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RESEARCH ARTICLE

Drawdown risk measures for asset portfolios with high frequency data

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Abstract
In this paper, we analyze Drawdown-based risk measures for an equity portfolio with high-frequency data. The returns of individual stocks are modeled through multivariate weighted-indexed semi-Markov chains with a copula dependence structure. Through this recently published model, we show that the estimate of Drawdown-based risk measures is more faithful than that obtained with the application of classic econometric models.

Keywords Drawdown risk measure · Weighted-indexed semi-Markov models · Asset portfolio · High-frequency data · Right censoring · GARCH models

JEL Classification C02 · G30

1 Introduction

Financial markets have always been characterized by sudden fluctuations, both up and down, such as happens in the prices of high-frequency assets. This makes investments in asset portfolios extremely risky, and it becomes extremely important for the investor to develop adequate risk measures to quantify the losses caused by adverse events in the asset markets. The choice of an appropriate risk measure, therefore, takes on an extremely relevant aspect. In this context, in addition to the classic risk measures such as the Value-at-Risk or the Expected Shortfall used in the Basel agreements, which

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represent quantile-based risk measures, new risk measures based on the concept of drawdown have been created. For example, we consider the drawdown of fixed level, the time to crash, the speed of crash, the recovery time, and the speed of recovery (D'Amico et al. 2020). More precisely, the drawdown process represents the distance of the price process from the so-called running maximum (see Zhang and Hadjiliadis 2012; Zhang 2018).

Risk measures based on drawdown have been addressed in the literature under various aspects and objectives that differ substantially from those applied in this work. For instance, Casati and Tabachnik (2012) analyze the properties of the maximum drawdown in time series. Charwand et al. (2017) applied optimization tools for solving the portfolio allocation problem in the framework of the electricity market through drawdown measure maximizing rate of return. An application in the energy markets sector was carried out by Gatfaoui (2019). The author applies the expected maximum drawdown as a measure of portfolio risk. Goldberg and Mahmoud (2017) develop a new tail risk measure called Conditional Expected Drawdown and they find evidence that this measure is more sensitive to serial correlation than the classical Expected Shortfall. The definition introduced by the authors has been taken up again in the following Sect. 2. Mendes and Lavrado (2017) investigate the Maximum Drawdown at Risk and they find evidence that this risk measure has some better properties than traditional measures, this is why we applied this risk measure in our application. Yao et al. (2013) approached the optimal portfolio selection problem in a mean-variance framework by inserting a drawdown constraint. The authors deduce remarkable properties of the efficient frontier obtained with this approach and this aspect could represent a future development of our work with the considered WISMC models. Since our work is aimed at drawdown-based risk measures applied to a high-frequency asset portfolio, we, therefore, propose to enrich the scope of these risk measures in an original way also through modelling with multivariate WISMC processes. After identifying the most suitable risk measures, the second fundamental aspect concerns the modeling of asset returns. In this regard, we can include in the literature traditional econometric models (see Engle 1982; Nelson 1991; Vrontos et al. 2001), or diffusive models (Andersen et al. 2010; Pospisil et al. 2009). Furthermore, a new model class based on weighted-indexed semi-Markov chains (WISMC) has been developed with good results (D'Amico and Petroni 2011, 2012a, b). The WISMC models were subsequently extended to the multivariate case, thanks to the use of copula functions, in order to faithfully replicate the complex dependency structure between asset returns (see D'Amico and Petroni 2020).

Regarding the specific case of high-frequency data, used in this work, the WISMC model is suitable to reproduce faithfully the statistical properties of one-dimensional return financial data (for example, the clustering of volatility and the autocorrelation of squared returns at a one-minute frequency). The WISMC model furnishes an accurate probabilistic description of the asset return evolution, which accounts for the serial dependence of asset returns by including past events (trade times and return sizes) through an index process that increases the memory of the process (see D'Amico and Petroni 2020) for more details and a comparison with other traditional models).

In this paper, we investigate some drawdown-based risk measures by considering a multivariate weighted-indexed semi-Markov chain model which is applied to an asset portfolio from the Dow Jones with high-frequency (1-min scale) data. In particular, we present an empirical analysis on 10 assets of the Dow Jones market covering the period February 10, 2022–July 19, 2022.

Specifically, taking an asset portfolio, we calculate the returns of the individual stocks with a multivariate model that foresees a WISMC process for the marginals, which will be linked by a copula function. We then determine, through Monte Carlo simulations, the prices and returns of our portfolio. We then estimate the risk measures based on the drawdown. At first, we compute the drawdown of the fixed level for several values of given thresholds. We deduce the optimal time to crash distribution and its associated parameters through the well-known AIC criterion. For this purpose, we take into account right-censored data in the maximum likelihood estimates of the parameters as data are censored as a consequence of the observation period. Next, we use the Kullback–Leibler divergence to select the distribution that best fits the empirical data. Next, we compute the speed of crash and the time to crash. Finally, we apply the same procedure for the recovery time and the speed of recovery.

To obtain a term of comparison, the entire procedure is compared with the results obtained with a classical econometric model, for which the marginals are estimated with various classes of GARCH processes with appropriate variants and linked by a copula function. In this way, a classic multivariate model is obtained which takes into account the characteristics of the individual series and their dependence structure.

The original aspect of our work consists in having considered drawdown-based risk measures to assess the riskiness of an equity portfolio, whose returns have been modeled through a multivariate WISMC model, and comparison with other classical models.

We find evidence that, in general, drawdown-based risk measures estimated with WISMC model more faithfully reproduce the similar risk measures estimated with classical econometric models used as a reference benchmark. The practical application, carried out using ten Dow Jones assets, is particularly significant as it is applied to a reference market for finance. Furthermore, these results support, in the more complex multivariate case, what was found in the univariate case (D'Amico et al. 2020). The proposed study is also extremely interesting because it represents a valid empirical support regarding the efficiency of the WISMC models developed in D'Amico and Petroni (2012b, 2020).

The paper is structured as follows. In Sect. 2, we describe the drawdown-based risk measures. In Sect. 3, we present the multivariate stochastic models (WISMC and classic econometric model). We present in Sect. 4 the empirical analysis with the database description, the model parameters estimation and the censoring problem. In Sect. 5, we present the results of our analysis and a comparison between the WISMC and the benchmark models. Finally, Sect. 6 concludes.

