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Behavior in time of solutions to a degenerate chemotaxis system with flux limitation

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A B S T R A C T

We study a new class of Keller–Segel models, which presents a limited flux and an optimal transport of cells density according to chemical signal density. As a prototype of this class we study radially symmetric solutions to the parabolic–elliptic system

$$\begin{cases} u_t = \nabla\cdot\left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \chi k_f \nabla\cdot\left(\frac{u\nabla v}{(1+|\nabla v|^2)^\alpha}\right), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu + u, & x\in\Omega,\ t>0 \end{cases}$$

under no flux boundary conditions in a ball $B = \Omega \subset R^N$ and initial condition $u(x,0) = u_0(x) > 0, \chi > 0, \alpha > 0,\ k_f > 0$ and $\mu = \frac{1}{|\Omega|}\int_\Omega u_0 dx$. Under suitable conditions on α and u_0 it is shown that the solution blows up in L^∞-norm at a finite time T_{max} and for some $p > 1$ it blows up also in L^p-norm. The proofs are mainly based on an helpful change of variables, on comparison arguments and some suitable estimates.

1. Introduction

Let us consider the chemotaxis system with nonlinear diffusion and flux limitation,

$$\begin{cases} u_t = \nabla\cdot\left(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \chi\nabla\cdot(uf(|\nabla v|^2)\nabla v), & x\in\Omega,\ t>0,\\ 0 = \Delta v - \mu + u, & x\in\Omega,\ t>0,\\ \frac{u\nabla u\cdot\nu}{\sqrt{u^2|\nabla u|^2}} = \nabla v\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x) > 0, & x\in\Omega, \end{cases} \tag{1.1}$$

with Ω a ball in $\mathbb{R}^N$, $N \geq 3$, the constant $\chi > 0$, $\int_\Omega v dx = 0$, the initial data u_0 such that

$$u_0 \in C^2(\overline{\Omega}),\ \text{radially symmetric and positive in } \overline{\Omega}, \tag{1.2}$$

where

$$\mu := \frac{1}{|\Omega|}\int_\Omega u_0(x)dx \tag{1.3}$$

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and

$$f(|\nabla v|^2) = k_f(1 + |\nabla v|^2)^{-\alpha} \tag{1.4}$$

with some $k_f > 0$ and $\alpha > 0$.

We point out that the general structure of the model is

$$\begin{cases} u_t = \nabla \cdot (D_u(u, v)\nabla u) - \nabla(S(u, v)u\nabla v) + H_1(u, v), \\ \tau v_t = D_v \Delta v + H_2(u, v), \end{cases} \tag{1.5}$$

where $u(x,t)$ represents the cells density, $v(x,t)$ is the density of the chemoattractant, S measures the chemotactic sensitivity, D_u, D_v are two positive functions, representing the diffusivity of the cells and of the chemoattractant respectively, H_1, H_2 model source terms related to interactions (see [1]).

The most investigated model has $D_u = S = 1$, $H_1 = 0$ or H_1 of logistic type and $H_2 = -v + u$ (see [2] and the survey paper [3]). Interesting results have been established also for the parabolic–elliptic chemotaxis systems ($\tau = 0$).

More recently nonlinear diffusion terms have been considered with D and S depending not only on u and v, but also on their gradient.

∘ Winkler and Djie in [4], with the aim to study simplified models in the theoretical description of chemotaxis phenomena under the influence of the volume-filling effect, considered (1.5) with $D_u = (u+1)^{-p}, S = (u+1)^{q-1}$, $D_v = 1$, $H_2 = u - \mu$, and proved that if $p+q < 2N$ then all solutions are global in time and bounded, whereas if $p+q > 2N$, $q > 0$, and Ω is a ball, then there exist solutions that become unbounded in finite time.

∘ In [5] (1.5) is investigated in $\Omega \times (0,T)$ bounded in $\mathbb{R}^N$ with $D_u = (u+\alpha)^{m_1-1}$ and $S = \chi(u+\alpha)^{m_2-2}$, $D_v = 1$, $H_2 = u - \mu$, under Neumann boundary conditions and initial conditions, $\alpha > 0$, $\chi > 0$, $m_1, m_2 \in \mathbb{R}$. It is proved that for some $p_0 > \frac{N}{2}(m_2 - m_1)$ any blowing up solution in $L^\infty(\Omega)$-norm, blows up also in $L^{p_0}(\Omega)$-norm and the blow-up time is estimated.

Recently the case $S = S(|\nabla v|)$ depending on the gradient of v (flux limitation term) received considerable attention in the biomathematical literature.

Here we report only the most important results on flux limitation.

• If $D_u = D_v = 1$ and $S = \chi|\nabla v|^{p-2}$, $\chi > 0$, $H_2 = u - \mu$, $\Omega \subset \mathbb{R}^N$,

$$p \in (1, \infty) \text{ if } N = 1; \qquad p \in \left(1, \frac{N}{N-1}\right) \text{ if } N \geq 2,$$

Negreanu and Tello in [6] obtained uniform bounds in $L^\infty(\Omega)$ and existence of global in time solutions of (1.5) with $\tau = 0$; moreover they prove that for the one-dimensional case there exist infinitely many non-constant steady-states for $p \in (1,2)$. As to its parabolic–parabolic case ($\tau = 1$), Yan and Li in [7] obtained global existence of weak solutions which are uniformly bounded provided that $1 < p < \frac{N}{N-1}$.

In [8], Kohatsu and Yokota established stability of constant equilibria for small initial data and $p \in (1, \infty)$.

• If $D_u = \frac{u}{\sqrt{u^2+|\nabla u|^2}}$ and $S = \frac{1}{\sqrt{1+|\nabla v|^2}}$, $H_1 = 0$, $H_2 = u - \mu$, Bellomo and Winkler [1] obtained global existence of bounded classical solutions for arbitrary positive radial initial data $u_0 \in C^3(\overline{\Omega})$ when either

$$N \geq 2 \text{ and } \chi\langle 1, \quad \text{or} \quad N = 1,\ \chi\rangle 0 \text{ and } m := \int_\Omega u_0 < m_c,$$

$$\text{with } m_c := \frac{1}{\sqrt{(\chi^2-1)}}, \quad \text{if } \chi > 1, \text{ and } m_c := \infty \text{ if } \chi \leq 1.$$

In [9], the authors showed that the above conditions are essentially optimal in the sense that if $\chi > 1$ and

$$m > \frac{1}{\sqrt{\chi^2-1}}, \text{ if } N = 1; \qquad m > 0 \text{ arbitrary, if } N \geq 2,$$

there exists $u_0 \in C^3(\overline{\Omega})$ with $\int_\Omega u_0 = m$, such that for some $T > 0$ there exists a unique classical solution blowing up at time T in $L^\infty(\Omega)$-norm.

• If $D_u = 1$ and $S(|\nabla v|^2) \geq K_f\left(1 + |\nabla v|^2\right)^{-\alpha}$, $K_f > 0$, $\chi = 1$, $0 < \alpha < \frac{N-2}{2(N-1)}$, Ω a ball in $\mathbb{R}^N$, with $N \geq 3$, for a considerably large set of radially symmetric initial data, Winkler in [10] proved that the problem admits solutions blowing up in finite time in L^∞-norm for the first component. Otherwise, if $S(|\nabla v|^2) \leq K_f\left(1 + |\nabla v|^2\right)^{-\alpha}$, $\chi = 1$ and α satisfies

$$\begin{cases} \alpha > \frac{N-2}{2(N-1)}, & \text{for } N \geq 2, \\ \alpha \in \mathbb{R}, & \text{for } N = 1, \end{cases}$$

in general (not symmetric setting), no blow-up solution can exist then a global bounded solution exists for arbitrary nonnegative continuous initial data.

