

Solving generic parametric linear matrix inequalities

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ABSTRACT

We consider linear matrix inequalities (LMIs) $A = A_0 + x_1 A_1 + \dots + x_n A_n \geq 0$ with the A_i 's being $m \times m$ symmetric matrices, with entries in a ring $\mathcal{R}$. When $\mathcal{R} = \mathbb{R}$, the feasibility problem consists in deciding whether the x_i 's can be instantiated to obtain a positive semi-definite matrix. When $\mathcal{R} = \mathbb{Q}[y_1, \dots, y_t]$, the problem asks for a formula on the parameters $y_1, \dots, y_t$, which describes the values of the parameters for which the specialized LMI is feasible. This problem can be solved using general quantifier elimination algorithms, with a complexity that is exponential in n . In this work, we leverage the LMI structure of the problem to design an algorithm that computes a formula Φ describing a dense subset of the feasible region of parameters, under genericity assumptions. The complexity of this algorithm is exponential in n, m and t but becomes polynomial in n when m and t are fixed. We apply the algorithm to a parametric sum-of-squares problem and to the convergence analyses of certain first-order optimization methods, which are both known to be equivalent to the feasibility of certain parametric LMIs, hence demonstrating its practical interest.

CCS CONCEPTS

• Computing methodologies → Algebraic algorithms.

KEYWORDS

Linear matrix inequality; Quantifier elimination; Optimization

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1 INTRODUCTION

Problem statement. For a set $\mathcal{E}$ and $m \in \mathbb{N}$, we denote by $\mathbb{S}_m(\mathcal{E})$ the set of $m \times m$ symmetric matrices with entries in $\mathcal{E}$. For n and t in $\mathbb{N}$, we consider sequences of variables $\mathbf{x} = (x_1, \dots, x_n)$ and parameters $\mathbf{y} = (y_1, \dots, y_t)$ and the ring $\tilde{\mathcal{R}} = \mathbb{Q}[\mathbf{y}][\mathbf{x}]$. The set of polynomials in $\tilde{\mathcal{R}}$ of degree at most 1 in $\mathbf{x}$ is denoted by $\tilde{\mathcal{R}}_{\leq 1}$.

A matrix $A = A_0 + x_1 A_1 + \dots + x_n A_n$ in $\mathbb{S}_m(\tilde{\mathcal{R}}_{\leq 1})$ is said to be a *parametric linear matrix*. For any $\mathbf{y} \in \mathbb{R}^t$, we denote by $A_{\mathbf{y}}$ the matrix obtained by specializing the parameters $\mathbf{y}$ to $\mathbf{y}$. The linear matrix inequality $A_{\mathbf{y}} \geq 0$ defines the spectrahedron $\text{spec}(A_{\mathbf{y}}) \subset \mathbb{R}^n$, i.e., the set of points $\mathbf{x}$ in $\mathbb{R}^n$ such that $A_{\mathbf{y}}(\mathbf{x})$ is positive semi-definite. It is well-known that $\text{spec}(A_{\mathbf{y}})$ is a convex semi-algebraic set, defined by $g_i(\mathbf{y}, \cdot) \geq 0$, where $g_i \in \tilde{\mathcal{R}}$ is the coefficient of λ^i in $\det(A + \lambda I_m)$ (see, e.g., [16, Sec. 5.2]).

PROBLEM 1 (GENERIC FEASIBILITY PROBLEM). *Given a parametric linear matrix $A \in \mathbb{S}_m(\tilde{\mathcal{R}}_{\leq 1})$, compute a semi-algebraic formula Φ in $\mathbb{Q}[\mathbf{y}]$, such that Φ defines a dense subset of the set of feasible parameters $\mathcal{P} := \{\mathbf{y} \in \mathbb{R}^t \mid \exists \mathbf{x} \in \mathbb{R}^n : A(\mathbf{y}, \mathbf{x}) \geq 0\}$.*

The problem is a natural generalization of the decisional feasibility problem for LMIs. Indeed, once the semi-algebraic formula Φ is computed, one can decide whether a given parameter $\mathbf{y}$ is in $\mathcal{P}$ by checking whether $\Phi(\mathbf{y})$ is true.

Prior works. In another perspective, the problem can be seen as a quantifier elimination (QE) problem, which consists of eliminating the unknowns from the LMI.

Quantifier elimination over the reals has a long history starting with Tarski's algorithm [30] with a non elementary recursive complexity. Collins' cylindrical algebraic decomposition algorithm [9] is the first practical algorithm. Its complexity is doubly exponential in the total number of unknowns.

It should be noted that, here, we have a quantifier elimination problem with a single block of variables to eliminate. The idea to exploit such a block structure to improve the complexity of quantifier elimination originates from [14] and culminates in [3] (see [4, Sec. 14.6] for more bibliographic notes). This yields algorithms which are doubly exponential in the number of alternates of quantifiers and exponential in the total number of unknowns. Still, putting into practice such algorithms is an open problem.

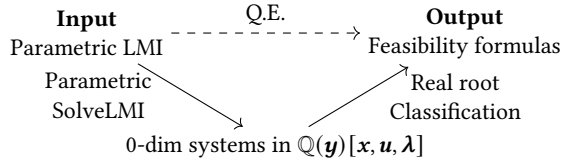
A weakened variant for one-block quantifier elimination is proposed in [20] but is restricted to systems of polynomial equations which enjoy regularity properties and do not involve inequalities.

Our study goes back to [17] which yields an *exact* algorithm for solving linear matrix inequalities, combining its principles with those of [20].

Contributions. We present an algorithm solving Problem 1, which can be seen as a generalization of the results in [16] to the parametric setting. The algorithm first computes a finite set of polynomial systems $(f_i)_i$ in $\tilde{\mathcal{R}}[\mathbf{u}, \boldsymbol{\lambda}]$, where $\mathbf{u}$ and $\boldsymbol{\lambda}$ are auxiliary variables, inspired by the subroutines in [16], used to solve *exactly* linear matrix inequalities. The polynomial systems $(f_i)_i$ satisfy the following



specialization property, which we prove. For generic values of the parameters $y \in \mathbb{R}^t$, each $f_i(y, \cdot) = 0$ has a finite number of complex solutions, and the non-emptiness of $\text{spec}(A_y)$ is determined by the existence of at least one solution to a certain $f_i(y, \cdot) = 0 \wedge g(y, \cdot) \geq 0$ where $g = (g_0, \dots, g_m)$ are defined above. To extract the formula Φ characterizing this condition, the algorithm applies to each relation $f_i = 0 \wedge g \geq 0$ the real root counting method described in [13], based on Hermite's quadratic form [18].



Note that for an input f_i , the algorithm in [13] is based on Gröbner bases computations, and, consequently, extra genericity assumptions on the input are required to control the complexity.

By contrast, in our case, the ideal generated by the input f_i turns out to be generically radical, but does not satisfy the genericity assumptions used in [13], because of the special structure of the f_i 's. Still, we show how to control the complexity in this case, replacing Gröbner bases computations with the computation of parametric geometric resolutions [27] of the input f_i . We obtain the following complexity result.

THEOREM 1.1. *There exists an algorithm which, given a generic parametric linear matrix $A \in \mathbb{S}_m(\tilde{R}_{\leq 1})$ of any fixed maximum total degree d as input, outputs a semi-algebraic formula Φ in $\mathbb{Q}[y]$ that solves Problem 1 for the input A , in*

$$2^{O(mt)} n^{O(1)} (md)^{O(t)} (\delta^* \Delta^*)^{O(t)} \quad (1)$$

arithmetic operations in $\mathbb{Q}$, where

$$\delta^* \in n^{O(m^2)} \quad \text{and} \quad \Delta^* \in e^{O(m^2 \log m)} n^{O(1)} d^{O(m^2+t)} t^{O(1)}.$$

Observe that for fixed m and t , the above complexity is polynomial in d and n , which is useful for the applications we have in mind (see below).

To obtain this result, we rely on techniques, based on the critical point method for quantifier elimination, introduced in [20]. These are restricted to the case of systems of polynomial constraints involving equations only. We use reductions to such a situation introduced in [16], but extending them to the parametric setting. As already sketched above, we are then bound to real root classification problems, which we solve by combining Hermite's quadratic forms as in [13]. However, the techniques used in [13] to control the complexity do not apply to our situation anymore because of the specific structure of the f_i 's. Instead, we show how to leverage this specific structure to obtain degree bounds on algebraic sets defined by the f_i 's (the quantities δ^* and Δ^* are maxima of these bounds), so that one can control adequately the cost of the calls to the parametric geometric resolution algorithm [27] in replacement of Gröbner bases in the real root classification algorithm.

We also report on practical experiments, with a first implementation of our algorithm. Tests include topical examples of parametric LMIs which are used to encode convergence properties of first-order

optimization algorithms [10, 31, 32]. This yields a new application area for computer algebra. As a matter of fact, these parametric LMIs appear to have a moderate size m and number of parameters t which justify our approach. These experiments show that our algorithm is able to tackle such problems which are out of reach of the state-of-the-art implementations of real quantifier elimination.

Plan of the paper. Section 2 presents the basic notions of determinantal and incidence varieties which are used to define the f_i 's. Section 3 presents the algorithm in [16, Sec. 3.3] for solving generic LMIs. Section 4 describes our algorithm for solving generic parametric LMIs, whose correctness is proven in Section 5, and whose complexity is analyzed in Section 6. Section 7 reports on practical experiments.

2 PRELIMINARIES

Throughout the paper, we denote by $\tilde{R} = \mathbb{Q}[y][x]$, and $R = \mathbb{Q}[x]$, as a special case of $\tilde{R}$ with $t = 0$.

2.1 Basic notions

Critical points. Let $\mathcal{V} \subset \mathbb{C}^n$ be a smooth and equidimensional algebraic set, and $\phi : \mathbb{C}^n \rightarrow \mathbb{C}^i$ be a regular map, with $i \in \llbracket 1, n \rrbracket$ (where $\llbracket 1, n \rrbracket := \{1, \dots, n\}$). We denote by $\text{crit}(\phi, \mathcal{V})$ the set of critical points of the restriction of ϕ to $\mathcal{V}$, i.e., the points where the Jacobian matrix $\text{jac}(f, \phi)$ of (f, ϕ) has rank $\leq n - \dim(\mathcal{V})$, where $f \subset R$ is a polynomial system generating the ideal $I(\mathcal{V})$ associated to $\mathcal{V}$ (see, e.g., [25, Sec. 3.1]).

Properness. Let E be a topological space, $\mathcal{V} \subset E^n$, and $\phi : \mathcal{V} \rightarrow \mathbb{C}^i$ be a map, with $i \in \llbracket 1, n \rrbracket$. We say that ϕ is proper if for all $y \in \phi(\mathcal{V})$, there exists a closed neighborhood $\mathcal{O}_y$ of y such that $\phi^{-1}(\overline{\mathcal{O}_y})$ is closed and bounded, where $\overline{\mathcal{O}_y}$ denotes the closure of $\mathcal{O}_y$.

Change of variables. Let F be a field of characteristic 0, and $M \in \text{GL}_n(\mathbb{Q})$ be an invertible matrix. For any polynomial $p \in F[x]$, we denote by p^M the polynomial satisfying $p^M(x) = p(Mx)$. Analogously, for any set $\mathcal{E} \subset F^n$, we denote by $\mathcal{E}^M := \{M^{-1}x \mid x \in \mathcal{E}\}$.

