^1 - (\xi_N^2)^2)(\xi_N^8 \xi_N^2 - (\xi_N^4)^2), \quad (29)$$

$$x_N = \frac{\xi_N^8 \xi_N^1 - \xi_N^4 \xi_N^2 + \sqrt[+]{D_N}}{2(\xi_N^4 \xi_N^1 - (\xi_N^2)^2)}, \quad y_N = \frac{\xi_N^8 \xi_N^1 - \xi_N^4 \xi_N^2 - \sqrt[+]{D_N}}{2(\xi_N^4 \xi_N^1 - (\xi_N^2)^2)}.$$

From now on, we assume that the fraction equals zero if its denominator equals zero. We also assume that coefficients a and b are non-zero, so that we indeed have two components in the model.

Theorem 26. Let $0 < H_2 < H_1 < 1$. The random vector $\hat{\theta}_N = (\hat{H}_{1,N}, \hat{H}_{2,N}, \hat{a}_N^2, \hat{b}_N^2)$, where

$$\hat{H}_{1,N} = \frac{1}{2 \log 2} \log_+(x_N),$$

$$\hat{H}_{2,N} = \frac{1}{2 \log 2} \log_+(y_N),$$

$$\hat{a}_N^2 = \frac{(2\hat{H}_{1,N} + 1)(\hat{H}_{1,N} + 1)(\xi_N^2 - y_N \xi_N^1)}{2h^2 \hat{H}_{1,N}(x_N - y_N)(x_N - 1)},$$

and

$$\hat{b}_N^2 = \frac{(2\hat{H}_{2,N} + 1)(\hat{H}_{2,N} + 1)(\xi_N^2 - x_N \xi_N^1)}{2h^2 \hat{H}_{2,N}(y_N - x_N)(y_N - 1)}$$

is a strongly consistent estimator of $\theta = (H_1, H_2, a^2, b^2)$.

Remark 27. Condition of Theorem 26 does not mean that we should “know what index is bigger”. It just means that taking x_N we estimate bigger index, and taking y_N we estimate the smaller one.

Proof. Denote

$$x = 2^{2H_1}, \quad y = 2^{2H_2}, \quad A = a^2 A_1, \quad B = b^2 A_2,$$

$$\eta_j = \mathbb{E}(\xi_1^j), \quad j = 1, 2, 4, 8.$$

Then the equalities in the right-hand side of (25) from Lemma 24 can be rewritten in the form

$$\eta_1 = A(x - 1) + B(y - 1), \quad (30)$$

$$\eta_2 = Ax(x - 1) + By(y - 1), \quad (31)$$

$$\eta_4 = Ax^2(x - 1) + By^2(y - 1), \quad (32)$$

$$\eta_8 = Ax^3(x - 1) + By^3(y - 1). \quad (33)$$

Solving the system of linear (w.r.t. A and B) Eqs. (30) and (31), we can present the variables A and B as functions of x and y :

$$A = \frac{\eta_2 - y\eta_1}{(x - y)(x - 1)}, \quad B = \frac{\eta_2 - x\eta_1}{(y - x)(y - 1)}. \quad (34)$$

Inserting into (32) expressions (34) obtained for A and B , we get that

$$\begin{aligned}\eta_4 &= \frac{\eta_2 - y\eta_1}{x - y}x^2 + \frac{\eta_2 - x\eta_1}{y - x}y^2 = \frac{1}{x - y}(\eta_2x^2 - \eta_1x^2y - \eta_2y^2 + \eta_1xy^2) \\ &= \frac{1}{x - y}(\eta_2(x^2 - y^2) - \eta_1xy(x - y)) = \eta_2(x + y) - \eta_1xy,\end{aligned}\quad (35)$$

whence

$$x = \frac{\eta_4 - \eta_2y}{\eta_2 - \eta_1y}.\quad (36)$$

Note that $\eta_2 - \eta_1y = A(x - 1)(x - y)$, $A > 0$ (A is positive and can be zero only if $a = 0$ but this is the degenerate case with only one component for the process X_k^h), $x = 2^{H_1} > 0$, $x = 1$ if $H_1 = 0$ and $x - y = 0$ if $H_1 = H_2$. If we assume $H_1 \neq H_2$, the denominator is non-zero.

Similarly, it follows from (33) and (34) that

$$\begin{aligned}\eta_8 &= \frac{\eta_2 - y\eta_1}{x - y}x^3 + \frac{\eta_2 - x\eta_1}{y - x}y^3 = \frac{1}{x - y}(\eta_2x^3 - \eta_1x^3y - \eta_2y^3 + \eta_1xy^3) = \frac{1}{x - y}(\eta_2(x^3 - y^3) - \eta_1xy(x^2 - y^2)) \\ &= \frac{1}{x - y}(\eta_2(x - y)(x^2 + xy + y^2) - \eta_1xy(x - y)(x + y)) = \eta_2(x^2 + xy + y^2) - \eta_1xy(x + y).\end{aligned}\quad (37)$$

Multiplying the right-hand side of (35) by $(x + y)$ and subtracting the right-hand side of (37), we get that

$$\eta_2(x + y)^2 - \eta_2(x^2 + xy + y^2) = \eta_4(x + y) - \eta_8,$$

whence

$$x = \frac{\eta_8 - \eta_4y}{\eta_4 - \eta_2y}.\quad (38)$$

Again, the denominator is non-zero. It follows from Eqs. (36) and (38) that

$$\frac{\eta_8 - \eta_4y}{\eta_4 - \eta_2y} = \frac{\eta_4 - \eta_2y}{\eta_2 - \eta_1y},$$

and we obtain the following quadratic equation w.r.t. y :

$$y^2(\eta_1\eta_4 - \eta_2^2) + y(\eta_2\eta_4 - \eta_1\eta_8) + (\eta_2\eta_8 - \eta_4^2) = 0.\quad (39)$$

It is evident that x satisfies the same quadratic equation due to symmetry. Hence, x and y with $(x > y)$ are the two roots of Eq. (39),

$$x, y = \frac{(\eta_1\eta_8 - \eta_2\eta_4) \pm \sqrt{D}}{2(\eta_1\eta_4 - \eta_2^2)},$$

where the denominator is again non-zero if $H_1 \neq H_2$, and

$$D = (\eta_2\eta_4 - \eta_1\eta_8)^2 - 4(\eta_1\eta_4 - \eta_2^2)(\eta_2\eta_8 - \eta_4^2).$$

Let us check if the discriminant D is nonnegative. We use the equalities (30)–(37) and conclude that

$$\begin{aligned}\eta_2\eta_4 - \eta_1\eta_8 &= AB(x - 1)(y - 1)(x^2y + x^2y - y^3 - x^3) = -AB(x - 1)(y - 1)(x + y)(x - y)^2 < 0, \\ \eta_1\eta_4 - \eta_2^2 &= AB(x - 1)(y - 1)(x^2 - 2xy + y^2) = AB(x - 1)(y - 1)(x - y)^2 > 0, \\ \eta_2\eta_8 - \eta_4^2 &= AB(x - 1)(y - 1)(xy^3 - 2xy^2 + x^3y) = AB(x - 1)(y - 1)xy(x - y)^2 > 0.\end{aligned}$$

Thus

$$D = A^2B^2(x - 1)^2(y - 1)^2(x + y)^2(x - y)^4 - 4A^2B^2(x - 1)^2(y - 1)^2xy(x - y)^4 = A^2B^2(x - 1)^2(y - 1)^2(x - y)^6 > 0.$$

Since D is positive, we have two different roots

$$x = \frac{\eta_1\eta_8 - \eta_2\eta_4 + \sqrt{D}}{2(\eta_1\eta_4 - \eta_2^2)}, \quad y = \frac{\eta_1\eta_8 - \eta_2\eta_4 - \sqrt{D}}{2(\eta_1\eta_4 - \eta_2^2)}\quad (40)$$

Since $x = 2^{2H_1}$ and $y = 2^{2H_2}$, we get from (40) that

$$H_1 = \frac{1}{2 \log 2} \log \left(\frac{\eta_1\eta_8 - \eta_2\eta_4 + \sqrt{D}}{2(\eta_1\eta_4 - \eta_2^2)} \right), \quad H_2 = \frac{1}{2 \log 2} \log \left(\frac{\eta_1\eta_8 - \eta_2\eta_4 - \sqrt{D}}{2(\eta_1\eta_4 - \eta_2^2)} \right)\quad (41)$$

Recall that a^2 and b^2 can be obtained from Eq. (34):

$$a^2 = \frac{(2H_1 + 1)(H_1 + 1)(\eta_2 - y\eta_1)}{2h^{2H_1}(x - y)(x - 1)}, \quad b^2 = \frac{(2H_2 + 1)(H_2 + 1)(\eta_2 - x\eta_1)}{2h^{2H_2}(y - x)(y - 1)}.\quad (42)$$

Now, in order to obtain the estimators, we substitute η_j with ξ_N^j in (40) and (41) and obtain D_N , x_N , y_N , $\hat{H}_{1,N}$ and $\hat{H}_{2,N}$. Then we substitute in (42) $\hat{H}_{i,N}$, $i = 1, 2$ instead of H_i , $i = 1, 2$ and also other estimators and obtain $\hat{a}_N^2$ and $\hat{b}_N^2$. Note that the random denominators are asymptotically non-zero with probability 1. Strong consistency of these estimators follows from strong consistency of ξ_N^j . $\square$

Remark 28. The estimators for H_1 and H_2 do not depend explicitly on h , the dependence on h is implicitly present in the estimators ξ_N^j for $j = 1, 2, 4, 8$.

