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A DEGENERATE ATTRACTION-REPULSION CHEMOTAXIS SYSTEM WITH LOGISTIC-TYPE SUPERLINEAR DEGRADATION

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Abstract. In this paper, we consider radially symmetric solutions of the degenerate cross-diffusion system with flux limitation diffusion and logistic source,

$$\begin{cases} u_t = \nabla \cdot \left(\frac{u \nabla u}{\sqrt{u^2+|\nabla u|^2}} \right) - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w) + f(u), \\ 0 = \Delta v + \alpha u - m_1(t), \\ 0 = \Delta w + \gamma u - m_2(t), \end{cases}$$

in $\Omega \times (0, \infty)$, with Ω a ball in $\mathbb{R}^N$, $N \geq 1$, and subjected to no-flux boundary conditions. The logistic dampening satisfies $f(u) = \lambda u - \mu u^k$ with λ, μ positive constants and $k \geq 1$. If $N \geq 3$ and $\chi\alpha - \xi\gamma > 0$, under certain smallness conditions on logistic degradation, we demonstrate that the solution $u(x,t)$ exhibits blow-up behavior in L^∞-norm at a finite time. Moreover, for some $p > N$, we prove that the solution also blows up in L^p-norm. On the other hand, if $\chi\alpha - \xi\gamma < 0$, or if $\chi\alpha - \xi\gamma > 0$ and k is large, we prove that the solution is global in time.

1. **Introduction.** In 1970, Keller and Segel (see [9] and [10]) introduced the first mathematical model that describes how chemotactic cells, like slime molds, move toward a chemical they release. They discovered that when these cells start to group together, it is due to an instability in how they spread out. The classical formulation of the chemotaxis model is

$$\begin{cases} u_t &= \Delta u - \chi \nabla \cdot (u \nabla v), \\ \tau v_t &= \Delta v + u - v, \end{cases} \tag{1}$$

where $\tau \in \{0,1\}$, $u = u(x,t)$ is the cell density, $v = v(x,t)$ is the chemoattractant concentration, and χ represents the chemotactic sensitivity. The sign of χ corresponds to chemoattraction if positive, and repulsion if negative.

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From 1970 until now, there have been many variants of system (1), and most of these models considered interactions between one cell type and one chemical.

An important extension of the classical Keller–Segel model to a more complex cell migration mechanism was proposed by Luca et al. in [13] in order to describe processes of the formation of senile plaques, i.e. abnormal foci that form in the brain during Alzheimer's disease,

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u \nabla v) + \xi \nabla \cdot (u \nabla w), & x \in \Omega,\ t > 0 \\ \tau v_t = \Delta v + \alpha u - \beta v, & x \in \Omega,\ t > 0 \\ \tau w = \Delta w + \gamma u - \delta w, & x \in \Omega,\ t > 0 \end{cases} \tag{2}$$

in a one-dimensional bounded domain Ω and under homogeneous Neumann boundary conditions. The parameters $\chi,\ \xi,\ \alpha,\ \beta,\ \delta$, and γ are assumed to be positive, and $\tau = 0, 1$. The unknown function $u(x,t)$ represents the density of the microglia cells, $v(x,t)$ denotes the concentration of attractive chemical signal, and $w(x,t)$ the repulsive cue (both signals are generated by the glial cells themselves or by other cells, in response to the presence of glial cells). A novel aspect of this model is that it includes both chemoattraction $\chi \nabla \cdot (u \nabla v)$ and chemorepulsion $\xi \nabla \cdot (u \nabla w)$.

In [18], Tao and Wang studied the existence of solutions which blow up in L^∞-norm at finite time. More precisely, in the two-dimensional setting, the authors proved that the solution component u of (2) with $\tau = 0$ blows up in finite time under the conditions $\chi\alpha - \xi\gamma > 0$, $\delta = \beta$, $\int_\Omega u_0(x)\,dx > \frac{8\pi}{\chi\alpha - \xi\gamma}$, and $\int_\Omega u_0(x)|x - x_0|^2 dx$ sufficiently small for some $x_0 \in \Omega$.