2.2 Regularity and auxiliary result

Let $\mathcal{V}$ be a locally closed set of $\mathbb{C}^n$. We say that $\mathcal{V}$ satisfies P (resp. P(d)) if the Zariski closure of $\mathcal{V}$ is either empty or smooth and equidimensional (resp. of dimension d).

We say that a polynomial system $f \subset \tilde{R}$ satisfies assumption A (resp. A(d)) if f generates a radical ideal, whose associated algebraic set $V(f)$ satisfies P (resp. P(d)).

The following lemma then serves as an equidimensional version of the Theorem on the Dimension of Fibres [28, Th. 1.25].

LEMMA 2.1. *Let $f \subset \mathbb{Q}[y, x]$ be a finite set of polynomials satisfying A(d), with $d \geq t$. Denote by $d' = d - t$. Then, there exists a non-empty Zariski open set $\mathcal{Y} \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}$, the specialization $f(y, \cdot)$ satisfies A(d').*

PROOF. If $V(f)$ is empty, it generates the ideal $\langle 1 \rangle$, as does $f(y, \cdot)$ for all $y \in \mathbb{C}^t$, and the lemma holds. We assume now $V(f) \neq \emptyset$.

If $\pi_y(V(f))$ is contained in a proper Zariski closed subset of $\mathbb{C}^t$, we take $\mathcal{Y}$ to be the complement of this subset. Then, for all $y \in \mathcal{Y}$, $V(f(y, \cdot))$ is empty, so the lemma also holds trivially.

Hence, assume that $\pi_{\mathbf{y}}(V(f))$ is Zariski dense in $\mathbb{C}^t$. By the Theorem on the Dimension of Fibres [28, Th. 1.25], there exists a non-empty Zariski open set $\mathcal{Y}'$ of $\mathbb{C}^t$ such that for all $y \in \mathcal{Y}'$, $\pi_{\mathbf{y}}^{-1}(y) \cap V(f)$ is an algebraic set of dimension d' .

Since f satisfies $A(d)$, $\text{crit}(\pi_{\mathbf{y}}, V(f))$ is defined by the vanishing of f and the $(n - d')$ -minors of $\text{jac}(f, \pi_{\mathbf{y}})$ [26, Lem. A.2]. By Sard's theorem [26, Prop. B.2], $\pi_{\mathbf{y}}(\text{crit}(\pi_{\mathbf{y}}, V(f)))$ is contained in a proper Zariski closed subset $Z \subset \mathbb{C}^t$. Let $\mathcal{Y} = (\mathbb{C}^t \setminus Z) \cap \mathcal{Y}'$. Then, for all $y \in \mathcal{Y}$, $x \in \pi_{\mathbf{x}}^{-1}(y) \cap V(f)$, $\text{jac}(f)(y, x)$ has rank $n - d'$. By the Jacobian criterion [11, Th. 16.19], $f(y, \cdot)$ satisfies $A(d')$. $\square$

2.3 Determinantal and incidence varieties

Let $A \in \mathbb{S}_m(\tilde{R}_{\leq 1})$ be a parametric linear matrix, and $r \in \llbracket 0, m - 1 \rrbracket$. The determinantal variety $\mathcal{D}_r \subset \mathbb{C}^{t+n}$ of A of rank r is defined as

$$\mathcal{D}_r := \{(y, x) \in \mathbb{C}^{t+n} \mid \text{rank}(A(y, x)) \leq r\}.$$

A desingularization of $\mathcal{D}_r$ in the sense of Room-Kempf [2] is given as follows: let $U = (u_{i,j})_{i \in \llbracket 1, m \rrbracket, j \in \llbracket 1, m-r \rrbracket}$ be a matrix of variables. For all $\iota \subset \llbracket 1, m \rrbracket$ with $|\iota| = m - r$, define the incidence variety $\mathcal{V}_{r,\iota} \subset \mathbb{C}^{t+n+m(m-r)}$ of A as

$$\mathcal{V}_{r,\iota} := \{(y, x, u) \in \mathbb{C}^{t+n+m(m-r)} \mid A(y, x) \cdot U = 0, U_{\iota} = \text{Id}_{m-r}\},$$

where U_{ι} is formed by the rows of U indexed by ι .

Following [16, Lem. 3.2], we construct a defining system $f_{r,\iota} \subset \tilde{R}[\mathbf{u}]$ for $\mathcal{V}_{r,\iota}$ as follows: let $G = (g_{i,j})_{i \in \llbracket 1, m \rrbracket, j \in \llbracket 1, m-r \rrbracket}$ the matrix obtained by substituting the entries of $U_{\iota} = \text{Id}_{m-r}$ into $A \cdot U$, so that G is independent of $u_{i,j}$'s for $i \in \iota$. Then, $f_{r,\iota}$ is defined by the polynomials in $(g_{i,j})_{i \notin \iota}$ and $U_{\iota} - \text{Id}_{m-r}$.

Properties of determinantal and incidence varieties. The dimension of $\mathcal{D}_r$ and its fibres $\mathcal{D}_r(y, \cdot)$ are classical results.

THEOREM 2.2 ([1, PROP. 3.2]). *For all $d \in \mathbb{N}$, there exists a non-empty Zariski open set $\mathcal{A}_d \subset \mathbb{S}_m(\mathbb{C}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$ such that for all $A \in \mathcal{A}_d \cap \mathbb{S}_m(\mathbb{Q}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$ and $r \in \llbracket 0, m - 1 \rrbracket$,*

- (1) $\mathcal{D}_r$ is empty or of dimension $t + n - \binom{m-r+1}{2}$;
- (2) In the case where $t \geq 1$, there exists a non-empty Zariski open set $\mathcal{Y}_A \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_A \cap \mathbb{Q}^t$, $\mathcal{D}_r(y, \cdot)$ is either empty or of dimension $n - \binom{m-r+1}{2}$.

The dimension of $\mathcal{V}_{r,\iota}$ and its fibres $\mathcal{V}_{r,\iota}(y, \cdot)$, as well as their regularity, are given by the following lemma, which is a generalization of [16, Prop. 3.4]. Its proof is given in Section 5.

LEMMA 2.3. *For all $d \in \mathbb{N}$, there exists a non-empty Zariski open set $\mathcal{A}_d \subset \mathbb{S}_m(\mathbb{C}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$ such that for all $A \in \mathcal{A}_d \cap \mathbb{S}_m(\mathbb{Q}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$, $r \in \llbracket 0, m - 1 \rrbracket$, $\iota \subset \llbracket 1, m \rrbracket$ with $|\iota| = m - r$,*

- (1) $f_{r,\iota}$ satisfies $P(t + n - \binom{m-r+1}{2})$;
- (2) when $t \geq 1$, there exists a non-empty Zariski open set $\mathcal{Y}_A \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_A \cap \mathbb{Q}^t$, $f_{r,\iota}(y, \cdot)$ satisfies $P(n - \binom{m-r+1}{2})$.

3 ALGORITHMS FOR LMIS

3.1 Principles

Let $A \in \mathbb{S}_m(R_{\leq 1})$. Algorithm from [16, Sec. 3.3] allows to decide whether the LMI $A \geq 0$ is feasible. For the ease of the reader, we recall in the following [16, Th. 1.2], which leverages the relation between the boundary structure of $\mathcal{S} := \text{spec}(A)$ and the rank stratification of A :

Let $A \in \mathbb{S}_m(R_{\leq 1})$ be a linear matrix, $r_{\min} = \min_{x \in \mathcal{S}} \text{rank}(A(x))$, and suppose that $\partial \mathcal{S} \neq \emptyset$. Let C be a connected component of $\mathcal{D}_{r_{\min}} \cap \mathbb{R}^n$ such that $C \cap \mathcal{S} \neq \emptyset$. Then $C \subset \mathcal{S}$.

Consequently, solving the LMI $A \geq 0$ boils down to:

- (1) Pick $x \in \mathbb{Q}^n$, and decide if $x \in \mathcal{S} \setminus \partial \mathcal{S}$. If yes, return true;
- (2) Otherwise, $\mathcal{S} \neq \mathbb{R}^n$, so that $\mathcal{S} \neq \emptyset$ if and only if $\partial \mathcal{S} \neq \emptyset$. We compute at least one point per connected component of $\mathcal{D}_r \cap \mathbb{R}^n$, for all $r \in \llbracket 0, m - 1 \rrbracket$, and then decide if the points intersect $\mathcal{S}$. If yes, return true, otherwise return false.

3.2 Critical point methods

For all $i \in \llbracket 1, n \rrbracket$, we denote by $\pi_i : \mathbb{C}^n \rightarrow \mathbb{C}^i$ the projection $x \mapsto x_{[1,i]} := (x_1, \dots, x_i)$, and by abuse of notation $\pi_0 : x \mapsto \bullet$. We start by recalling [25, Th. 2]. Let $f \in R$ and $V = V(f)$.

Assume that f satisfies $A(d)$. There exists a non-empty Zariski open set $\mathcal{M} \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M} \cap \text{GL}_n(\mathbb{Q})$,

- (1) *The restriction of π_{i-1} to $\text{crit}(\pi_i, \mathcal{V}^M)$ is proper;*
- (2) *For all $i \in \llbracket 1, d + 1 \rrbracket$, $\text{crit}(\pi_i, \mathcal{V}^M)$ satisfies $P(i - 1)$;*
- (3) *For all $\tau = (\tau_1, \dots, \tau_d) \in \mathbb{R}^d$, the set*

$$\bigcup_{i=1}^{d+1} (\text{crit}(\pi_i, \mathcal{V}^M) \cap \pi_{i-1}^{-1}(\tau_{[1,i-1]}))$$

is finite and meets all the connected components of $\mathcal{V}^M \cap \mathbb{R}^n$.

We use [12, Sec. 5.2] to derive Lagrange polynomial systems defining $\text{crit}(\pi_i, V(f^M))$.

LEMMA 3.1 ([12, SEC. 5.2]). *Assume that f satisfies $A(d)$. There exists a non-empty Zariski open set $\mathcal{M} \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M} \cap \text{GL}_n(\mathbb{Q})$, it holds that*

- (4) *There exists a non-empty Zariski open set $\mathcal{T}_M \subset \mathbb{C}^n$ such that for all $\tau \in \mathcal{T}_M \cap \mathbb{Q}^n$, $i \in \llbracket 1, d + 1 \rrbracket$,*

$$\text{crit}(\pi_i, V(f^M)) \cap \pi_{i-1}^{-1}(\tau_{[1,i-1]}) = \pi_{\mathbf{x}}(\text{Lag}(f^M(\tau_{[1,i-1]}, \cdot))),$$

where $\text{Lag}(f) = f \cup (\lambda^T \text{jac}(f) - (1, 0, \dots, 0))$ is the Lagrange system associated with f , with λ a vector of variables.

We denote by POLARVARIETIES an algorithm which takes as input polynomials $f = (f_1, \dots, f_s)$ and returns the Lagrange systems defined in Lemma 3.1. Under the assumptions of [25, Th. 2], their solutions meet all connected components of $V(f) \cap \mathbb{R}^n$.