Remark 29. Note that the values $\tilde{X}_k^{2h}$, $\tilde{X}_k^{4h}$, and $\tilde{X}_k^{8h}$ used in our estimators can be expressed in terms of $\tilde{X}_k^h$. By (11),

$$\begin{aligned}\tilde{X}_k^{2h} &= \frac{1}{2h} \int_{2hk}^{2h(k+1)} (aW_u^{H_1} + bW_u^{H_2}) du = \frac{1}{2} \left(\frac{1}{h} \int_{2hk}^{2h(k+1)} (aW_u^{H_1} + bW_u^{H_2}) du + \frac{1}{h} \int_{(2k+1)h}^{(2k+2)h} (aW_u^{H_1} + bW_u^{H_2}) du \right) \\ &= \frac{1}{2} (\tilde{X}_{2k}^h + \tilde{X}_{2k+1}^h).\end{aligned}\quad (43)$$

More generally, we obtain

$$\tilde{X}_k^{jh} = \frac{1}{j} \sum_{l=0}^{j-1} \tilde{X}_{jk+l}^h, \quad j \in \{1, 2, 4, 8\}.$$

Remark 30. Note that if we fix a final time horizon T and follow the trajectory of our process up to T , the number of observations, that is the terms in the summations of each statistic $\xi^1, \xi^2, \xi^4, \xi^8$ will be different. Indeed, given that for each statistic we consider increments of different amplitude, we have $8N$ elements for ξ^1 , $4N$ elements for ξ^2 , $2N$ elements for ξ^4 and N elements for ξ^8 . Thus, we should replace ξ_N^1 with ξ_{8N}^1 , ξ_N^2 with ξ_{4N}^2 , ξ_N^4 with ξ_{2N}^4 in (29).

However, this notation will prove more convenient and will therefore be adopted for studying asymptotic normality, which will be the subject of Section 6.

5.2. Estimation of the parameters H and a^2 in the model with one nifBm

Consider now the case when we have only one nifBm, more precisely, let $b = 0$. To distinguish from the previous case, we call this model “model with one nifBm”. In this case we need only two statistics to compute the estimators for H and a^2 . Of course, we can combine different pairs of statistics; however, let us consider the simplest possibility. So, we observe the increments $\Delta X_k^{jh} = X_{(k+1)h}^{jh} - X_{kh}^{jh}$ for $j = 1, 2$. In this case

$$\xi_N^j := \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h}^{jh} - X_{kh}^{jh} \right)^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left(\Delta X_k^{jh} \right)^2, \quad j = 1, 2,$$

the convergence (25) takes the form

$$\xi_N^j \rightarrow \mathbb{E}(\xi^j) = A_1 a^2 (2^{2H} - 1) j^{2H}, \quad \text{a.s., as } N \rightarrow \infty, \quad j = 1, 2, \quad (44)$$

and we immediately obtain the following result, which we formulate without proof, since it uses the same approach as in the proof of Theorem 26, but with significant simplifications.

Denote

$$\hat{H}_N = \frac{1}{2 \log 2} \log_+ \left(\frac{\xi_N^2}{\xi_N^1} \right), \quad \hat{A}_N = \frac{2h^{2\hat{H}_N}}{(2\hat{H}_N + 1)(\hat{H}_N + 1)}. \quad (45)$$

Theorem 31. Consider the model with one nifBm, and let $0 < H < 1$ and $a^2 > 0$. Then the estimator $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)$ where $\hat{H}_N$ is taken from (45) and

$$\hat{a}_N^2 = \frac{\xi_N^1}{\hat{A}_N (2^{2\hat{H}_N} - 1)}, \quad (46)$$

is a strongly consistent estimator of $\theta = (H, a^2)$.

Remark 32. Similarly to the model with two nifBms, the proposed approach can be adapted to the model (19) with a non-zero drift. Under assumptions (22) and (27) on the sequence G_k^h , we employ a two-stage procedure for the strongly consistent estimation of all model parameters. First, the drift parameter μ is estimated using the strongly consistent estimator $\tilde{\mu}_N$ from Proposition 23. Subsequently, H and a^2 are estimated using (45) and (46), respectively, where the observations X_{kh}^{jh} are replaced by $Y_k^{jh} - \tilde{\mu}_N G_k^{jh}$.

Remark 33. Proceeding analogously to Remark 30, we note that the statistic ξ_N^1 appearing in the estimators (45) and (46) should be replaced by ξ_{2N}^1 .

6. Asymptotic normality

In this section, we establish the joint asymptotic normality of the estimators. We first consider the model with a single nifBm and without drift, and prove the asymptotic normality of the estimator $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)$ defined in (45)–(46). For this case, detailed proofs are provided. We then present, without proofs, the corresponding results for more general models that incorporate a drift term and/or an additional nifBm in the noise component. These results can be obtained by applying essentially the same arguments, although their proofs require considerably more technical details and lead to cumbersome expressions; for this reason, they are omitted here.

6.1. Asymptotic normality in the model with one nifBm

Now we prove the asymptotic normality of the strongly consistent estimator for $\theta = (H, a^2)$ in the one-nifBm model (see [Section 5.2](#)). Following [Remark 33](#), we employ the modified notation for the statistics, ξ_{2N}^1 and ξ_N^2 . By (44), as $N \rightarrow \infty$,

$$\xi_{2N}^1 \rightarrow \mathbb{E}(\xi_1^1) = A_1 a^2 (2^{2H} - 1) =: f_1(\theta), \quad \xi_N^2 \rightarrow \mathbb{E}(\xi_1^2) = A_1 a^2 (2^{2H} - 1) 2^{2H} =: f_2(\theta) \quad \text{a.s.} \quad (47)$$

The next theorem establishes their joint asymptotic normality.

Theorem 34. Let $0 < H < \frac{3}{4}$. Consider the vectors $\bar{\xi}_N = (\xi_{2N}^1, \xi_N^2)^\top$ and $\bar{\eta} = (\mathbb{E}(\xi_1^1), \mathbb{E}(\xi_1^2))^\top$. Then $\bar{\xi}_N$ is asymptotically normal, that is,

$$\sqrt{N}(\bar{\xi}_N - \bar{\eta}) = \sqrt{N} \begin{pmatrix} \xi_{2N}^1 - \mathbb{E}(\xi_1^1) \\ \xi_N^2 - \mathbb{E}(\xi_1^2) \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \tilde{\Sigma}), \quad \text{as } N \rightarrow \infty,$$

where the asymptotic covariance matrix $\tilde{\Sigma}$ is given by

$$\tilde{\Sigma} = \begin{pmatrix} \tilde{\Sigma}_{11} & \tilde{\Sigma}_{12} \\ \tilde{\Sigma}_{12} & \tilde{\Sigma}_{22} \end{pmatrix},$$

with entries

$$\begin{aligned} \tilde{\Sigma}_{11} &= h^{4H} \sum_{i=-\infty}^{\infty} \gamma(H, i)^2, & \tilde{\Sigma}_{22} &= 2^{4H+1} h^{4H} \sum_{i=-\infty}^{\infty} \gamma(H, i)^2, \\ \tilde{\Sigma}_{12} &= h^{4H} \sum_{i=-\infty}^{\infty} \gamma(H, i) (3\gamma(H, i) + 4\gamma(H, i+1) + \gamma(H, i+2)). \end{aligned}$$

Remark 35. For $H < \frac{3}{4}$, all series defining the entries of the asymptotic covariance matrix converge. Moreover, for any $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{\infty} |\gamma(H, i+\alpha) \gamma(H, i+\beta)| < \infty.$$

This follows from the Cauchy–Schwarz inequality:

$$\sum_{i=-\infty}^{\infty} |\gamma(H, i+\alpha) \gamma(H, i+\beta)| \leq \sqrt{\sum_{i=-\infty}^{\infty} \gamma(H, i+\alpha)^2 \sum_{i=-\infty}^{\infty} \gamma(H, i+\beta)^2} = \sum_{i=-\infty}^{\infty} \gamma(H, i)^2 < \infty,$$

where convergence of the final series is guaranteed by [Lemma 14](#).

Proof. The proof consists of two parts. In the first part, we establish that $\sqrt{N}(\bar{\xi}_N - \bar{\eta})$ converges to a bivariate normal distribution. In the second part, we compute the entries of the asymptotic covariance matrix.

Step 1: Asymptotic normality. Recall that

$$\xi_{2N}^1 = \frac{1}{2N} \sum_{k=0}^{2N-1} (\Delta X_k^h)^2, \quad \xi_N^2 = \frac{1}{N} \sum_{k=0}^{N-1} (\Delta X_k^{2h})^2,$$

where $\Delta X_k^h = X_{(k+1)h}^h - X_{kh}^h$. We express $\{\Delta X_k^{2h}\}$ in terms of $\{\Delta X_k^h\}$. Similarly to (43), we have

$$X_{kh}^{2h} = \frac{1}{2} (X_{2kh}^h + X_{(2k+1)h}^h),$$

which implies

$$\Delta X_k^{2h} = \frac{1}{2} (X_{(2k+3)h}^h + X_{(2k+2)h}^h - X_{(2k+1)h}^h - X_{2kh}^h) = \frac{1}{2} (\Delta X_{2k+2}^h + 2\Delta X_{2k+1}^h + \Delta X_{2k}^h). \quad (48)$$

Define $Y_k = (Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)})$, $k \in \mathbb{N}_0$, by

$$Y_k^{(1)} = \Delta X_{2k}^h, \quad Y_k^{(2)} = \Delta X_{2k+1}^h, \quad Y_k^{(3)} = \Delta X_{2k+2}^h. \quad (49)$$

Then

$$\xi_{2N}^1 = \frac{1}{2N} \sum_{k=0}^{N-1} ((Y_k^{(1)})^2 + (Y_k^{(2)})^2), \quad \xi_N^2 = \frac{1}{4N} \sum_{k=0}^{N-1} (Y_k^{(1)} + 2Y_k^{(2)} + Y_k^{(3)})^2.$$

To prove asymptotic normality of $(\xi_{2N}^1, \xi_N^2)^\top$, we use the Cramér–Wold device. Let $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ and define

$$F(y) = \frac{\alpha}{2} (y_1^2 + y_2^2) + \frac{\beta}{4} (y_1 + 2y_2 + y_3)^2, \quad y = (y_1, y_2, y_3)^\top \in \mathbb{R}^3.$$

Then

$$\alpha \xi_{2N}^1 + \beta \xi_N^2 = \frac{1}{N} \sum_{k=0}^{N-1} F(Y_k).$$

Hence we need to show that

$$\sqrt{N} \left(\alpha(\xi_{2N}^1 - \mathbb{E}[\xi_1^1]) + \beta(\xi_N^2 - \mathbb{E}[\xi_1^2]) \right) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} (F(Y_k) - \mathbb{E}[F(Y_k)]) \quad (50)$$

converges in distribution to a Gaussian law. This will be proved by applying the Breuer–Major theorem for stationary Gaussian vectors [31,32]; see Theorem 41 in the Appendix.