The case $\alpha = \frac{N-2}{2(N-1)}$ plays the role of a critical exponent and it is still an open problem.

• If $D_u = 1$ and $S(|\nabla v|^2) = K_f\left(1 + |\nabla v|^2\right)^{-\alpha}$, $K_f > 0$, $\chi = 1$, $0 < \alpha < \frac{N-2}{2(N-1)}$, $\Omega = B_R(0) \subset \mathbb{R}^N$, with $N \geq 3$, Marras et al. in [11], for suitable initial data, proved that a solution which blows up in L^∞-norm blows up also in L^p-norm for some $p > \frac{N}{2}$. Moreover, a safe time interval of existence of the solution $[0,T]$ is obtained, with T a lower bound of the blow-up time. Moreover the same authors in [12] extended the results in [11] when a source term of logistic type is present in the first equation.

• If $D_u = \frac{u^p}{\sqrt{u^2+|\nabla u|^2}}$ and $S(u,|\nabla v|^2) = \chi \frac{u^{q-1}}{\sqrt{1+|\nabla v|^2}}$, Chiyoda et al. in [13] considered the system (1.5) in a ball in $\mathbb{R}^N$, $N \in \mathbb{N}$, under no-flux boundary conditions and initial condition $u_0(x) > 0$. Assuming suitable conditions for χ and u_0 when $1 \le p \le q$, they obtained existence of blow-up solutions. When $p = q = 1$, the system reduces to (1.1) when $\alpha = \frac{1}{2}$.

If in (1.5), the source $H_1 \neq 0$, many interesting results on local, global existence and blow up of solutions have been derived (see for instance [14–17] and references therein).

Main Results The present work is addressed to study the behavior in time of the solutions of (1.1) where the diffusion term is nonlinear, depending on u and ∇u, as well as the drift term, depending on v and ∇v.

In particular in Section 2 we construct an initial data such that the solution of problem (1.1) blows up in L^∞-norm. We prove that the solutions of (1.1) blow up at finite time in L^p-norm, for some $p > 1$, if they blow up in L^∞-norm (Section 3).

For the proof of our results we use suitable change of variables, comparison arguments and some helpful estimates.

We want to prove the following theorems.

Theorem 1.1 (*Finite-time blow-up in L^∞-norm*). *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \ge 3$ and $R > 0$, f satisfy (1.4). Assume*

$$0 < \alpha < \frac{N-2}{2(N-1)}. \tag{1.6}$$

Then for all $\mu > 0$ there exists a radially symmetric positive initial data $u_0 \in C^\infty(\bar{\Omega})$ such that satisfies (1.3), and such that there exists a radially symmetric positive classical solution (u, v) of (1.1) in $\Omega \times (0, T_{max})$ for some $T_{max} > 0$ which blows up at T_{max} in the sense that $\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty$.

Theorem 1.2 (*Finite-time blow-up in L^p-norm*). *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \ge 3$ and $R > 0$. Then, a classical solution (u, v) of (1.1), provided by Theorem 1.1, is such that for all $p > N$,*

$$\limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty.$$

2. Blow-up in L^∞-norm

Before discussing blow-up in L^∞-norm, we give a result on local existence of classical solutions to (1.1). We note that blow-up criterion via the standard manner (see [18, Lemma 2.1]) says that

$$\text{if } T_{max} < \infty, \text{ then either } \liminf_{t \nearrow T_{max}} \inf_{x\in\Omega} u(x,t) = 0 \text{ or } \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{W^{1,\infty}(\Omega)} = \infty.$$

This includes possibility of extinction and gradient blow-up of solutions, whereas the following lemma presents a simple criterion ruling out this possibility.

Lemma 2.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \ge 1$ and $R > 0$, f satisfy (1.4) with $\alpha > 0$. Assume u_0 complies (1.2) and (1.3). Then there exist $T_{max} \in (0, \infty]$ and a pair (u, v) of positive radially symmetric functions $u \in C^{2,1}(\overline{\Omega} \times [0, T_{max}))$ and $v \in C^{2,0}(\overline{\Omega} \times [0, T_{max}))$ which solve (1.1) classically in $\Omega \times (0, T_{max})$, and moreover u satisfies that*

$$\textit{if} \quad T_{max} < \infty, \quad \textit{then} \quad \limsup_{t \nearrow T_{max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty.$$

Proof. The claim can be proved by a pointwise lower estimate for u and by a uniform estimate for $|\nabla u|$ almost in the same way as in [18]. □

In order to obtain the blow-up phenomena in finite time of L^∞-norm of solutions on (1.1), we follow the proof of Theorem 1.1 in [11] .

So we first transform the system (1.1) in a non local scalar parabolic equation.

Transformation of (1.1) in nonlocal scalar parabolic equation.

Assume $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \ge 3$, $R > 0$ and $u_0 \in C^0(\overline{\Omega})$ is radially symmetric with respect to $x = 0$. If (u, v) is the corresponding radial solution in $\Omega \times (0, T_{max})$, we write $u = u(r, t)$ and $v = v(r, t)$ with $r = |x| \in [0, R]$.

Following Jäger–Luckhaus [19] we introduce the mass accumulation function

$$w(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} u(\rho,t)d\rho, \quad s = r^N \in [0, R^N], \quad t \in [0, T_{max}). \tag{2.1}$$

Then we have

$$w_s(s,t) = \frac{1}{N} u(s^{\frac{1}{N}}, t) > 0, \quad w_{ss}(s,t) = \frac{1}{N^2} s^{\frac{1}{N}-1} u_r(s^{\frac{1}{N}}, t).$$

From the second equation in (1.1) we deduce

$$\frac{1}{r^{N-1}} \left(r^{N-1} v_r(r,t)\right)_r = \mu - u(r,t), \quad \left(r^{N-1} v_r(r,t)\right)_r = \mu r^{N-1} - u(r,t)\, r^{N-1} \tag{2.2}$$

and by integrating (2.2),

$$r^{N-1}v_r(r,t) = \mu\int_0^r \rho^{N-1}d\rho - \int_0^r \rho^{N-1}u(\rho,t)d\rho = \frac{\mu r^N}{N} - w.$$

Using (1.1), we obtain

$$\begin{aligned} w_t(s,t) &= \int_0^{s^{\frac{1}{N}}} r^{N-1}u_t(r,t)dr \\ &= \int_0^{s^{\frac{1}{N}}} \Big(r^{N-1}\frac{uu_r}{\sqrt{u^2+u_r^2}}\Big)_r dr - \chi\int_0^{s^{\frac{1}{N}}}\Big(r^{N-1}u(r,t)v_r f(v_r^2)\Big)_r dr \\ &= s^{1-\frac{1}{N}}\frac{u\,u_r}{\sqrt{u^2+u_r^2}} - \chi k_f s^{1-\frac{1}{N}}\frac{u\,v_r}{(1+v_r^2)^{\alpha}}. \end{aligned}$$