3.3 Bridging the regularity gap

Let $A \in \mathbb{S}_m(R_{\leq 1})$ be a linear matrix. As $\mathcal{D}_r$ is not smooth in general, we cannot apply POLARVARIETIES directly to its defining system, but only to its desingularizations $\mathcal{V}_{r,\iota}$, where $\iota \subset \llbracket 1, m \rrbracket$ with $|\iota| = m - r$.

To compute at least one point per connected component of $\mathcal{D}_r \cap \mathbb{R}^n$ outside $\mathcal{D}_{r-1}$, [16, Prop. 3.5, 3.6] show that it suffices to compute critical points on the incidence varieties $\mathcal{V}_{r,\iota}$, or the points given by the Lagrange systems of $f_{r,\iota}$. We restate the result as follows:

PROPOSITION 3.2 ([16, PROP. 3.5 AND 3.6]). *Assume that $\mathcal{V}_{r,\iota}$ satisfies $P(n - \binom{m-r+1}{2})$. Then, there exists a non-empty Zariski open set $\mathcal{M} \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M} \cap \text{GL}_n(\mathbb{Q})$, for any connected component $C \subset \mathcal{D}_r \cap \mathbb{R}^n$, the following holds:*

- (1) *For all $i \in \llbracket 1, d \rrbracket$, $\pi_i(C^M)$ is closed; furthermore, for all $\tau \in \mathbb{R}$ lying on the boundary of $\pi_1(C^M)$, $\pi_1^{-1}(\tau) \cap C^M$ is finite.*

- (2) Let τ in the boundary of $\pi_1(C^M)$. Then, for all $x \in \pi_1^{-1}(\tau) \cap C^M$ such that $x \in \mathcal{D}_r^M \setminus \mathcal{D}_{r-1}^M$, there exists $u \in \mathbb{R}^{m(m-r)}$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m - r$, such that $(x, u) \in \text{crit}(\pi_1, \mathcal{V}_{r,\iota}^M)$.
- (3) The saturation of the ideal generated by $\text{Lag}(f_{r,\iota}^M)$ by an ideal associated with $\mathcal{D}_{r-1}^M$ is radical, and the locally closed set $\mathcal{E}_{r,\iota}^M := V(\text{Lag}(f_{r,\iota}^M)) \cap \pi_{\mathbf{x}}^{-1}(\mathcal{D}_r^M \setminus \mathcal{D}_{r-1}^M)$ satisfies $P(\binom{m-r}{2})$.
- (4) $\pi_{\mathbf{x}}(\mathcal{E}_{r,\iota}^M)$ is either empty or finite.
- (5) Let $(x, u) \in \text{crit}(\pi_1, \mathcal{V}_{r,\iota}^M)$ such that $x \in \mathcal{D}_r^M \setminus \mathcal{D}_{r-1}^M$. Then, there exists $\lambda \in \mathbb{R}^{m(m-r) + \binom{m-r+1}{2}}$ such that $(x, u, \lambda) \in \mathcal{E}_{r,\iota}^M$.

We denote by **REALDET** an algorithm which takes as input $A \in \mathbb{S}_m(R_{\leq 1})$, generates for all $r \in \llbracket 0, m-1 \rrbracket$ and $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m - r$ the systems $f_{r,\iota}$ describing the aforementioned incidence varieties and returns $(f_{r,\iota})_{i \in \llbracket 1, d+1 \rrbracket}$, the output of **POLARVARIETIES**, when called on these systems with their associated rank r . The algorithm therefore consists purely of ring operations over R .

The routine **PSDMATRIXCOND** simply takes as input A and returns the coefficients of $\det(A + \lambda I_m)$ in R (where λ is a new variable).

Finally, the algorithm **SOLVE LMI** takes as input $A \in \mathbb{S}_m(R_{\leq 1})$ and calls **REALDET** and **PSDMATRIXCOND**, to decide the existence of solutions to the systems returned by **REALDET** outside the appropriate determinantal varieties $\mathcal{D}_{r-1}$ and which make non-negative the polynomials output from **PSDMATRIXCOND**.

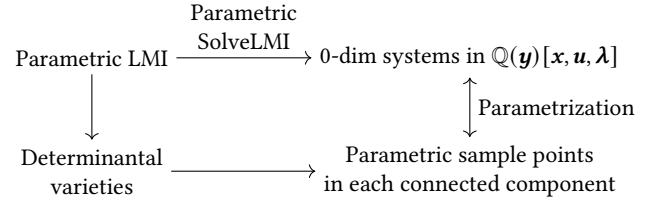
4 ALGORITHMS FOR PARAMETRIC LMIS

4.1 Principles

We consider now a parametric linear matrix $A \in \mathbb{S}_m(\tilde{R}_{\leq 1})$. We have remarked that all operations performed by **REALDET** are ring operations over the coefficients of the entries of A . Hence, it can be used with entries in $\tilde{R}$. Hence, it returns a set of polynomial systems F in $\tilde{R}[\mathbf{u}, \boldsymbol{\lambda}]$, which, specialized at any parameter $y \in \mathbb{Q}^t$, gives us the same result if we pass the linear matrix $A(y, \cdot) \in \mathbb{S}_m(R_{\leq 1})$ to the algorithm. Furthermore, we have the expected properties for generic choices of y , as stated in the following proposition.

PROPOSITION 4.1. *For all $d \in \mathbb{N}$, there exists a non-empty Zariski open set $\mathcal{A}_d \subset \mathbb{S}_m(\mathbb{C}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$ such that for all $A \in \mathcal{A}_d \cap \mathbb{S}_m(\mathbb{Q}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$, there exists a non-empty Zariski open set $\mathcal{M}_A \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M}_A \cap \text{GL}_n(\mathbb{Q})$, there exists a non-empty Zariski open set $\mathcal{T}_{A,M} \subset \mathbb{C}^n$ such that for all $\tau \in \mathcal{T}_{A,M} \cap \mathbb{Q}^n$, there exists a non-empty Zariski open set $\mathcal{Y}_{A,M,\tau} \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_{A,M,\tau} \cap \mathbb{Q}^t$, $r \in \llbracket 0, m-1 \rrbracket$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m - r$, $\bigcup_{r,\iota} \pi_{\mathbf{x}}(V(f_{r,\iota}^M(y, \cdot)))$ is finite and contains at least one point per connected component of each $\mathcal{D}_r^M(y, \cdot)$.*

The proof of the proposition is technical and is given in Section 5. Let us instead focus on the consequence of the proposition. Intuitively, the polynomial systems $f \in F$ are zero-dimensional in $\mathbb{Q}(\mathbf{y})[\mathbf{x}, \mathbf{u}, \boldsymbol{\lambda}]$ for generic choices of A .



Our goal then becomes to determine semi-algebraic formulas on $y \in \mathbb{R}^t$ such that $f(y, \cdot) = 0$ and $g(y, \cdot) \geq 0$ has a finite number of solutions. Assume that we have the following subroutine:

- **CLASSIFICATION.** Input: $f = (f_1, \dots, f_p) \subset \tilde{R}[\mathbf{u}, \boldsymbol{\lambda}]$ such that $\dim(\langle f \rangle_{\mathbb{Q}(\mathbf{y})}) = 0$, $g = (g_1, \dots, g_s) \subset \tilde{R}$. Output: a list of tuples of the form (Φ_i, y_i, r_i) , with Φ_i a semi-algebraic formula in $\mathbb{Q}[\mathbf{y}]$ defining $\mathcal{T}_i \subset \mathbb{R}^t$, $y_i \in \mathcal{T}_i$, $r_i \in \mathbb{N}$, such that
 - For all $y \in \mathcal{T}_i$, $f(y, \cdot) = 0 \wedge g(y, \cdot) \geq 0$ has r_i real solutions,
 - The union of $\mathcal{T}_i$ is dense in $\mathbb{R}^t$.

Then, it suffices to take the disjunction of Φ_i for which $r_i > 0$ to obtain the desired semi-algebraic formulas. We deduce Algorithm 1 for solving the generic feasibility problem for parametric LMIs.

Algorithm 1 PARAMETRICESOLVE LMI

Input: $A \in \mathbb{S}_m(\tilde{R}_{\leq 1})$

Output: A semi-algebraic formula Φ in $\mathbb{Q}[\mathbf{y}]$ defining a dense subset of the set $\{y \in \mathbb{R}^t \mid A(y, \cdot) \geq 0\}$.

```

1: choose  $M \in \text{GL}_n(\mathbb{Q})$ 
2: choose  $\tau \in \mathbb{Q}^n$ 
3:  $F^M \leftarrow \text{REALDET}(A, M, \tau)$ 
4:  $g^M \leftarrow \text{CHANGEVARS}(\text{PSDMATRIXCOND}(A), M)$ 
5:  $\Phi \leftarrow \text{PSDMATRIXCOND}(A(\cdot, 0)) > 0$ 
6: for all  $f_{r,\iota}^M \in F^M$  do
7:    $\Phi_{r,\iota,i} \leftarrow \text{CLASSIFICATION}(f_{r,\iota}^M, g^M)$ 
8:    $\Phi \leftarrow \Phi \vee \bigvee_{j:r_j>0} \Phi_{r,\iota,i,j}$ 
9: end for
10: return  $\Phi$ 

```

Proof of correctness of Algorithm 1.

THEOREM 4.2. *Let $A \in \mathbb{S}_m(\tilde{R}_{\leq 1})$ be a parametric linear matrix. Assume that A and all parameters chosen in Algorithm 1 satisfy the assumptions of Proposition 4.1. Then, Algorithm 1 is correct.*

PROOF. For any $f_{r,\iota}^M \in F^M$ given by Line 3 of Algorithm 1, let $\mathcal{Y}_{r,\iota,i} \subset \mathbb{C}^t$ be the Zariski open set in Proposition 4.1 such that for all $y \in \mathcal{Y}_{r,\iota,i} \cap \mathbb{Q}^t$, $\Phi_{r,\iota,i}(y)$ if and only if there exists an $x \in \pi_{\mathbf{x}}(V(f_{r,\iota}^M(y, \cdot)))$ such that $g^M(y, x) \geq 0$. Let $\mathcal{Y} = \bigcap_{r,\iota,i} \mathcal{Y}_{r,\iota,i}$, so that for all $y \in \mathcal{Y}$, $\Phi(y)$ if and only if either $A(y, 0) \geq 0$, or there exists an $x^M \in \bigcup_{r,\iota,i} \pi_{\mathbf{x}}(V(f_{r,\iota}^M(y, \cdot)))$ such that $g^M(y, x) \geq 0$, i.e., **SOLVE LMI** returns true with input $A(y, \cdot)$. Finally, for all $y \in \mathcal{Y} \cap \mathcal{Y}_{A,M,\tau}$ defined in Proposition 4.1, the above condition is equivalent to $\text{spec}(A(y, \cdot)) \neq \emptyset$, hence the correctness of Algorithm 1. $\square$

Finally, we explain how to implement the **CLASSIFICATION** subroutine. We first take [13, Algorithm 2], which deals with the condition $f(y, \cdot) = 0 \wedge g(y, \cdot) > 0$. Then, in Line 26, we change the definition of r_y , from the entry corresponding to $(1, \dots, 1) \in \{0, 1, -1\}^s$,

to the sum of entries corresponding to $\{0, 1\}^s$. Then, r_y is the number of real solutions such that $\text{sign}(g(y, \cdot)) \in \{0, 1\}^s$.