2 Risk measures

Risk measures play a fundamental role in the theory of financial risk and count an ever-increasing number in the literature. Traditional risk measures such as Value-at-Risk (see Jorion 2007) and Conditional Value-at-Risk (see Rockafellar and Uryasev 2002) have been used fundamentally by the Basel Committee on Banking Supervision for the drafting of the "Market risk capital requirements" (see BCBS 2013). These measures have the disadvantage of being defined through the quantiles of the distributions and therefore do not take into account the time sequence of the data. To remedy this problem, new ad hoc risk measures have been created based on the concept of Drawdown (see D'Amico et al. 2020; Maier-Paape and Zhu 2018; Zhang and Hadjiliadis 2012; Zhang 2018).

Let us consider the discrete-time varying asset price process $X(t)$ and introduce its running maximum process $Y(t)$ as

$$Y(t) = \max_{s \in \{0,1,\ldots,t\}} \{X(s)\}.$$

The Drawdown process

$$D(t) = Y(t) - X(t), \quad t \geq 0 \tag{1}$$

represents then the correction of the asset price with respect to a previous relative maximum. Given a time horizon $\{0, 1, \ldots, t\}$, the Maximum Drawdown is defined as the highest value of the drawdown achieved in the predetermined horizon, that is:

$$MD_{\{0,1,\ldots,t\}} = \sup_{s \in \{0,1,\ldots,t\}} \{D(s)\}. \tag{2}$$

We are now introducing several Drawdown-based risk measures related to market crashes: the Drawdown of fixed level, the time to crash, the speed of crash, the recovery time and the speed of recovery (see at this purpose (D'Amico et al. 2020)).

(1) The *Drawdown of fixed level* $\tau(K)$ represents the first time that the Drawdown process attains or overcomes a given threshold K:

$$\tau(K) := \min\{t \geq 0 \mid D(t) \geq K\} \text{ where } K \geq 0$$

(2) Given the *last visit time* of the maximum before the stopping time $\tau(K)$ as

$$\rho(K) := \max\{t \in [0, \tau(K)] \mid Y(t) = X(t)\}$$

we deduce the *time to crash* $T_c(K)$ as

$$T_c(K) := \tau(K) - \rho(K)$$

that represents the time the Drawdown process needs to experience the first drop of level K.

(3) The *speed of crash* $S_c(K)$ is the speed at which the first K-change occurs, that is

$$S_c(K) := \frac{K}{\tau(K) - \rho(K)} = \frac{K}{T_c(K)}.$$

Given now two thresholds K and K', we define the quantity

$$\gamma\left(K, K'\right) := \min\left\{t > \tau(K) \mid D(t) \leq K'\right\} \text{ with } K > K'$$

that represents the first time in which the Drawdown process drops below the threshold K' after crossing the threshold K for the first time. This simply means that the asset goes from a more risky situation (the threshold K is exceeded), to a less risky one (the threshold K' is attained).

(4) The *recovery time* $R_t\left(K, K'\right)$ is given as

$$R_t\left(K, K'\right) := \gamma\left(K, K'\right) - \tau(K)$$

and represents the time to experience the first K'-descent in the Drawdown process following the first K-ascent.

(5) The *speed of recovery* $S_r\left(K, K'\right)$ is

$$S_r\left(K, K'\right) := \frac{K - K'}{\gamma\left(K, K'\right) - \tau(K)} = \frac{K - K'}{R_t\left(K, K'\right)}$$

and represents the speed at which a $\left(K - K'\right)$ variation takes place. These drawdown-based risk measures provide risk managers with an alternative tool with which to assess the riskiness of an investment portfolio. For example, $\tau(K)$ represents the riskness of an asset and it is a function of the threshold K-value (a low value of K denotes low risk events while a high value of K denotes riskier events). Consequently, $T_c(K)$ and $S_c(K)$ measure how long these risky events last and how quickly they take place. Contrary to the previous measures, $R_t\left(K, K'\right)$ and $S_r\left(K, K'\right)$ represent the behavior of an asset after reaching a certain threshold of K in its drawdown, therefore they are linked to both the K threshold and the second K' threshold. More specifically, the recovery time and recovery speed denote how long the drawdown process holds the first K change before experiencing a $\left(K - K'\right)$ drop and the speed with which this drop takes place. Furthermore, to define the measure $R_t\left(K, K'\right)$ we must consider the K' threshold in addition to the K threshold. Thus, the speed of recovery $S_r\left(K, K'\right)$ indicates the speed at which we cross the space $\left(K - K'\right)$ in the range $\left[\tau(K), \gamma\left(K, K'\right)\right]$. We observe that the crash is interpreted as a decline in the price process whose magnitude is represented by the K threshold chosen in a subjective way. For example, a large value for K means a very large decrease in prices. Finally, we introduce some tail risk measures based on the drawdown process $D(t)$ introduced in (1) but defined this time on the returns of the portfolio rather than on its price.

(6) The Maximum Drawdown distribution, introduced in (2), plays a crucial role for risk management. To build this distribution, we can calculate the MD on a time window of fixed width and then execute a rolling window in order to cover the entire available dataset. In this way we get a set of values for the MD. The quantile of this distribution, for a confidence level $\alpha \in (0, 1)$, represents the *Maximum Drawdown*

at Risk defined as (see Goldberg and Mahmoud 2017)

$$DaR_\alpha = \inf \{m \mid \mathbb{P}(MD > m) \leq 1 - \alpha\}.$$

This definition is very similar to the classical Value-at-Risk, and it represents the smallest maximum drawdown m such that the probability that the maximum drawdown MD exceeds m is at most $1 - \alpha$.

(7) The *Conditional Expected Drawdown* for a confidence level $\alpha \in (0, 1)$ is the expected maximum drawdown given that the maximum drawdown threshold at α is breached. We can set

$$CDaR_\alpha = \frac{1}{1-\alpha} \int_\alpha^1 DaR_u \, du = \mathbb{E}(MD \mid MD > DaR_\alpha).$$

The latter definition is similar to the Conditional Value-at-Risk. A detailed analysis with related properties was conducted by Goldberg and Mahmoud (2017).

