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Embeddings of metric Boolean algebras in $\mathbb{R}^N$

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A R T I C L E I N F O

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A B S T R A C T

A Boolean algebra $\mathbf{A}$ equipped with a (finitely-additive) positive probability measure m can be turned into a metric space $(\mathbf{A}, d_m)$, where $d_m(a, b) = m((a \wedge \neg b) \vee (\neg a \wedge b))$, for any $a, b \in A$, sometimes referred to as *metric Boolean algebra*. In this paper, we study under which conditions the space of atoms of a finite metric Boolean algebra can be isometrically embedded in $\mathbb{R}^N$ (for a certain N) equipped with the Euclidean metric. In particular, we characterize the topology of the positive measures over a finite algebra $\mathbf{A}$ such that the metric space $(\mathsf{At}(\mathbf{A}), d_m)$ embeds isometrically in $\mathbb{R}^N$ (with the Euclidean metric).

1. Introduction

Every Boolean algebra $\mathbf{A}$ equipped with a *positive* probability measure m is a metric space, where the Kolmogorov distance between two elements $a, b \in A$ is given by the value (in $[0, 1]$) assigned by m to the symmetric difference between a and b, namely the element $(a \wedge \neg b) \vee (\neg a \wedge b)$. Algebras "metrized" by positive measures are called *metric Boolean algebras*, a nomenclature introduced by Kolmogorov [15], or *normed Boolean algebras* [21].

The main mathematical interest around metric (or normed) Boolean algebras mainly arises from probability theory and its subfield of stochastic geometry [8]. Moreover, these structures have recently found interesting applications in the theory of random sets (see e.g. [18]), which can be represented as a random element taking values in a normed Boolean algebra [7,21]. However, the observation that positive probability measures turn an algebra into a metric space adds an importance going far beyond logic and probability. Indeed, in the last few years, the interest around the geometry of discrete metric spaces has relevantly increased. Due to innovative ideas of Gromov [9] and others [1,3,13], the study of traditionally relevant

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geometrical properties usually conceived for Riemannian manifolds, such as length of curves, measures, sectional and Ricci curvatures, has been fruitfully extended to discrete spaces. In this context, the present work focuses on expanding the geometry of metric Boolean algebras and, in particular, to understand for which measures m these metric spaces resemble Euclidean spaces, in the sense that the subspace of atoms of a metric Boolean algebra can be isometrically embedded in the prototypical example of metric space, $\mathbb{R}^N$ (for some N) equipped with the Euclidean metric. The choice of the space of the atoms (forcing to confine our attention to atomic algebras only) is driven, on the one side, by its relevance in probability theory, as, in an atomic σ-algebra of events, the probability of any event depends on the probability distribution on the set of atoms and, on the other, by the impossibility of getting an isometric embedding of the entire algebra (see Remark 5).

The paper is structured as follows. In Section 2 are introduced all the preliminary notions necessary to go through the reading of the whole paper. Section 3 discusses the details of isometric embeddings of generic metric spaces into Euclidean spaces (ruled by Morgan's theorem) and shows the existence of a probability measure allowing the embedding of the atoms of a metric Boolean algebra in $\mathbb{R}^N$, for an appropriate N (Corollary 13). Section 4 contains the main contribution of the paper, namely the study of the topology of the space of the measures for which the metric space of the atoms of a finite Boolean algebra can be isometrically embedded in $\mathbb{R}^N$, for some N: the main finding is that this space is contractible, while its complement is simply connected but not contractible. Finally, Section 6 contains the proof of the very useful technical Lemma 11, which is applied throughout the whole paper, for establishing the existence/non-existence of isometric embeddings in $\mathbb{R}^N$.

2. Preliminaries: metric Boolean algebras

Let $\mathbf{A}$ be a Boolean algebra equipped with a *strictly positive* (finitely additive) probability measure, i.e. a map $m\colon \mathbf{A} \to [0,1]$ such that:

(1) $m(\top) = 1$,
(2) $m(a \vee b) = m(a) + m(b)$, for every $a, b \in A$ such that $a \wedge b = \perp$,
(3) $m(a) > 0$, for every $a \in A$, $a \neq \perp$,

where we indicate with $\top$ and $\perp$ the top and bottom element, respectively, of a Boolean algebra (to avoid confusion with the numbers $0, 1$).

The following recalls the well-known properties of probability measures.

Proposition 1. *Let m be a finitely additive probability measure over a Boolean algebra $\mathbf{A}$, then the following hold, for every $a, b \in A$:*

(1) m is monotone, i.e. if $a \leq b$ then $m(a) \leq m(b)$;
(2) $m(a) + m(b) = m(a \vee b) + m(a \wedge b)$;
(3) $m(a \vee b) \leq m(a) + m(b)$;
(4) $m(\neg a) = 1 - m(a)$;
(5) if m is strictly positive, $a < b$ implies $m(a) < m(b)$;
(6) If $\mathbf{A}$ is finite then $\sum_{i=1}^{k} m(a_i) = 1$, where $\{a_1, \ldots, a_k\}$ is the set of atoms of $\mathbf{A}$.

Remark 2. Let $\mathbf{A}$ be a Boolean algebra equipped with a strictly positive (finitely additive) probability measure m. Then $(\mathbf{A}, d_m)$ is a metric space (as observed by Kolmogorov [15]), where

$$d_m(a,b) \coloneqq m(a \vartriangle b) = m((a \wedge \neg b) \vee (\neg a \wedge b)),$$

for every $a, b \in A$. □

From now on, we always intend a Boolean algebra $\mathbf{A}$ equipped with a strictly positive (finitely additive) probability measure m and refers to it as *metric* Boolean algebra, namely as the metric space $(\mathbf{A}, d_m)$ (we will not make explicit reference to the metric d_m). Observe that the assumption that m is strictly positive is crucial to have a metric space: in case m is *not* strictly positive then $(\mathbf{A}, d_m)$ is just a pseudo-metric space.