4.2 Hermite's quadratic form

Let $\mathbb{K}$ be a base field of characteristic 0, and $f \in \mathbb{K}[x]$ be a polynomial system generating a zero-dimensional ideal $\langle f \rangle_{\mathbb{K}}$. By [4, Th. 4.86], the vector space $\mathcal{A}_{\mathbb{K}} := \mathbb{K}[x]/\langle f \rangle_{\mathbb{K}}$ has finite dimension, denoted δ . For all $g \in \mathcal{A}_{\mathbb{K}}$, the Hermite's bilinear map is defined by

$$\begin{aligned} \text{herm}(f, g) : \mathcal{A}_{\mathbb{K}} \times \mathcal{A}_{\mathbb{K}} &\rightarrow \mathbb{K} \\ (p, q) &\mapsto \text{Tr}(L_{gpq}), \end{aligned}$$

where L_{gpq} is the multiplication map $h \mapsto gpqh$ on $\mathcal{A}_{\mathbb{K}}$, and Tr denotes the trace. The associated quadratic form $\text{Herm}(f, g) : p \mapsto \text{herm}(f, g)(p, p)$ is then called Hermite's quadratic form.

Hermite's theorem [4, Th. 4.102]. When $\mathbb{K} \subset \mathbb{R}$, for all $g \in \mathbb{K}[x]$, denote by $\text{TaQ}(g, f)$ the Tarski-query of g for f , defined as $\text{TaQ}(g, f) := |\{x \in V(f) \mid g(x) > 0\}| - |\{x \in V(f) \mid g(x) < 0\}|$.

The following identities hold:

- $\text{rank}(\text{Herm}(f, g)) = |\{x \in V(f) \mid g(x) \neq 0\}|$.
- $\text{Sign}(\text{Herm}(f, g)) = \text{TaQ}(g, f)$,

where $\text{Sign}(\Phi)$ denotes the signature of the quadratic form Φ .

Hermite matrices. Let $\mathcal{B} = (b_1, \dots, b_{\delta})$ be a basis of $\mathcal{A}_{\mathbb{K}}$. For all $g \in \mathcal{A}_{\mathbb{K}}$, the Hermite matrix associated with g with respect to $\mathcal{B}$ is then defined as the matrix $\mathcal{H}_g = (h_{i,j})_{i,j \in [1, \delta]} \in \mathbb{S}_{\delta}(\mathbb{K})$, where $h_{i,j} = \text{herm}(f, g)(b_i, b_j) = \text{Tr}(L_{gb_i b_j})$.

To compute $h_{i,j}$, let $M_{gb_i b_j} = (m_{k,\ell})_{k,\ell \in [1, \delta]}$ be the multiplication matrix of $L_{gb_i b_j}$ with respect to $\mathcal{B}$. By definition, $m_{k,\ell}$ is the coefficient of b_{ℓ} in the representative of $gb_i b_j b_k$ in $\mathcal{A}_{\mathbb{K}}$, which can be obtained by a normal form computation if f is given by a Gröbner basis. We can then proceed by $h_{i,j} = \text{Tr}(M_{gb_i b_j}) = \sum_{k \in [1, \delta]} m_{k,k}$.

Specialization property of parametric Hermite matrices. Let $\mathbb{K} = \mathbb{Q}(\mathbf{y})$, $f \in \mathbb{K}[x]$ generating a zero-dimensional ideal $\langle f \rangle_{\mathbb{K}}$, and $g \in \mathbb{K}[x]$. Let $>_{\mathbf{x}}$ and $>_{\mathbf{y}}$ be monomial orders for $\mathbf{x}$ and $\mathbf{y}$, respectively, and $>$ be an elimination order which eliminates $\mathbf{x}$, built with $>_{\mathbf{x}}$ and $>_{\mathbf{y}}$. Let $\mathcal{G}$ be the reduced Gröbner basis of $\langle f \rangle$ with respect to $>$, and $\mathcal{B}$ be the monomial basis formed by the monomials of $\mathbb{K}[x]$ irreducible by $\mathcal{G}$.

The following generalizes [13, Prop. 3.1], by dropping the assumption that $>_{\mathbf{x}}$ and $>_{\mathbf{y}}$ are graded reverse lexicographic orders.

THEOREM 4.3. *Let $\mathcal{W}_{\infty} := \bigcup_{y \in \mathcal{G}} V(\text{lc}_{\mathbf{x}}(y))$, where $\text{lc}_{\mathbf{x}}(y)$ is the leading coefficient of the polynomial $y \in \mathbb{K}[x]$ with respect to $>_{\mathbf{x}}$. Then, for all $y \in \mathbb{C}^t \setminus \mathcal{W}_{\infty}$,*

- (1) *The specialization $\mathcal{G}(y, \cdot)$ is a Gröbner basis of $\langle f(y, \cdot) \rangle$ with respect to $>_{\mathbf{x}}$.*
- (2) *Consequently, $\text{Herm}(f, g)(y, \cdot) = \text{Herm}(f(y, \cdot), g(y, \cdot))$.*

PROOF. For $y \in \mathbb{C}^t \setminus \mathcal{W}_{\infty}$, $\text{lc}_{\mathbf{x}}(g)$ does not vanish at y for every $g \in \mathcal{G}$. By [19, Th. 3.1], $\mathcal{G}(y, \cdot)$ is then a Gröbner basis of $\langle f(y, \cdot) \rangle$ with respect to $>_{\mathbf{x}}$. We then conclude by the same proof as in [21, Prop. 10]. $\square$

4.3 Real root counting

Let $\mathbb{K} = \mathbb{R}$, $f \in \mathbb{R}[x]$ be a polynomial system, and $g \in \mathbb{R}[x]$. Denote by $c(\sigma, f) := |\{x \in V(f) \mid \text{sign}(g(x)) = \sigma\}|$, we have

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c(0, f) \\ c(1, f) \\ c(-1, f) \end{bmatrix} = \begin{bmatrix} \text{TaQ}(g^0, f) \\ \text{TaQ}(g^1, f) \\ \text{TaQ}(g^2, f) \end{bmatrix}.$$

The identity can be generalized to a tensor identity, in the case where $\mathbf{g} = (g_1, \dots, g_s) \in \mathbb{R}[x]$. Denote by $\text{SIGN}(\mathbf{g}, f) \subset \{0, 1, -1\}^s$ the set of realizable signs of $\mathbf{g}$ on $V(f)$.

THEOREM 4.4 ([4, PROP. 10.70]). *Let $A = (\alpha_1, \dots, \alpha_{|A|}) \subset \{0, 1, 2\}^s$, $\Sigma = (\sigma_1, \dots, \sigma_{|\Sigma|}) \subset \{0, 1, -1\}^s$, each in the lexicographic order. Suppose that $\Sigma \supset \text{SIGN}(\mathbf{g}, f)$. Then,*

$$\text{Mat}(A, \Sigma) \cdot c(\Sigma, f) = \text{TaQ}(\mathbf{g}^A, f),$$

where

- $\text{Mat}(A, \Sigma) = (\sigma_j^{\alpha_i})_{i \in [1, |A|], j \in [1, |\Sigma|]}$,
- $c(\Sigma, f) = (c(\sigma_j, f))_{j \in [1, |\Sigma|]}$,
- $\text{TaQ}(\mathbf{g}^A, f) = (\text{TaQ}(g^{\alpha_i}, f))_{i \in [1, |A|]}$.

A trivial choice of Σ is $\Sigma = \{0, 1, -1\}^s$, which gives c simply by inverting the square matrix $\text{Mat}(A, \Sigma)$. By summing the coefficients corresponding to $\sigma \in \{0, 1\}^s$, we obtain $a_1, \dots, a_{|A|} \in \mathbb{Q}$ such that

$$|\{x \in V(f) \mid g(x) \geq 0\}| = \sum_{i=1}^{|A|} a_i \text{Sign}(\text{Herm}(\mathbf{g}^{\alpha_i}, f)). \quad (2)$$

It suffices to extract a semi-algebraic formula corresponding to the positivity of the linear combination of signatures. For this, we have several options:

- (O1) We can directly return the positivity assertion (2), which is itself a semi-algebraic formula.
- (O2) Let $\mathcal{S} \subset \mathbb{R}^t$ be the semi-algebraic set defined by the non-vanishing of the non-zero principal minor of maximal size of all Hermite matrices. Then, on each connected component of $\mathcal{S}$, the signatures of the Hermite matrices are constant. Thus, we can then return the *realizable* combinations of signatures after computing one point per connected component of $\mathcal{S}$ and examining the signatures on these points.
- (O3) Furthermore, the signature of a Hermite matrix is generically determined by the sign variation of its leading principal minors (see, e.g., [13, Lem. 4.8]). If we add the condition of their non-vanishing in the definition of $\mathcal{S}$, we can refine the formula into a combination of signs of the leading principal minors of the Hermite matrices.

When $\mathbb{K} = \mathbb{R}(\mathbf{y})$, by Theorem 4.3, Equation (2) holds for generic specializations of $y \in \mathbb{R}^t$, if we further intersect $\mathcal{S}$ with $\mathbb{R}^t \setminus \mathcal{W}_{\infty}$. This leads us to the use of [13, Algorithm 2] which performs real root classification for parametric zero-dimensional systems using Hermite matrices, combined with the following subroutine:

- **SAMPLEPOINTS.** Input: a semi-algebraic formula Φ in $\mathbb{Q}[\mathbf{y}]$. Output: a finite set of points in $\mathbb{R}^t$ containing at least one point per connected component of the semi-algebraic set defined by Φ . We refer to the algorithm of [21] for an algorithm having such a specification (see also references therein).

5 PROOF OF GENERICITY

This section is dedicated to the proof of Proposition 4.1, which gives us the last piece of the puzzle to prove the correctness of Algorithm 1. The main difficulty is to show that the choices of $M \in \text{GL}_n(\mathbb{Q})$ and $\tau \in \mathbb{Q}^n$ in Proposition 4.1 do not depend on the choice of parameters $y \in \mathbb{R}^t$. The proof requires a careful analysis of all the propositions and lemmas presented in the previous sections.

5.1 Expected dimension of incidence varieties

PROOF OF LEMMA 2.3 (1). For all $d \in \mathbb{N}$, denote by n_d the dimension of $\mathbb{Q}[\mathbf{y}]_{\leq d}$. Observe that, as a vector space, $\mathbb{S}_m(\mathbb{Q}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$ is isomorphic to $\mathbb{S}_m^{(n+1)n_d}(\mathbb{Q})$, so that we can rewrite A as

$$A(y, x) = \sum_{\substack{k \in \mathbb{N}^t, |k| \leq d \\ \ell \in \mathbb{N}^n, |\ell| \leq 1}} A_{k,\ell} y^k x^\ell,$$

where $A_{k,\ell} = (a_{k,\ell,i,j})_{i,j \in \llbracket 1, m \rrbracket} \in \mathbb{S}_m(\mathbb{Q})$. Denote then, by abuse of notation, $A \in \mathbb{S}_m^{(n+1)n_d}(\mathbb{Q})$ the parametric linear matrix.