3 Asset returns' model

3.1 The weighted-indexed semi-Markov Chain Model

Following D'Amico and Petroni (2011, 2012a, b) we use, in this paper, a Weighted-Indexed Semi-Markov Chain (hereafter "WISMC") to model the marginal distribution of asset returns. According to D'Amico and Petroni (2020), the dependency structure between asset returns is implemented by using a copula function. The WISMC model has been chosen because it is able to reproduce faithfully the long-term dependence of stock returns which is an important aspect when dealing with drawdown-based risk measures. Here we give just a short description of the model, for further details the reader can refer to the original papers where the model has been introduced D'Amico and Petroni (2020). If $X(t)$ is the price of an asset at time $t \in \mathbb{N}$, the associated log return is defined as $R(t) = \log(X(t)/X(t-1))$. Then, returns are discretized into a returns' series $\{J_n\}_{n\in\mathbb{N}}$ with the associated jump times $\{T_n\}_{n\in\mathbb{N}}$ of the asset returns. The index process is defined as follows:

$$V_n^\lambda = \sum_{k=0}^{n-1} \sum_{a=T_{n-1-k}}^{T_{n-k}-1} \left(\frac{\lambda^{T_n - a} J_{n-1-k}^2}{\sum_{a=1}^{T_n} \lambda^a} \right). \tag{3}$$

were λ is a memory parameter estimated from the data. With these definitions, the probability structure of the processes takes the form:

$$\begin{aligned} &\mathbb{P}\left[J_{n+1} = j, T_{n+1} - T_n \leq t | J_n, T_n, V_n^\lambda, J_{n-1}, T_{n-1}, V_{n-1}^\lambda, \ldots\right] \\ &= \mathbb{P}\left[J_{n+1} = j, T_{n+1} - T_n \leq t | J_n, V_n^\lambda\right] := Q_{J_n j}^\lambda \left(V_n^\lambda; t\right). \end{aligned} \tag{4}$$

Then, a copula function is used to build the joint distribution of asset returns (see at this purpose Joe (1997) and Nelsen (2006)). The marginal distributions is compatible with a WISMC process and the dependence structure among m assets is respected.

3.2 Traditional econometric models

To compare the WISMC model with traditional econometric models, we use the family of GARCH processes (see for example Engle 1982; Nelson 1991) with some variants of it such as EGARCH (exponential generalized autoregressive conditional heteroscedastic, see Tsay 2010) and GJR (Glosten, Jagannathan, and Runkle, see Glosten et al. 1993) processes. A more general framework was considered by Burda and Bélisle (2019) where the authors applied a C-MGARCH model with a dynamic copula. Parameter estimation can be easily carried out with the Matlab 'estimate' command. We will be able to identify, for each marginal, the best process through the AIC (Akaike's Information Criterion) minimization (see Ljung 1999). Once the appropriate process has been chosen, we can proceed to Monte Carlo simulations using this time the 'simulate' Matlab command. We note that the returns of the series used are correlated therefore we cannot independently simulate the individual series. We will have to use a multivariate model that takes into account not only the characteristics of the individual marginals but also their complex dependency structure. An efficient and well-established way in the literature to achieve the intended purpose is to make use of the copula functions.

4 Empirical analysis

4.1 Database

The dataset used in our application consists of ten assets that are part of the DJ index and were downloaded through Refinitiv. The 10 assets are chosen to represent the most important industry sectors. We consider a time horizon from February 10, 2022, to July 19, 2022 with a frequency equal to one minute. Regarding the calculation of returns, we have removed the opening and closing values for each trading day to avoid a discontinuity in the calculation of returns. We also reworked the dataset to match the trading days for each asset. At the end 41,730 data are obtained for the high frequency return for each asset; the main statistics are inserted in Table 1. We show in Fig. 1 the histogram of the price series.

Table 1 Main statistics of the assets returns with 1-min frequency

Stock	Mean	Std. dev.	Skewness	Kurtosis
UnitedHealth Group Inc	4.57E−06	7.75E−04	0.15	4.64
Johnson & Johnson	1.89E−06	5.78E−04	0.06	7.42
Walmart Inc	1.51E−06	6.64E−04	0.08	5.61
Verizon communications Inc	1.16E−06	6.03E−04	−0.57	27.89
McDonald's Corp	5.61E−07	6.04E−04	0.06	4.22
Nextera energy Inc	4.21E−07	7.97E−04	0.11	5.19
Duke energy corp	1.58E−06	6.64E−04	−0.01	13.36
Southern Co	3.00E−06	6.88E−04	0.09	12.58
Dominion energy Inc	7.12E−07	6.73E−04	0.08	12.13
Consolidated Edison Inc	2.50E−06	7.12E−04	0.12	14.08

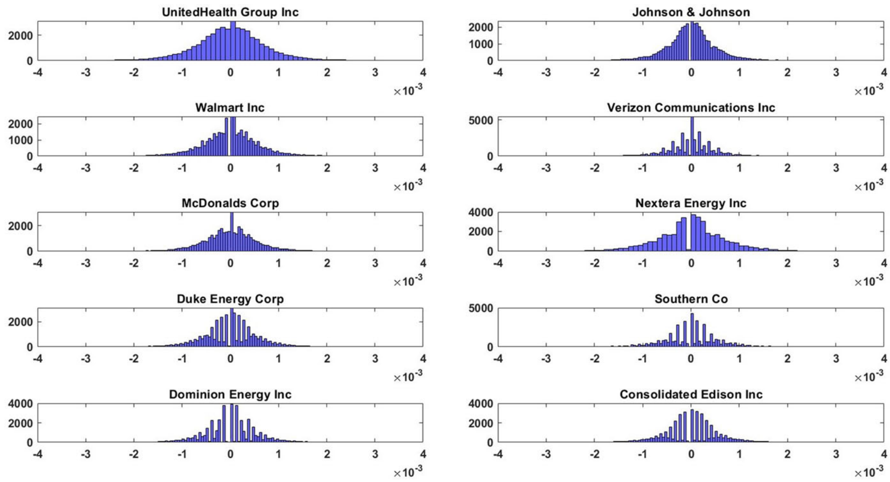

Fig. 1 Histograms of returns for each asset

The chosen asset are correlated to each other, the correlation matrix is shown below:

$$\rho = \begin{pmatrix} 1.00 & 0.48 & 0.39 & 0.34 & 0.42 & 0.39 & 0.37 & 0.38 & 0.38 & 0.35 \\ 0.48 & 1.00 & 0.43 & 0.41 & 0.38 & 0.37 & 0.45 & 0.47 & 0.46 & 0.43 \\ 0.39 & 0.43 & 1.00 & 0.36 & 0.42 & 0.37 & 0.38 & 0.38 & 0.40 & 0.36 \\ 0.34 & 0.41 & 0.36 & 1.00 & 0.35 & 0.34 & 0.43 & 0.45 & 0.44 & 0.42 \\ 0.42 & 0.38 & 0.42 & 0.35 & 1.00 & 0.38 & 0.34 & 0.35 & 0.36 & 0.33 \\ 0.39 & 0.37 & 0.37 & 0.34 & 0.38 & 1.00 & 0.49 & 0.51 & 0.51 & 0.46 \\ 0.37 & 0.45 & 0.38 & 0.43 & 0.34 & 0.49 & 1.00 & 0.77 & 0.74 & 0.72 \\ 0.38 & 0.47 & 0.38 & 0.45 & 0.35 & 0.51 & 0.77 & 1.00 & 0.75 & 0.72 \\ 0.38 & 0.46 & 0.40 & 0.44 & 0.36 & 0.51 & 0.74 & 0.75 & 1.00 & 0.72 \\ 0.35 & 0.43 & 0.36 & 0.42 & 0.33 & 0.46 & 0.72 & 0.72 & 0.72 & 1.00 \end{pmatrix}$$

From the descriptive statistics it can be observed that the asset returns do not follow Gaussian distributions. In fact, the kurtosis is always far from 3 that is the value for a Gaussian distribution.