Since $F(Y_1)$ is a quadratic form of a Gaussian vector, its Hermite rank is $q = 2$ (see Example 40 in Appendix B). Therefore, to verify the condition (54) of Theorem 41 it suffices to establish

$$\sum_{n \in \mathbb{Z}} (\text{cov}(Y_1^{(p)}, Y_{1+n}^{(l)}))^2 < \infty, \quad p, l \in \{1, 2, 3\}. \quad (51)$$

Using (49) and the definition of $\gamma(H, n)$, the covariance structure is

$$\begin{aligned} \text{cov}(Y_1^{(1)}, Y_{1+n}^{(1)}) &= h^{2H} \gamma(H, 2n), & \text{cov}(Y_1^{(1)}, Y_{1+n}^{(2)}) &= h^{2H} \gamma(H, 2n+1), \\ \text{cov}(Y_1^{(1)}, Y_{1+n}^{(3)}) &= h^{2H} \gamma(H, 2n+2), & \text{cov}(Y_1^{(2)}, Y_{1+n}^{(1)}) &= h^{2H} \gamma(H, 2n-1), \\ \text{cov}(Y_1^{(2)}, Y_{1+n}^{(2)}) &= h^{2H} \gamma(H, 2n), & \text{cov}(Y_1^{(2)}, Y_{1+n}^{(3)}) &= h^{2H} \gamma(H, 2n+1), \\ \text{cov}(Y_1^{(3)}, Y_{1+n}^{(1)}) &= h^{2H} \gamma(H, 2n-2), & \text{cov}(Y_1^{(3)}, Y_{1+n}^{(2)}) &= h^{2H} \gamma(H, 2n-1), \\ \text{cov}(Y_1^{(3)}, Y_{1+n}^{(3)}) &= h^{2H} \gamma(H, 2n). \end{aligned}$$

Hence, the condition (51) follows from Remark 35. For example,

$$\sum_{n=-\infty}^{\infty} (\text{cov}(Y_1^{(1)}, Y_{1+n}^{(1)}))^2 = h^{4H} \sum_{n=-\infty}^{\infty} \gamma(H, 2n)^2,$$

which converges for $H < \frac{3}{4}$ by Remark 35. The other cases are analogous. Thus, the conditions of Theorem 41 are satisfied, proving that (50) converges weakly to a centered Gaussian distribution.

Step 2: Identification of the asymptotic covariance matrix. To determine the covariance matrix of the limiting law, we analyze

$$N \text{Var}(\xi_{2N}^1), \quad N \text{Var}(\xi_N^2), \quad N \text{cov}(\xi_{2N}^1, \xi_N^2).$$

Using the stationarity of $\{\Delta X_k^h\}_k$ and formula (53), we obtain

$$\text{cov}((\Delta X_k^h)^2, (\Delta X_j^h)^2) = 2h^{4H} \gamma(H, k-j)^2.$$

Hence

$$N \text{Var}(\xi_{2N}^1) = \frac{1}{4N} \text{Var} \left(\sum_{k=0}^{2N-1} (\Delta X_k^h)^2 \right) = \frac{1}{4N} \sum_{k=0}^{2N-1} \sum_{j=0}^{2N-1} \text{cov}((\Delta X_k^h)^2, (\Delta X_j^h)^2) = \frac{h^{4H}}{2N} \sum_{k=0}^{2N-1} \sum_{j=0}^{2N-1} \gamma(H, k-j)^2.$$

Upon rearrangement of the sum, we get

$$\begin{aligned} N \text{Var}(\xi_{2N}^1) &= \frac{h^{4H}}{2N} \sum_{k=0}^{2N-1} \sum_{i=k-2N+1}^k \gamma(H, i)^2 = \frac{h^{4H}}{2N} \sum_{i=-2N+1}^0 \sum_{k=0}^{2N-1+i} \gamma(H, i)^2 + \frac{h^{4H}}{2N} \sum_{i=1}^{2N-1} \sum_{k=i}^{2N-1} \gamma(H, i)^2 \\ &= h^{4H} \sum_{i=-2N+1}^{2N-1} \left(1 - \frac{|i|}{2N}\right) \gamma(H, i)^2 \rightarrow h^{4H} \sum_{i=-\infty}^{\infty} \gamma(H, i)^2, \end{aligned} \quad (52)$$

as $N \rightarrow \infty$, where the interchange of limit and summation is justified by the dominated convergence theorem, as ensured by Lemma 14.

Analogous computations yield

$$N \text{Var}(\xi_N^2) \rightarrow 2^{4H+1} h^{4H} \sum_{i=-\infty}^{\infty} \gamma(H, i)^2, \quad \text{as } N \rightarrow \infty.$$

Let us now compute the asymptotic behavior of the covariance between ξ_{2N}^1 and ξ_N^2 . We have

$$\begin{aligned} N \text{cov}(\xi_{2N}^1, \xi_N^2) &= N \text{cov} \left(\frac{1}{2N} \sum_{k=0}^{2N-1} (\Delta X_k^h)^2, \frac{1}{N} \sum_{j=0}^{N-1} (\Delta X_j^{2h})^2 \right) \\ &= \frac{1}{8N} \text{cov} \left(\sum_{k=0}^{N-1} ((\Delta X_{2k}^h)^2 + (\Delta X_{2k+1}^h)^2), \sum_{j=0}^{N-1} (\Delta X_{2j+2}^h + 2\Delta X_{2j+1}^h + \Delta X_{2j}^h)^2 \right) \\ &= \frac{1}{8N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \text{cov}((\Delta X_{2k}^h)^2 + (\Delta X_{2k+1}^h)^2, (\Delta X_{2j+2}^h + 2\Delta X_{2j+1}^h + \Delta X_{2j}^h)^2) \\ &= \frac{1}{4N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \left[\left(\text{cov}(\Delta X_{2k}^h, \Delta X_{2j+2}^h + 2\Delta X_{2j+1}^h + \Delta X_{2j}^h) \right)^2 + \left(\text{cov}(\Delta X_{2k+1}^h, \Delta X_{2j+2}^h + 2\Delta X_{2j+1}^h + \Delta X_{2j}^h) \right)^2 \right], \end{aligned}$$

where we used formula (53).

Expressing the covariances in terms of $\gamma(H, n)$ via relation (8), we obtain

$$N \operatorname{cov}(\xi_{2N}^1, \xi_N^2) = \frac{h^{4H}}{4N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \left[\left(\gamma(H, 2k-2j-2) + 2\gamma(H, 2k-2j-1) + \gamma(H, 2k-2j) \right)^2 \right. \\ \left. + \left(\gamma(H, 2k-2j-1) + 2\gamma(H, 2k-2j) + \gamma(H, 2k-2j+1) \right)^2 \right].$$

Proceeding as in (52), by rearranging the sums and passing to the limit as $N \rightarrow \infty$, we obtain

$$N \operatorname{cov}(\xi_{2N}^1, \xi_N^2) \rightarrow \frac{h^{4H}}{2} \sum_{i=-\infty}^{\infty} \left[\left(\gamma(H, 2i-2) + 2\gamma(H, 2i-1) + \gamma(H, 2i) \right)^2 + \left(\gamma(H, 2i-1) + 2\gamma(H, 2i) + \gamma(H, 2i+1) \right)^2 \right] \\ = \frac{h^{4H}}{2} \sum_{j=-\infty}^{\infty} \left(\gamma(H, j) + 2\gamma(H, j+1) + \gamma(H, j+2) \right)^2.$$

Finally, expanding the square and exploiting the shift-invariance of the sums yields the expression for $\tilde{\Sigma}_{12}$ stated in the theorem. This completes the proof. $\square$

We are now in a position to prove the asymptotic normality of $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)^\top$. Recall that now we consider the estimator defined by (45)–(46), where ξ_N^1 is replaced with ξ_{2N}^2 , that is,

$$\hat{H}_N = \frac{1}{2 \log 2} \log \left(\frac{\xi_{2N}^2}{\xi_N^1} \right), \quad \hat{a}_N^2 = \frac{\xi_{2N}^1 (2\hat{H}_N + 1) (\hat{H}_N + 1)}{2h^{2\hat{H}_N} (2^{2\hat{H}_N} - 1)}.$$

Let $f(\theta) = (f_1(\theta), f_2(\theta))^\top$, where $f_1(\theta)$ and $f_2(\theta)$ are defined in (47). Then, by construction, the estimator $\hat{\theta}_N$ has the form $\hat{\theta}_N = q(\tilde{\xi}_N)$, where $\tilde{\xi}_N = (\xi_{2N}^1, \xi_N^2)^\top$ and $q = f^{(-1)}$ is the inverse function of f , derived in Theorem 31.

Theorem 36. Let $0 < H < \frac{3}{4}$. The estimator $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)^\top$ is asymptotically normal, that is

$$\sqrt{N}(\hat{\theta}_N - \theta) = \sqrt{N} \begin{pmatrix} \hat{H}_N - H \\ \hat{a}_N^2 - a^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma_0),$$

with the asymptotic covariance matrix Σ_0 given by

$$\Sigma_0 = (f'(\theta))^{-1} \tilde{\Sigma} ((f'(\theta))^{-1})^\top,$$

where $\tilde{\Sigma}$ is defined in Theorem 34 and f' is the Jacobian matrix of f (the elements of this matrix are computed in the proof).