In [3], in a ball $\Omega \in \mathbb{R}^N,\ N \geq 3$, if the attraction dominates over the repulsion in the sense that $\chi\alpha - \xi\gamma > 0$, conditions of blow-up phenomena were derived, and a lower bound of the blow up time was obtained for the solutions of (2) with logistic source and $\tau = 0$, while in [5] boundedness and finite-time blow-up was studied in the case of quasilinear attraction-repulsion. In [21], in the presence of nonlinear production in the second and third equation of (2), in $\Omega \times (0, T_{\max})$ ($\Omega \in \mathbb{R}^N,\ N \geq 2$), the author derived conditions on data to produce a unique classical solution (u, v, w) which is global, i.e. $T_{\max} = \infty$, and such that u, v, and w are uniformly bounded.

In [11], the author proved that the fully parabolic attraction-repulsion chemotaxis system (2) admits radially symmetric solutions which blow up in finite time in a bounded three-dimensional domain, if $\chi\alpha - \xi\gamma > 0$. Recently, Chiyo et al. in [4] studied the quasilinear fully parabolic attraction-repulsion system

$$\begin{cases} u_t = \nabla \cdot \Big((u+1)^{m-1} \nabla u - \chi u (u+1)^{p-2} \nabla v + \xi u (u+1)^{p-2} \nabla w \Big), \\ v_t = \Delta v + \alpha u - \beta v, \\ w_t = \Delta w + \gamma u - \delta w, \end{cases}$$

in Ω, a ball in $\mathbb{R}^N$, where $N \in \{2, 3\}$, $m, p \in \mathbb{R}, \chi, \xi, \alpha, \beta, \gamma, \delta > 0$ are constants, and u_0, v_0, and w_0 are positive functions in $\overline{\Omega}$. The authors proved that the solutions to this system blow up in finite-time under some positive initial data when $\chi\alpha - \xi\gamma > 0,\ p \geq 2$, and $p - m > \frac{2}{N}$. (See also the references therein).

In [8], the authors considered an attraction-repulsion chemotaxis model with nonlinear diffusion and nonlinear signal production. Under alternative conditions on the data, they established global existence and boundedness of classical solutions. Moreover, in the absence of the source term and under suitable conditions on the data, finite-time blow-up was proved. Interesting results on the existence and boundedness of solutions to a quasilinear chemotaxis-Navier-Stokes system were presented in [12] and [23].

Motivated by biological and medical phenomena, in this paper we focus our attention on the following parabolic-elliptic-elliptic attraction-repulsion chemotaxis system with flux limitation diffusion and superlinear logistic degradation

$$\begin{cases} u_t = \nabla\cdot\Big(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}} - \chi u\nabla v + \xi u\nabla w\Big) + \lambda u - \mu u^k, & x\in\Omega,\ t>0,\\ 0 = \Delta v + \alpha u - m_1(t), \quad m_1(t) = \frac{\alpha}{|\Omega|}\int_\Omega u(x,t)dx, & x\in\Omega,\ t>0,\\ 0 = \Delta w + \gamma u - m_2(t), \quad m_2(t) = \frac{\gamma}{|\Omega|}\int_\Omega u(x,t)dx, & x\in\Omega,\ t>0,\\ \dfrac{u\nabla u\cdot\nu}{\sqrt{u^2+|\nabla u|^2}} = \nabla v\cdot\nu = \nabla w\cdot\nu = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{3}$$

where $\Omega = B_R(0)\subset\mathbb{R}^N$ $(N\geq 1)$ is a ball, $R>0$, with χ, ξ, λ, μ, α, and γ positive constants, $k>1$, and ν is the outward normal vector to $\partial\Omega$. The initial data u_0 satisfies

$$u_0\in C^2(\overline{\Omega}),\ \text{radially symmetric and positive in } \overline{\Omega}. \tag{4}$$

The interest on the flux limitation diffusion term $\nabla\cdot\Big(\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\Big)$ is due to the fact that particles do not move (diffuse) arbitrarily in a space, but rather move along specific pathways, such as the borders of cells and with a finite speed of propagation, which is a key characteristic of the model. The corresponding degenerate chemotaxis system with flux limitation diffusion was investigated by [1] and [2], which were extended to the quasilinear case by [6] and [16].

After some preliminary results in Section 2, we prove, in Section 3, that the blow-up phenomena in L^∞-norm appears in the case $\chi\alpha-\xi\gamma>0$ (see Theorem 1.1 below), while in Section 4 the blow-up is proved in L^p-norm (see Theorem 1.2 below). Finally, Section 5 is devoted to the study of the global existence and boundedness of the solution of (3) in both repulsion-dominant and attraction-dominant cases (see Theorem 1.3 below). The main results are presented below.