In general, not every Boolean algebra can be equipped with a strictly positive (finitely additive) probability measure: a characterization of those Boolean algebras for which such measures exist is due to Kelley [14, Theorem 4] (see also [12]). For the purpose of the present paper, we observe that every atomic Boolean algebra whose sets of atoms are numerable admits a strictly positive measure [11, Theorem 2.5].

The basic properties of the term-operation $\vartriangle$ are recalled in the following.

Proposition 3. *Let $\mathbf{A}$ be a Boolean algebra and $a \vartriangle b \coloneqq (a \wedge \neg b) \vee (\neg a \wedge b)$, for every $a, b \in A$. Then the following hold:*

(1) $a \vartriangle \bot = a$,
(2) $a \vartriangle \top = \neg a$,
(3) $\neg a \vartriangle \neg b = a \vartriangle b$,
(4) $a \vartriangle \neg a = \top$.

We now prove an easy fact about atoms of (atomic) Boolean algebras that will be used in the next section. We indicate by $\mathsf{At}(\mathbf{A})$ the set of atoms of an atomic Boolean algebra $\mathbf{A}$.

Lemma 4. *Let $\mathbf{A}$ be an atomic metric Boolean algebra and let $a, b \in \mathsf{At}(\mathbf{A})$ (with $a \neq b$). Then $d_m(a,b) = m(a) + m(b)$.*

Proof. Let $a, b \in \mathsf{At}(\mathbf{A})$, with $a \neq b$. Since $a \wedge b = \bot$, then $a \leq \neg b$ and $b \leq \neg a$; thus $d_m(a,b) = m((a \wedge \neg b) \vee (\neg a \wedge b)) = m(a \wedge \neg b) + m(\neg a \wedge b) = m(a) + m(b)$. □

3. Isometric embeddings in $\mathbb{R}^N$

It is natural to wonder whether the metric space $(\mathbf{A}, d_m)$ embeds isometrically into $\mathbb{R}^N$ (for some N) with the Euclidean metric. Unfortunately this is never the case.

Remark 5. A Boolean algebra $\mathbf{A}$ (with $|A| > 2$) can *not* be isometrically embedded in $\mathbb{R}^N$, for any $N \in \mathbb{N}$, equipped with the Euclidean distance. Indeed, let $a \in A$ be any element different from the constants. Then $d_m(a, 0) = m(a)$, $d_m(a, 1) = m(\neg a) = 1 - m(a)$ and $d_m(0, 1) = 1$, which implies that any isometric embedding maps $a, 0, 1$ on the same line, where (for analogous reasoning) should lie also $\neg a$. Since $d_m(a, \neg a) = 1$, the only possibility is setting $\iota\colon \mathbf{A} \to \mathbb{R}^N$, $\iota(a) = \iota(0)$ and $\iota(\neg a) = \iota(1)$ (or, viceversa), but such an embedding can not be isometric, as $|\iota(a) - \iota(0)| = 0$ while $d_m(a, 0) \neq 0$. □

We now turn our attention to a relevant subspace of a metric atomic Boolean algebra, namely the space $\mathsf{At}(\mathbf{A})$ of its atoms (with the metric d_m).

Question. Is there a (finitely additive) strictly positive probability measure m such that the space $(\mathsf{At}(\mathbf{A}), d_m)$ can be isometrically embedded in $\mathbb{R}^N$, for some N?

Embeddings of generic metric spaces into $\mathbb{R}^N$ are ruled by a theorem of Morgan [19]. We recall some notions relevant for its introduction.

Definition 6. A metric space (X, d) is flat if the determinant of the $n \times n$ matrix $M(x_0, \ldots, x_n)$, whose generic entry is $\langle x_i, x_j, x_0\rangle = \frac{1}{2}(d(x_0, x_i)^2 + d(x_0, x_j)^2 - d(x_i, x_j)^2)$, is non-negative for every *n-simplex*, namely every choice of $n+1$ points $\{x_0, \ldots, x_n\}$ in X.

Example 7. The space $\mathbb{R}^N$ with the Euclidean metric is flat; the same holds for any subset of $\mathbb{R}^N$ with the metric induced by the Euclidean one. So, for example, the unit circle $S^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ is flat with the distance between two points given by the length of the segment in $\mathbb{R}^2$ connecting them. On the other hand, S^1 equipped with the geodesic distance is not flat.

Definition 8. The dimension of a space (X, d) is the greatest N (if exists) such that there exists a N-simplex whose determinant (according to Definition 6) is positive.

Theorem 9 *(Morgan [19]). A metric space (X, d) embeds in $\mathbb{R}^N$ if and only if it is flat and has dimension less or equal to N.*

Morgan's theorem is constructive. Indeed, given (X, d) flat and with dimension N, then the embedding into $\mathbb{R}^N$ is given by:

$$\begin{aligned} f \colon X &\to \mathbb{R}^N \\ x &\mapsto (\langle x, x_1, x_0\rangle, \ldots, \langle x, x_N, x_0\rangle), \end{aligned}$$

for a generic n-simplex $\{x_0, x_1, \ldots, x_N\}$.

From now on, our analysis will be confined to *finite* metric Boolean algebras.

Remark 10. In order to simplify notation, given a (finite) metric Boolean algebra $\mathbf{A}$ with $k+1$ atoms $\mathsf{At}(\mathbf{A}) = \{a_0, a_1, \ldots, a_k\}$, we set $x_\alpha = m(a_\alpha)$ (thus $x_\alpha > 0$, for every $\alpha \in \{0, 1, \ldots, k\}$). The matrix $M(x_0, \ldots, x_n)$, $2 \le n \le k$, introduced in Definition 6, associated to $\mathbf{A}$ has generic entry

$$\langle x_i, x_j, x_0\rangle = (x_0 + x_i)^2\delta_{ij} + (x_0^2 + x_0x_i + x_0x_j - x_ix_j)(1 - \delta_{ij}). \tag{1}$$