Since $u = N\ w_s$ and $u_r = N^2 s^{1-\frac{1}{N}} w_{ss}$, we derive

$$\begin{aligned} w_t(s,t) =&\ s^{1-\frac{1}{N}}\frac{Nw_s\ N^2 s^{1-\frac{2}{N}}w_{ss}}{\sqrt{N^2w_s^2+N^4s^{2-\frac{2}{N}}w_{ss}^2}} - \chi k_f\frac{Nw_s\ (\frac{\mu}{N}s-w)}{(1+(\frac{\mu}{N}s^{\frac{1}{N}}-s^{\frac{1}{N}-1}w)^2)^{\alpha}} \\ =&\ N^2\frac{s^{2-\frac{2}{N}}w_s\ w_{ss}}{\sqrt{w_s^2+N^2s^{2-\frac{2}{N}}w_{ss}^2}} + \chi k_f\cdot N\frac{w_s\ (w-\frac{\mu}{N}s)}{(1+(s^{\frac{1}{N}-1}w-\frac{\mu}{N}s^{\frac{1}{N}})^2)^{\alpha}} \end{aligned}$$

for $s\in(0,R^N)$ and $t\in(0,T_{max})$. Thus we can conclude that w satisfies the parabolic problem

$$\begin{cases} w_t = N^2\dfrac{s^{2-\frac{2}{N}}w_s\ w_{ss}}{\sqrt{w_s^2+N^2s^{2-\frac{2}{N}}w_{ss}^2}} + \chi k_f\cdot N\dfrac{w_s\ (w-\frac{\mu}{N}s)}{(1+(s^{\frac{1}{N}-1}w-\frac{\mu}{N}s^{\frac{1}{N}})^2)^{\alpha}}, \\ w(0,t)=0, \qquad w(R^N,t)=\dfrac{\mu R^N}{N}, \qquad t\in(0,T_{max}), \\ w(s,0)=w_0(s), \qquad s\in(0,R^N) \end{cases} \tag{2.3}$$

with

$$w_0(s) = \int_0^{s^{\frac{1}{N}}}\rho^{N-1}u_0(\rho)d\rho, \quad s\in[0,R^N].$$

The next step is to prove that the functional $\int_0^{R^N} s^{-a}w^b(s,t)ds$, with $b\in(0,1)$, satisfies a differential inequality.

To this end we first prove the following lemma.

Lemma 2.2. *Assume $\Omega = B_R(0)\subset\mathbb{R}^N$ with some $R>0$ and $N\geq 2$. Let $u_0\in C^0(\overline{\Omega})$ be radial and let (u,v) be the solution of (1.1). Then for all $a>0$, $b\in(0,1)$ and $\epsilon>0$ the function w defined in (2.1) satisfies*

$$\begin{aligned} &\frac{1}{b}\frac{d}{dt}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)ds \\ &\geq -N\int_0^{R^N}s^{1-\frac{1}{N}}(s+\epsilon)^{-a}w^{b-1}w_s\ ds \\ &\quad + Nk_f\overline{C}\int_0^{R^N}s^{(2-\frac{2}{N})\alpha}(s+\epsilon)^{-a}w^bw_sds - \mu\chi k_f\int_0^{R^N}s(s+\epsilon)^{-a}w^{b-1}w_sds \end{aligned} \tag{2.4}$$

for all $t\in(0,T_{max})$ and $\bar{C} := \bar{C}(R,N,\mu,\alpha)$ a positive constant.

Proof. Following Lemma 3.2 in [12], we multiply the first equation in (2.3) by $(s+\epsilon)^{-a}w^{b-1}(s,\tau)$ and integrating from 0 to R^N we obtain

$$\begin{aligned} &\frac{1}{b}\frac{d}{dt}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)ds \\ &= N^2\int_0^{R^N}s^{2-\frac{2}{N}}(s+\epsilon)^{-a}w^{b-1}\frac{w_sw_{ss}}{\sqrt{w_s^2+N^2s^{2-\frac{2}{N}}w_{ss}^2}}ds \\ &\quad + N\chi k_f\int_0^{R^N}(s+\epsilon)^{-a}w^{b-1}\frac{w_s(w-\frac{\mu}{N}s)}{\left[1+s^{\frac{2}{N}-2}(w-\frac{\mu}{N}s)^2\right]^{\alpha}}ds \end{aligned}$$

$$=: \ \mathcal{I}_1 + \mathcal{I}_2. \tag{2.5}$$

To estimate $\mathcal{I}_1$, since $w_s \geq 0$, we observe that $w_s w_{ss} \geq -|w_{ss}|w_s$ and thus we obtain

$$\frac{w_s w_{ss}}{\sqrt{w_s^2 + N^2 s^{2-\frac{2}{N}} w_{ss}^2}} \geq \frac{-w_s|w_{ss}|}{N s^{1-\frac{1}{N}} |w_{ss}|}.$$

Then we can write

$$\begin{aligned}\mathcal{I}_1 &= N^2 \int_0^{R^N} s^{2-\frac{2}{N}} (s+\epsilon)^{-a} w^{b-1} \frac{w_s w_{ss}}{\sqrt{w_s^2 + N^2 s^{2-\frac{2}{N}} w_{ss}^2}} \, ds \\ &\geq -N \int_0^{R^N} s^{1-\frac{1}{N}} (s+\epsilon)^{-a} w^{b-1} w_s \, ds.\end{aligned} \tag{2.6}$$

We now consider $\mathcal{I}_2$ and we obtain that

$$\begin{aligned}\mathcal{I}_2 &= N\chi k_f \int_0^{R^N} (s+\epsilon)^{-a} w^{b-1} \frac{w_s(w - \frac{\mu}{N}s)}{\left[1 + s^{\frac{2}{N}-2}(w - \frac{\mu}{N}s)^2\right]^{\alpha}} ds \\ &= N\chi k_f \int_0^{R^N} (s+\epsilon)^{-a} w^{b} \frac{w_s}{\left[1 + s^{\frac{2}{N}-2}(w - \frac{\mu}{N}s)^2\right]^{\alpha}} ds \\ &\quad - \mu\chi k_f \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} \frac{w_s}{\left[1 + s^{\frac{2}{N}-2}(w - \frac{\mu}{N}s)^2\right]^{\alpha}} ds \\ &=: \ \mathcal{I}_{21} + \mathcal{I}_{22}.\end{aligned} \tag{2.7}$$

To estimate the term $\mathcal{I}_{21}$ we observe that since $w \leq \frac{\mu}{N} R^N$ and $s \leq R^N$ we obtain $(w - \frac{\mu}{N}s)^2 \leq w^2 + \frac{\mu^2}{N^2}s^2 \leq 2\frac{\mu^2}{N^2}R^{2N}$. So we can write

$$\frac{1}{[1 + s^{\frac{2}{N}-2}(w - \frac{\mu}{N}s)^2]^{\alpha}} = \frac{s^{(2-\frac{2}{N})\alpha}}{[s^{2-\frac{2}{N}} + (w - \frac{\mu}{N}s)^2]^{\alpha}} \geq \frac{s^{(2-\frac{2}{N})\alpha}}{[R^{2N-2} + 2\frac{\mu^2}{N^2}R^{2N}]^{\alpha}} =: \ \overline{C} s^{(2-\frac{2}{N})\alpha},$$

with $\overline{C} = \frac{1}{[R^{2N-2} + 2\frac{\mu^2}{N^2}R^{2N}]^{\alpha}}$. By substituting the previous estimate in $\mathcal{I}_{21}$ we arrive at

$$\mathcal{I}_{21} \geq N k_f \overline{C} \int_0^{R^N} s^{(2-\frac{2}{N})\alpha} (s+\epsilon)^{-a} w^b w_s ds. \tag{2.8}$$

Now we estimate $\mathcal{I}_{22}$.