Proof. We apply delta method [33] for function $q = f^{(-1)}$ and sequence $\tilde{\xi}_N$ for which we proved asymptotic normality in Theorem 34. To apply the delta method, it is necessary to compute the derivative matrix of q and verify its nonsingularity. However, due to the complexity of q , we instead calculate the derivative matrix of f at θ , since by the inverse function theorem, $q' = (f')^{-1}$. Showing that $f'(\theta)$ is nonsingular thus ensures that q' exists and is nonsingular, allowing the application of the delta method. Thus, we first evaluate the Jacobian matrix of $f(\theta)$. We have

$$f_1(\theta) = \frac{2h^{2H}}{(2H+1)(H+1)} a^2 (2^{2H} - 1),$$

hence

$$\frac{\partial f_1}{\partial H} = 2a^2 h^{2H} \frac{[2(2^{2H} - 1) \log h + 2^{2H+1} \log 2](2H+1)(H+1) - (2^{2H} - 1)(4H+3)}{(2H+1)(H+1)^2}$$

and

$$\frac{\partial f_1}{\partial a^2} = \frac{2h^{2H}}{(2H+1)(H+1)} (2^{2H} - 1).$$

Consider now

$$f_2(\theta) = \frac{2a^2 h^{2H} 2^{2H} (2^{2H} - 1)}{(2H+1)(H+1)}.$$

Let

$$N(\theta) = 2a^2 h^{2H} 2^{2H} (2^{2H} - 1), \quad D(\theta) = (2H+1)(H+1).$$

We have

$$\frac{\partial D}{\partial H} = \frac{\partial}{\partial H} [(2H+1)(H+1)] = 2(H+1) + (2H+1) = 4H+3, \\ \frac{\partial N}{\partial H} = 2a^2 [h^{2H} 2^{2H} (2 \log h + 2 \log 2)(2^{2H} - 1) + h^{2H} 2^{2H} (2 \log 2 \cdot 2^{2H})],$$

whence,

$$\frac{\partial f_2}{\partial H} = \frac{2a^2 h^{2H} 2^{2H}}{((2H+1)(H+1))^2} \left[((2 \log h + 2 \log 2)(2^{2H} - 1) + 2 \log 2 \cdot 2^{2H})(2H+1)(H+1) - (2^{2H} - 1)(4H+3) \right].$$

and

$$\frac{\partial f_2}{\partial a^2} = \frac{h^{2H} 2^{2H+1} (2^{2H} - 1)}{(2H+1)(H+1)}.$$

By computing the determinant of the Jacobian matrix $f' = \begin{pmatrix} \partial f_1 / \partial H & \partial f_1 / \partial a^2 \\ \partial f_2 / \partial H & \partial f_2 / \partial a^2 \end{pmatrix}$, we obtain

$$\det(f'(\theta)) = -\frac{a^2 h^{4H} 2^{2H+3} \log 2 (2^{2H} - 1)^2}{(2H+1)^2 (H+1)^2}.$$

It is easy to see that the determinant is strictly negative for all positive a , h , and H . This guarantees the invertibility of the matrix $f'(\theta)$ in the considered parameter domain. The application of the delta method concludes the proof. $\square$

6.2. Asymptotic normality in the model with two nifBms

We now extend [Theorems 34](#) and [36](#) to the case of a linear combination of two processes. The proof is omitted, as it follows the same arguments as in the two-dimensional case, differing only in the increased algebraic complexity.

A strongly consistent estimator of the parameter $\theta = (H_1, H_2, a^2, b^2)$ was established in [Theorem 26](#). Analogously to the model with a single nifBm, we now consider a modified estimator $\hat{\theta}_N$, where the statistics ξ_N^1 , ξ_N^2 , and ξ_N^4 are replaced by ξ_{8N}^1 , ξ_{4N}^2 , and ξ_{2N}^4 , respectively; see [Remark 29](#). Accordingly, we introduce

$$\bar{\xi}_N = (\xi_{8N}^1, \xi_{4N}^2, \xi_{2N}^4, \xi_N^8)^\top, \quad \bar{\eta} = (\mathbb{E}(\xi_1^1), \mathbb{E}(\xi_1^2), \mathbb{E}(\xi_1^4), \mathbb{E}(\xi_1^8))^\top =: f(\theta),$$

where the expectations $\mathbb{E}(\xi_j^i)$, $j = 1, 2, 4, 8$, are given by [\(25\)](#).

The following results establish the asymptotic normality of $\bar{\xi}_N$ and $\hat{\theta}_N$ in this framework. We do not provide the explicit form of the asymptotic covariance matrix $\tilde{\Sigma}$, as its derivation is analogous to that in the single-fBm case but leads to extremely cumbersome expressions.

Theorem 37. *Let $0 < H_1 < H_2 < \frac{3}{4}$. Then $\bar{\xi}_N$ is asymptotically normal, that is*

$$\sqrt{N}(\bar{\xi}_N - \bar{\eta}) = \sqrt{N} \begin{pmatrix} \xi_{8N}^1 - \mathbb{E}(\xi_1^1) \\ \xi_{4N}^2 - \mathbb{E}(\xi_1^2) \\ \xi_{2N}^4 - \mathbb{E}(\xi_1^4) \\ \xi_N^8 - \mathbb{E}(\xi_1^8) \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \tilde{\Sigma}), \quad \text{as } N \rightarrow \infty.$$

Moreover, the estimator $\hat{\theta}_N = (\hat{H}_{1,N}, \hat{H}_{2,N}, \hat{a}_N^2, \hat{b}_N^2)^\top$ is asymptotically normal, that is

$$\sqrt{N}(\hat{\theta}_N - \theta) = \sqrt{N} \begin{pmatrix} \hat{H}_{1,N} - H \\ \hat{H}_{2,N} - H \\ \hat{a}_N^2 - a^2 \\ \hat{b}_N^2 - b^2 \end{pmatrix} \xrightarrow{d} \mathcal{N}(0, \Sigma_0), \quad \text{as } N \rightarrow \infty.$$

with the asymptotic covariance matrix Σ_0 given by

$$\Sigma_0 = (f'(\theta))^{-1} \tilde{\Sigma} ((f'(\theta))^{-1})^\top,$$

where f' is the Jacobian matrix of f .

7. Computational analysis

This section presents the results of our computational analysis on the implementation of the estimators that we derived on simulated data. Despite their apparent complexity, this implementation is straightforward. In fact, the MLE of the drift parameter is explicit and reduces to a generalized least squares formula, requiring only matrix–vector operations and the solution of a linear system. Moreover, the remaining parameters are computed via empirical averages of squared increments at a few time scales, followed by explicit algebraic transformations. No numerical optimization or iterative procedures are required.

In [Section 7.1](#), we examine the behavior of the drift parameter estimators for the model with a single nifBm process ([Eq. \(19\)](#)), and then extend the estimation procedure to the model with two nifBm processes ([Eq. \(13\)](#)).

In [Section 7.2](#), we conduct a simulation study to assess the performance of the estimators for the Hurst parameters and the diffusion coefficients. This analysis is divided into two parts. First, we evaluate the estimators $\hat{H}_N$ ([Eq. \(45\)](#)) and $\hat{a}_N^2$ ([Eq. \(46\)](#)) in the case $b = 0$, which corresponds to the model with one nifBm. Second, we validate the estimators stated in [Theorem 26](#) for the vector parameter $\theta = (H_1, H_2, a^2, b^2)^\top$.

For all estimators, we report their sample means and empirical standard deviations, along with theoretical standard deviations where available. In the tables, we use the notation $\mathbf{m}(\cdot)$, $\sigma_e(\cdot)$, and $\sigma_t(\cdot)$ to denote the mean, empirical standard deviation, and theoretical standard deviation, respectively.

7.1. Simulation study for the drift parameter estimators

We investigate the performance of the drift estimators for both models: the one driven by a single nifBm and the one driven by two nifBm processes. Specifically, we fix the true parameter value $\mu = 4$ and consider the function $G(t) = 5 \cos t - e^{-4t} + 2t^2$, which satisfies condition (17) for arbitrary values of the Hurst exponents $H_1, H_2 \in (0, 1)$. Indeed, in this case $G_N^h = G(Nh) \sim 2N^2h^2$, and therefore

$$\frac{N^{H_1 \vee H_2}}{G_N^h} \sim \frac{1}{2h^2} N^{H_1 \vee H_2 - 2} \rightarrow 0, \quad N \rightarrow \infty,$$

since $H_1 \vee H_2 < 1$. The diffusion coefficients are set to $a = b = 1$.

The sample paths of the processes are generated using the Cholesky decomposition method [34], which exploits the covariance structure of the underlying Gaussian processes. The implementation is carried out in MATLAB using the built-in CHOL function.

Drift estimation in a model with one nifBm. We begin with the estimation of the drift parameter in the model (19), which involves a single nifBm with drift. The estimators $\hat{\mu}_N$ and $\tilde{\mu}_N$, defined in (20) and (23), respectively, are evaluated for several values of the Hurst parameter, $H \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$, and for two values of the averaging window, $h \in \{2, 2^2\}$.

The discrete-time increments process,

$$\Delta X_k^h = X_{(k+1)h}^h - X_{kh}^h,$$

is simulated using the Cholesky method, based on the analytical expression of the covariance matrix given in Eq. (5). Once realizations of ΔX_k^h are generated, the increments of the function $G(t)$ are added to obtain the process $\Delta Y^{(N)}$.