Indeed, $\langle x_i, x_j, x_0\rangle = \frac{1}{2}(d_m(a_0, a_i)^2 + d_m(a_0, a_j)^2 - d_m(a_i, a_j)^2)$, and, for $i = j$, $d_m(a_i, a_j) = 0$, hence, by Lemma 4, $\langle x_i, x_j, x_0\rangle = \frac{1}{2}(2(x_0 + x_i)^2) = (x_0 + x_i)^2$. Else, for $i \ne j$, $\langle x_i, x_j, x_0\rangle = \frac{1}{2}((x_0 + x_i)^2 + (x_0 + x_j)^2 - (x_i + x_j)^2) = \frac{1}{2}(x_0^2 + x_i^2 + 2x_0x_i + x_0^2 + x_j^2 + 2x_0x_j - x_i^2 - x_j^2 - 2x_ix_j) = x_0^2 + x_0x_i + x_0x_j - x_ix_j$. □

Lemma 11. *Let $\mathbf{A}$ be a finite metric atomic Boolean algebra with $k+1$ atoms and $M(x_0, \ldots, x_n)$, $2 \le n \le k$ be the matrix defined above. Then*

$$\det(M(x_0, \ldots, x_n)) = 2^{n-1}\left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \cdots \cdot \hat{x}_\alpha \cdot \cdots \cdot x_n\right)^2 - (n-1)\left(\sum_{\alpha=0}^{n} x_0^2 \cdot \cdots \cdot \hat{x}_\alpha^2 \cdots \cdot x_n^2\right)\right], \tag{2}$$

where $\hat{x}_i$ means that x_i has to be omitted.

Proof. It is displayed in Section 6. □

Remark 12. In the proof of Lemma 11, it is actually enough to assume that $x_0, \dots, x_n$ satisfies the property introduced in (1); thus from now on, we will weaken the assumption that $x_0, \dots, x_n$ are probabilities: we will just assume that they are elements in $(0, 1)$ satisfying equation (1).

Lemma 11 allows to provide a positive answer to the above stated question. Indeed, as a corollary we get the embedding of the atoms from a finite metric Boolean algebra which are assigned with the same probability, according to the *principle of indifference*.

Corollary 13. *Let* $\mathbf{A}$ *be a finite metric Boolean algebra with* $k+1$ *atoms* ($|A| = 2^{k+1}$) *and* m *a finitely additive probability measure such that* $x_i = m(a_i) = \frac{1}{k+1}$, *for every* $a_i \in \mathrm{At}(\mathbf{A})$. *Then* $\mathrm{At}(\mathbf{A})$ *embeds in* $\mathbb{R}^k$ *with the Euclidean metric.*

Proof. In virtue of Theorem 9, it is enough to show that $\mathrm{At}(\mathbf{A})$ is flat and has dimension k. By assumption $x_0 = x_1 = \dots = x_k = \frac{1}{k+1} > 0$. Hence, applying Lemma 11, we get

$$\begin{aligned}\det(M(x_0, \dots, x_n)) =& 2^{n-1}\left[\left((n+1)x_0^n\right)^2 - (n-1)(n+1)x_0^{2n}\right] \\ =& 2^{n-1}\left[(n+1)^2 x_0^{2n} - (n^2-1)x_0^{2n}\right] \\ =& 2^{n-1} 2n x_0^{2n} > 0,\end{aligned}$$

for every $2 \leq n \leq k$. $\square$

It follows from Theorem 9 and Lemma 11 that the space $\mathrm{At}(\mathbf{A})$ of a Boolean algebra $\mathbf{A}$ with 3 atoms ($|A| = 8$) embeds in $\mathbb{R}^2$ (it always true that a metric space of cardinality 3 embeds isometrically in $\mathbb{R}^2$!).

Remark 14. It is easy to check that $\det(M(x_0, x_1, x_2)) > 0$, a property that we will apply insofar with no explicit mention.

Remark 15. Observe that for any $\lambda \in \mathbb{R}$, $\det(M(\lambda x_0, \dots, \lambda x_n)) = \lambda^{2n} \det(M(x_0, \dots, x_n))$. Moreover, $\forall n$ ($2 \leq n \leq k$), $M(x_0, \dots, x_n) \geq 0$ if and only if $\left(\sum_{\alpha=0}^{n} \frac{1}{x_\alpha}\right)^2 - (n-1)\sum_{\alpha=0}^{n} \frac{1}{x_\alpha^2} \geq 0$. Indeed, by applying equation (2) in Lemma 11, we get

$$\left(\sum_{\alpha=0}^{n} \frac{x_0 \cdots \hat{x}_\alpha \cdots x_n}{x_0 \cdots x_n}\right)^2 - (n-1)\sum_{\alpha=0}^{n}\left(\frac{x_0 \cdots \hat{x}_\alpha \cdots x_n}{x_0 \cdots x_n}\right)^2 = \left(\frac{\sum_{\alpha=0}^{n} x_0 \cdots \hat{x}_\alpha \cdots x_n}{x_0 \cdots x_n}\right)^2 -$$

$$-(n-1)\frac{\sum_{\alpha=0}^{n}(x_0 \cdots \hat{x}_\alpha \cdots x_n)^2}{(x_0 \cdots x_n)^2} = \left(\sum_{\alpha=0}^{n} \frac{1}{x_\alpha}\right)^2 - (n-1)\sum_{\alpha=0}^{n} \frac{1}{x_\alpha^2}. \quad \square$$

It is not always the case that the space of atoms of a (finite) metric Boolean algebra embeds in $\mathbb{R}^N$. In the following we consider the probability assignment in accordance with the *binomial distribution*.