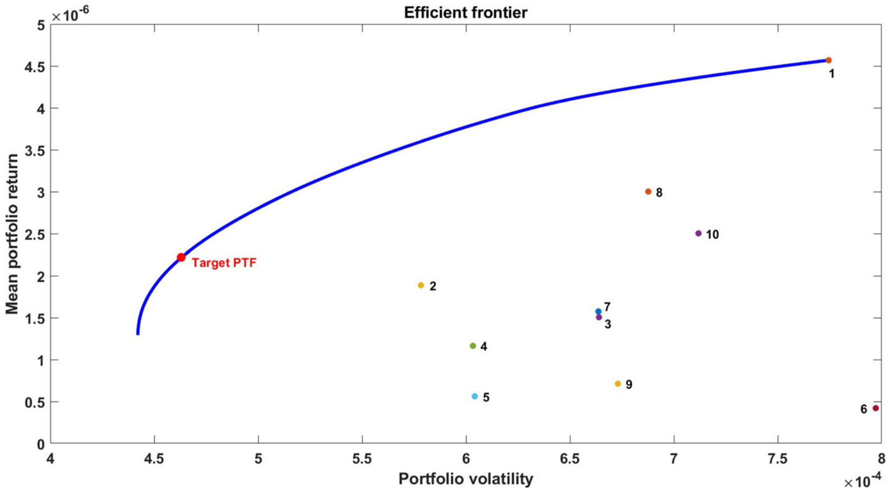

Fig. 2 Efficient frontier and target portfolio

To choose the target portfolio of our study, we constructed the classic efficient Markowitz frontier based on the criteria of mean and variance (without short sales) and arbitrarily selected an efficient portfolio. The efficient frontier and the chosen portfolio are shown in Fig. 2.

4.2 Parameters' estimation

We apply the multivariate WISMC model to the target portfolio described in the previous section. Following D'Amico and Petroni (2020) returns are discretized into five symmetrical states. Also, the index process V^λ is discretized into five states (low, medium-low, medium, medium-high and high volatility). Using the optimization procedure described in D'Amico and Petroni (2020) we found the optimal parameter $\lambda = 0.97$ for all stocks. We used a t-copula to reproduce the cross-correlation between the assets. The parameters of the copula are estimated by maximum likelihood estimators. Next, we apply Monte Carlo simulations to derive synthetic time series.

4.3 Censoring problem and divergence measure

The problem of censored data has already been highlighted in the univariate case in D'Amico et al. (2020). We briefly report below the description of this problem as it will be clearly addressed also in the multivariate case.

We observe that drawdown-based risk measures are right-censored as a consequence of the observation period. Specifically, the Type-1 right censoring occurs. This happens when an experiment, where a certain number of subjects or objects are observed, is stopped after reaching a given observation time limit. Subjects still alive at the given time limit (in our case, the number of minutes in a typical trading day) are called censored on the right. The problem, therefore, occurs in the estimation of the following risk measures: the drawdown of fixed level, the time to crash, and the recovery time. Furthermore, to determine the distributions with relative parameters that represent these risk measures, we will need to take into account that we are using right-censored data. In this regard, we use special algorithms present in Matlab (specifically, the *fitdist* command admits 'censoring' as option). Finally, to compare the relative distributions of risk measures for the different models used (i.e., to make a comparison between the WISMC model and the traditional models used as a benchmark), we use the Kullback–Leibler measure Kullback and Leibler (1951).

The Kullback–Leibler divergence of the distribution Q from the distribution P is denoted $D_{KL}(P||Q)$ and it represents the measure of the information lost when we approximate P with Q. It is defined as:

$$D_{KL}(P||Q) = \int_{-\infty}^{+\infty} p(x)\log_2\left(\frac{p(x)}{q(x)}\right) dx$$

where p and q represent the probability densities of P and Q respectively. Here, p and q denote respectively the simulated and the empirical distributions of the specified risk measures.

5 Results

We consider the target portfolio described before. After having determined the log returns for each asset, the prices of the assets are rescaled considering an initial value equal to 100. The initial value of the portfolio will therefore be equal to 100. Let's compare the characteristics of the empirical portfolio with the portfolio simulated with the WISMC model, as opposed to the classic multivariate model used as a benchmark, which foresees the marginals modelled with an $EGARCH(1, 1)$ process and linked by a t-copula.

We show in Fig. 3 the price, running maximum and Drawdown processes for the empirical series and a single simulation through the multivariate models (WISMC and econometric). From this figure, we can get an idea and a comparison of the price trend, running maximum and Drawdown process for the empirical series and those simulated with the various multivariate models.

5.1 Risk measures

(1) *The Drawdown of fixed level* $\tau(K)$ Descriptive statistics of Drawdown of fixed level τ and related censored units computed on real data as a function of K are given in Table 2. We observe that the average value for the drawdown of fixed level τ increases as the K threshold increases. This is a predictable result as extreme events take longer to occur. For example, we go from an average of 81.3646 min. for $K = 0.005$ to 108.6538 min. for $K = 0.0065$. The percentage of right-censored values clearly increases with the K threshold, in fact, more extreme events occur less frequently in

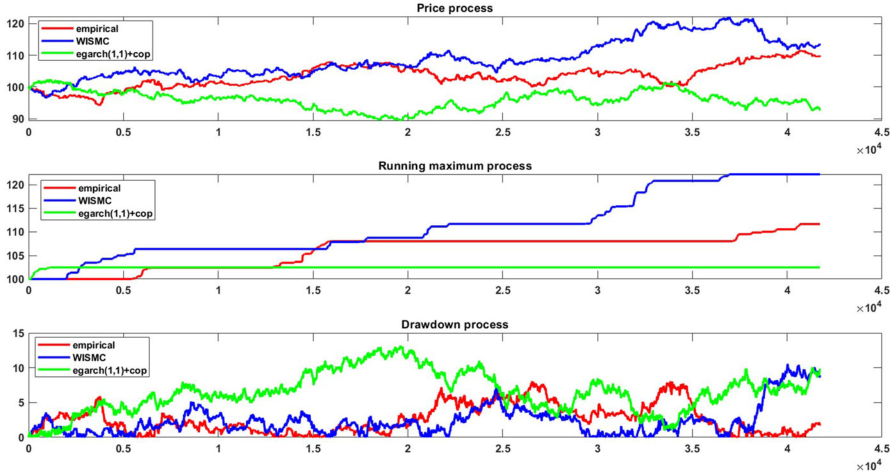

Fig. 3 Price process $X(t)$, Running maximum process $Y(t)$ and Drawdown process $D(t)$