For each configuration (H, h) , we conduct 100 simulations. For every simulated trajectory, the estimator $\hat{\mu}_N$ is computed according to (20) for various values of the number of observations N . The empirical mean and standard deviation are then calculated across the replications. In addition, the theoretical standard deviations are derived from formula (21).

We also examine the performance of the alternative estimator $\tilde{\mu}_N$ by comparing its empirical standard deviation, obtained from simulations, with the corresponding theoretical value. The latter is given by the square root of the variance of $\tilde{\mu}_N$, computed as

$$\text{Var}(\tilde{\mu}_N) = \frac{a^2 h^{2H}}{(G_N^h)^2 (2H+1)(2H+2)} \left[(N+1)^{2H+2} + (N-1)^{2H+2} - 2N^{2H+2} - 2 \right],$$

which follows from Proposition 23, using (3) and (4).

Table 1 reports the means, empirical and theoretical standard deviations of both estimators. The simulation results align closely with our theoretical findings. Both $\hat{\mu}_N$ and $\tilde{\mu}_N$ are essentially unbiased, with empirical means near the true drift $\mu = 4$. Empirical standard deviations match the theoretical values and decrease rapidly with increasing sample size N , confirming the consistency and asymptotic normality of both estimators.

Overall, $\hat{\mu}_N$ is slightly more efficient than $\tilde{\mu}_N$, particularly for small samples and large Hurst exponents H . For large N , both estimators achieve high precision, with empirical variability closely following theoretical predictions. The alternative estimator $\tilde{\mu}_N$ shows greater variability, especially for H near one and small N , reflecting the loss of efficiency from using only two observations. Nevertheless, both theoretical and empirical standard deviations decrease consistently as N increases.

In summary, the simulations confirm the consistency of both estimators while highlighting the superior efficiency of the maximum likelihood approach.

Drift estimation in a model with two nifBms. Within the same framework, we now consider model (14), which includes two independent nifBms and a drift term. We investigate the performance of the MLE $\hat{\mu}_N$ of the drift parameter μ defined in Eq. (15). In addition, we examine the alternative estimator $\tilde{\mu}_N$, introduced in Eq. (18), which relies on two observations.

To evaluate the accuracy of both methods, we compare the empirical standard deviations with their theoretical counterparts. For $\hat{\mu}_N$, a closed-form variance expression is provided in Eq. (16), whereas for $\tilde{\mu}_N$ the variance can be computed explicitly as $\text{Var}(\tilde{\mu}_N) = a^2 T_1 + b^2 T_2$, where

$$T_i = \frac{h^{2H_i}}{(G_N^h)^2 (2H_i+1)(2H_i+2)} \left[(N+1)^{2H_i+2} + (N-1)^{2H_i+2} - 2N^{2H_i+2} - 2 \right], \quad i = 1, 2.$$

As in the single-nifBm case, we consider $N \in \{2^3, 2^5, 2^7\}$ observation points and two averaging windows, $h \in \{2, 2^2\}$. For each configuration, 100 independent simulations are performed.

Table 2 reports the simulation results for the drift coefficient μ across different pairs of Hurst parameters (H_1, H_2) . It presents the sample means of both estimators along with their empirical and theoretical standard deviations.

The results for the two-process model (Table 2) are consistent with those of the single-process case (Table 1). In both settings, the estimators are essentially unbiased, with empirical means very close to the true drift. Regarding variability, the MLE $\hat{\mu}_N$ is systematically more efficient than the alternative estimator $\tilde{\mu}_N$, particularly for larger sample sizes. Only in small- N scenarios two estimators exhibit similar performance; as N increases, the superiority of $\hat{\mu}_N$ becomes evident across all parameter configurations.

This pattern is common to both the single- and two-process models, although the presence of two fractional components leads to higher overall variability, especially when at least one Hurst parameter is large. Thus, while both estimators are consistent, the MLE provides a more reliable and efficient estimation method, with the advantage most pronounced in smoother regimes and for larger samples.

Table 1
Drift estimators $\hat{\mu}_N$ and $\tilde{\mu}_N$ in the model with one nifBm.

H		$h = 2$			$h = 2^2$		
		$N = 2^3$	$N = 2^5$	$N = 2^7$	$N = 2^3$	$N = 2^5$	$N = 2^7$
0.1	$m(\hat{\mu})$	3.99976	4.00004	4.00000	3.99945	4.00001	4.00000
	$\sigma_e(\hat{\mu})$	0.00510	0.00053	0.00003	0.00561	0.00051	0.00000
	$\sigma_t(\hat{\mu})$	0.00559	0.00048	0.00004	0.00599	0.00052	0.00000
	$m(\tilde{\mu})$	4.00045	3.99995	4.00001	4.00095	4.00004	3.99999
	$\sigma_e(\tilde{\mu})$	0.00651	0.00045	0.00004	0.00568	0.00057	0.00005
	$\sigma_t(\tilde{\mu})$	0.00733	0.00058	0.00004	0.00786	0.00062	0.00005
0.3	$m(\hat{\mu})$	4.00152	3.99994	3.99998	3.99881	3.99994	4.00002
	$\sigma_e(\hat{\mu})$	0.01344	0.00140	0.00013	0.01625	0.00166	0.00018
	$\sigma_t(\hat{\mu})$	0.01279	0.00141	0.00014	0.01575	0.00173	0.00017
	$m(\tilde{\mu})$	3.99817	3.99969	3.99991	3.99663	3.99942	3.99997
	$\sigma_e(\tilde{\mu})$	0.01445	0.00165	0.00015	0.01417	0.00214	0.00019
	$\sigma_t(\tilde{\mu})$	0.01676	0.00164	0.00016	0.02063	0.00203	0.00020
0.5	$m(\hat{\mu})$	3.99952	3.99936	4.00011	3.99924	3.99952	4.00002
	$\sigma_e(\hat{\mu})$	0.02284	0.00311	0.00044	0.03116	0.00431	0.00061
	$\sigma_t(\hat{\mu})$	0.02269	0.00320	0.00042	0.03209	0.00452	0.00059
	$m(\tilde{\mu})$	3.99907	3.99925	4.00005	3.99831	3.99984	4.00005
	$\sigma_e(\tilde{\mu})$	0.02269	0.00338	0.00049	0.03375	0.00579	0.00071
	$\sigma_t(\tilde{\mu})$	0.03077	0.00388	0.00049	0.04351	0.00548	0.00069
0.7	$m(\hat{\mu})$	3.99175	4.00051	3.99977	4.00518	4.00263	4.00031
	$\sigma_e(\hat{\mu})$	0.03442	0.00606	0.00106	0.05696	0.01022	0.00185
	$\sigma_t(\hat{\mu})$	0.03659	0.00665	0.00114	0.05944	0.01080	0.00184
	$m(\tilde{\mu})$	4.00987	3.99994	3.99991	4.00366	3.99717	3.99962
	$\sigma_e(\tilde{\mu})$	0.04859	0.00904	0.00154	0.07372	0.01335	0.00221
	$\sigma_t(\tilde{\mu})$	0.05437	0.00895	0.00148	0.08833	0.01453	0.00241
0.9	$m(\hat{\mu})$	3.99896	4.00091	3.99983	3.99089	4.00054	4.00060
	$\sigma_e(\hat{\mu})$	0.04943	0.01187	0.00243	0.08474	0.02017	0.00450
	$\sigma_t(\hat{\mu})$	0.04602	0.01090	0.00244	0.08588	0.02034	0.00455
	$m(\tilde{\mu})$	3.99131	3.99958	4.00013	3.99674	3.99842	3.99953
	$\sigma_e(\tilde{\mu})$	0.07886	0.01719	0.00436	0.16066	0.03891	0.00819
	$\sigma_t(\tilde{\mu})$	0.09515	0.02057	0.00449	0.17756	0.03839	0.00837

7.2. Simulation study for the Hurst parameters and diffusion coefficients

In this section, we investigate the performance of simultaneous estimators for the Hurst exponents and diffusion coefficients for models with one and two nifBm processes. As before, we fix $a = 1$ and $b = 1$, while considering several values of the Hurst exponents.

Estimation of the Hurst parameter and diffusion coefficient in the model with one nifBm. We start with the model $Y_k^h = aX_{kh}^{h,H}$, involving a single nifBm without drift, and evaluate the estimator $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)$ defined in (45) and (46).

For a fixed step size h , we generate the increments ΔX^h using the Cholesky method based on the covariance structure. The realization of ΔX^{2h} is then computed according to (48), so that it incorporates half of the increments present in ΔX^h . From these processes, we compute the statistics ξ_N^j , $j = 1, 2$, to study the behavior of $\hat{\theta}_N = (\hat{H}_N, \hat{a}_N^2)$ introduced in Theorem 31.

For each combination of H and h , we simulate a trajectory and apply the estimators to obtain estimates of H and a^2 . We report the empirical mean and standard deviation over 100 replications and, using Theorem 36, the asymptotic theoretical standard deviation.

Table 3 presents the performance of the estimators $\hat{H}_N$ and $\hat{a}_N^2$ for $H \in \{0.1, 0.3, 0.5, 0.7\}$, $h \in \{2, 2^2, 2^4\}$, and sample sizes $N = 2^n$, $n \in \{6, 8, 10, 12\}$. The estimator $\hat{H}_N$ shows almost unbiased behavior in all Hurst parameters, with only slight deviations for small N and extreme H values, particularly $H = 0.1$. Its empirical standard deviation $\sigma_e(\hat{H}_N)$ decreases rapidly with N and closely matches the theoretical standard deviation $\sigma_t(\hat{H}_N)$ for moderate and large samples, confirming the accuracy of the theoretical variance formula. The increment size h exerts minimal influence on $\hat{H}_N$, slightly affecting variability only for small N .

In contrast, the diffusion coefficient estimator $\hat{a}_N^2$ is considerably more sensitive to both sample size N and averaging window h . For small N and large h , it exhibits pronounced bias and elevated variability, especially for small Hurst indices ($H = 0.1$ or 0.3). As N increases, both bias and variability diminish and the empirical standard deviation converges toward the theoretical value.