Since $\frac{1}{[1+s^{\frac{2}{N}-2}(w-\frac{\mu}{N}s)^2]^{\alpha}} \leq 1$, we obtain

$$\begin{aligned}\mathcal{I}_{22} &= -\mu\chi k_f \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} \frac{w_s}{\left[1 + s^{\frac{2}{N}-2}(w - \frac{\mu}{N}s)^2\right]^{\alpha}} ds \\ &\geq -\mu\chi k_f \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s ds.\end{aligned} \tag{2.9}$$

Substituting (2.6), (2.7) (with (2.8) and (2.9)) in (2.5) we obtain (2.4). □

In order to obtain a differential inequality for the functional $\int_0^{R^N} s^{-a} w^b(s,t)dt$, we take into account of Lemma 2.2 and we prove the following lemma.

Lemma 2.3. *Suppose that α fulfills (1.6). Then there exist $a > 0$, $b \in (0,1)$ satisfying*

$$\begin{aligned}&1 < a < 2(1-\alpha)\frac{N-1}{N}, \\ &b_0 < b < 1, \quad b_0 := \frac{\frac{N}{N-1}a - 1}{1 - 2\alpha}.\end{aligned} \tag{2.10}$$

and some positive $\delta_1(a,b,\mu,\chi,k_f)$, $\delta_2(a,b,N,k_f)$ and $\delta_3(a,b,R,N,\mu,\chi,k_f)$ such that

$$\begin{aligned}\frac{1}{b}\int_0^{R^N} s^{-a} w^b(s,t) ds \geq \frac{1}{b}\int_0^{R^N} s^{-a} w_0^b(s) ds - \delta_1 \int_0^t \int_0^{R^N} s^{-a} w^b(s,\tau) ds d\tau \\ + \delta_2 \int_0^t \left(\int_0^{R^N} s^{-a} w^b(s,\tau) ds\right)^{\frac{b+1}{b}} d\tau - \delta_3 t.\end{aligned} \tag{2.11}$$

Proof. We recall the inequality (2.4):

$$\frac{1}{b}\frac{d}{dt}\int_0^{R^N}(s+\epsilon)^{-a}w^b(s,t)ds \geq -N\int_0^{R^N} s^{1-\frac{1}{N}}(s+\epsilon)^{-a}w^{b-1}w_s\,ds \\ + Nk_f\overline{C}\int_0^{R^N} s^{2(1-\frac{1}{N})\alpha}(s+\epsilon)^{-a}w^b w_s ds \\ -\mu\chi k_f\int_0^{R^N} s(s+\epsilon)^{-a}w^{b-1}w_s ds \\ =: I_1+I_2+I_3. \tag{2.12}$$

Integration by parts of I_1 of (2.12) leads to

$$\begin{aligned} I_1 &= -N\int_0^{R^N} s^{1-\frac{1}{N}}(s+\epsilon)^{-a}w^{b-1}w_s\,ds \\ &= -\frac{N}{b}\left[s^{1-\frac{1}{N}}(s+\epsilon)^{-a}w^b\right]_0^{R^N} + \frac{N}{b}\int_0^{R^N}\frac{d}{ds}\left(s^{1-\frac{1}{N}}(s+\epsilon)^{-a}\right)w^b ds \\ &= -\frac{N}{b}R^{N-1}(R^N+\epsilon)^{-a}w^b(R^N,t) + \frac{N}{b}\int_0^{R^N}\frac{d}{ds}\left(s^{1-\frac{1}{N}}(s+\epsilon)^{-a}\right)w^b ds \\ &= -\frac{N}{b}R^{N-1}(R^N+\epsilon)^{-a}\frac{\mu^b R^{bN}}{N^b} + \frac{N}{b}\int_0^{R^N}\frac{d}{ds}\left(s^{1-\frac{1}{N}}(s+\epsilon)^{-a}\right)w^b ds \\ &= -A(\epsilon) + \frac{N}{b}\left(1-\frac{1}{N}\right)\int_0^{R^N} s^{-\frac{1}{N}}(s+\epsilon)^{-a}w^b ds - a\frac{N}{b}\int_0^{R^N} s^{1-\frac{1}{N}}(s+\epsilon)^{-a-1}w^b ds, \end{aligned}$$

with $A(\epsilon) = \frac{N}{b}R^{N-1}(R^N+\epsilon)^{-a}\frac{\mu^b R^{bN}}{N^b}$.

Thus we can write

$$I_1 \geq -A(\epsilon) + \frac{N}{b}\left(1-\frac{1}{N}\right)\int_0^{R^N} s^{-\frac{1}{N}}(s+\epsilon)^{-a}w^b ds \\ -a\frac{N}{b}\int_0^{R^N} s^{-\frac{1}{N}}(s+\epsilon)^{-a}w^b ds \\ = -A(\epsilon) - \frac{N}{b}\left[a-(1-\frac{1}{N})\right]\int_0^{R^N} s^{-\frac{1}{N}}(s+\epsilon)^{-a}w^b ds. \tag{2.13}$$

Here, taking into account of (2.10) and (2.15) we observe that $\gamma < 1-\frac{2}{N} < 1-\frac{1}{N} < a$, so that $a-\gamma > 0$, and hence we can define $\delta > 0$ such that $\delta := \frac{Nk_f\overline{C}}{2(b+1)}(a-\gamma)$. Since $a > 1$ by (2.10), using the Young inequality in the last term of (2.13), we have

$$I_1 \geq -A(\epsilon) - C_\delta\int_0^{R^N} s^{-\frac{b+1}{N}}(s+\epsilon)^{-a(b+1)+b(a+1-\gamma)}ds \\ -\frac{b}{b+1}\delta\int_0^{R^N}(s+\epsilon)^{-(a+1-\gamma)}w^{b+1}ds \geq -A(\epsilon) \\ -C_\delta\int_0^{R^N} s^{-\frac{b+1}{N}}(s+\epsilon)^{(-a+b-b\gamma)}ds - \delta\int_0^{R^N}(s+\epsilon)^{-(a+1-\gamma)}w^{b+1}ds, \tag{2.14}$$

where

$$\gamma := 2(1-\frac{1}{N})\alpha < 2(1-\frac{1}{N})\frac{N-2}{2(N-1)} = \frac{N-2}{N} < 1, \tag{2.15}$$

and

$$C_\delta := \frac{1}{b+1}\left\{\frac{N}{b}\left[a-(1-\frac{1}{N})\right]\right\}^{b+1}\delta^{-b}. \tag{2.16}$$

We now estimate I_2. Integrating by parts, we have

$$\begin{aligned} I_2 &= Nk_f\overline{C}\int_0^{R^N} s^{2(1-\frac{1}{N})\alpha}(s+\epsilon)^{-a}w^b w_s ds \\ &= \frac{Nk_f\overline{C}}{b+1}[(s^\gamma(s+\epsilon)^{-a}w^{b+1})]_0^{R^N} - \frac{Nk_f\overline{C}}{b+1}\int_0^{R^N}\frac{d}{ds}(s^\gamma(s+\epsilon)^{-a})w^{b+1}ds \\ &\geq -\frac{Nk_f\overline{C}}{b+1}\int_0^{R^N}[\gamma s^{\gamma-1}(s+\epsilon)^{-a} - as^\gamma(s+\epsilon)^{-a-1}]w^{b+1}ds \end{aligned} \tag{2.17}$$