Suppose w.l.o.g. that r is fixed, $\iota \in \llbracket 1, m-r \rrbracket$. Consider the map

$$\varphi : \mathbb{C}^{t+n+m(m-r)} \times \mathbb{S}_m^{(n+1)n_d}(\mathbb{C}) \rightarrow \mathbb{C}^{m(m-r)+\binom{m-r+1}{2}}, \\ (y, x, u, A) \mapsto f_{r,\iota},$$

and, for $A \in \mathbb{S}_m^{(n+1)n_d}(\mathbb{Q})$, its induced map $\varphi_A : \mathbb{C}^{t+n+m(m-r)} \rightarrow \mathbb{C}^{m(m-r)+\binom{m-r+1}{2}}$ defined by $\varphi_A(y, x, u) = \varphi(y, x, u, A) = f_{r,\iota}$, so that $\varphi_A^{-1}(0) = V(f_{r,\iota}) = \mathcal{V}_{r,\iota}$.

If $\varphi^{-1}(0) = \emptyset$, for all $A \in \mathbb{S}_m^{(n+1)n_d}(\mathbb{Q})$, $\varphi_A^{-1}(0)$ is empty and we are done. Otherwise, assume that $\varphi^{-1}(0) \neq \emptyset$. Note that for all $(y, x, u, A) \in \varphi^{-1}(0)$, $\text{jac}(\varphi)(y, x, u, A)$ has full rank, given by the derivatives w.r.t. $(a_{0,0,i,j})_{i,j \in \llbracket 1, m-r \rrbracket \vee j \in \llbracket 1, m-r \rrbracket}$ of the entries of G , and, the derivatives w.r.t. $(u_{i,j})_{i \in \llbracket 1, m-r \rrbracket}$ of the entries of $U_i - \text{Id}_{m-r}$. Hence, 0 is a regular value of φ . By Thom's weak transversality theorem [26, Prop. B.3], there exists a non-empty Zariski open set $\mathcal{A}_d \subset \mathbb{S}_m^{(n+1)n_d}(\mathbb{C})$ such that for all $A \in \mathcal{A}_d \cap \mathbb{S}_m^{(n+1)n_d}(\mathbb{Q})$, 0 is a regular value of φ_A . By the Jacobian criterion [11, Th. 16.19], $f_{r,\iota}$ satisfies $P(t+n-\binom{m-r+1}{2})$. $\square$

PROOF OF LEMMA 2.3 (2). As $f_{r,\iota}$ satisfies $P(t+n-\binom{m-r+1}{2})$, by Lemma 2.1, there exists a non-empty Zariski open set $\mathcal{Y} \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y} \cap \mathbb{Q}^t$, $f_{r,\iota}(y, \cdot)$ satisfies $P(n-\binom{m-r+1}{2})$. $\square$

We then generalize Lemma 2.3 (2), so that the choice of $\tau \in \mathbb{Q}^n$ in REALDET becomes independent of the choice of $y \in \mathbb{Q}^t$ in the following proposition.

PROPOSITION 5.1. *Under the assumptions of Lemma 2.3, the following additional result holds:*

- (3) *In the case where $t \geq 1$, there exists a non-empty Zariski open set $\mathcal{T} \subset \mathbb{C}^n$ such that for all $\tau \in \mathcal{T} \cap \mathbb{Q}^n$, there exists a non-empty Zariski open set $\mathcal{Y}_\tau \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_\tau \cap \mathbb{Q}^t$, for all $i \in \llbracket 1, n+1 \rrbracket$, $f_{r,\iota}(y, \tau_{\llbracket 1, i-1 \rrbracket}, \cdot)$ satisfies $P(n-\binom{m-r+1}{2}-i)$.*

PROOF. As $f_{r,\iota}$ satisfies $P(t+n-\binom{m-r+1}{2})$, by Lemma 2.1, for all $i \in \llbracket 1, n+1 \rrbracket$, there exists a non-empty Zariski open set $\mathcal{T}_i \subset \mathbb{C}^n$ such that for all $\tau \in \mathcal{T}_i \cap \mathbb{Q}^n$, there exists a non-empty Zariski open set $\mathcal{Y}_{i,\tau} \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_{i,\tau} \cap \mathbb{Q}^t$, $f_{r,\iota}(y, \tau_{\llbracket 1, i-1 \rrbracket}, \cdot)$ satisfies $P(n-\binom{m-r+1}{2}-i)$. We then take $\mathcal{T} = \cap_{i=1}^{n+1} \mathcal{T}_i$ and $\mathcal{Y}_\tau = \cap_{i=1}^{n+1} \mathcal{Y}_{i,\tau}$. $\square$

5.2 Dancing with genericity

We now give parametric generalizations of Proposition 3.2. These will imply that one can run over a fraction field the algorithms of [16] to reduce to real root classification problems the one of solving a parametric LMI.

LEMMA 5.2. *Assume that for all $r \in \llbracket 0, m-1 \rrbracket$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m-r$, $\mathcal{V}_{r,\iota}$ satisfies $P(t+n-\binom{m-r+1}{2})$. Then, there exists a non-empty Zariski open set $\mathcal{M} \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M} \cap \text{GL}_n(\mathbb{Q})$, there exists a non-empty Zariski open set $\mathcal{Y}_M \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_M \cap \mathbb{Q}^t$, for any connected component $C \subset \mathcal{D}_r(y, \cdot) \cap \mathbb{R}^n$, the results of Proposition 3.2 hold.*

PROOF. The proof of Proposition 3.2 follows from [17, Prop. 17, 18]. As the proof of [17, Prop. 17] is purely algebraic and thus valid over the base field $\mathbb{C}(\mathbf{y})$, we can then invert the order of choosing y and M as in [20, Prop. 6]. $\square$

LEMMA 5.3. *Assume that for all $r \in \llbracket 0, m-1 \rrbracket$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m-r$, $\mathcal{V}_{r,\iota}$ satisfies $P(t+n-\binom{m-r+1}{2})$. Then, there exists a non-empty Zariski open set $\mathcal{M} \subset \text{GL}_n(\mathbb{C})$ such that for all $M \in \mathcal{M} \cap \text{GL}_n(\mathbb{Q})$, there exists a non-empty Zariski open set $\mathcal{Y}_M \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_M \cap \mathbb{Q}^t$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m-r$, the results of Proposition 3.2 hold.*

PROOF. The proof of Proposition 3.2 follows from [16, Lem. 4.2 (i)]. Applying the same proof shows that for generic choice of M , the system (f, g, h) in [16, Equation 4.3] satisfies Q in $\mathcal{O} = \{(x, y, z) : \text{rank } A(x) = r\}$, i.e., the rank of $D(f, g, h)$ is the codimension of $V((f, g, h))$ in $V((f, g, h)) \cap \mathcal{O}$. The choice of y is then generic and follows from the Th. on the Dimension of Fibres [28, Th. 1.25]. $\square$

PROOF OF PROPOSITION 4.1. For all $d \in \mathbb{N}$, let $\mathcal{A}_d$ be the intersection of the Zariski open sets defined in Theorem 2.2 and Lemma 2.3. Then, for all $A \in \mathcal{A}_d \cap \mathbb{S}_m(\mathbb{Q}[\mathbf{y}]_{\leq d}[\mathbf{x}]_{\leq 1})$, the assumptions of the above lemmas are satisfied. It is then straightforward to show that for generic choices of $M \in \text{GL}_n(\mathbb{Q})$ and $\tau \in \mathbb{Q}^n$, there exists a non-empty Zariski open set $\mathcal{Y}_{A,M,\tau} \subset \mathbb{C}^t$ such that for all $y \in \mathcal{Y}_{A,M,\tau} \cap \mathbb{Q}^t$, the results of Proposition 3.2 hold for $A(y, \cdot) \in \mathbb{S}_m(R_{\leq 1})$, and that for all $r \in \llbracket 0, m-1 \rrbracket$, $\iota \in \llbracket 1, m \rrbracket$ with $|\iota| = m-r$, $i \in \llbracket 1, n+1 \rrbracket$, $\mathcal{D}_r(y, \cdot)$, $\mathcal{V}_{r,\iota}(y, \cdot)$, and $f_{r,\iota}(y, \cdot)$ satisfy all the results in [16]. The correctness of Proposition 4.1 then follows from the correctness of [16, Th. 3.7]. $\square$

6 COMPLEXITY ANALYSIS

We give now a complexity analysis of [13, Algorithm 2] (and hence, by construction, of CLASSIFICATION), without the extra genericity condition assumed in [13], i.e., the homogeneous components of f of highest degree form a regular sequence. Denote $\mathcal{M}_{a,b} = \binom{a+b}{a}$.

6.1 Parametric geometric resolutions

Let $\mathbf{f} = (f_1, \dots, f_p) \in \tilde{R}[\mathbf{u}, \boldsymbol{\lambda}]$ generating a zero-dimensional ideal $\langle \mathbf{f} \rangle_{\mathbb{Q}(\mathbf{y})}$, and $\mathbf{g} = (g_1, \dots, g_s) \in \tilde{R}$. In the case where $\mathbf{f}$ generates a radical ideal, we can use parametric geometric resolutions.

Definition 6.1. Let $P \in \mathbb{Q}(\mathbf{y})[\mathbf{x}]$ be a polynomial. We denote by $\text{denom}(P) \in \mathbb{Q}[\mathbf{y}]$ the least common multiple of the denominators of the coefficients of P , and by $\text{num}(P) = P \text{denom}(P) \in \tilde{R}$.

PROPOSITION 6.2 (PARAMETRIC GEOMETRIC RESOLUTION, [27, Th. 1]). Denote by $\Delta := \deg(V(f))$ the affine degree (see, e.g., [15]) of the algebraic set $V(f)$, and $\delta := \deg_\pi(f)$ the generic cardinality of the fibers of the canonical projection $\pi : (\mathbf{y}, \mathbf{x}) \mapsto \mathbf{y}$ to $V(f)$. Then,

- (1) There exists a monic irreducible polynomial $q \in \mathbb{Q}(\mathbf{y})[T]$ and polynomials $v_1, \dots, v_n \in \mathbb{Q}(\mathbf{y})[T]$, with

$$\begin{aligned} \deg_T(q) &= \delta, & \deg_T(v_i) &\leq \delta - 1, \\ \deg(\text{denom}(q)) &\leq \Delta, & \deg(\text{denom}(v_i)) &\leq \Delta, \\ \deg_{\mathbf{y}}(\text{num}(q)) &\leq 2\Delta, & \deg_{\mathbf{y}}(\text{num}(v_i)) &\leq 2\Delta, \end{aligned}$$

such that the parametrization

$$q(\mathbf{y}, T) = 0, \quad x_i = \frac{v_i(\mathbf{y}, T)}{q'(\mathbf{y}, T)} \quad \forall i \in \llbracket 1, n \rrbracket$$

gives the solutions of the system $f(\mathbf{y}, \mathbf{x}) = 0$ when both systems are specialized at all parameters $\mathbf{y} = (y_1, \dots, y_t) \in \mathbb{Q}^t$ such that $q, v_1, \dots, v_n$ are well-defined.