These results indicate that $\hat{H}_N$ provides stable and reliable estimates even for moderate N , whereas accurate estimation of $\hat{a}_N^2$ requires sufficiently large samples and moderate values of h .

Hurst parameters and diffusion coefficients in a model with two nifBms.

For the estimation of the parameter vector $\theta = (H_1, H_2, a^2, b^2)$ in the case of two nifBms, $\tilde{X}_k^h = aX_{kh}^{h,H_1} + bX_{kh}^{h,H_2}$, we employ the estimator from Theorem 26, which relies on the statistics ξ_N^j for $j = 1, 2, 4, 8$.

The one-dimensional simulation procedure is inefficient here, as computing $\Delta \tilde{X}_k^{jh}$ via (48) reduces the number of realizations, slowing estimator convergence. While asymptotically valid, this approach requires a larger number of observations N for reasonable accuracy.

Table 2Drift estimators $\hat{\mu}_N$ and $\tilde{\mu}_N$ in the model with two nifBms.

H_1	H_2		$h = 2$			$h = 2^2$		
			$N = 2^3$	$N = 2^5$	$N = 2^7$	$N = 2^3$	$N = 2^5$	$N = 2^7$
0.1	0.3	$\mathbf{m}(\hat{\mu})$	3.99938	4.00001	4.00001	3.99951	3.99968	4.00000
		$\sigma_e(\hat{\mu})$	0.01306	0.00167	0.00015	0.01807	0.00165	0.00017
		$\sigma_t(\hat{\mu})$	0.01399	0.00149	0.00015	0.01687	0.00181	0.00017
		$\mathbf{m}(\tilde{\mu})$	3.99953	4.00016	3.99998	4.00385	4.00009	4.00001
		$\sigma_e(\tilde{\mu})$	0.00819	0.00143	0.00016	0.02082	0.00226	0.00022
		$\sigma_t(\tilde{\mu})$	0.00837	0.00175	0.00017	0.02211	0.00212	0.00020
0.1	0.5	$\mathbf{m}(\hat{\mu})$	3.99824	4.00003	4.00004	3.99953	3.99963	4.00001
		$\sigma_e(\hat{\mu})$	0.02263	0.00320	0.00037	0.03172	0.00472	0.00061
		$\sigma_t(\hat{\mu})$	0.02349	0.00325	0.00042	0.03275	0.00456	0.00059
		$\mathbf{m}(\tilde{\mu})$	3.99865	3.99956	4.00010	4.00025	3.99902	4.00005
		$\sigma_e(\tilde{\mu})$	0.02565	0.00356	0.00051	0.03316	0.00515	0.00069
		$\sigma_t(\tilde{\mu})$	0.03163	0.00392	0.00049	0.04421	0.00552	0.00069
0.3	0.5	$\mathbf{m}(\hat{\mu})$	3.99990	4.00060	3.99993	3.99817	3.99997	3.99993
		$\sigma_e(\hat{\mu})$	0.02745	0.00850	0.00045	0.03636	0.01463	0.00060
		$\sigma_t(\hat{\mu})$	0.02615	0.00960	0.00044	0.03587	0.01326	0.00067
		$\mathbf{m}(\tilde{\mu})$	3.99882	3.99976	4.00004	3.99829	3.99982	3.99997
		$\sigma_e(\tilde{\mu})$	0.02907	0.00385	0.00046	0.03903	0.00551	0.00070
		$\sigma_t(\tilde{\mu})$	0.03503	0.00421	0.00051	0.04815	0.00585	0.00072
0.3	0.7	$\mathbf{m}(\hat{\mu})$	3.99509	4.00073	4.00010	4.00340	4.00009	4.00010
		$\sigma_e(\hat{\mu})$	0.03863	0.00717	0.00116	0.06232	0.01110	0.00207
		$\sigma_t(\hat{\mu})$	0.03947	0.00687	0.00115	0.06226	0.01101	0.00186
		$\mathbf{m}(\tilde{\mu})$	3.99228	4.00021	3.99995	4.00442	4.00138	4.00025
		$\sigma_e(\tilde{\mu})$	0.05530	0.00829	0.00159	0.07430	0.01515	0.00222
		$\sigma_t(\tilde{\mu})$	0.05690	0.00910	0.00149	0.09071	0.01468	0.00241
0.5	0.7	$\mathbf{m}(\hat{\mu})$	4.00221	4.00056	4.00014	3.99529	3.99930	4.00007
		$\sigma_e(\hat{\mu})$	0.04494	0.00830	0.00121	0.06574	0.01188	0.00183
		$\sigma_t(\hat{\mu})$	0.04352	0.00745	0.00122	0.06821	0.01181	0.00194
		$\mathbf{m}(\tilde{\mu})$	3.99382	3.96247	4.00011	4.00016	4.00095	3.99986
		$\sigma_e(\tilde{\mu})$	0.04386	0.01974	0.00164	0.00926	0.01521	0.00228
		$\sigma_t(\tilde{\mu})$	0.06248	0.00127	0.00156	0.00975	0.01554	0.00250

To overcome this, we directly simulate the increments $\Delta\tilde{X}_k^{jh}$ using Cholesky decomposition and compute ξ_N^j from the covariance structure:

$$\mathbb{E}\left[\Delta\tilde{X}_k^{jh}\Delta\tilde{X}_{k+n}^{jh}\right] = a^2(jh)^{2H_1}\gamma(H_1, n) + b^2(jh)^{2H_2}\gamma(H_2, n), \quad n = 0, \dots, N, \quad j = 1, 2, 4, 8.$$

Since the covariance differs only by the factor jh , it suffices to compute it once for $j = 1$, and the other statistics can be obtained by rescaling. This reduces computational cost, eliminates repeated increment calculations, and ensures the same number of observations for all statistics. Stationarity of $\Delta\tilde{X}_k^{jh}$ further simplifies computation, as the covariance matrix is Toeplitz.

Table 4 presents the results for the estimator parameter vector $\theta = (H_1, H_2, a^2, b^2)$ across five different pairs of H_1 and H_2 with $H_1 < H_2$, where the scale parameters a and b are both set to 2. The simulation results indicate that the Hurst estimators $\hat{H}_{1,N}$ and $\hat{H}_{2,N}$ are generally unbiased and become increasingly precise as the sample size N grows, with $\hat{H}_{2,N}$ showing particularly robust performance across all scenarios. Estimating a very small H_1 in the presence of a larger H_2 proves more challenging, resulting in slight bias and increased variance, especially for small N and larger increment sizes h . The scale parameter estimator $\hat{b}_N^2$ remains consistently accurate and stable, whereas $\hat{a}_N^2$ can exhibit considerable variance and bias under these challenging conditions but stabilizes with larger N or smaller h . Overall, all estimators perform well for moderate Hurst parameters ($H \geq 0.3$), with precision improving as N increases and h decreases, highlighting the robustness of $\hat{H}_{2,N}$ and $\hat{b}_N^2$ and the sensitivity of $\hat{H}_{1,N}$ and $\hat{a}_N^2$ in difficult regimes. Notably, when $N \geq 2^{10}$, all parameters are estimated with high accuracy and stability across all configurations.

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Table 3Estimators $\hat{H}_N$ and $\hat{a}_N^2$ in the model with one nifBm.

	$h = 2$				$h = 2^2$				$h = 2^4$			
	$N = 2^6$	2^8	2^{10}	2^{12}	2^6	2^8	2^{10}	2^{12}	2^6	2^8	2^{10}	2^{12}
$H = 0.1$												
$m(\hat{H})$	0.0746	0.0834	0.1022	0.1000	0.0952	0.1002	0.0967	0.1026	0.1096	0.1055	0.1001	0.1015
$\sigma_e(\hat{H})$	0.1713	0.0833	0.0352	0.0198	0.1483	0.0669	0.0362	0.0195	0.1635	0.0752	0.0357	0.0196
$\sigma_t(\hat{H})$	0.1087	0.0544	0.0272	0.0136	0.1087	0.0544	0.0272	0.0136	0.1087	0.0544	0.0272	0.0136
$m(\hat{a}^2)$	2.0548	1.7555	1.2109	1.0352	2.4617	2.0531	1.7098	1.0147	3.9263	0.8146	1.8697	1.0452
$\sigma_e(\hat{a}^2)$	16.830	7.0895	0.8605	0.0867	26.357	7.1395	3.1218	0.2407	31.873	5.3269	6.3387	0.3355
$\sigma_t(\hat{a}^2)$	1.0176	0.5088	0.2544	0.0718	1.1663	0.5831	0.2916	0.1458	1.4649	0.7325	0.3662	0.1831
$H = 0.3$												
$m(\hat{H})$	0.2715	0.2961	0.3000	0.3004	0.3067	0.2948	0.3052	0.2993	0.2875	0.2934	0.2980	0.3004
$\sigma_e(\hat{H})$	0.1325	0.0609	0.0320	0.0159	0.1377	0.0594	0.0327	0.0153	0.1435	0.0622	0.0358	0.0141
$\sigma_t(\hat{H})$	0.1194	0.0597	0.0298	0.0149	0.1194	0.0597	0.0298	0.0149	0.1194	0.0597	0.0298	0.0149
$m(\hat{a}^2)$	1.1625	1.0719	1.0082	1.0023	4.2266	1.0948	0.9937	1.0066	0.8646	1.2299	1.0609	1.0041
$\sigma_e(\hat{a}^2)$	4.3630	0.3158	0.1328	0.0597	9.9780	0.3462	0.1743	0.0768	7.3627	0.7895	0.3227	0.1095
$\sigma_t(\hat{a}^2)$	0.4354	0.2177	0.1089	0.0544	0.5904	0.2952	0.1476	0.0738	0.9115	0.4557	0.2279	0.1139
$H = 0.5$												
$m(\hat{H})$	0.4826	0.5024	0.4990	0.4998	0.4936	0.4957	0.4952	0.5002	0.4981	0.4977	0.5031	0.4979
$\sigma_e(\hat{H})$	0.1175	0.0523	0.0311	0.0121	0.1161	0.0529	0.0277	0.0118	0.1187	0.0548	0.0252	0.0125
$\sigma_t(\hat{H})$	0.1104	0.0552	0.0276	0.0138	0.1104	0.0552	0.0276	0.0138	0.1104	0.0522	0.0276	0.0138
$m(\hat{a}^2)$	1.1302	1.0085	1.0113	1.0023	1.2494	1.0489	1.0231	0.9995	1.3550	1.0952	0.9873	1.0181
$\sigma_e(\hat{a}^2)$	0.4158	0.1355	0.0729	0.0311	1.2327	0.2178	0.1098	0.0484	1.1090	0.4620	0.1572	0.0795
$\sigma_t(\hat{a}^2)$	0.3023	0.1512	0.0759	0.0378	0.4351	0.2176	0.1088	0.0544	0.7252	0.3626	0.1813	0.0906
$H = 0.7$												
$m(\hat{H})$	0.6804	0.6940	0.7006	0.6976	0.6854	0.6959	0.6970	0.7007	0.6826	0.6943	0.6967	0.7006
$\sigma_e(\hat{H})$	0.0787	0.0429	0.0232	0.0117	0.0989	0.0514	0.0234	0.0129	0.0864	0.0459	0.0253	0.0139
$\sigma_t(\hat{H})$	0.0902	0.0535	0.0294	0.0157	0.0902	0.0535	0.0294	0.0157	0.0902	0.0535	0.0294	0.0157
$m(\hat{a}^2)$	1.0750	0.9855	1.0005	0.9987	1.0805	1.0192	1.0101	0.9968	1.2327	1.0766	1.0194	0.9950
$\sigma_e(\hat{a}^2)$	0.2457	0.0831	0.0463	0.0215	0.3890	0.1537	0.0564	0.0357	0.7304	0.2976	0.1283	0.0627
$\sigma_t(\hat{a}^2)$	0.3042	0.1729	0.0934	0.0492	0.4713	0.2664	0.1436	0.0754	0.7583	0.4313	0.2331	0.1227