A similar calculation of I_3 of (2.12) leads to

$$\begin{aligned} I_3 &= -\mu\chi k_f \int_0^{R^N} s(s+\epsilon)^{-a} w^{b-1} w_s ds = -\frac{\mu\chi k_f}{b}\left[s(s+\epsilon)^{-a}w^b\right]_0^{R^N} \\ &+ \frac{\mu\chi k_f}{b}\int_0^{R^N} \frac{d}{ds}\Big(s(s+\epsilon)^{-a}\Big) w^b ds = -\frac{\mu\chi k_f}{b} R^N (R^N+\epsilon)^{-a}\frac{\mu^b R^{bN}}{N^b} \\ &+ \frac{\mu\chi k_f}{b}\int_0^{R^N} [(s+\epsilon)^{-a} - as(s+\epsilon)^{-a-1}] w^b ds \\ &\geq -A_1(\epsilon) - \frac{\mu\chi k_f}{b}\big(a-1\big)\int_0^{R^N} (s+\epsilon)^{-a} w^b ds, \end{aligned} \tag{2.18}$$

with $A_1(\epsilon) = \frac{\mu^{1+b}\chi k_f}{b} R^N (R^N+\epsilon)^{-a}\frac{R^{bN}}{N^b}$.

By replacing (2.14), (2.17) and (2.18) into (2.12) we have

$$\begin{aligned} &\frac{1}{b}\frac{d}{dt}\int_0^{R^N} (s+\epsilon)^{-a} w^b(s,t) ds \\ &\geq -C_\delta \int_0^{R^N} s^{-\frac{b+1}{N}} (s+\epsilon)^{(-a+b-b\gamma)} ds - \delta\int_0^{R^N} (s+\epsilon)^{-(a+1-\gamma)} w^{b+1} ds \\ &\quad - \frac{Nk_f\overline{C}}{b+1}\int_0^{R^N} [\gamma s^{\gamma-1}(s+\epsilon)^{-a} - as^\gamma (s+\epsilon)^{-a-1}] w^{b+1} ds \\ &\quad - \frac{\mu\chi k_f}{b}\big(a-1\big)\int_0^{R^N} (s+\epsilon)^{-a} w^b ds - (A(\epsilon)+A_1(\epsilon)). \end{aligned}$$

Integrating from 0 to $t \in (0, T_{max})$ we arrive to

$$\begin{aligned} &\frac{1}{b}\int_0^{R^N} (s+\epsilon)^{-a} w^b(s,t) ds - \frac{1}{b}\int_0^{R^N} (s+\epsilon)^{-a} w_0^b(s) ds \\ &\geq -C_\delta t\int_0^{R^N} s^{-\frac{b+1}{N}} (s+\epsilon)^{(-a+b-b\gamma)} ds - \delta\int_0^t\int_0^{R^N} (s+\epsilon)^{-(a+1-\gamma)} w^{b+1} ds d\tau \\ &\quad - \frac{Nk_f\overline{C}}{b+1}\int_0^{R^N} [\gamma s^{\gamma-1}(s+\epsilon)^{-a} - as^\gamma (s+\epsilon)^{-a-1}] w^{b+1} ds \\ &\quad - \frac{\mu\chi k_f}{b}\big(a-1\big)\int_0^t\int_0^{R^N} (s+\epsilon)^{-a} w^b ds d\tau - \tilde{A}_\epsilon t, \end{aligned}$$

where $\tilde{A}_\epsilon = A(\epsilon) + A_1(\epsilon)$.

Now, from the monotone convergence theorem, taking $\epsilon \searrow 0$, we arrive to

$$\begin{aligned} &\frac{1}{b}\int_0^{R^N} s^{-a} w^b(s,t) ds - \frac{1}{b}\int_0^{R^N} s^{-a} w_0^b(s) ds \\ &\geq -C_\delta t\int_0^{R^N} s^{b-b\gamma-\frac{b+1}{N}-a} ds + \Big(\frac{Nk_f\overline{C}}{b+1}(a-\gamma) - \delta\Big)\int_0^t\int_0^{R^N} s^{\gamma-a-1} w^{b+1} ds d\tau \\ &\quad - \frac{\mu\chi k_f}{b}\big(a-1\big)\int_0^t\int_0^{R^N} s^{-a} w^b ds d\tau - \tilde{A} t, \end{aligned}$$

with $\tilde{A} = \lim_{\epsilon\searrow 0} \tilde{A}_\epsilon$.

Note that $\beta := (b - b\gamma - \frac{b+1}{N} - a) + 1 > 0$ by (2.10), so that $\int_0^{R^N} s^{b-b\gamma-\frac{b+1}{N}-a} ds = \frac{R^{N\beta}}{\beta}$. Hence we have

$$\begin{aligned} &\frac{1}{b}\int_0^{R^N} s^{-a} w^b(s,t) ds - \frac{1}{b}\int_0^{R^N} s^{-a} w_0^b(s) ds \\ &\geq \Big(\frac{Nk_f\overline{C}}{b+1}(a-\gamma) - \delta\Big)\int_0^t\int_0^{R^N} s^{\gamma-a-1} w^{b+1} ds d\tau \\ &\quad - \frac{\mu\chi k_f}{b}\big(a-1\big)\int_0^t\int_0^{R^N} s^{-a} w^b ds d\tau - \overline{A} t \end{aligned} \tag{2.19}$$

with

$$\overline{A} := \tilde{A} + C_\delta \frac{R^{N\beta}}{\beta} = R^{N-1-a}\frac{\mu^b}{b}\frac{R^{bN}}{N^b}\left[N + \mu\chi k_f R\right] + C_\delta\frac{R^{N\beta}}{\beta}.$$

Here, recalling the definition $\delta := \frac{Nk_f\overline{C}}{2(b+1)}(a-\gamma)$, we note that $\frac{Nk_f\overline{C}}{b+1}(a-\gamma) - \delta > 0$.