Table 2 Drawdown of fixed level as a function of K

K	Mean	Std. dev.	Skew.	Kurtosis	Cens. (%)
0.0050	81.3646	100.3007	1.6135	4.3905	10.28
0.0060	104.1954	114.7291	1.2984	3.3540	18.69
0.0065	108.6538	113.8960	1.3287	3.5551	27.10

the chosen horizon. The percentage of right-censored data is 10.28% for $K = 0.005$ while it rises to 27.10% for $K = 0.0065$. (2) *The time to crash* $T_c(K)$ Descriptive statistics of the time to crash T_c and related censored units computed on real data as a function of K are given in Table 3. We deduce from the given table that the average value for the time to crash T_c increases as the K threshold increases. This is, here again, a predictable result as extreme events take longer to occur. For example, we go from an average of 37.3333 min. for $K = 0.005$ to 62.8590 min. for $K = 0.0065$. The percentage of right-censored values clearly increases with the K threshold, in fact, more extreme events occur less frequently in the chosen horizon. The percentage of right-censored data is 10.28% for $K = 0.005$ while it rises to 27.10% for $K = 0.0065$, exactly as in the case of the drawdown of fixed level. From the analysis of the skewness coefficient and kurtosis, it is evident that these distributions are far from Gaussianity therefore it is extremely important to determine the most appropriate distributions to represent these data.

The selection of the best parametric model for the measure T_c on both real and simulated data is performed by means of the AIC criterion, considering several K values. In this regard, we use Matlab 'fitdist' command which allows examining the right censored data across a set of traditional distributions. The results are given in Table 4 ('EV' is the Extreme Value distribution, 'Exp' is the exponential distribution). We note that the EV distribution has two parameters (which we have denoted 'param1'

Table 3 Time to crash as a function of K

K	Mean	Std. dev.	Skew.	Kurtosis	Cens. (%)
0.0050	37.3333	43.8302	2.2959	8.6336	10.28
0.0060	62.2529	77.8366	2.1218	7.0817	18.69
0.0065	62.8590	72.1079	2.0340	6.4715	27.10

Table 4 Best parametric model for the measure T_c

Model	AIC ($K = 0.005$)	Distrib.	Param. 1	Param. 2
Empirical	−1302.6211	EV	150.6920	171.3742
cop+garch(1,1)	−1301.1419	EV	98.9753	130.2638
cop+garch(2,2)	−1244.5315	EV	72.7516	95.4108
cop+egarch(1,1)	−940.1275	EV	40.8430	19.1214
cop+gjr(1,1)	−1244.5119	EV	74.6161	95.1342
WISMC	−1310.5669	EV	122.1279	143.8698
Model	AIC ($K = 0.006$)	Distrib.	Param. 1	Param. 2
Empirical	−1235.7997	EV	228.4803	194.7849
cop+garch(1,1)	−1302.7175	EV	136.9011	144.0285
cop+garch(2,2)	−1297.9397	EV	123.2429	131.1367
cop+egarch(1,1)	−1024.6219	EV	56.8701	28.9030
cop+gjr(1,1)	−1293.2421	EV	118.6123	123.8756
WISMC	−1286.9165	EV	176.7879	164.9346
Model	AIC ($K = 0.0065$)	Distrib.	Param. 1	Param. 2
Empirical	−1149.0292	EV	282.2619	214.0872
cop+garch(1,1)	−1270.8293	EV	187.3219	168.6682
cop+garch(2,2)	−1293.8433	EV	131.0815	128.0311
cop+egarch(1,1)	−1243.5072	EV	83.8218	93.5161
cop+gjr(1,1)	−1290.8825	EV	164.3094	153.8274
WISMC	−1228.4062	EV	232.0630	187.0452

and ‘param2’) while the exponential distribution has only one parameter (denoted ‘param1’).

We find evidence that for the selected values of K ($K = 0.005$, $K = 0.006$, and $K = 0.0065$), both for the empirical data and for the data simulated with the model WISMC and with classical econometric models, the optimal distribution is always the ‘Extreme Value’.

The Kullback–Leibler divergence computed for the risk measure T_c, considering different levels of K, is given in Table 5.

We have indicated in bold, for each value of the K threshold, the minimum value of the Kullback–Leibler measure. We note that for all the selected values of the K threshold (i.e., $K = 0.005$, $K = 0.006$ and $K = 0.0065$), the WISMC model is more

Table 5 Kullback–Leibler indicator for the measure T_c

Model	$K = 0.005$	$K = 0.006$	$K = 0.0065$
cop+garch(1,1)	0.1176	0.2001	0.1490
cop+garch(2,2)	0.4191	0.3013	0.5392
cop+egarch(1,1)	8.2224	6.8541	1.3778
cop+gjr(1,1)	0.4112	0.3627	0.2555
WISMC	0.0374	0.0272	0.0415

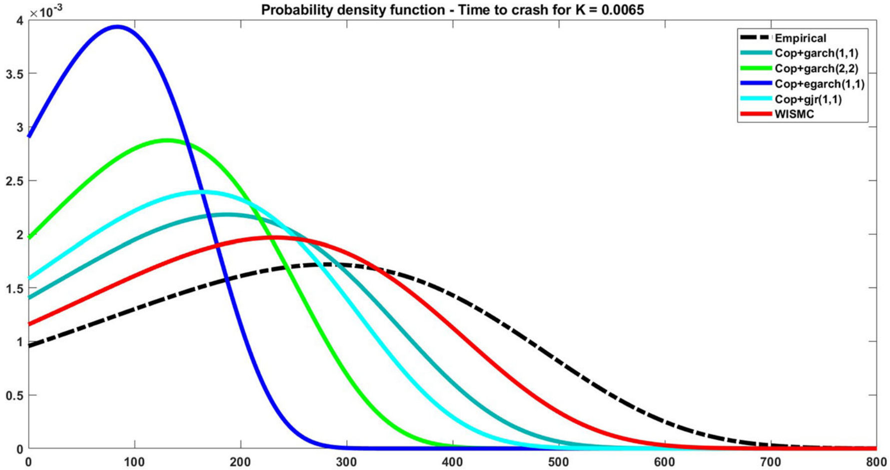

Fig. 4 Probability density function for T_c with $K = 0.0065$

suitable and therefore preferable in the case of situations with greater risk (denoted with a high value of the K threshold). We represent in Fig. 4 the probability density functions for the various models in the case $K = 0.0065$. In this case, we highlight an optimal fitting between the WISMC model and empirical values.