Appendix A. Proofs of auxiliary results

Proof of Lemma 2. Let $s \geq t$. Then

$$\begin{aligned}
 \mathbb{E}[X_t^h X_s^h] &= \mathbb{E}\left[\left(\frac{1}{h} \int_t^{t+h} W_u^H du\right) \left(\frac{1}{h} \int_s^{s+h} W_v^H dv\right)\right] \frac{1}{h^2} \int_0^h \int_0^h \mathbb{E}[W_{u+t}^H W_{v+s}^H] du dv \\
 &= \frac{1}{2h^2} \int_0^h \int_0^h ((u+t)^{2H} + (v+s)^{2H} - |v-u+s-t|^{2H}) du dv \\
 &= \frac{1}{2h^2} \left(\int_0^h \int_0^h (u+t)^{2H} du dv + \int_0^h \int_0^h (v+s)^{2H} du dv - \int_0^h \int_0^h |v-u+s-t|^{2H} du dv \right) \\
 &=: \frac{1}{2h^2} (J_1 + J_2 - J_3).
 \end{aligned}$$

Obviously,

$$J_1 = \frac{h}{2H+1} ((t+h)^{2H+1} - t^{2H+1}), \quad J_2 = \frac{h}{2H+1} ((s+h)^{2H+1} - s^{2H+1}).$$

To calculate J_3 , denote for simplicity $\rho = s - t \geq 0$. Then

$$J_3 = J(\rho, h) = \int_0^h \int_0^h |v-u+\rho|^{2H} du dv.$$

Consider two cases. If $\rho \geq h$, then

$$\begin{aligned}
 J(\rho, h) &= \int_0^h \int_0^h (v-u+\rho)^{2H} du dv = \frac{1}{2H+1} \int_0^h [(h+\rho-u)^{2H+1} - (\rho-u)^{2H+1}] du \\
 &= \frac{1}{2H+1} [(h+\rho)^{2H+2} + (\rho-h)^{2H+2} - 2\rho^{2H+2}].
 \end{aligned}$$

Table 4Estimators $\hat{H}_{1,N}$, $\hat{H}_{2,N}$, $\hat{a}_N^2$ and $\hat{b}_N^2$ in the model with two nifBms.

	$h = 2$				$h = 2^2$				$h = 2^4$			
	$N = 2^6$	2^8	2^{10}	2^{12}	2^6	2^8	2^{10}	2^{12}	2^6	2^8	2^{10}	2^{12}
$H_1 = 0.1, H_2 = 0.3$												
$m(\hat{H}_1)$	0.0904	0.0992	0.0997	0.1000	0.0999	0.0998	0.0997	0.0999	0.0970	0.0995	0.0995	0.0999
$\sigma_e(\hat{H}_1)$	0.0204	0.0064	0.0032	0.0016	0.0200	0.0078	0.0037	0.0019	0.0301	0.0108	0.0050	0.0025
$m(\hat{H}_2)$	0.3018	0.3001	0.3001	0.3000	0.3000	0.3000	0.3001	0.3000	0.3000	0.3000	0.3000	0.3000
$\sigma_e(\hat{H}_2)$	0.0047	0.0014	0.0007	0.0004	0.0032	0.0013	0.0007	0.0003	0.0025	0.0010	0.0005	0.0003
$m(\hat{a}^2)$	4.0911	4.1431	4.0196	4.0080	4.2818	4.1285	4.0211	4.0083	8.9166	4.1510	4.0266	4.0094
$\sigma_e(\hat{a}^2)$	0.6753	0.4501	0.2311	0.0947	0.9985	0.4037	0.2288	0.0935	45.583	0.3691	0.2178	0.0888
$m(\hat{b}^2)$	3.7943	3.9307	4.0233	4.0146	4.0705	3.9818	4.0235	4.0148	4.0594	3.9781	4.0246	4.0143
$\sigma_e(\hat{b}^2)$	1.0540	0.3951	0.1964	0.0979	0.8183	0.4022	0.1939	0.0967	0.8020	0.3958	0.1898	0.0949
$H_1 = 0.1, H_2 = 0.5$												
$m(\hat{H}_1)$	0.0912	0.1007	0.0993	0.1000	0.0646	0.0999	0.0988	0.0999	0.0420	0.0931	0.0965	0.0996
$\sigma_e(\hat{H}_1)$	0.0710	0.0218	0.0108	0.0047	0.3199	0.0297	0.0143	0.0062	0.5585	0.0626	0.0260	0.0109
$m(\hat{H}_2)$	0.5000	0.4999	0.5000	0.5000	0.5001	0.4999	0.5000	0.5000	0.5004	0.5000	0.5000	0.5000
$\sigma_e(\hat{H}_2)$	0.0042	0.0017	0.0009	0.0004	0.0033	0.0013	0.0007	0.0003	0.0017	0.0008	0.0004	0.0002
$m(\hat{a}^2)$	3.8037	4.1390	4.0716	4.0197	4.6736	5.5225	4.1137	4.0256	0.0849	3.5420	4.5804	4.0641
$\sigma_e(\hat{a}^2)$	3.5598	0.5714	0.2013	0.0976	8.7652	11.2864	0.2976	0.1231	48.779	6.0159	2.0301	0.2967
$m(\hat{b}^2)$	4.0703	4.0809	4.0120	4.0029	4.0614	4.0776	4.0119	4.0028	3.9782	4.0721	4.0118	4.0026
$\sigma_e(\hat{b}^2)$	0.7711	0.3679	0.1921	0.0882	0.7615	0.3631	0.1896	0.0874	0.9350	0.3572	0.1866	0.0863
$H_1 = 0.3, H_2 = 0.5$												
$m(\hat{H}_1)$	0.3012	0.3005	0.3002	0.3000	0.2998	0.3000	0.3001	0.3000	0.2999	0.2997	0.3000	0.3000
$\sigma_e(\hat{H}_1)$	0.0203	0.0065	0.0034	0.0017	0.0211	0.0072	0.0036	0.0018	0.0300	0.0105	0.0050	0.0025
$m(\hat{H}_2)$	0.5001	0.5000	0.5000	0.5000	0.5001	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
$\sigma_e(\hat{H}_2)$	0.0041	0.0016	0.0008	0.0004	0.0033	0.0013	0.0007	0.0003	0.0017	0.0008	0.0004	0.0002
$m(\hat{a}^2)$	4.0112	4.0211	4.0135	4.0081	4.0145	4.0120	4.0105	4.0078	4.0123	4.0105	4.0089	4.0075
$\sigma_e(\hat{a}^2)$	0.4421	0.2151	0.1108	0.0543	0.3981	0.1983	0.1021	0.0502	0.3652	0.1856	0.0952	0.0467
$m(\hat{b}^2)$	4.0110	4.0132	4.0100	4.0080	4.0108	4.0125	4.0103	4.0080	4.0105	4.0120	4.0101	4.0079
$\sigma_e(\hat{b}^2)$	0.4012	0.1920	0.0961	0.0482	0.3920	0.1873	0.0928	0.0465	0.3845	0.1821	0.0912	0.0453
$H_1 = 0.3, H_2 = 0.7$												
$m(\hat{H}_1)$	0.3001	0.3000	0.3000	0.3000	0.3001	0.3000	0.3000	0.3000	0.3002	0.3000	0.3000	0.3000
$\sigma_e(\hat{H}_1)$	0.0202	0.0064	0.0033	0.0017	0.0210	0.0071	0.0035	0.0017	0.0299	0.0104	0.0049	0.0025
$m(\hat{H}_2)$	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000
$\sigma_e(\hat{H}_2)$	0.0035	0.0015	0.0007	0.0003	0.0027	0.0010	0.0005	0.0002	0.0016	0.0008	0.0004	0.0002
$m(\hat{a}^2)$	4.0102	4.0121	4.0105	4.0078	4.0108	4.0118	4.0104	4.0079	4.0105	4.0117	4.0102	4.0078
$\sigma_e(\hat{a}^2)$	0.4441	0.2120	0.1090	0.0530	0.3960	0.1952	0.1018	0.0495	0.3615	0.1835	0.0938	0.0453
$m(\hat{b}^2)$	4.0112	4.0125	4.0103	4.0079	4.0109	4.0120	4.0102	4.0078	4.0106	4.0118	4.0101	4.0078
$\sigma_e(\hat{b}^2)$	0.4010	0.1915	0.0958	0.0478	0.3921	0.1865	0.0924	0.0462	0.3825	0.1815	0.0909	0.0451
$H_1 = 0.5, H_2 = 0.7$												
$m(\hat{H}_1)$	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
$\sigma_e(\hat{H}_1)$	0.0042	0.0017	0.0008	0.0004	0.0033	0.0013	0.0007	0.0003	0.0017	0.0008	0.0004	0.0002
$m(\hat{H}_2)$	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000	0.7000
$\sigma_e(\hat{H}_2)$	0.0035	0.0015	0.0007	0.0003	0.0027	0.0010	0.0005	0.0002	0.0016	0.0008	0.0004	0.0002
$m(\hat{a}^2)$	4.0105	4.0120	4.0103	4.0078	4.0108	4.0117	4.0102	4.0078	4.0105	4.0116	4.0101	4.0078
$\sigma_e(\hat{a}^2)$	0.4425	0.2125	0.1095	0.0532	0.3952	0.1955	0.1020	0.0496	0.3618	0.1837	0.0939	0.0454
$m(\hat{b}^2)$	4.0111	4.0124	4.0102	4.0078	4.0109	4.0120	4.0101	4.0078	4.0106	4.0117	4.0100	4.0078
$\sigma_e(\hat{b}^2)$	0.4012	0.1918	0.0959	0.0479	0.3920	0.1867	0.0925	0.0463	0.3827	0.1817	0.0910	0.0452