Example 16. The binomial distribution (with parameters n and p) is the probability distribution of the number of successes in a sequence of n independent experiments (Bernoulli process), each asking a "yes-no" question, and each with a two-valued outcome: success (with probability p) or failure (with probability

$q = 1 - p$). Thus the Boolean algebra of events is $\mathcal{P}(\Omega)$, where $\Omega = \{1, \dots, n\}$ and atoms consist of all the sequences (regardless of the order) of successes and failures. The probability $x_\alpha = m(a_\alpha)$ of an atom a_α of $\mathcal{P}(\Omega)$ is:

$$x_\alpha = \binom{n}{\alpha} p^\alpha (q)^{n-\alpha},$$

with $p \in (0, 1)$ and $q = 1 - p$. For the sake of simplicity, we set $p = q = \frac{1}{2}$. Relying on Remark 15, it is easy to check that $M(x_0, \dots, x_3), M(x_0, \dots, x_4) > 0$. On the other hand, for $n = 5$, we have $(1 + \frac{1}{5} + \frac{1}{10} + \frac{1}{10} + \frac{1}{5} + 1)^2 - 4(1 + \frac{1}{25} + \frac{1}{100} + \frac{1}{100} + \frac{1}{25} + 1) = -\frac{41}{25}$.

4. Main results

Let $\mathcal{M}(\mathbf{A})$ be the space of probability measures over a finite Boolean algebra $\mathbf{A}$, with $|A| = 2^{k+1}$ and $\mathcal{M}(\mathrm{At}(\mathbf{A}))$ the space of probability measures over the atoms of $\mathbf{A}$ ($|\mathrm{At}(\mathbf{A})| = k+1$). $\mathcal{M}(\mathrm{At}(\mathbf{A})) \subset (0, 1)^{k+1}$ is a convex (open) subspace of $(0, 1)^{k+1}$ (we are considering only the positive measures). $\mathcal{M}(\mathrm{At}(\mathbf{A}))$ can be naturally identified with the space $(0, 1)^{k+1} = (0, 1) \times \cdots \times (0, 1)$ by identifying a measure $m \in \mathcal{M}(\mathrm{At}(\mathbf{A}))$ with its values $\vec{x} = (x_0, x_1, \dots, x_k) \in \mathbb{R}_+^{k+1}$, $x_\alpha = m(a_\alpha)$, $\alpha = 0, \dots, k$.

The space $\mathcal{M}(\mathrm{At}(\mathbf{A}))$ can be described as:

$$\mathcal{M}(\mathrm{At}(\mathbf{A})) = (0, 1)^{k+1} \cap \Pi_k,$$

where Π_k is the interior of the standard k-simplex (or probability simplex) of $\mathbb{R}^{k+1}$, namely

$$\Pi_k = \{\vec{x} \in (0, 1)^{k+1} \mid \sum_{\alpha=0}^{k} x_\alpha = 1\}.$$

At the light of the above discussion, we define the space of measures $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ induced by the flat metric of $\mathbb{R}^N$, namely those measures m such that $(\mathrm{At}(\mathbf{A}), d_m)$ admits an isometric embedding into some Euclidean space $\mathbb{R}^N$. By Morgan's theorem, $m \in \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ if and only if the metric space $(\mathrm{At}(\mathbf{A}), d_m)$ is flat, hence:

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k, \tag{3}$$

where

$$C_n = \{\vec{x} \in \mathbb{R}_+^{k+1} \mid \det(M(x_0, \dots, x_n)) \geq 0\}, \text{ with } 3 \leq n \leq k. \tag{4}$$

Notice that we are taking $\vec{x} \in \mathbb{R}_+^{k+1}$ and not $\vec{x} \in (0, 1)^{k+1}$. We are interested in the solution of the following.

Problem. Study the topology of $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$, and of its complement $\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$, with the topology induced by $(0, 1)^{k+1} \subset \mathbb{R}_+^{k+1}$.

The main contribution of the present paper is the solution to the above mentioned problem (see Theorem 21). In order to tackle it, we begin by analyzing the topology of C_n.

Lemma 17. *For each $3 \leq n \leq k$, the space $C_n \cong H_n \times \mathbb{R}_+^{k-n}$ where H_n is a solid half-hypercone in $\mathbb{R}_+^{n+1}$.*

Proof. Consider the involutive homeomorphism

$$\Phi : \mathbb{R}_+^{k+1} \to \mathbb{R}_+^{k+1}, \vec{x} = (x_0, \dots, x_k) \mapsto (\frac{1}{x_0}, \dots, \frac{1}{x_k}). \tag{5}$$

In view of Lemma 11 and Remarks 14 and 15 the image of C_n via Φ is given by

$$\Phi(C_n) = \{(z_0, \dots, z_k) \in \mathbb{R}_+^{k+1} \mid (z_0 + \cdots + z_n)^2 - (n-1)(z_0^2 + \cdots + z_n^2) \geq 0,\ 3 \leq n \leq k\}.$$

Equivalently,

$$\Phi(C_n) = \{(z_0, \dots, z_k) \in \mathbb{R}_+^{k+1} \mid (n-2)\sum_{\alpha=0}^{n} z_\alpha^2 - 2 \sum_{\substack{\alpha,\beta=0 \\ \alpha<\beta}}^{n} z_\alpha z_\beta \leq 0\}.$$

We claim that, for every $3 \leq n \leq k$, $\Phi(C_n)$ is affinely homeomorphic the product of a solid half-hypercone $H_n \subset \mathbb{R}_+^{n+1}$ with $\mathbb{R}_+^{k-n}$, i.e. $\Phi(C_n) \cong H_n \times \mathbb{R}_+^{k-n} \subset \mathbb{R}_+^{k+1}$. In order to show the claim, observe that

$$\Phi(C_n) = \{(z_0, \dots, z_k) \in \mathbb{R}_+^{k+1} \mid \vec{z}^{\,t} A \vec{z} \leq 0\},$$

where A is the matrix (of order $n+1$)

$$A = \begin{pmatrix} n-2 & -1 & \cdots & -1 \\ -1 & n-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-2 \end{pmatrix}.$$