Theorem 1.1 (Finite-time blow-up in L^∞-norm)**.** *Let $\Omega = B_R(0)\subset\mathbb{R}^N$, $N\geq 3$, $R>0$, and $\alpha,\ \gamma,\ \lambda,\ \mu>0$, and $k>1$. If*

$$\chi\alpha-\xi\gamma>0 \tag{5}$$

and one of the following conditions is satisfied,

$$\text{i)}\quad 1<k<2,\quad \mu>0; \tag{6}$$

$$\text{ii)}\quad k=2,\quad 0<\mu<(1-\tfrac{2}{N})(1-\tfrac{1}{N})(\chi\alpha-\xi\gamma), \tag{7}$$

then for all $m_0>0$ there is a positive radially decreasing initial data $u_0\in C^2(\overline{\Omega})$ with

$$\frac{1}{|\Omega|}\int_\Omega u_0(x)dx = m_0 \tag{8}$$

such that (3) *possesses a classical solution* (u, v, w) *in* $\Omega \times (0, T_{\max})$*, for some* $T_{\max} \in (0, \infty)$*, which blows up at* $T_{\max}$ *in the sense that*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{9}$$

Theorem 1.2 (Finite-time blow-up in L^p-norm)**.** *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$*,* $N \geq 3$*, and* $R > 0$*. Then, the classical solution* (u, v, w) *of* (3)*, provided by Theorem 1.1, is such that for all* $p > N$*,*

$$\limsup_{t \nearrow T_{\max}} \|u(\cdot, t)\|_{L^p(\Omega)} = \infty.$$

Theorem 1.3 (Global existence and boundedness)**.** *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$*,* $N \geq 2$*, and* $R > 0$*. Suppose*

- *Case 1:* **Repulsion-dominant case** $\chi\alpha - \xi\gamma < 0$

$$k \geq 1, \quad \mu > 0, \tag{10}$$

or

- *Case 2:* **Attraction-dominant case** $\chi\alpha - \xi\gamma > 0$

$$k > 2, \quad \mu > 0, \tag{11}$$

$$k = 2, \quad \mu > (1 - \tfrac{1}{N})(\chi\alpha - \xi\gamma). \tag{12}$$

Then, for all initial data u_0 *fulfilling* (4)*, system* (3) *possesses a global classical positive solution* (u, v, w) *in* $\Omega \times (0, \infty)$*, which is bounded in the sense that*

$$\sup_{t \in (0,\infty)} \|u(\cdot, t)\|_{L^\infty(\Omega)} < \infty.$$

Blow-up	Open	Boundedness
$0 < \mu < (1 - \frac{2}{N})(1 - \frac{1}{N})\tilde{\chi}$	$(1 - \frac{2}{N})(1 - \frac{1}{N})\tilde{\chi} \leq \mu \leq (1 - \frac{1}{N})\tilde{\chi}$	$\mu > (1 - \frac{1}{N})\tilde{\chi}$

Table 1. The case $k = 2$, attraction-dominant case with $\tilde{\chi} = \chi\alpha - \xi\gamma > 0$

2. **Preliminaries.** We begin with the following lemma concerning local existence of classical solutions to (3). This lemma can be proved in the radial setting in the same way as in [1] and [20].

Lemma 2.1. *Let* $\Omega = B_R(0) \subset \mathbb{R}^N$*,* $N \geq 2$*, and* $R > 0$*, and let* $\lambda \in \mathbb{R}$*,* $\mu > 0$*,* $k > 1$*,* $\chi, \xi, \alpha, \gamma, > 0$*. Then, for all nonnegative* $u_0 \in C^0(\overline{\Omega})$*, there exists* $T_{\max} \in (0, \infty]$ *such that* (3) *possesses a unique classical solution* (u, v, w) *such that*

$$u \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

$$v, w \in \bigcap_{\vartheta > n} C^0([0, T_{\max}); W^{1,\vartheta}(\Omega)) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})),$$

and

$$u > 0, \quad v > 0, \quad w > 0 \quad \textit{for all } t \in (0, T_{\max}).$$

Moreover,

$$\textit{if } T_{\max} < \infty, \quad \textit{then } \limsup_{t \nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} = \infty. \tag{13}$$