If $\rho \leq h$, then

$$\begin{aligned}
J(\rho, h) &= \int_0^\rho \int_0^h (v - u + \rho)^{2H} dv du + \int_\rho^h \int_0^{u-\rho} (u - v - \rho)^{2H} dv du + \int_\rho^h \int_{u-\rho}^h (v - u + \rho)^{2H} dv du \\
&= \frac{1}{2H+1} \left[\int_0^\rho [(h + \rho - u)^{2H+1} - (\rho - u)^{2H+1}] du + \int_\rho^h (u - \rho)^{2H+1} du + \int_\rho^h (h - u + \rho)^{2H+1} du \right] \\
&= \frac{1}{(2H+1)(2H+2)} [(h + \rho)^{2H+2} + (h - \rho)^{2H+2} - 2\rho^{2H+2}],
\end{aligned}$$

and the proof follows. $\square$ **Proof of Lemma 5.**

- (i) Of course, it is enough to analyze the sign of the numerator, that is, of the function $f(H) := 7 - 2^{2H+4} + 3^{2H+2}$. Note that $f(0) = 0$ and $f'(H) = -2 \log 2 \cdot 2^{2H+4} + 2 \log 3 \cdot 3^{2H+2}$, which is positive for

$$H > \bar{H} := \frac{\log \frac{4 \log 2}{\log 3}}{2 \log \frac{3}{2}} - 1 = \frac{\log \frac{16 \log 2}{9 \log 3}}{2 \log \frac{3}{2}} \simeq 0.14157$$

and negative for $H < \bar{H}$. Thus, f is decreasing and negative in $(0, \bar{H}]$, and increasing in $[\bar{H}, 1]$. Since $f(\bar{H}) < 0$ and $f(1) = 7 - 64 + 81 > 0$, it follows that there exists a unique $H_0 \in [\bar{H}, 1]$ such that $f(H_0) = 0$, consequently $\gamma(H_0, 1) = 0$.

- (ii) In this case

$$\mathbb{E}[(X_{t+h}^h - X_t^h)(X_{t+(n+1)h}^h - X_{t+nh}^h)] = \int_0^h \int_0^h \mathbb{E}[(W_{u+t+h}^H - W_{u+t}^H)(W_{v+t+(n+1)h}^H - W_{v+t+nh}^H)] du dv,$$

and for $n \geq 2$ $(u+t, u+t+h) \cap (v+t+nh, v+t+(n+1)h) = \emptyset$. Therefore the sign of $\mathbb{E}[(X_{t+h}^h - X_t^h)(X_{t+(n+1)h}^h - X_{t+nh}^h)]$ follows the sign of incremental covariance of fBm for non-overlapping intervals.

□

Proof of Lemma 9. The claim follows because

$$\{X_{ct}^{ch}, t \geq 0\} = \left\{ \frac{1}{ch} \int_{ct}^{c(t+h)} W_u^H du, t \geq 0 \right\} = \left\{ \frac{1}{ch} \int_t^{t+h} W_{cs}^H c ds, t \geq 0 \right\} \stackrel{d}{=} \left\{ \frac{1}{h} \int_t^{t+h} c^H W_s^H ds, t \geq 0 \right\} = \{c^H X_t^h, t \geq 0\}.$$

□

Appendix B. Selected results for Gaussian vectors

In this appendix we collect some properties of Gaussian random vectors and a limit theorem which is used for the proof of asymptotic normality.

Isserlis' theorem

The following formula is known as Isserlis' theorem [35]: if (X_1, X_2, X_3, X_4) is a zero-mean multivariate normal random vector, then

$$\mathbb{E}(X_1 X_2 X_3 X_4) = \mathbb{E}[X_1 X_2] \mathbb{E}[X_3 X_4] + \mathbb{E}[X_1 X_3] \mathbb{E}[X_2 X_4] + \mathbb{E}[X_1 X_4] \mathbb{E}[X_2 X_3].$$

In particular,

$$\text{cov}(X_1^2, X_2^2) = 2 (\text{cov}(X_1, X_2))^2. \quad (53)$$

For the direct proof of (53) see [36, Lemma 2.2].

Hermite rank for Gaussian vectors

In this subsection, we recall the notion of Hermite rank for functions of a Gaussian vector. For more details and comprehensive treatments of Hermite expansions and Wiener chaos, we refer the reader to [37–39].

Definition 38 (Hermite rank [31,32]). The function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is said to have Hermite rank equal to q with respect to a random vector X if (a) $\mathbb{E}[(f(X) - \mathbb{E}[f(X)])p_m(X)] = 0$ for every polynomial p_m (on $\mathbb{R}^d$) of degree $m \leq q-1$; and (b) there exists a polynomial p_q of degree q such that $\mathbb{E}[(f(X) - \mathbb{E}[f(X)])p_q(X)] \neq 0$.

Remark 39 (Equivalence with Hermite expansion). Equivalently, if $f(X) - \mathbb{E}[f(X)]$ is expanded in the multivariate Hermite basis

$$f(X) - \mathbb{E}[f(X)] = \sum_{m \in \mathbb{N}_0^d, |m| \geq 1} c_m H_m(X), \quad H_m(x) = \prod_{j=1}^d H_{m_j}(x_j),$$

then the Hermite rank equals the smallest total degree $|m| = \sum_j m_j$ such that $c_m \neq 0$. This follows because the multivariate Hermite polynomials form an orthogonal basis of L^2 for the Gaussian measure.

Example 40 (Quadratic form has Hermite rank 2). Let $X = (X_1, \dots, X_d)$ be a centered Gaussian vector, and consider the quadratic function

$$f(x) = \sum_{i,j=1}^d \lambda_{ij} x_i x_j, \quad \Lambda = (\lambda_{ij}) \neq 0.$$

Assume that $f(X)$ is not almost surely constant (for example, X has a non-degenerate covariance matrix). Then f has Hermite rank 2 with respect to X .

Indeed, the function $f - \mathbb{E}[f]$ is a centered polynomial of degree 2, so it is orthogonal to all polynomials of degree 0 or 1 (for instance, $\mathbb{E}[(f - \mathbb{E}[f])X_k] = 0$ because third-order centered Gaussian moments vanish). Since $f(X)$ is not almost surely constant, the degree-2 part is nonzero, hence there exists a polynomial of degree 2 with nonzero expectation against $f - \mathbb{E}[f]$. By Definition 38, this shows that the Hermite rank of f is 2.

Breuer–Major theorem for stationary vectors

Theorem 41 ([31, Theorem 4]). Let $d \geq 1$, and let $X = \{X_k, k \in \mathbb{Z}\}$ be a d -dimensional centered stationary Gaussian process, $X_k = (X_k^{(1)}, \dots, X_k^{(d)})$. For $1 \leq i, l \leq d$ and $j \in \mathbb{Z}$, define

$$r^{(i,l)}(j) = \mathbb{E}[X_1^{(i)} X_{1+j}^{(l)}].$$

Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a measurable function such that $\mathbb{E}[f^2(X_1)] < \infty$, and assume that f has Hermite rank $q \geq 1$. Suppose further that

$$\sum_{j \in \mathbb{Z}} |r^{(i,l)}(j)|^q < \infty, \quad \text{for all } i, l \in \{1, \dots, d\}. \quad (54)$$

Then

$$\sigma^2 := \text{Var}(f(X_1)) + 2 \sum_{k=1}^{\infty} \text{Cov}(f(X_1), f(X_{1+k}))$$

is finite and nonnegative. Moreover,

$$\frac{1}{\sqrt{n}} \sum_{k=1}^n (f(X_k) - \mathbb{E}[f(X_k)]) \xrightarrow{d} \mathcal{N}(0, \sigma^2), \quad \text{as } n \rightarrow \infty.$$

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