- (2) There exists polynomials $\tilde{v}_1, \dots, \tilde{v}_s \in \mathbb{Q}(\mathbf{y})[T]$, with

$$\begin{aligned} \deg_T(\tilde{v}_i) &\leq \delta - 1, & \deg(\text{denom}(\tilde{v}_i)) &\leq \Delta \deg(g_i), \\ \deg_{\mathbf{y}}(\text{num}(\tilde{v}_i)) &\leq 2\Delta \deg(g_i), \end{aligned}$$

such that, when denoting by

$$\tilde{g}_i := \frac{\tilde{v}_i(\mathbf{y}, T)}{q'(\mathbf{y}, T)} \quad \forall i \in \llbracket 1, s \rrbracket,$$

the sign of $\mathbf{g}(\mathbf{y}, \mathbf{x})$ is the same as the sign of $\tilde{\mathbf{g}}_i$ for all solutions of the parametrization.

PROOF. (1) Let u be a primitive element of the extension $\mathbb{Q}(\mathbf{y}) \rightarrow \mathbb{Q}(\mathbf{y})[\mathbf{x}]/\langle f \rangle_{\mathbb{Q}(\mathbf{y})}$ with coefficients in $\mathbb{Q}$, so that $q, v_1, \dots, v_n$, as well as the degree bounds, are given by the parametric geometric resolution of $f(\mathbf{y}, \mathbf{x}) = 0$ for u [27, Th. 1].

(2) Consider the system $f(\mathbf{y}, \mathbf{x}) = 0$, $g_i(\mathbf{y}, \mathbf{x}) - \tilde{g}_i = 0$, where $\tilde{g}_i$ is a new variable. As the value of $\tilde{g}_i$ is uniquely determined by other variables, u is a primitive element of the extension $\mathbb{Q}(\mathbf{y}) \rightarrow \mathbb{Q}(\mathbf{y})[\mathbf{x}, \tilde{\mathbf{g}}]/\langle f, g_i - \tilde{g}_i \rangle_{\mathbb{Q}(\mathbf{y})}$, and $\deg_\pi(f, g_i - \tilde{g}_i) = \deg_\pi(f)$. The parametrization of $\tilde{g}_i$ is then given by the parametric geometric resolution of the system for u . Finally, Bézout's theorem gives $\deg(V(f, g_i - \tilde{g}_i)) \leq \Delta \deg(g_i)$. The degree bounds follow. $\square$

We recall that the Lagrange systems with saturation of rank defects generically satisfy the assumptions above. Hence, we compute the param. geometric resolution, and pass $\mathbf{f} = q$ and $\mathbf{g} = (\tilde{g}_1, \dots, \tilde{g}_t)$ to CLASSIFICATION instead. We then have $\mathcal{B} = \{1, T, \dots, T^{\delta-1}\}$, as the unique monomial basis of $\mathbb{Q}(\mathbf{y})[T]/\langle q \rangle_{\mathbb{Q}(\mathbf{y})}$. For $\alpha \in \{0, 1, 2\}^s$, the Hermite matrix $\mathcal{H}_{\mathbf{g}^\alpha}$ w.r.t. $\mathcal{B}$ is, using the notations of [4],

$$(\mathcal{H}_{\mathbf{g}^\alpha})_{i,j} = \text{Tr}(M_{T^{i+j}} \mathbf{g}^\alpha) = \sum_{k=1}^{\delta} \text{cof}(T^k, \text{Rem}_T(T^{i+j+k} \mathbf{g}^\alpha, q)). \quad (3)$$

Complexity analysis. Let $\deg(\mathbf{f}) := \max_i \deg(f_i)$ and $\deg(\mathbf{g}) := \max_i \deg(g_i)$. By [27, Th. 2], there exists a probabilistic algorithm which computes the polynomials in Proposition 6.2 (1) using

$$\tilde{O}\left((n\Gamma_f + n^4 + t^2n)\delta \mathcal{M}_{t,4\Delta}\right)$$

arithmetic operations in $\mathbb{Q}$, where Γ_f is the length of a straight-line program evaluating $\mathbf{f}$, and the notation $g = \tilde{O}(f)$ refers to $g = O(f \log^\kappa(f))$ for some $\kappa > 0$. For Proposition 6.2 (2), by simply reapplying the algorithm for each $i \in \llbracket 1, s \rrbracket$, we arrive at

$$\tilde{O}\left(\sum_{i=1}^s (n(\Gamma_f + \Gamma_{g_i}) + n^4 + t^2n)\delta \mathcal{M}_{t,4\Delta} \deg(\mathbf{g})\right). \quad (4)$$

Bounds on p, s . Let $\mathbf{f} = f_{r,t,i} \in \tilde{R}[\mathbf{u}, \lambda]$ be one of the polynomial systems given by Line 3 of Algorithm 1, and $\mathbf{g} \in \tilde{R}$ be given by PSDMATRIXCOND. By definition, each Lagrange system of $f_{r,t}$ has at most $O(n + m^2)$ polynomials, on which we add $\binom{m}{r}$ polynomials for the saturation of rank defects. We then denote by $p^* \in O(n + 2^m)$ a global bound on p , and similarly, $s^* \in O(m)$.

Bounds on Γ_f, Γ_{g_i} . Let $\mathbf{f} = f_{r,t,i} \in \tilde{R}[\mathbf{u}, \lambda]$ be one of the polynomial systems given by Line 3 of Algorithm 1, and $\mathbf{g} \in \tilde{R}$ be given by PSDMATRIXCOND. To evaluate each polynomial in $f_{r,t,i}$ or $\mathbf{g}$, we can first evaluate the linear matrix is in $O(m^2 n \mathcal{M}_{d,t})$ arithmetic operations in $\mathbb{Q}$, and then compute the evaluation of $\mathbf{f}$ and $\mathbf{g}$ purely in $\mathbb{Q}$. Hence, we have $\Gamma_f^* \in \tilde{O}(2^m n^2 \mathcal{M}_{d,t})$ and $\Gamma_{g_i}^* \in O(m^{O(1)} n \mathcal{M}_{d,t})$.

Bound on $\deg(\mathbf{g})$. Let $\mathbf{g} \in \tilde{R}$ be the polynomial system given by PSDMATRIXCOND. Then, we have $\deg(\mathbf{g}) \leq \deg(\det(\lambda I_m + A)) = m(d+1)$, where $d := \deg_{\mathbf{y}}(A)$ is the maximum degree of the entries of A in $\mathbf{y}$. We then have $\deg(\mathbf{g})^* \in O(md)$.

Bounds on $\deg(f), \delta$. Let $\mathbf{f} = f_{r,t,i} \in \tilde{R}[\mathbf{u}, \lambda]$ be one of the polynomial systems given by Line 3 of Algorithm 1. By [16, Prop. 5.1], for all $\mathbf{y} \in \mathbb{R}^t$ such that $f_{r,t,i}(\mathbf{y}, \cdot)$ is well-defined, the specialized Lagrange system is of maximum degree 2, hence of total degree $d+2$, and that $\delta \leq \delta^* := \mathcal{M}_{n,m(m+1)/2}^3$ when $n \leq m(m+1)/2$, and 0 otherwise. Furthermore, the saturation of rank defects consists of polynomials of total degree $r(d+1)$, so that $\deg(\mathbf{f}) = \max(d+2, r(d+1))$, i.e., $\deg(\mathbf{f})^* \in O(md)$.

Bound on Δ . Let $\mathcal{U} = (u_{ij})_{i,j \in \llbracket 1, m \rrbracket}$ be the general symmetric matrix with entries variables u_{ij} (with $u_{ji} = u_{ij}$), and $m^* := m(m+1)/2$. One has $\text{sing}(\mathcal{D}_r) = \mathcal{D}_{r-1}$ and $\mathcal{D}_r \subset \mathbb{C}^{m^*}$ is equidimensional of codimension $c := \binom{m-r+1}{2}$ [6]. Thus, $\mathcal{D}_r \setminus \mathcal{D}_{r-1}$ is a constructible set of local pure codimension c and locally defined by c equations $\mu = (\mu_1, \dots, \mu_c)$ of degree $\leq r+1$ in the entries of $\mathcal{U}$.

Consider the groups of variables $\mathbf{u} = (u_{ij}, 1 \leq i \leq j \leq m)$, $\mathbf{y} = (y_1, \dots, y_t)$ and $\mathbf{x} = (x_1, \dots, x_n)$. Let $\mathbf{v} = [u_{ij} - a_{ij}(\mathbf{y}, \mathbf{x}), 1 \leq i \leq j \leq m]$ be the list of polynomials defining the entries of the parametric LMI, of degree 1 with respect to $\mathbf{u}$ and $\mathbf{x}$, and of degree d with respect to $\mathbf{y}$.

Define the system $\mathbf{f} = [\mu, \mathbf{v}]$ consisting of $c + m^*$ polynomials in $n + t + m^*$ variables. The Jacobian matrix of $\mathbf{f}$ with respect to the three variable groups above is:

$$\text{jac}(\mathbf{f}) = \begin{bmatrix} \text{jac}_{\mathbf{u}}(\mu) & 0 & 0 \\ \text{Id}_{m^*} & \text{jac}_{\mathbf{y}}(\mathbf{v}) & \text{jac}_{\mathbf{x}}(\mathbf{v}) \end{bmatrix}.$$

Remark that $\text{jac}_{\mathbf{x}}(\mathbf{v})$ is constant in $\mathbf{x}$. Next we introduce Lagrange multipliers $\lambda = (\lambda_1, \dots, \lambda_{c+m^*})$ and consider the augmented system $[\mathbf{f}, \lambda^T \text{jac}(\mathbf{f})]$. With respect to $(\mathbf{u}, \mathbf{y}, \mathbf{x}, \lambda)$, one has:

$$\begin{aligned} \mu &\rightarrow c \text{ polynomials of mdeg} \leq (r+1, 0, 0, 0) \\ \mathbf{v} &\rightarrow m^* \text{ polynomials of mdeg} \leq (1, d, 1, 0) \\ \lambda^T \text{jac}_{\mathbf{u}}(\mathbf{f}) &\rightarrow m^* \text{ polynomials of mdeg} \leq (r, 0, 0, 1) \\ \lambda^T \text{jac}_{\mathbf{y}}(\mathbf{f}) &\rightarrow t \text{ polynomials of mdeg} \leq (0, d-1, 1, 1) \\ \lambda^T \text{jac}_{\mathbf{x}}(\mathbf{f}) &\rightarrow n \text{ polynomials of mdeg} \leq (0, d, 0, 1) \end{aligned}$$

The value of Δ , by [26, Prop. I.1], is bounded from above by a multilinear Bézout bound given by the sum of coefficients of

$$((r+1)\theta_u)^c (\theta_u + d\theta_y + \theta_x)^{m^*} (r\theta_u + \theta_\lambda)^{m^*} \\ ((d-1)\theta_y + \theta_x + \theta_\lambda)^t (d\theta_y + \theta_\lambda)^n$$

modulo $\langle \theta_u^{m^*+1}, \theta_y^{t+1}, \theta_x^{n+1}, \theta_\lambda^{c+m^*+1} \rangle$. Since the polynomial is homogeneous of degree $t+n+2m^*+c$, this is also $(r+1)^c$ times the coefficient of $\theta_u^{m^*-c} \theta_y^t \theta_x^n \theta_\lambda^{c+m^*}$ of the polynomial

$$P := (\theta_u + d\theta_y + \theta_x)^{m^*} (r\theta_u + \theta_\lambda)^{m^*} \\ ((d-1)\theta_y + \theta_x + \theta_\lambda)^t (d\theta_y + \theta_\lambda)^n.$$