It is not difficult to check that the eigenvalues of A are $\lambda_0 = \lambda_1 = \cdots = \lambda_{n-1} = n-1$ and $\lambda_n = -2$ and an orthonormal basis of eigenvectors

$$v_0 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ v_i = c_i \begin{pmatrix} 1 \\ \vdots \\ 1 \\ -(i+1) \\ 0 \\ \vdots \end{pmatrix},\ v_n = \frac{1}{\sqrt{n+1}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix},$$

with $c_i = \frac{1}{\sqrt{(i+1)(i+2)}}$, for $i = \{1, \dots, n-1\}$, and 1 in the first $i+1$ entries. Thus if $P = (v_0 \dots v_n)$ is the associated orthogonal matrix and D the diagonal matrix of eigenvalues we can write

$$\begin{aligned} \Phi(C_n) &= \{(z_0, \dots, z_k) \in \mathbb{R}_+^{k+1} \mid \vec{z}^{\,t} P D P^t \vec{z} \leq 0\} = \{(y_0, \dots, y_k) \in \mathbb{R}^{k+1} \mid \vec{y}^{\,t} D \vec{y} \leq 0\} \cap \{P\vec{y} > 0\} \\ &= \{(y_0, \dots, y_k) \in \mathbb{R}^{k+1} \mid (n-1)^2(y_0^2 + \cdots + y_{n-1}^2) - 2y_n^2 \leq 0\} \cap \{P\vec{y} > 0\}, \end{aligned}$$

with $\vec{y} = P^t \vec{z}$. The affine transformation $P^{-1} = P^t \colon \mathbb{R}^{k+1} \to \mathbb{R}^{k+1}$ shows that, for every $3 \leq n \leq k$, $\Phi(C_n)$ is affinely homeomorphic to

$$(H_n \cap \mathbb{R}_+^{n+1}) \times \mathbb{R}_+^{k-n},$$

where $H_n = \{(y_0, \dots, y_n) \in \mathbb{R}^{n+1} \mid (n-1)^2(y_0^2 + \cdots + y_{n-1}^2) - 2y_n^2 \leq 0\}$ is a solid hypercone in $\mathbb{R}^{n+1}$. In order to prove the claim we have to show that $H_n \subset \mathbb{R}_+^{n+1}$.

Observe that H_n is indeed obtained by a rotation around the line generated by $(0,\dots,0,1)$ of angle α, with $\cos(\alpha) = \sqrt{\frac{n-2}{n}}$. Hence $P(H_n)$ is a solid hypercone obtained by a rotation around the line generated by $(1,\dots,1,1) \in \mathbb{R}^{n+1}$ of the same angle. On the other hand, the angle α_i between the vector $(1,\dots,1)$ and each vectors of the canonical basis e_i of $\mathbb{R}^{n+1}$ satisfies $\cos(\alpha_i) = \frac{1}{\sqrt{n+1}}$, for every $i \in \{0,\dots,n\}$. Thus, since $n \geq 3$,

$$\sqrt{\frac{n-2}{n}} = \cos(\alpha) > \cos(\alpha_i) = \frac{1}{\sqrt{n+1}},\ i = 0,\dots,n.$$

It follows that $\alpha < \alpha_i$ therefore $P(H_n) \subset \mathbb{R}^{n+1}_+$ proving our claim and concluding the proof of the lemma. $\square$

In order to provide an answer to the above stated problem, we need to prove some rather technical lemmas, whose proofs consist of adaptations of the proof techniques used in a standard results for compact convex subsets (with non-empty interior) in $\mathbb{R}^k$ (see e.g. [16, Proposition 5.1]).

Recall that a set $X \subseteq S^k$ in the unit k-dimensional sphere $S^k \subset \mathbb{R}^{k+1}$ is geodesically convex if, for every $x_1, x_2 \in X$ there exits a (unique) minimal geodesic connecting them in X.

Lemma 18. *Let $X \subseteq S^k_+ = S^k \cap \mathbb{R}^{k+1}_+$ be a compact, geodesically convex set with $\mathrm{Int}(X) \neq \emptyset$. Then any geodesic ray starting at $p \in \mathrm{Int}(X)$ intersects X in exactly one point.*

Proof. Let g be a geodesic ray starting at $p \in \mathrm{Int}(X)$. Since X is compact, then $X \cap g$ is compact and thus bounded. So let x_0 be the point having the maximal (spherical) distance from p, $x_0 = \max\{d(x,p) \mid x \in X\}$. It is immediate to verify that $x_0 \in \partial X$ (x_0 belongs to the boundary of X). Thus, in order to show that $X \cap g = \{x_0\}$ one needs to verify that $g \setminus \{x_0\} \subset \mathrm{Int}(X)$. Let $B_r(p)$ an open (spherical) ball contained in X and $I_c = \{\overline{yx_0} \mid y \in B_r(p)$ and $\overline{yx_0}$ the geodesic segment connecting y to $x_0\}$ (I_c is the "ice-cream cone" formed by the geodesic from $B_r(p)$ to x_0). Clearly, $g \subset I_c \subset X$, where the last inclusion holds since X is geodesically convex. For every $x \in g \setminus \{x_0\}$, there is an open ball $B_{r_x}(x) \subseteq I_c \setminus \{x_0\} \subset X$. This shows that $x \in \mathrm{Int}(X)$ and concludes the proof. $\square$

Lemma 19. *Let $K \subset \mathbb{R}^{k+1}$ be a compact, star-shaped space with respect to 0, with $\mathrm{Int}(K) \neq \emptyset$ and such that any ray from 0 intersects K in exactly one point. Then there exists a homeomorphism $F\colon \overline{B_1(0)} \to K$ such that $F(S^k) = \partial K$, where $B_1(0) \subset \mathbb{R}^{k+1}$ denotes the open unit ball centered at the origin.*

Proof. Define $f\colon \partial K \to S^k$, $x \mapsto f(x) = \frac{x}{|x|}$, i.e. $f(x)$ is the intersection of the ray (from O) with the sphere S^k. By definition, f is continuous and, since any ray from 0 intersects S in exactly one point, is also bijective. Since ∂K is compact, f is a homeomorphism by the closed map lemma. We then define $F\colon \overline{B_1(0)} \to K$ as $x \mapsto \begin{cases} |x| f^{-1}(\frac{x}{|x|}) & \text{if } x \neq 0, \\ 0, & \text{if } x = 0. \end{cases}$ F takes every radial segment $\overline{0x}$ with $x \in S^{k-1}$ in the radial segment $\overline{0f^{-1}(x)}$ in K, with $f^{-1}(x) \in \partial K$ (it is well defined since S is star-shaped in 0). F is continuous as f^{-1} is and $\lim_{x\to 0} F(x) = 0$; it is injective as any ray from 0 intersects K in exactly one point and surjective as any $y \in K$ belongs to some ray. Finally, since K is compact, F is a homeomorphism by the closed map lemma. $\square$