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded and smooth domain, and $\lambda > 0$, $\mu > 0$, $k > 1$. Then, for a solution (u, v, w) of (3), we have*

$$\int_\Omega u\,dx \leq \bar{m} \quad \textit{for all } t \in (0, T_{\max}), \tag{14}$$

with

$$\bar{m} = \max\Big\{ \int_\Omega u_0\,dx,\ \Big(\frac{\lambda}{\mu}\,|\Omega|^{k-1}\Big)^{\frac{1}{k-1}} \Big\}. \tag{15}$$

Proof. From the first equation in (3), we obtain

$$\frac{d}{dt}\int_\Omega u\,dx = \lambda \int_\Omega u\,dx - \mu \int_\Omega u^k\,dx \leq \lambda \int_\Omega u\,dx - \mu|\Omega|^{1-k}\Big(\int_\Omega u\,dx\Big)^k, \tag{16}$$

where in the last term we used Hölder's inequality:

$$\int_\Omega u\,dx \leq |\Omega|^{\frac{k-1}{k}}\Big(\int_\Omega u^k\,dx\Big)^{\frac{1}{k}}.$$

From (16), we infer that $z := \int_\Omega u dx$ satisfies

$$\begin{cases} z'(t) \leq \lambda z(t) - \bar{\mu} z^k(t), \quad \bar{\mu} = \mu|\Omega|^{1-k}, \quad \text{for all } t \in [0, T_{\max}). \\ z(0) = z_0. \end{cases}$$

Upon an ODE comparison argument, this entails that

$$z(t) \leq \bar{m} \quad \text{for all } t \in (0, T_{\max}).$$

This clearly proves the lemma. □

Remark 2.3. We observe that from Lemma 2.2 and the second and third equations of (3) the following important property on the mass can be easily derived

$$m_1(t) \leq \frac{\alpha}{|\Omega|}\bar{m} \tag{17}$$

and

$$m_2(t) \leq \frac{\gamma}{|\Omega|}\bar{m} \tag{18}$$

Lemma 2.4. *Let u_0 be a decreasing function in $\overline{\Omega}$ satisfying (4). Then,*

$$u_r \leq 0 \quad \textit{in } (0, R) \times (0, T_{\max}).$$

Proof. Following the steps in [15, Lemmas 2.3], we can write

$$\begin{aligned} u_t = {} & \frac{u^3 u_{rr}}{\sqrt{u^2 + u_r^2}^3} + \frac{u_r^4}{\sqrt{u^2 + u_r^2}^3} + \frac{N-1}{r}\frac{u u_r}{\sqrt{u^2 + u_r^2}} \\ & - \chi u_r v_r - \chi u(m_1(t) - \alpha u) + \xi u_r w_r + \xi u(m_2(t) - \gamma u) + \lambda u - \mu u^k. \end{aligned} \tag{19}$$

By differentiation of (19) with respect to r, we obtain

$$\begin{aligned} u_{rt} = & \frac{u^3 u_{rrr}}{\sqrt{u^2+u_r^2}^3} + 3\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u^3 u_r u_{rr}^2}{\sqrt{u^2+u_r^2}^5} + 4\frac{u^2 u_r^3 u_{rr}}{\sqrt{u^2+u_r^2}^5} \\ & + \frac{u_r^5 u_{rr}}{\sqrt{u^2+u_r^2}^5} - 3\frac{u u_r^5}{\sqrt{u^2+u_r^2}^5} - \frac{N-1}{r^2}\frac{u u_r}{\sqrt{u^2+u_r^2}} \\ & + \frac{N-1}{r}\frac{u^3 u_{rr}}{\sqrt{u^2+u_r^2}^3} + \frac{N-1}{r}\frac{u_r^4}{\sqrt{u^2+u_r^2}^3} - \chi u_r v_{rr} \\ & - \chi u_{rr} v_r - \chi m_1(t) u_r + 2\chi\alpha u u_r + \xi u_{rr} w_r + \xi u_r w_{rr} \\ & + \xi m_2(t) u_r - 2\xi\gamma u u_r + \lambda u_r - \mu k u^{k-1} u_r. \end{aligned} \tag{20}$$