The multilinear bound, denoted by $\text{MBB}(\Delta)$, is thus

$$(r+1)^c \text{cof}(\theta_u^{m^*-c} \theta_y^t \theta_x^n \theta_\lambda^{c+m^*}, P) \\ = (r+1)^c \sum_{\Theta} d^{\alpha_y} r^{\beta_u} (d-1)^{\gamma_y} d^{\delta_y} \binom{m^*}{\alpha_u, \alpha_y, \alpha_x} \binom{m^*}{\beta_u} \binom{t}{\gamma_y, \gamma_x, \gamma_\lambda} \binom{n}{\delta_y},$$

where Θ is the set of $(\alpha_u, \alpha_y, \alpha_x, \beta_u, \gamma_y, \gamma_x, \gamma_\lambda, \delta_y)$ satisfying

$$\begin{aligned} \alpha_u + \alpha_y + \alpha_x &= m^* & \gamma_y + \gamma_x + \gamma_\lambda &= t \\ \alpha_u + \beta_u &= m^* - c & \alpha_y + \gamma_y + \delta_y &= t \\ \alpha_x + \gamma_x &= n & m^* - \beta_u + \gamma_\lambda + n - \delta_y &= c + m^*. \end{aligned} \quad (5)$$

LEMMA 6.3. *For $n > m^* + t$, we have $\text{MBB}(\Delta) = 0$. In particular, if m and t are fixed, the bound is 0 for large n .*

PROOF. The contribution of θ_x in P is only given by factors $(\theta_u + d\theta_y + \theta_x)^{m^*}$ and $((d-1)\theta_y + \theta_x + \theta_\lambda)^t$, thus any monomial of P has degree $\leq m^* + t$ in θ_x , so that $\Theta = \emptyset$ if $n > m^* + t$. If m and t are fixed, so does $m^* + t$, hence for large n the bound is 0. $\square$

We remark that this is also true for the bound on δ in [16, Prop. 5.1]. Let us now bound the size of Θ .

LEMMA 6.4. $|\Theta| \leq \min\{(t+1)^3, (t+1)(n+1)^2\}$.

PROOF. Equation (5) can be reduced to $L = [\alpha_u = \gamma_y + \delta_y + \gamma_x + m^* - t - n, \alpha_y = -\gamma_y - \delta_y + t, \alpha_x = -\gamma_x + n, \gamma_\lambda = -\gamma_y - \gamma_x + t, \beta_u = -\gamma_y - \delta_y - \gamma_x + t + n - c]$, with free $\gamma_y, \gamma_x, \delta_y$; these latter verify the bounds $0 \leq \gamma_y \leq t, 0 \leq \gamma_x \leq \min\{t, n\}$, and $0 \leq \delta_y \leq \min\{t, n\}$. Thus $|\Theta| \leq (t+1)(\min\{t, n\} + 1)^2$ as claimed. $\square$

LEMMA 6.5. *Let Δ^* be defined as*

$$(6r+6)^{m^*} d^{2t+m^*} \left(e \frac{m^* + c + t}{m^* + c} \right)^{m^*+c} 3^t \min\{(t+1)^3, (t+1)(n+1)^2\}.$$

Then, $\text{MBB}(\Delta)$ is bounded from above by Δ^ .*

PROOF. Recall that $\delta_y \leq t$ in Θ . Let $D = m^* + c - (n - \delta_y)$, which is nonnegative (from the last equation in (5), using $\beta_u \leq m^*$). One has $n + D = m^* + c + \delta_y \leq m^* + c + t$ and hence

$$\binom{n}{\delta_y} = \binom{n}{n - \delta_y} = \binom{n}{m^* + c - D} \leq \binom{n + D}{m^* + c} \leq \binom{m^* + c + t}{m^* + c},$$

and the last binomial is bounded by $\left(e \frac{m^* + c + t}{m^* + c} \right)^{m^*+c}$. Thus from $\binom{m^*}{\alpha_u, \alpha_y, \alpha_x} \leq 3^{m^*}$ and $\binom{m^*}{\beta_u} \leq 2^{m^*}$, $\text{MBB}(\Delta)$ is at most

$$(r+1)^c d^{m^*} r^{m^*-c} (d-1)^t d^t \sum_{\Theta} \binom{m^*}{\alpha_u, \alpha_y, \alpha_x} \binom{m^*}{\beta_u} \binom{t}{\gamma_y, \gamma_x, \gamma_\lambda} \binom{n}{\delta_y} \\ \leq (r+1)^c d^{m^*} r^{m^*-c} (d-1)^t d^t \left(e \frac{m^* + c + t}{m^* + c} \right)^{m^*+c} 3^{m^*} 2^{m^*} 3^t |\Theta|,$$

and we conclude by Lemma 6.4. $\square$

6.2 Degree bounds on Hermite matrices

We recall that a Hermite matrix $\mathcal{H}_{g^\alpha}$ has its entries given by Equation (3). To compute the remainders in the equation, as each $\tilde{g}_i$ has denominator q' , we need to find the inverse of q' modulo q , or the inverse of $\text{num}(q')$ modulo $\text{num}(q)$. Let $c = \text{Res}(\text{num}(q), \text{num}(q')) \in \mathbb{Q}[\mathbf{y}]$, and denote by $a, b \in \mathbb{Q}[\mathbf{y}][T]$ the polynomials of degree at most $\deg_T(q)$ such that $a \text{num}(q) + b \text{num}(q') = c$. Once b is known, $c^{-1}b$ is the inverse of $\text{num}(q')$ by $\text{num}(q)$.

LEMMA 6.6. *Let $P, Q \in \mathbb{Q}[\mathbf{y}][T]$ be polynomials. Denote by $M = \max(\deg_{\mathbf{y}}(P), \deg_{\mathbf{y}}(Q))$. Then,*

$$\deg_{\mathbf{y}}(\text{Res}(P, Q)) \leq M(\deg_T(P) + \deg_T(Q)),$$

$$\deg_{\mathbf{y}}(\text{Prem}_T(P, Q)) \leq M \max(\deg_T(P) - \deg_T(Q) + 2, 1)$$

Moreover, there exists $A, B \in \mathbb{Q}[\mathbf{y}][T]$, with $\deg_T(A) \leq \deg_T(Q)$, $\deg_T(B) \leq \deg_T(P)$, such that $AP + BQ = \text{Res}(P, Q)$, and

$$\max(\deg_{\mathbf{y}}(A), \deg_{\mathbf{y}}(B)) \leq M(\deg_T(P) + \deg_T(Q)).$$

Finally, all the items above can be computed in $O(\deg_T(P) \deg_T(Q))$ operations in $\mathbb{Q}[\mathbf{y}]$.

PROOF. Suppose w.l.o.g. that $\deg_T(P) \geq \deg_T(Q)$. From [4, Equation 4.3, Notation 8.55], using the notations of [4], we have

$$\text{Res}(P, Q) = \varepsilon_{\deg_T(P)} \text{sRes}_0(P, Q),$$

$$\text{Prem}(P, Q) = \varepsilon_{\deg_T(P) - \deg_T(Q) + 1} \text{sRes}_{\deg_T(Q) - 1}(P, Q),$$

where $\varepsilon_i = (-1)^{i(i-1)/2}$. The degree bounds then follow from [4, Prop. 8.71]. Moreover, [4, Prop. 8.64] gives

$$\text{sRes}_0(P, Q) = \text{sRemU}_0(P, Q)P + \text{sRemV}_0(P, Q)Q,$$

with sRemU_0 and sRemV_0 satisfying the same degree bounds as sRes_0 . Finally, all the items can be computed in [4, Algorithm 8.81], hence the complexity. $\square$

As a direct consequence, $\deg_T(b) \leq \delta$, $\deg_{\mathbf{y}}(b) \leq 2\Delta(2\delta - 1)$, and $\deg(c) \leq 2\Delta(2\delta - 1)$. We then factor out the denominator from the sum in Equation (3), and hence deduce a degree bound on the entries of Hermite matrices and their principal minors.

PROPOSITION 6.7. *For $\alpha \in \{0, 1, 2\}^s$, there exists $\rho \in \mathbb{Q}[\mathbf{y}]$, with*

$$\deg(\rho) \leq |\alpha| \Delta(6\delta - 3 + \deg(\mathbf{g})) + \Delta(2\delta - 2),$$

where $|\alpha| := \sum_{i=1}^s \alpha_i$, such that for all $i, j \in \llbracket 0, \delta - 1 \rrbracket$, $\rho M_{T^{i+j}} g^\alpha$ is a matrix with entries in $\mathbb{Q}[\mathbf{y}] \subseteq \mathcal{D}(|\alpha|)$, where

$$\mathcal{D}(|\alpha|) := 2\Delta(2\delta - 1) \max(1, |\alpha|(|\alpha| + 1)(2\delta - 1 + \deg(\mathbf{g}))) \\ + \Delta(3\delta - 3) + |\alpha| \Delta \in O(|\alpha|^2 (\delta + md)\delta\Delta).$$

PROOF. We rewrite the remainder as

$$\begin{aligned} \text{Rem}_T(T^{i+j+k} \mathbf{g}^\alpha, q) &= \text{Rem}_T(T^{i+j+k} \tilde{\mathbf{g}}^\alpha, q) \\ &= \frac{\text{denom}(q')^{|\alpha|}}{c^{|\alpha|} \text{denom}(p)} \frac{\text{Prem}_T(T^{i+j+k} \text{num}(p), \text{num}(q))}{\text{denom}(q)^{\max(i+j+k+\deg_T(p)-\deg_T(q)+1, 0)}}, \end{aligned}$$

where $p = b^{|\alpha|} \tilde{v}^\alpha \in \mathbb{Q}(\mathbf{y})[T]$, with the degree bounds

$$\begin{aligned} \deg_T(p) &\leq |\alpha| (2\delta - 1), \\ \deg_{\mathbf{y}}(\text{num}(p)) &\leq 2|\alpha| \Delta (2\delta - 1 + \deg(\mathbf{g})), \\ \deg(\text{denom}(p)) &\leq |\alpha| \Delta \deg(\mathbf{g}), \end{aligned}$$

so that

$$\begin{aligned} &\deg_{\mathbf{y}}(\text{Prem}(T^{i+j+k} \text{num}(p), \text{num}(q))) \\ &\leq 2\Delta \max(|\alpha| (2\delta - 1 + \deg(\mathbf{g})), 1) \\ &\quad \cdot \max(i + j + k + |\alpha| (2\delta - 1) - \delta + 2, 1) \\ &\leq 2\Delta (2\delta - 1) \max(1, |\alpha| (|\alpha| + 1) (2\delta - 1 + \deg(\mathbf{g}))). \end{aligned}$$

It suffices then to take

$$\rho = c^{|\alpha|} \text{denom}(p) \text{denom}(q)^{\max(3\delta - 2 + \deg_T(p) - \deg_T(q), 0)},$$

hence the degree bound on ρ .