To estimate the term $\int_0^{R^N} s^{\gamma-a-1}w^{b+1}ds$ we apply the Hölder inequality:

$$\begin{aligned}
\int_0^{R^N} s^{-a}w^b ds &= \int_0^{R^N} s^{-a+\frac{b}{b+1}(a+1-\gamma)}\Big(s^{\gamma-a-1}w^{b+1}\Big)^{\frac{b}{b+1}}ds\\
&\le \Big(\int_0^{R^N} s^{-a(b+1)+b(a+1-\gamma)}ds\Big)^{\frac{1}{b+1}}\Big(\int_0^{R^n} s^{\gamma-a-1}w^{b+1}ds\Big)^{\frac{b}{b+1}}\\
&= \Big(\int_0^{R^N} s^{b-b\gamma-a}ds\Big)^{\frac{1}{b+1}}\Big(\int_0^{R^n} s^{\gamma-a-1}w^{b+1}ds\Big)^{\frac{b}{b+1}}\\
&= \overline{c}^{\,-1}\Big(\int_0^{R^n} s^{\gamma-a-1}w^{b+1}ds\Big)^{\frac{b}{b+1}}
\end{aligned}$$

with $\overline{c} = \frac{b-b\gamma-a+1}{R^{N(b-b\gamma-a+1)}}$ and $b-b\gamma-a > -1$ (since $b-b\gamma-a > b-b\gamma-\frac{b+1}{N}-a > -1$). From which we derive

$$\int_0^{R^N} s^{\gamma-a-1}w^{b+1}ds \ge \overline{c}\Big(\int_0^{R^N} s^{-a}w^b ds\Big)^{\frac{b+1}{b}}. \tag{2.20}$$

Substituting (2.20) in (2.19) we obtain

$$\begin{aligned}
\frac{1}{b}\int_0^{R^N} s^{-a}w^b(s,t)ds \ge{}& \frac{1}{b}\int_0^{R^N} s^{-a}w_0^b(s)ds - \frac{\mu\chi k_f}{b}\Big(a-1\Big)\int_0^t\int_0^{R^N} s^{-a}w^b ds d\tau\\
&+\Big(\frac{Nk_f\overline{C}}{b+1}(a-\gamma)-\delta\Big)\overline{c}\int_0^t\Big(\int_0^{R^N} s^{-a}w^b ds\Big)^{\frac{b+1}{b}}d\tau - \overline{A}t,
\end{aligned}$$

i.e. (2.11) with $\delta_1 := \frac{\mu\chi k_f}{b}\Big(a-1\Big)$, $\delta_2 := \Big(\frac{Nk_f}{b+1}\overline{C}(a-\gamma)-\delta\Big)\overline{c}$ and $\delta_3 := \overline{A}$. □

Proof of Theorem 1.1. Set $y(t) = \int_0^{R^N} s^{-a}w^b ds$. In light of Lemma 2.3, we notice that the inequality (2.11) can be rewritten in the following form :

$$y(t) \ge \delta_0 - \delta_1\int_0^t y(\tau)d\tau + \delta_2\int_0^t y^{1+d}(\tau)d\tau - \delta_3 t, \quad \forall t \in (0, T_{max}), \tag{2.21}$$

with $d = \frac{1}{b}$ and $\delta_0 = \frac{1}{b}\int_0^{R^N} s^{-a}w_0^b(s)ds$.

On the other hand, there exists $\epsilon_1 \in (0, \delta_2/\delta_1)$ such that

$$y(t) = \int_0^{R^N} s^{-a}w^b ds \le \epsilon_1\Big(\int_0^{R^N} s^{-a}w^b ds\Big)^{1+d} + c(\epsilon_1) = \epsilon_1 y^{1+d}(t) + c(\epsilon_1). \tag{2.22}$$

Using (2.22) in (2.21) we obtain

$$y(t) \ge \delta_0 + C_3\int_0^t y^{1+d}(\tau)d\tau - C_4 t, \quad \forall t \in (0, T_{max}),$$

where $C_3 := \delta_2 - \delta_1\epsilon_1 > 0$ (due to the choice of ϵ_1), $C_4 := \delta_1 c(\epsilon_1) + \delta_3$.

Let us introduce the function

$$z(t) = \delta_0 - \Big(\frac{C_4}{C_3}\Big)^{\frac{1}{1+d}} + C_3\int_0^t y^{1+d}d\tau - C_4 t,$$

which satisfies

$$\begin{aligned}
&z'(t) = C_3 y^{1+d}(t) - C_4 \ge C_3\Big(z(t) + \Big(\frac{C_4}{C_3}\Big)^{\frac{1}{1+d}}\Big)^{1+d} - C_4 \ge C_3 z^{1+d}(t),\\
&z(0) = \delta_0 - \Big(\frac{C_4}{C_3}\Big)^{\frac{1}{1+d}}.
\end{aligned} \tag{2.23}$$

Following the step in the proof of Theorem 0.1 in [20] we can conclude that $z(0) = \delta_0 - \Big(\frac{C_4}{C_3}\Big)^{\frac{1}{1+d}} > 0$ for the hypothesis on initial data. In fact, let us introduce the following nonnegative function

$$\psi_\varepsilon(s) := \frac{\mu}{N}\frac{R^N+\varepsilon}{s+\varepsilon}s, \quad s \in [0, R^N],\ \varepsilon > 0,$$

satisfying $\psi_\varepsilon \nearrow \frac{\mu R^N}{N}$ as $\varepsilon \searrow 0$. From the monotone convergence theorem, since $a > 1$, we obtain

$$\int_0^{R^N} s^{-a}\psi_\varepsilon^b(s)ds \to \infty, \quad as\ \varepsilon \searrow 0.$$

Finally, for some sufficiently small $\varepsilon > 0$ we define $w_0(s) := \psi_\varepsilon(s)$, $s \in [0, R^N]$ and we easily obtain $z(0) > 0$.

We observe that w_0 belongs to $C^\infty([0, R^N])$ with $w_0(0) = 0$, $w_0(R^N) = \frac{\mu R^N}{N}$ and $w_{0s}(s) > 0$ for all $s \in [0, R^N]$. As a consequence $u_0(x) := N w_{0s}(|x|^N)$ for $x \in \bar{\Omega}$ is radially symmetric, smooth and positive in $\bar{\Omega}$ with $\frac{1}{|\bar{\Omega}|}\int_\Omega u_0 dx = \mu$.

Since $z(0) > 0$, by comparison theorem $z(t)$ is positive. The inequality (2.23) is equivalent to

$$-\frac{1}{d}\left(z^{-d}(t)\right)' \geq C_3,$$

from which, integrating from 0 to t, we obtain

$$-z^{-d}(0) \leq z^{-d}(t) - z^{-d}(0) \leq -dC_3 t, \quad t \in (0, T_{max}),$$

and $t \leq \frac{z^{-d}(0)}{dC_3}$. If $t \to T_{max}$ then $z \to +\infty$ with $T_{max} \leq \frac{z^{-d}(0)}{dC_3}$.

The proof of Theorem 1.1 is completed by the blow-up criterion in Lemma 2.1. □

3. Blow-up in L^p-norm

In this section we prove by contradiction that $u(x,t)$ blows up in $L^p(\Omega)$-norm for all $p > N$. We have the following lemma.

Lemma 3.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 3$ be a bounded and smooth domain. Let (u, v) be a classical solution of system (1.1). Then if there exists $C > 0$ such that for some $p_0 > N$,*

$$\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C \quad \text{for any } t \in (0, T_{max}),$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \hat{C} \quad \text{for any } t \in (0, T_{max}).$$

Proof. Let us consider the L^p-norm of $u(\cdot,t)$ of (1.1) for all $p > \max\{p_0, 2N\}$. Using the first equation of (1.1) and then integrating by parts, we obtain

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= \int_\Omega u^{p-1}u_t dx\\ &= -\int_\Omega \nabla u^{p-1}\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}dx + \chi k_f\int_\Omega \nabla u^{p-1}\frac{u\nabla v}{(1+|\nabla v|^2)^\alpha}dx\\ &= -(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx + \chi k_f(p-1)\int_\Omega u^{p-1}\frac{\nabla u\nabla v}{(1+|\nabla v|^2)^\alpha}dx.\end{aligned}$$

We use the following inequality proved by Bellomo and Winkler in [1, Lemma 6.1]:

$$\int_\Omega u^{p-1}|\nabla u|dx \leq \int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx + \int_\Omega u^p dx.$$