Lemma 20. *Let $K_1, K_2 \subseteq \mathbb{R}^{k+1}$ be compact, star-shaped spaces with respect to 0, with $\mathrm{Int}(K_i) \neq \emptyset$ (for $i = 1, 2$), $K_1 \subset \mathrm{Int}(K_2)$ and such that any ray from 0 intersects ∂K_i (for $i = 1, 2$) in exactly one point. Then there exists a homeomorphism $F\colon \overline{B_2(0)} \setminus B_1(0) \to K_2 \setminus \mathrm{Int}(K_1)$ such that $F(S_1^{k-1}) = \partial K_1$ and $F(S_2^{k-1}) = \partial K_2$. In particular*

$$\mathrm{Int}K_2 \setminus K_1 \cong B_2(0) \setminus \overline{B_1(0)} \cong S^k \times (0,1).$$

Proof. By Lemma 19, there are two homeomorphisms $F_1\colon S_1^{k-1} \to \partial K_1$, $F_2\colon S_2^{k-1} \to \partial K_2$. Define

$$F\colon \overline{B_2(0)} \setminus B_1(0)) \to K_2 \setminus \mathrm{Int}(K_1),\ x \mapsto F(x) = (2-|x|)F_1(\frac{x}{|x|}) + (|x|-1)F_2(\frac{2x}{|x|}).$$

By definition (and the continuity of F_1, F_2), F is continuous and surjective. To see that F is injective, observe that, for points x_1, x_2 belonging to different rays, this follows from the assumption that every ray from 0 intersects ∂K_i $(i = 1, 2)$ in exactly one point. Differently, let $x_2 = \lambda x_1$ (w.l.o.g. $\lambda > 0$) and $F(x_1) = F(x_2)$. Upon setting $F_1(\frac{x}{|x|}) = a \in \partial K_1$ and $F_2(\frac{x}{|x|}) = b \in \partial K_1$, we have $(2-|x_1|)a + (|x_1|-1)b = (2-|x_2|)a + (|x_2|-1)b$, thus $(|x_2|-|x_1|)(a-b) = 0$ and, since $a \neq b$, $|x_1| = |x_2|$, i.e. $x_1 = x_2$ (as $x_2 = \lambda x_1$). Finally, F is a homeomorphism by the closed map lemma. The last part follows by restricting F to $\mathrm{Int}K_2 \setminus K_1$ and by the fact that the annulus $B_2(0) \setminus \overline{B_1(0)}$ is homeomorphic to $S^k \times (0, 1)$. □

The solution to the above presented problem is given by the following.

Theorem 21. *Let $k \geq 3$. Then:*

(1) $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is contractible.
(2) $\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is simply-connected (not contractible).

Proof. (1) Consider the (open) retraction

$$s : \mathbb{R}_+^{k+1} \to \Pi_k \subset (0,1)^{k+1},\ \vec{x} = (x_0, \dots, x_k) \mapsto \frac{\vec{x}}{s(\vec{x})}, \tag{6}$$

where $s(\vec{x}) := \sum_{\alpha=0}^{k} x_\alpha$. Then

$$\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A})) = \bigcap_{n=3}^{k} C_n \cap \Pi_k = s\left(\bigcap_{n=3}^{k} C_n\right)$$

is contractible being a strong deformation retract of $\bigcap_{n=3}^{k} C_n$ which is contractible (it is homeomorphic to $\bigcap_{n=3}^{k} H_n \times \mathbb{R}_+^{k-n}$ by Lemma 17).

(2) Let $C = \bigcap_{n=3}^{k} C_n$ and $H = \bigcap_{n=3}^{k} H_n$ (the solid half-hypercones introduced in Lemma 17). Then, using (6), $\mathcal{M}(\mathrm{At}(\mathbf{A})) \setminus \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ is a strong deformation retract of $\mathbb{R}_+^{k+1} \setminus C \cong \mathbb{R}_+^{k+1} \setminus H$. Let $X = H \cap \overline{S_+^k}$. Then X is compact (the intersection of closed sets in the compact $\overline{S_+^k}$), $H \cong X \times (0, +\infty)$ (by Remark 15) and $\mathbb{R}_+^{k+1} \setminus H \cong (S_+^k \setminus X) \times (0, +\infty)$. Moreover, X has non-empty interior (it follows from Corollary 13 and the proof of Lemma 17 that, for instance, $p = (1, \dots, 1) \in Int(X)$) and it is geodesically convex as a subset of S_+^k. To see this, consider $x_1, x_2 \in X \subset H$. Let $\overline{x_1 x_2} \in H$ be the segment connecting x_1, x_2 (H is convex) and $Hyp \subset C$ the hypercone in $\mathbb{R}^{k+1}$ generated by $\overline{x_1 x_2}$ (the inclusion $Hyp \subset C$ follows by Remark 15). Then $Hyp \cap S_+^k \subset X$ is the minimal geodesic segment in S^k, since it is the intersection of S_+^k with an hyperplane containing 0, x_1 and x_2. Thus, by Lemma 18, the geodesic from $p = (1, \dots, 1)$ intersects ∂X exactly in one point. Consider the stereographic projection π from the $S^k \setminus \{-p\}$ to the tangent space $T_p S^k$ of the sphere S^k at the point p, namely the homeomorphism which to a point x of $S^k \setminus \{-p\}$ associates

the intersection of the line joining x to $-p$ with T_pS^k. Thus $K_1 := \pi(X)$ and $K_2 := \pi(\overline{S^k_+})$ are subsets in $\mathbb{R}^k \cong T_pS^k$ satisfying the assumptions of Lemma 20. It follows that

$$\mathrm{Int}K_2 \setminus K_1 \cong S^k_+ \setminus X \cong S^{k-1} \times (0,1)$$

and therefore $\mathbb{R}^{k+1}_+ \setminus C = S^k_+ \setminus X \times (0,+\infty) \cong S^{k-1} \times (0,1) \times (0,+\infty)$ is homotopically equivalent to S^{k-1}, thus simply connected ($k \geq 3$) and not contractible. □

Remark 22. Consider the set of measure which can be induced in $\mathbf{R}^N$ with a fixed N. Then one can show that this set is open in Π_k for $N \geq k$ since each C_n with $n \leq k$ is open.