(3) *The speed of crash* $S_c(K)$ The Descriptive statistics of the speed of crash S_c and related censored units computed on real data as a function of K are given in Table 6.

We deduce from this table that the average value for the speed of crash S_c decreases as the K threshold increases. This is a predictable result as assets reach high thresholds slower than low thresholds. For example, we go from an average of 0.000381 for $K = 0.005$ to 0.000264 for $K = 0.0065$. The percentage of right-censored values follows the same profile as in previous cases as we always have a single K threshold to reach.

(4) *The recovery time* $R_t(K, K')$ Descriptive statistics of the recovery time R_t and related censored units for real data as a function of K and K' are given in Table 7.

Compared to the previous risk measures, this time we must consider the two thresholds to be reached called K and K'. We highlight that the mean value for the recovery time R_t depends on both thresholds K and K'. In these applications, we have chosen the threshold pairs $K = 0.005$ & $K' = 0.0045$, $K = 0.006$ & $K' = 0.0055$, and

Table 6 Speed of crash as a function of K

K	Mean	Std. dev.	Skew.	Kurtosis	Cens. (%)
0.0050	0.000381	0.000415	2.287602	9.728872	10.28
0.0060	0.000326	0.000356	1.737450	5.547645	18.69
0.0065	0.000264	0.000242	1.468253	4.697550	27.10

Table 7 Recovery time as a function of K and K'

K–K' pair	Mean	Std. dev.	Skew.	Kurtosis	Cens. (%)
0.005−0.0045	42.3976	76.3987	2.4490	8.3439	22.43
0.006−0.0055	37.3600	70.3535	2.9842	11.7699	29.91
0.0065−0.00625	26.5385	60.5808	3.4080	14.4554	39.25

$K = 0.0065$ & $K' = 0.00625$. We also deduce that the percentage of right-censored values is higher with respect to the time to crash T_c because both thresholds K and K' must be attained.

The selection of the best parametric model for the measure R_t on both real and simulated data by mean of AIC criterion, fixing K and K' is given in Table 8.

We observe that for the selected values of K and K' ($K = 0.005$ & $K' = 0.0045$, $K = 0.006$ & $K' = 0.0055$, and $K = 0.0065$ & $K' = 0.00625$), both for the empirical data and for the data simulated with the model WISMC and with classical econometric models, the optimal distribution is always the 'Extreme Value' distribution.

The Kullback–Leibler divergence computed for the risk measures R_t is given in Table 9.

We have indicated in bold, for each pair of values of the K and K' thresholds, the minimum value of the Kullback–Leibler measure. We note that for all pairs of thresholds $K = 0.005$ & $K' = 0.0045$, $K = 0.006$ & $K' = 0.0055$, and $K = 0.0065$ & $K' = 0.00625$ the WISMC model is more suitable. Indeed, the WISMC is preferable in the case of situations with high risk (denoted with high values of the K threshold). We represent in Fig. 5 the probability density functions for the various models in the case $K = 0.0065$ and $K' = 0.00625$.

(5) *The speed of recovery* $S_r(K, K')$ The descriptive statistics of the speed of recovery S_r and related censored units for real data as a function of K and K' are given in Table 10.

We observe that the mean value for the speed of recovery S_r is rather stable. We also note that the percentage of right-censored values is higher with respect to the time to crash T_c because both thresholds K and K' must be attained, exactly as for the recovery time R_t (we actually used the same K and K' thresholds).

The Selection of the best parametric model for the measure S_r on both real and simulated data by mean of AIC criterion, fixing K and K' is given in Table 11.

We highlight that for the selected values of K and K' ($K = 0.005$ & $K' = 0.0045$, $K = 0.006$ & $K' = 0.0055$, and $K = 0.0065$ & $K' = 0.00625$), both for the empirical data and for the data simulated with the WISMC model and with classical econometric

Table 8 Best parametric model for the measure R_t

Model	AIC ($K = 0.005$; $K' = 0.0045$)	Distrib.	Param. 1	Param. 2
Empirical	−1325.6255	EV	218.9804	187.8786
cop+garch(1,1)	−1367.9196	EV	141.5697	171.0066
cop+garch(2,2)	−1387.9332	EV	147.2492	163.4690
cop+egarch(1,1)	−1382.5227	EV	125.5154	156.7407
cop+gjr(1,1)	−1375.9840	EV	124.6280	158.5580
WISMC	−1366.3313	EV	183.6025	181.4620
Model	AIC ($K = 0.006$; $K' = 0.0055$)	Distrib.	Param. 1	Param. 2
Empirical	−1245.5752	EV	261.8739	205.7650
cop+garch(1,1)	−1358.5374	EV	181.5568	183.8243
cop+garch(2,2)	−1366.3014	EV	156.2543	175.7412
cop+egarch(1,1)	−1382.3260	EV	124.5572	15.6872
cop+gjr(1,1)	−1369.8892	EV	149.9985	171.5238
WISMC	−1321.7333	EV	239.6906	193.5962
Model	AIC ($K = 0.0065$; $K' = 0.00625$)	Distrib.	Param. 1	Param. 2
Empirical	−1154.0111	EV	313.6571	221.8711
cop+garch(1,1)	−1305.7149	EV	205.0306	202.0902
cop+garch(2,2)	−1363.6874	EV	150.0867	174.2163
cop+egarch(1,1)	−1377.9828	EV	120.7965	160.3426
cop+gjr(1,1)	−1325.5283	EV	154.6649	183.0795
WISMC	−1247.7591	EV	270.9300	208.3139

Table 9 Kullback–Leibler indicator for the measure R_t

Model	$K = 0.005$ $K' = 0.0045$	$K = 0.006$ $K' = 0.0055$	$K = 0.0065$ $K' = 0.00625$
cop+garch(1,1)	0.1169	0.1002	0.1329
cop+garch(2,2)	0.1075	0.1681	0.3322
cop+egarch(1,1)	0.1755	38.6607	0.5007
cop+gjr(1,1)	0.1753	0.1918	0.2923
WISMC	0.0345	0.0027	0.0069

Table 10 Speed of recovery as a function of K and K'

K–K' pair	Mean	Std. dev.	Skew.	Kurtosis	Cens.
0.005−0.0045	0.0000778	0.0000735	1.0053	3.1207	22.43%
0.006−0.0055	0.0000746	0.0000710	1.0393	3.1815	29.91%
0.0065−0.00625	0.0000538	0.0000429	0.5067	1.9148	39.25%