Thus we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \leq -(p-1)\int_\Omega u^{p-1}|\nabla u|dx + (p-1)\int_\Omega u^p dx + \mathcal{J}, \tag{3.1}$$

for all $t \in (0, T_{max})$, where

$$\mathcal{J} := \chi k_f(p-1)\int_\Omega u^{p-1}\frac{\nabla u\nabla v}{(1+|\nabla v|^2)^\alpha}dx.$$

Integrating by parts and using the boundary conditions and the second equation of (1.1), we obtain

$$\begin{aligned}\mathcal{J} &= \chi k_f\frac{p-1}{p}\int_\Omega \frac{\nabla u^p\nabla v}{(1+|\nabla v|^2)^\alpha}dx \\ &= -\chi k_f\frac{p-1}{p}\int_\Omega \frac{u^p\Delta v}{(1+|\nabla v|^2)^\alpha}dx + \alpha\chi k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx\\ &= -\chi k_f\frac{p-1}{p}\int_\Omega \frac{u^p(\mu-u)}{(1+|\nabla v|^2)^\alpha}dx + \alpha\chi k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx\\ &\leq \chi k_f\frac{p-1}{p}\int_\Omega u^{p+1}dx + \alpha\chi k_f\frac{p-1}{p}\int_\Omega u^p\frac{\nabla v\cdot\nabla(|\nabla v|^2)}{(1+|\nabla v|^2)^{\alpha+1}}dx,\end{aligned} \tag{3.2}$$

where in the last step we neglected the negative term $-\mu\chi k_f\frac{p-1}{p}\int_\Omega u^p dx$ and we used the inequality $\frac{1}{(1+|\nabla v|^2)^\alpha} \leq 1$ as $\alpha > 0$.

To estimate the second term in (3.2) we follow the step in the proof of [12, (5.7)] and by using the radially symmetric setting we can obtain

$$\mathcal{J} \leq 2\mu\alpha\chi k_f\frac{p-1}{p}\int_\Omega u^p dx + \chi k_f\frac{p-1}{p}\left(1 + 2\alpha N(N-1)\bar{c}_1\right)\int_\Omega u^{p+1}dx \tag{3.3}$$

$$+ 2\alpha N(N-1)\overline{c}_2 \chi k_f \frac{p-1}{p}\Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p+1}{p+1+\epsilon}}$$

with some constants $\overline{c_1}, \overline{c_2} > 0$ depending on small $\epsilon > 0$.

Taking into account of (3.3) in (3.1), we can write

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le -\frac{p-1}{p}\int_\Omega |\nabla u^p| dx + \mathcal{A}\int_\Omega u^p dx + \mathcal{B}\int_\Omega u^{p+1} dx + \mathcal{C}\Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p+1}{p+1+\epsilon}} \tag{3.4}$$

for all $t \in (0, T_{max})$, where

$$\begin{cases} \mathcal{A} &= p-1+2\mu\alpha\chi k_f \frac{p-1}{p}, \\ \mathcal{B} &= \chi k_f \frac{p-1}{p}\big(1+2\alpha N(N-1)\overline{c}_1\big), \\ \mathcal{C} &= 2\alpha N(N-1)\overline{c}_2 \chi k_f \frac{p-1}{p}. \end{cases} \tag{3.5}$$

By using the Gagliardo–Nirenberg inequality and the Young inequality we estimate the term $\int_\Omega u^{p+1} dx$ as

$$\begin{aligned} \int_\Omega u^p dx = \|u^p\|_{L^1(\Omega)} &\le C_{GN}\|\nabla u^p\|^\theta_{L^1(\Omega)}\|u^p\|^{1-\theta}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|_{L^{\frac{1}{2}}(\Omega)}, \\ &\le C_{GN}\Big(\int_\Omega |\nabla u^p| dx\Big)^\theta\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^{2(1-\theta)} + C_{GN}\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^2 \\ &\le \epsilon_1\int_\Omega |\nabla u^p| dx + c(\epsilon_1)\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^2, \end{aligned} \tag{3.6}$$

where $\theta = \frac{N}{N+1} \in (0,1)$ and $\epsilon_1 > 0$ to be fixed later. Similarly, we obtain

$$\begin{aligned} \int_\Omega u^{p+1} dx = \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1}{p}}(\Omega)} &\le C_{GN}\|\nabla u^p\|^{\frac{p+1}{p}\bar{\theta}}_{L^1(\Omega)}\|u^p\|^{\frac{p+1}{p}(1-\bar{\theta})}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^{\frac{1}{2}}(\Omega)} \\ &\le \epsilon_2\int_\Omega |\nabla u^p| dx + c(\epsilon_2)\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^{\frac{2(p+1-N)}{p-2N}} + C_{GN}\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^{\frac{2(p+1)}{p}} \end{aligned} \tag{3.7}$$

where $\bar{\theta} = \frac{1+\frac{1}{p+1}}{1+\frac{1}{N}} = \frac{N(p+2)}{(N+1)(p+1)} \in (0,1)$ and $\epsilon_2 > 0$ to be fixed later. Also, we have

$$\begin{aligned} \Big(\int_\Omega u^{p+1+\epsilon} dx\Big)^{\frac{p+1}{p+1+\epsilon}} &= \|u^p\|^{\frac{p+1}{p}}_{L^{\frac{p+1+\epsilon}{p}}(\Omega)} \\ &\le C_{GN}\|\nabla u^p\|^{\tilde{\theta}\frac{p+1}{p}}_{L^1(\Omega)}\|u^p\|^{(1-\tilde{\theta})\frac{p+1}{p}}_{L^{\frac{1}{2}}(\Omega)} + C_{GN}\|u^p\|^{\frac{p+1}{p}}_{L^{\frac{1}{2}}(\Omega)} \\ &\le \epsilon_3\int_\Omega |\nabla u^p| dx + c(\epsilon_3)\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^\beta + C_{GN}\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^{\frac{2(p+1)}{p}}, \end{aligned} \tag{3.8}$$

where $\tilde{\theta} = \frac{1+\frac{1+\epsilon}{p+1+\epsilon}}{1+\frac{1}{N}} = \frac{N(p+2+2\epsilon)}{(N+1)(p+1+\epsilon)} \in (0,1)$ and $\tilde{\theta}\frac{p+1}{p} \in (0,1)$ for sufficiently small $\epsilon > 0$, and $\beta = \frac{2(p+1)[(p+1-N)-(N-1)\epsilon]}{(p+1)(p-2N)-((N-1)p+2N)\epsilon}$ and $\epsilon_3 > 0$ to be fixed later.

Substituting (3.6), (3.7), (3.8) in (3.4), and noting that $2 < \frac{2(p+1)}{p} < \frac{2(p+1-N)}{p-2N} < \beta$, we infer from the Young inequality that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le -\Big(\frac{p-1}{p} - \mathcal{A}\epsilon_1 - \mathcal{B}\epsilon_2 - \mathcal{C}\epsilon_3\Big)\int_\Omega |\nabla u^p| dx + c_1(\epsilon_1,\epsilon_2,\epsilon_3)\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^2 + c_2(\epsilon_1,\epsilon_2,\epsilon_3)\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^\beta.$$

Taking $\epsilon_1, \epsilon_2, \epsilon_3$ suitably small and using (3.6) in the form

$$\int_\Omega |\nabla u^p| dx \ge c_4\int_\Omega u^p dx - c_5\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^2,$$

we arrive at

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_6 p^{1+N}\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^2 + c_7 p\Big(\int_\Omega u^{\frac{p}{2}} dx\Big)^\beta. \tag{3.9}$$

Now we define

$$p := p_k = p_0 2^k, \tag{3.10}$$

for nonnegative integers k and let introduce

$$M_k := \sup_{t\in(0,T)} \int_\Omega u^{p_k}\,dx, \tag{3.11}$$

with $k \geq 1$ and $T \in (0, T_{max})$ is fixed.