Moreover, suppose w.l.o.g. that $\deg_T(p) \geq \deg_T(q)$ holds. Then, for $k \in \llbracket 0, \delta - 1 \rrbracket$, we have

$$\begin{aligned} &\text{cof}(T^k, \text{Rem}_T(T^{i+j+k} \mathbf{g}^\alpha, q)) \\ &= \rho^{-1} \text{cof}(T^k, \text{Prem}_T(T^{i+j+k} \text{num}(p), \text{num}(q))) \\ &\quad \text{denom}(q)^{3\delta - 3 - i - j - k} \text{denom}(q')^{|\alpha|}, \end{aligned} \quad (6)$$

hence the degree bound on the entries of $\rho M_{T^{i+j} \mathbf{g}^\alpha}$. $\square$

Hence, the numerator of each $(\mathcal{H}_{\mathbf{g}^\alpha})_{i,j}$ has degree $\leq \mathcal{D}(|\alpha|)$, and the numerator of each minor of $\mathcal{H}_{\mathbf{g}^\alpha}$ has degree $\leq \delta \mathcal{D}(|\alpha|)$.

6.3 Arithmetic complexity

Recall that one can evaluate a multivariate polynomial P in k variables at any point within $O(\mathcal{M}_{\deg(P),k})$ arithmetic operations, using [7, Sec. 3a, 3b]. Similarly, with $O(\mathcal{M}_{k,\deg(P)})$ interpolation points and the value of P at these points, one can recover P within $\tilde{O}(\mathcal{M}_{k,\deg(P)})$ arithmetic operations. Below, we don't use $\mathcal{M}_{\deg(P),k}$ and $\mathcal{M}_{k,\deg(P)}$ interchangeably, even though they are equal.

PROPOSITION 6.8. *Each call to [13, Algorithm 2] with input $\mathbf{f} = (f_1, \dots, f_p)$, $\mathbf{g} = (g_1, \dots, g_s)$ takes*

$$\tilde{O}(s \mathcal{M}_{t,\delta \mathcal{D}(2s)} (2\delta s \beta)^{t+1} 2^{3t} (\delta \mathcal{D}(2s))^{2t+1}) \quad (7)$$

arithmetic operations in $\mathbb{Q}$, where β is the number of realizable signs of $\mathbf{g}$ on $V(\mathbf{f})$, and outputs at most $(4\delta s \beta \mathcal{D}(2s))^t$ formulas that consist of $O(\delta s \beta)$ polynomials of degree at most $\delta \mathcal{D}(2s)$.

PROOF. By [13, Prop. 4.11], each call to SAMPLEPOINTS involves at most $2\delta s \beta$ polynomials, and hence outputs a set of at most $(4\delta^2 s \beta \mathcal{D}(2s))^t$ points in $\tilde{O}(\mathcal{M}_{t,\delta \mathcal{D}(2s)} (2\delta s \beta)^{t+1} 2^{3t} (\delta \mathcal{D}(2s))^{2t+1})$ arithmetic operations in $\mathbb{Q}$. There are at most s calls to SAMPLEPOINTS, hence the complexity in Equation (7).

We show that the complexity of computing the minors (Line 9 - 13) and evaluating the signatures (Line 16, 25) is dominated by the complexity of computing sample points; as observed in [13, Sec. 4.3], the other steps are simply linear algebra and negligible.

Line 9 - 13. For $\mathcal{H}_{\mathbf{g}^\alpha}$ with $|\alpha| \geq 1$, the numerator of $(\mathcal{H}_{\mathbf{g}^\alpha})_{i,j}$, denoted by P , is given by Equation (3). Suppose w.l.o.g. that $\deg_T(p) \leq \deg_T(q) - 1$, so that the coefficients are given by Equation (6).

To compute P , we need $\mathcal{M}_{t,\deg(P)}$ interpolation points, where $\deg(P) \leq \mathcal{D}(|\alpha|)$. For each interpolation point $\mathbf{y} \in \mathbb{Q}^t$, we first compute $\text{num}(p)(\mathbf{y})$, $\text{num}(q)(\mathbf{y})$, $\text{denom}(q)(\mathbf{y})$, and $\text{denom}(q')(\mathbf{y})$, in $O(\mathcal{M}_{\deg(\text{num}(p)),t+1})$ operations; then, for each $k \in \llbracket 0, \delta - 1 \rrbracket$, we compute $\text{Prem}_T(T^{i+j+k} \text{num}(p), \text{num}(q))(\mathbf{y})$ in $O(|\alpha| \delta^2)$ operations; finally, we compute $P(\mathbf{y})$ in $O(\delta)$ operations. Hence, we need $\tilde{O}(\mathcal{M}_{t,\mathcal{D}(|\alpha|)} (\mathcal{M}_{\deg(\text{num}(p)),t+1} + |\alpha| \delta^3))$ operations.

Consequently, computing $\mathcal{H}_{\mathbf{g}^\alpha}$ takes the same complexity as P , as we can take the same points for all traces, and computing ρ has a negligible cost in front of P . Also, there are at most $2\delta \beta$ principal minors to compute, and computing each takes at most

$$\tilde{O}(\mathcal{M}_{t,\delta \mathcal{D}(2s)} (\mathcal{M}_{4s\Delta(2\delta-1+\deg(\mathbf{g}))+2s(2\delta-1),t+1} + 2s\delta^3))$$

arithmetic operations in $\mathbb{Q}$, as it takes at most $O(\delta^3)$ operations for $\mathcal{H}_{\mathbf{g}^\alpha}(\mathbf{y})$, and $O(\delta^\omega)$ operations for the determinant. However, all the above bounds are negligible in front of the complexity of computing sample points, as

$$\begin{aligned} &\mathcal{M}_{4s\Delta(2\delta-1+\deg(\mathbf{g}))+2s(2\delta-1),t+1} \\ &< 2(4s\Delta(2\delta-1+\deg(\mathbf{g}))+2s(2\delta-1))^{t+1} \\ &\ll (2s+1)(2\delta-1)(4s\Delta(2\delta-1+\deg(\mathbf{g})))^{t+1} \\ &< (\delta \mathcal{D}(2s))^{t+1} \ll (\delta \mathcal{D}(2s))^{2t+1}. \end{aligned}$$

Line 16, 25. For the evaluation of the signatures, we recall that there are $3\delta \beta$ minors, whose numerators have deg. at most $\delta \mathcal{D}(2s)$, to evaluate at at most $(4\delta^2 s \beta \mathcal{D}(2s))^t$ points. This is then in

$$O(\mathcal{M}_{t,\delta \mathcal{D}(2s)} \delta \beta (4\delta^2 s \beta \mathcal{D}(2s))^t)$$

arithmetic operations in $\mathbb{Q}$. $\square$

PROOF OF THEOREM 1.1. For all $\mathbf{f} = f_{r,i} \in \tilde{R}[\mathbf{u}, \boldsymbol{\lambda}]$, we have $s \leq m$, $\mathcal{D}(2s) \leq 32(m+1)^2(\delta + md + m)\delta \Delta$, and $\beta \leq 3^m$, so that the complexity of each call to CLASSIFICATION, given by Equation (7) in Proposition 6.8, is in

$$\tilde{O}\left(2^{3t} m (3^m 2m \delta)^{t+1} (32m^2(\delta + md)\delta^2 \Delta)^{3t+1}\right)$$

operations in $\mathbb{Q}$. As REALDET gives at most $\sum_{r=0}^{m-1} \binom{m-1}{r} (n - \binom{m-r+1}{2}) = 2^{m-3}(8n - m^2 - 3m + 8) - n - 1$ polynomial systems, computing real root classifications for all $\mathbf{f}$ then take the complexity in Equation (1).

By [16, Th. 5.6], REALDET is in $n^{O(m^2)} 2^{O(m)} (n + m^2)^{O(1)}$ arithmetic operations in $\mathbb{Q}[\mathbf{y}]$, which is negligible compared to the terms of Equation (1) without Δ^* , let alone the complexity in $\mathbb{Q}$ by interpolation. The same argument holds for PSDMATRIXCOND and CHANGEVARS. Finally, the complexity of computing each parametric geometric resolution, given by Equation (4), is in $2^{O(t+m)} (nt)^{O(1)} (md)^{O(t)} \delta^* \Delta^{*t}$ arithmetic operations in $\mathbb{Q}$, which is also negligible compared to the terms of Equation (1). $\square$

7 PRACTICAL EXPERIMENTS

We present practical experiments with our implementation on LMIs with background in multiple domains. Due to their large size, we refer to <https://www.polsys.lip6.fr/~weijia/generic-param-lmi/> for

the exact form of the LMIs. In the following, we denote by PRB_tn the LMI for the problem PRB with t parameters and n variables.

Sum-of-squares problem. Decomposing a polynomial with real coefficients as a sum of squares (to certify its non-negativity) is done by solving an LMI. When the coefficients of the input polynomial depend on some parameters, one obtains a parametric LMI. In the following, MKN11 is extracted from a perturbation of Motzkin's polynomial [22], and RBN11 from Robinson's polynomial [23].

Algorithm analyses. We also consider LMIs extracted from the analysis of classical first-order optimization algorithms. In the following, GRD12 to GRD23 corresponds to different choices of parameters and specializations of a same LMI of size $m = 3$, with 2 parameters and 4 free variables, generated from the gradient descent method. Similarly, PPM21 and PPM31 are generated from the proximal point method, and DRS32 to DRS43 comes from [24, SM3.1.1, SM3.2.1] in the analysis of the Douglas-Rachford splitting.

Experimental setup. The timings are given in hours (h.), minutes (min.) and seconds (s.), and the computations have been performed on a server Intel Xeon Gold 6246R with 1.5 TB of RAM. We compute Hermite matrices using *msolve* [5] for Gröbner basis computation with *graded reverse lexicographical* orderings and the Maple package *Groebner* for elimination orderings. These are used as a replacement of the parametric geometric resolution algorithm to build our Hermite matrices (see [13]). The columns O1 and O2 correspond to the timings of the first two options in Section 4.3. The columns QE1 and QE2 correspond to the timings of the quantifier elimination in Maple 2024.2 [8] and Wolfram 14.2 [29].

	O1	O2	QE1	QE2
MKN11	5.0 s	1.5 s	5.7 s	0.06 s
RBN11	5.0 s	1.6 s	7.1 s	0.04 s
GRD12	1.0 s	3.7 s	∞	0.5 s
GRD13	19 s	17 s	∞	∞
GRD14	∞	∞	∞	∞
GRD21	0.5 s	1.7 s	1.3 s	0.1 s
GRD22	5.8 s	2 min	∞	42 min
GRD23	∞	∞	∞	∞
PPM21	0.3 s	0.3 s	0.3 s	0.005 s
PPM31	0.3 s	0.4 s	0.4 s	0.007 s
DRS32	2.2 s	8 h	∞	∞
DRS33	18 min	∞	∞	∞
DRS42	52 s	∞	∞	∞
DRS43	∞	∞	∞	∞

In the table, the symbol ∞ indicates that the computation is unfortunately intractable either because no result was obtained within 48 hours or due to lack of memory.

When $n \geq 2$, our implementation succeeds for examples where quantifier elimination in Maple 2024.2 and Wolfram 14.2 either fails due to lack of memory or does not terminate after 48 hours. This illustrates the benefit of our approach on this range of problems.

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