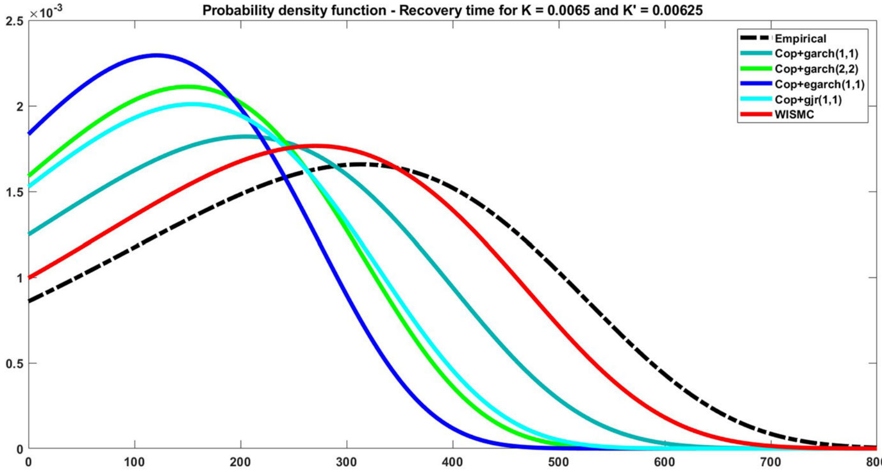

Fig. 5 Probability density function for R_t with $K = 0.002$ & $K' = 0.0015$

Table 11 Best parametric model for the measure S_r

Model	AIC ($K = 0.005$; $K' = 0.0045$)	Distrib.	Param. 1	Param. 2
Empirical	−1339.5401	EV	191.2040	204.6934
cop+garch(1,1)	−1377.0100	EV	126.2270	179.6442
cop+garch(2,2)	−1395.1800	EV	117.8461	172.8009
cop+egarch(1,1)	−1392.7414	EV	105.2921	166.0346
cop+gjr(1,1)	−1385.9192	EV	106.0475	167.4439
WISMC	−1377.5219	EV	165.2634	193.1554
Model	AIC ($K = 0.006$; $K' = 0.0055$)	Distrib.	Param. 1	Param. 2
Empirical	−1258.6551	EV	243.2014	223.1514
cop+garch(1,1)	−1369.2965	EV	166.4749	194.8876
cop+garch(2,2)	−1375.3702	EV	138.3117	185.1247
cop+egarch(1,1)	−1392.7414	EV	105.2921	166.0346
cop+gjr(1,1)	−1380.1930	EV	131.8173	181.6805
WISMC	1332.9637	EV	228.0861	206.0922
Model	AIC ($K = 0.0065$; $K' = 0.00625$)	Distrib.	Param. 1	Param. 2
Empirical	−1163.1222	EV	305.1799	235.8772
cop+garch(1,1)	−1313.2006	EV	196.0062	210.7437
cop+garch(2,2)	−1372.6038	EV	132.8293	183.3052
cop+egarch(1,1)	−1385.9192	EV	106.0475	167.4440
cop+gjr(1,1)	−1333.8220	EV	138.6158	192.1594
WISMC	−1256.3923	EV	263.3080	219.3289

Table 12 Kullback–Leibler indicator for the measure S_r

Model	$K = 0.005$ $K' = 0.0045$	$K = 0.006$ $K' = 0.0055$	$K = 0.0065$ $K' = 0.00625$
cop+garch(1,1)	0.0960	0.0957	0.1319
cop+garch(2,2)	0.1208	0.1692	0.3462
cop+egarch(1,1)	0.1629	0.3112	0.5081
cop+gjr(1,1)	0.1587	0.1916	0.3036
WISMC	0.0235	0.0031	0.0247

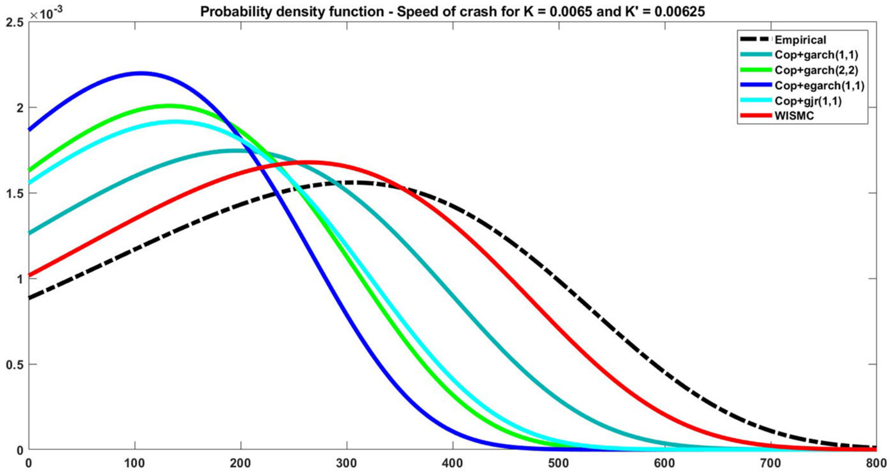

Fig. 6 Probability density function for S_r with $K = 0.0065$ & $K' = 0.00625$

models, the optimal distribution is always the 'Extreme Value' distribution as we have also deduced for the previous risk measures, with no exceptions.

Finally, the Kullback–Leibler divergence computed for the risk measures S_r is given in Table 12.

We have indicated in bold, for each pair of values of the K and K' thresholds, the minimum value of the Kullback–Leibler measure. We note that for all the given pairs of thresholds, the WISMC model is always the more suitable one. We represent in Fig. 6 the probability density functions for the various models in the case $K = 0.0065$ and $K' = 0.00625$.

Overall, through the analysis of the Kullback–Leibler measure, the multivariate WISMC model is on the whole the most suitable for representing the drawdown-based risk measures for an asset portfolio.

(6) The **Maximum Drawdown at Risk** and the **Conditional Expected Drawdown** Below, in Tables 13, 14, 15, 16 and 17, the results of the DaR_α and $CDaR_\alpha$ are reported with confidence levels of 95%, 97.5% and 99% and with a time horizon of one to 5 days.