From the definition of p in (3.10) and of M_k in (3.11) we have

$$\int_\Omega u^{\frac{p}{2}}\,dx = \int_\Omega u^{p_0 2^{k-1}} \leq M_{k-1}, \quad t \in (0,T).$$

This in conjunction with (3.9) implies that

$$\frac{d}{dt}\int_\Omega u^{p_k}\,dx + \int_\Omega u^{p_k}\,dx \leq c_6 p_k^{1+N} M_{k-1}^2 + c_7 p_k M_{k-1}^\beta.$$

By comparison arguments we obtain

$$\int_\Omega u^{p_k}\,dx = M_k \leq \max\left\{\int_\Omega u_0^{p_k}\,dx,\ c_6 p_k^{1+N} M_{k-1}^2 + c_7 p_k M_{k-1}^\beta\right\} \text{ for all } k \geq 1.$$

Now, if there exists a sequence $(k_j)_{j\in\mathbb{N}} \subset \mathbb{N}$ such that $k_j \to \infty$ as $j \to \infty$ and

$$M_{k_j} \leq \int_\Omega u_0^{p_{k_j}}\,dx \quad \text{for all } j \in \mathbb{N},$$

then we have

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^{p_{k_j}}(\Omega)} \leq \|u_0\|_{L^{p_{k_j}}(\Omega)},$$

and taking $j \to \infty$, we obtain

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)}.$$

Conversely, if no such sequence exists, then for some large $k_0 \in \mathbb{N}$ we have

$$M_k \leq c_6 p_k^{1+N} M_{k-1}^2 + c_7 p_k M_{k-1}^\beta \quad \text{for all } k \geq k_0.$$

Since $p_k = p_0 2^k$ and $2 < \beta \leq 2 + \frac{c_8}{2^k}$ for all $k \geq k_0$ with some large $k_0 \geq 1$, there exists a number $\delta > 1$ independent of T such that

$$M_k \leq \delta^k M_{k-1}^{2+\frac{c_8}{2^k}} \quad \text{for all } k \geq k_0,$$

and by induction and limiting procedure as in [21, p. 714] we obtain

$$\limsup_{k\to\infty} M_k^{\frac{1}{p_k}} \leq c_9$$

for some $c_9 > 0$. In view of the definition (3.11), this proves

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_9.$$

Thus in both cases we obtain that if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)}$ is bounded for some $p_0 > N$ then $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. □

Proof of Theorem 1.2. Taking into account of Lemma 3.1, if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C$ for some $p_0 > N$, then also $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. This is a contradiction since Theorem 1.1 holds. Therefore $\limsup_{t\nearrow T_{max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty$ for all $p > N$. □

Remark 3.1. The investigation on blow-up solutions of system (1.1) goes on with the study of the behavior near the blow-up time Tmax. Since it is not always possible to compute T_{max} we think that deriving a lower bound is a matter of great importance as in [12, Theorem 1.3].

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References

[1] N. Bellomo, M. Winkler, A degenerate Chemotaxis system with flux limitation: Maximally extended solutions and absence of gradient blow-up, Comm. Partial Differential Equations 42 (3) (2017) 436–473.
[2] E.F. Keller, L.A. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol. 26 (1970) 399–415.
[3] N. Bellomo, A. Belloquid, Y. Tao, M. Winkler, Toward a mathematical theory of Keller–Segel model of pattern formation in biological tissues, Math. Models Methods Appl. Sci. 25 (9) (2015) 1663–1763.
[4] M. Winkler, K.C. Djie, Boundedness and finite time collapse in a Chemotaxis system with volume filling effect, Nonlinear Anal. 72 (2) (2010) 1044–1074.
[5] M. Marras, T. Nishino, G. Viglialoro, A refined criterion and lower bounds for the blow-up time in a parabolic–elliptic Chemotaxis system with nonlinear diffusion, Nonlinear Anal. (2019) 111725.
[6] M. Negreanu, J.I. Tello, On a parabolic–elliptic system with gradient dependent chemotactic coefficient, J. Differential Equations 265 (2018) 733–751.
[7] J. Yan, Y. Li, Existence and boundedness of solutions for a Keller–Segel system with gradient dependent chemotactic sensitivity, Electron. J. Differential Equations 2020 (122) (2020) 1–14.
[8] S. Kohatsu, T. Yokota, Stability of constant equilibria in a Keller–Segel system with gradient dependent chemotactic sensitivity, Matematiche (Catania) 78 (1) (2023) 213–237.
[9] N. Bellomo, M. Winkler, Finite-time blow-up in a degenerate Chemotaxis system with flux limitation, Trans. Amer. Math. Soc. Ser. B 4 (2017) 31–67.
[10] M. Winkler, A critical blow-up exponent for flux limitation in a Keller–Segel system, preprint: arXiv:2010.01553.
[11] M. Marras, S. Vernier-Piro, T. Yokota, Blow-up phenomena for a Chemotaxis system with flux limitation, J. Math. Anal. Appl. 515 (2022) 126376.
[12] M. Marras, S. Vernier-Piro, T. Yokota, Behavior in time of solutions of a Keller–Segel system with flux limitation and source term, NoDEA Nonlinear Differential Equations Appl. 30 (2023) 65.
[13] Y. Chiyoda, M. Mizukami, T. Yokota, Finite-time blow-up in a quasilinear degenerate Chemotaxis system with flux limitation, Acta Appl. Math. 167 (2020) 231–259.
[14] M. Fuest, Approaching optimality in blow-up results for Keller–Segel systems with logistic-type dampening, NoDEA Nonlinear Differential Equations Appl. (2021) 1–17.
[15] M. Marras, S. Vernier-Piro, Finite time collapse in Chemotaxis systems with logistic-type superlinear source, Math. Methods Appl. Sci. 43 (17) (2020) 10027–10040.
[16] M. Marras, S. Vernier-Piro, G. Viglialoro, Decay in Chemotaxis systems with a logistic term, Discrete Contin. Dyn. Syst. Series S 13 (2) (2020) 257–268.
[17] M. Winkler, Finite-time blow-up in low-dimensional Keller–Segel systems with logistic-type superlinear degradation, Z. Angew. Math. Phys. 69 (40) (2018) 1–25.
[18] M. Mizukami, T. Ono, T. Yokota, Extensibility criterion ruling out gradient blow-up in a quasilinear degenerate Chemotaxis system with flux limitation, J. Differential Equations 267 (9) (2019) 5115–5164.
[19] W. Jäger, S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc. 329 (2) (1992) 819–824.
[20] M. Winkler, Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction, J. Math. Anal. Appl. 384 (2011) 261–272.
[21] Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic–parabolic Keller–Segel system with subcritical sensitivity, J. Differential Equations 252 (2012) 692–715.