5. Conclusion and future work

The space of *strictly positive* probability measures $\mathcal{M}(\mathbf{A})$ of a finite Boolean algebra $\mathbf{A}$ is an open convex sets (the boundary being given by all probability measures) in $[0,1]^n$, with $n = \mid A \mid$. The main contribution of the present work consists in "splitting" such convex space into the (disjoint) union of the contractible set $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$, corresponding to those measures m for which the space $(\mathrm{At}(\mathbf{A}), d_m)$ embeds isometrically in $\mathbb{R}^N$, and its complement. We have given examples of measures belonging to each of the two components: the measure corresponding to the principle of indifference (see Corollary 13) and the binomial distribution (see Example 16), respectively. As a remarkable consequence of our topological characterization, we get that $m \in \mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ if and only if m is homeotopically equivalent to the measure of indifference; on the other hand, all measures m for which $(\mathrm{At}(\mathbf{A}), d_m)$ can not be isometrically embedded in $\mathbb{R}^N$ are path connected (though not homeotopically equivalent) to the binomial distribution.

It is natural to wonder in which component of $\mathcal{M}(\mathrm{At}(\mathbf{A}))$ can be located well-known distributions finding applications in probability (e.g. the hypergeometrical distribution) and exploits the potential applications of the fact that measures belonging to the component $\mathcal{M}_{ind}(\mathrm{At}(\mathbf{A}))$ have the same homotopy. In order to tackle the former problem, refinements of the useful Lemma 11 shall be found to ease calculations.

Our main result (Theorem 21) relies on the topological characterization (see Lemma 17) of the objects $C_n = \{\vec{x} \in \mathbb{R}^{k+1}_+ \mid \det(M(x_0, \ldots, x_n)) \geq 0\}$ $(3 \leq n \leq k)$, for which the assumption about the finiteness of the Boolean algebra considered is crucial (the existence of atoms is only a byproduct). We leave to future work the extension of the present setting to infinite (atomic) Boolean algebras.

Finally, we confined our attention to metric Boolean algebras. However, in the last decades the theory of probability has been extended to algebraic semantics of several non-classical logics, via the development of the so-called theory of states. We will dedicate future work to the study of the metric properties, for instance, of MV-algebras (a study initiated in [20,17]) equipped with a faithful state (see e.g. [6,5,4]) or involutive bisemilattices (see [2]) and their isometric embeddability into Euclidean spaces.

6. Appendix

This section is dedicated to the proof of Lemma 11, whose technicalities are not so important (we believe) for the reading of the whole message of the paper.

Recall that, given a (square) matrix A of order n, the *adjugate* $Adj(A)$ of A is the transpose of the cofactor matrix of A. Equivalently, $Adj(A)$ is the matrix of order n such that $A \cdot Adj(A) = Adj(A) \cdot A = \det(A) \cdot \mathbb{I}_n$, where $\mathbb{I}_n$ is the identity matrix of order n. We recall here a result from linear algebra (see e.g. [10]) that we will use in the proof of Lemma 11.

Lemma 23 *(Matrix determinant lemma). Let A be a matrix of order n and u, v column vectors in $\mathbb{R}^n$. Then*

$$\det(A + uv^t) = \det(A) + v^t Adj(A) u.$$

Lemma 10. *Let* $\mathbf{A}$ *be a finite metric atomic Boolean algebra with* $k+1$ *atoms and* $M(x_0,\dots,x_n)$*,* $2 \le n \le k$ *be the matrix defined above. Then*

$$\det(M(x_0,\dots,x_n)) = 2^{n-1}\left[\left(\sum_{\alpha=0}^{n} x_0\cdot\dots\cdot\hat{x}_\alpha\cdot\dots\cdot x_n\right)^2 - (n-1)\left(\sum_{\alpha=0}^{n} x_0^2\cdot\dots\cdot\hat{x}_\alpha^2\cdot\dots\cdot x_n^2\right)\right],$$

where $\hat{x}_i$ *means that* x_i *has to be omitted.*

Proof. Preliminarily observe that $M(x_0,\dots,x_n) = A + vv^t$, where A is the matrix whose generic entry is $A_{ij} = -2(1-\delta_{ij})x_i x_j$ and $v = (x_0+x_1,\dots,x_0+x_n) \in \mathbb{R}^n$.

Claim 1. $\det(A) = -2^n x_1^2 \dots x_n^2(n-1)$.

Observe that $\det(A) = (-2)^n x_1^2\dots x_n^2 \det(B)$, where $B = \begin{pmatrix} 0 & 1 & \cdots & 1\\ 1 & 0 & \cdots & 1\\ \vdots & \vdots & \ddots & \vdots\\ 1 & 1 & \cdots & 0\end{pmatrix}$, i.e. $B_{ij} = 1-\delta_{ij}$. Moreover, $B = cc^t - \mathbb{I}_n$, for $c = (1,\dots,1)$. Thus, by Lemma 23, $\det(B) = \det(-\mathbb{I}_n) + c\mathrm{Adj}(-\mathbb{I}_n)c^t$. Observe that $\mathrm{Adj}(-\mathbb{I}_n) = (-1)^{n-1}\mathbb{I}_n$ (indeed $-\mathbb{I}_n(-1)^{n-1}\mathbb{I}_n = (-1)^n\mathbb{I}_n = \det(-\mathbb{I}_n)\cdot\mathbb{I}_n$), hence

$$\begin{aligned}
\det(B) &= \det(-\mathbb{I}_n) + c^t\mathrm{Adj}(-\mathbb{I}_n)c && \text{(Lemma 23)}\\
&= (-1)^n + c^t(-1)^{n-1}\mathbb{I}_n c\\
&= (-1)^n + (-1)^{n-1}c^t c\\
&= (-1)^n + (-1)^{n-1}n\\
&= (-1)^{n-1}(n-1).
\end{aligned}$$

It then follows that $\det(A) = (-2)^n x_1^2\cdot\dots\cdot x_n^2\det(B) = (-2)^n x_1^2\cdot\dots\cdot x_n^2(-1)^{n-1}(n-1) = -2^n x_1^2\cdot\dots\cdot x_n^2(n-1)$, showing Claim 1.