The empirical results of the tail risk measures are compared with the results obtained with the WISMC model and with the classic econometric models. From these results,

Table 13 DaR_α and $CDaR_\alpha$ for 1 day horizon

Measure	Empirical (%)	WISMC (%)	egarch(1,1) (%)	gjr(1,1) (%)	garch(1,1) (%)	garch(2,2) (%)
$DaR_{0.95}$	0.55	0.69	0.57	0.43	0.47	0.52
$DaR_{0.975}$	0.57	0.76	0.59	0.45	0.54	0.56
$DaR_{0.99}$	0.83	0.83	0.62	0.49	0.59	0.61
$CDaR_{0.95}$	0.67	0.77	0.62	0.47	0.54	0.59
$CDaR_{0.975}$	0.83	0.82	0.65	0.49	0.60	0.62
$CDaR_{0.99}$	0.85	0.86	0.72	0.54	0.63	0.69

Table 14 DaR_α and $CDaR_\alpha$ for 2 days horizon

Measure	Empirical (%)	WISMC (%)	egarch(1,1) (%)	gjr(1,1) (%)	garch(1,1) (%)	garch(2,2) (%)
$DaR_{0.95}$	0.57	0.78	0.62	0.49	0.54	0.57
$DaR_{0.975}$	0.83	0.84	0.66	0.53	0.59	0.68
$DaR_{0.99}$	0.85	0.87	0.75	0.54	0.67	0.69
$CDaR_{0.95}$	0.85	0.84	0.72	0.52	0.61	0.66
$CDaR_{0.975}$	0.89	0.87	0.75	0.55	0.66	0.70
$CDaR_{0.99}$	0.93	0.87	0.75	0.56	0.70	0.76

Table 15 DaR_{α} and $CDaR_{\alpha}$ for 3 days horizon

Measure	Empirical (%)	WISMC (%)	egarch(1,1) (%)	gjr(1,1) (%)	garch(1,1) (%)	garch(2,2) (%)
$DaR_{0.95}$	0.83	0.84	0.66	0.53	0.59	0.61
$DaR_{0.975}$	0.85	0.87	0.75	0.54	0.63	0.69
$DaR_{0.99}$	0.93	0.87	0.75	0.56	0.70	0.76
$CDaR_{0.95}$	0.90	0.87	0.74	0.55	0.68	0.71
$CDaR_{0.975}$	0.93	0.88	0.75	0.56	0.70	0.75
$CDaR_{0.99}$	0.93	0.88	0.75	0.56	0.70	0.76

Table 16 DaR_{α} and $CDaR_{\alpha}$ for 4 days horizon

Measure	Empirical (%)	WISMC (%)	egarch(1,1) (%)	gjr(1,1) (%)	garch(1,1) (%)	garch(2,2) (%)
$DaR_{0.95}$	0.83	0.87	0.75	0.54	0.59	0.68
$DaR_{0.975}$	0.93	0.87	0.75	0.56	0.70	0.72
$DaR_{0.99}$	0.93	0.88	0.75	0.56	0.70	0.76
$CDaR_{0.95}$	0.91	0.92	0.75	0.56	0.68	0.73
$CDaR_{0.975}$	0.93	0.92	0.75	0.56	0.70	0.76
$CDaR_{0.99}$	0.93	1.03	0.75	0.56	0.70	0.76

Table 17 DaR_α and $CDaR_\alpha$ for 5 days horizon

Measure	Empirical (%)	WISMC (%)	egarch(1,1) (%)	gjr(1,1) (%)	garch(1,1) (%)	garch(2,2) (%)
$DaR_{0.95}$	0.83	0.87	0.75	0.55	0.59	0.72
$DaR_{0.975}$	0.93	0.88	0.75	0.56	0.70	0.76
$DaR_{0.99}$	0.93	1.03	0.75	0.56	0.70	0.76
$CDaR_{0.95}$	0.91	0.94	0.79	0.57	0.69	0.76
$CDaR_{0.975}$	0.93	1.01	0.79	0.60	0.70	0.76
$CDaR_{0.99}$	0.93	1.03	0.79	0.60	0.70	0.76

Table 18 Backtesting results for 1-day horizon: number and percentage of exceptions

α	WISMC	egarch(1,1)	gjr(1,1)	garch(1,1)	garch(2,2)
95%	14 (2.8%)	22 (4.4%)	163 (32.6%)	119 (23.8%)	57 (11.4%)
97.5%	14 (2.8%)	14 (2.8%)	117 (23.4%)	96 (19.2%)	22 (4.4%)
99%	12 (2.4%)	14 (2.8%)	80 (16.0%)	95 (19.0%)	14 (2.8%)

it is clear that the estimates obtained with the WISMC model more faithfully reproduce the empirical results. Regarding the benchmark models, the copula-egarch(1,1) model is generally the best while the more traditional copula-gjr(1,1) model gives the worst results.

To verify the correctness of the DaR_α estimate, we set up a backtesting analysis as in the typical case of Value-at-Risk. For this purpose, we estimate the DaR_α in a time window of 30 days and compare it with the Maximum Drawdown actually achieved and deduced from the empirical data. An exception occurs whenever the empirical Maximum Drawdown exceeds the DaR_α value. So let's consider a rolling window with a forward shift of a certain amount. At the end of the process, we deduce the percentage of exceptions, which should ideally be equal to $1 - \alpha$ where α represents the chosen confidence threshold. We report the results in Table 18 for a 1-day DaR_α. We perform a total of 500 backtesting with confidence thresholds of 95%, 97.5% and 99%.

We note that for the higher threshold of 99%, the WISMC model has a more reliable number of exceptions than the classic models. Regarding the estimation of the DaR, the WISMC model is therefore more reliable than the benchmark models for the higher thresholds, which are of greater interest from a risk management perspective.

6 Discussion

In this paper, we presented an empirical analysis with the joint use of the multivariate WISMC model for asset returns and drawdown-based risk measures. In particular, we investigate several drawdown-based risk measures useful in managing market crises (namely, the drawdown of fixed level, the time to crash, the speed of crash, the recovery time, and the speed of recovery). We applied these risk measures to an equity portfolio with high-frequency data, included in the Dow Jones and representative of the most important industrial sectors. This target portfolio was selected amongst the efficient portfolios built with available assets. We modelled the returns of our portfolio through a multivariate WISMC model and a Student copula from which we deduced synthetic series through Monte Carlo simulations.

Our elaborations show that the risk measures estimated through our multivariate WISMC model turned out, in general, to be more reliable than the analogous measures estimated through a classical model used as a reference benchmark. Indeed, the distributions used to represent the drawdown-based risk measures turned out to be more similar to their respective empirical distributions in the case of the WISMC model. Similarly, drawdown-based tail risk measures such as DaR_α and $CDaR_\alpha$ are estimated more effectively in the case of the WISMC model, especially with high confidence thresholds, as we could also observe from the backtesting analyzes.

Declarations

Conflict of interest The authors declare that they have no conflict of interest and that they did not receive support from any organization for the submitted work.

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