Claim 2. $\mathrm{Adj}(A) = D$*, with generic entry*

$$D_{ij} = 2^{n-1}x_1^2\dots x_n^2\left(\frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2}(n-1)\right).$$

By definition of adjugate, $A\cdot D = D\cdot A = \det(A)\cdot\mathbb{I}_n$, equivalently $\sum_{k=1}^{n} A_{ik}D_{kj} = \det(A)\cdot\delta_{ij}$.

$$\begin{aligned}
\sum_{k=1}^{n} A_{ik}D_{kj} &= \sum_{k=1}^{n} -2(1-\delta_{ik})x_i x_k\cdot\left(2^{n-1}x_1^2\dots x_n^2\left(\frac{1}{x_i x_j} - \frac{\delta_{kj}}{x_k^2}(n-1)\right)\right)\\
&= -2^n x_1^2\dots x_n^2\sum_{k=1}^{n}(1-\delta_{ik})\left(\frac{x_i}{x_j} - \frac{x_i\delta_{kj}}{x_k}(n-1)\right)\\
&= -2^n x_1^2\dots x_n^2\left(\frac{x_i}{x_j}\sum_{k=1}^{n}(1-\delta_{ik}) - x_i(n-1)\sum_{k=1}^{n}\frac{(1-\delta_{ik})\delta_{kj}}{x_k}\right)\\
&= -2^n x_1^2\dots x_n^2\left(\frac{x_i}{x_j}(n-1) - \frac{x_i}{x_j}(n-1) + x_i(n-1)\sum_{k=1}^{n}\frac{\delta_{ik}\delta_{kj}}{x_k}\right)
\end{aligned}$$

$$= -2^n x_1^2 \dots x_n^2 (n-1) x_i \sum_{k=1}^{n} \frac{\delta_{ik}\delta_{kj}}{x_k} = \det(A) \cdot \delta_{ij},$$

which shows Claim 2.

To simplify notation, let us fix $E_{ij} = \frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2}(n-1)$ (a part of the generic entry of D). In order to conclude, observe that, by Lemma 23,

$$\begin{aligned}
&\det(M(x_0, \dots, x_n)) = \det(A) + vDv^t \\
&= -2^n x_1^2 \dots x_n^2 (n-1) + 2^{n-1} x_1^2 \dots x_n^2 vEv^t \text{ (Claims 1, 2)} \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + vEv^t\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + \sum_{i,j=1}^{n} (x_0 + x_i) E_{ij} (x_0 + x_j)\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + \sum_{i,j=1}^{n} (x_0 + x_i) \left(\frac{1}{x_i x_j} - \frac{\delta_{ij}}{x_i^2}(n-1)\right)(x_0 + x_j)\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + \sum_{i,j=1}^{n} \left(\frac{x_0}{x_i} + 1\right)\left(\frac{x_0}{x_j} + 1\right) - \sum_{i,j=1}^{n} \left(\frac{x_0}{x_i} + 1\right)(x_0 + x_j)\frac{\delta_{ij}(n-1)}{x_i}\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + \left[\sum_{i=1}^{n} \left(\frac{x_0}{x_i} + 1\right)\right]^2 - (n-1)\sum_{i=1}^{n} \left(\frac{x_0}{x_i} + 1\right)^2\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left(-2(n-1) + \left[n - 1 + 1 + \sum_{i=1}^{n} \frac{x_0}{x_i}\right]^2 - (n-1)\left(\sum_{i=1}^{n} \frac{x_0^2}{x_i^2} + 2\sum_{i=1}^{n} \frac{x_0}{x_i} + n + 1 - 1\right)\right) \\
&= -2^{n-1} x_1^2 \dots x_n^2 (-2(n-1) + \left(1 + \sum_{i=1}^{n} \frac{x_0}{x_i}\right)^2 + (n-1)^2 + 2(n-1)\sum_{i=1}^{n} \frac{x_0}{x_i} + 2(n-1) - \\
&- (n-1)\sum_{i=1}^{n} \frac{x_0^2}{x_i^2} - 2(n-1)\sum_{i=1}^{n} \frac{x_0}{x_i} - (n-1)^2 - (n-1)) \\
&= -2^{n-1} x_1^2 \dots x_n^2 \left[\left(1 + \sum_{i=1}^{n} \frac{x_0}{x_i}\right)^2 - (n-1)\left(1 + \sum_{i=1}^{n} \frac{x_0^2}{x_i^2}\right)\right] \\
&= -2^{n-1} \left[\left(x_1 \dots x_n + \sum_{i=1}^{n} \frac{x_0 x_1 \dots x_n}{x_i}\right)^2 - (n-1)\left(x_1^2 \dots x_n^2 + \sum_{i=1}^{n} \frac{x_0^2 x_1^2 \dots x_n^2}{x_i^2}\right)\right] \\
&= 2^{n-1} \left[\left(\sum_{\alpha=0}^{n} x_0 \cdot \dots \cdot \hat{x}_j \cdot \dots \cdot x_n\right)^2 - (n-1)\left(\sum_{\alpha=0}^{n} x_0^2 \cdot \dots \cdot \hat{x}_j^2 \cdots x_n^2\right)\right]. \quad \square
\end{aligned}$$

Ethical approval

It does not apply.

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All the authors contributed equally.

Data availability

No data set has been used.

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