Taking into account the facts that by the radial symmetry $u_r(0,\cdot)=0$, $u_r(R,\cdot)\le 0$, and that $u_{0r}\le 0$ by the radial decreasing of u_0, then, fixing $T\in(0,T_{\max})$, we see that the function $\varphi=u_r$ belongs to $C^0([0,R]\times[0,T])$ as well as to $C^{2,1}((0,R)\times(0,T))$, and satisfies the parabolic problem

$$\begin{cases} \varphi_t = c_1(r,t)\varphi_{rr} + c_2(r,t)\varphi_r + c_3(r,t)\varphi & \text{in } (0,R)\times(0,T), \\ \varphi \le 0 & \text{on } \{0,R\}\times(0,T), \\ \varphi(\cdot,0)\le 0 & \text{in } (0,R), \end{cases} \tag{21}$$

with

$$c_1(r,t) = \frac{u^3}{\sqrt{u^2+u_r^2}^3},$$

$$\begin{aligned} c_2(r,t) = & 3\frac{u^2 u_r^3}{\sqrt{u^2+u_r^2}^5} - 3\frac{u^3 u_r u_{rr}}{\sqrt{u^2+u_r^2}^5} + 4\frac{u^2 u_r^3}{\sqrt{u^2+u_r^2}^5} + \frac{u_r^5}{\sqrt{u^2+u_r^2}^5} \\ & + \frac{N-1}{r}\frac{u^3}{\sqrt{u^2+u_r^2}^3} - \chi v_r + \xi w_r, \end{aligned}$$

$$\begin{aligned} c_3(r,t) = & -3\frac{u u_r^4}{\sqrt{u^2+u_r^2}^5} - \frac{N-1}{r^2}\frac{u}{\sqrt{u^2+u_r^2}} + \frac{N-1}{r}\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} \\ & + 2\chi\alpha u - \chi v_{rr} - \chi m_1(t) + \xi w_{rr} + \xi m_2(t) - 2\xi\gamma u + \lambda - \mu k u^{k-1}. \end{aligned}$$

We can obtain $\sup_{(0,R)\times[0,T]} c_3(r,t) < \infty$. In fact, neglecting the negative terms in $c_3(r,t)$ and taking into account that $\frac{N-1}{r}\frac{u_r^3}{\sqrt{u^2+u_r^2}^3} \le \frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}}$, we can write

$$\begin{aligned} c_3(r,t) \le & \frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}} + 2\chi\alpha u - \chi v_{rr} \\ & + \xi w_{rr} + \xi m_2(t) + \lambda. \end{aligned} \tag{22}$$

Since $u\in C^{2,1}(\overline{\Omega}\times[0,T])$, $T<T_{\max}$, and $u_0>0$, $u>0$ in $\overline{\Omega}$, there exists a constant $C_0>0$ such that $u(r,t)\ge \inf_{[0,R]\times[0,T]} u(r,t) \ge C_0$. Moreover from the mean value theorem, there exists $\theta \in (0,1)$ such that $u_r(r,t) = u_r(0,t) + r u_{rr}(r\theta,t) = r u_{rr}(r\theta,t)$ with $|u_{rr}| \le M$ with some constant $M > 0$, and we can write

$$\frac{N-1}{r}\frac{|u_r|}{\sqrt{u^2+u_r^2}} \le (N-1)\frac{|u_{rr}(r\theta,t)|}{u} \le (N-1)\frac{M}{C_0}. \tag{23}$$

Finally, by considering the second and third equations of (3) in radial coordinates,

$$v_{rr} = -\alpha u + m_1(t) - \frac{N-1}{r}v_r \ge -\alpha u - \frac{N-1}{r}|v_r|,$$

$$w_{rr} = -\gamma u + m_2(t) - \frac{N-1}{r}w_r \le m_2(t) + \frac{N-1}{r}|w_r| \le \frac{\gamma}{|\Omega|}\bar{m} + \frac{N-1}{r}|w_r|,$$

we obtain $v_{rr} \ge -\alpha u + C_1$ and $w_{rr} \le C_2$, with $C_1,\ C_2 > 0$. In fact, since $v \in C^{2,1}(\bar{\Omega}) \times [0,T]$, $T < T_{\max}$, from the mean value theorem, there exists $\theta_1 \in (0,1)$ such that $v_r(\theta_1,t) = v_r(0,t) + r v_{rr}(r\theta_1,t) = r v_{rr}(r\theta_1,t)$ with $|v_{rr}| \le M_1$ with some constant $M_1 > 0$, and we obtain

$$\frac{N-1}{r}|v_r| \le (N-1)|v_{rr}(r\theta_1,t)| \le (N-1)M_1, \quad M_1 > 0$$

and thus

$$v_{rr} \ge -\alpha u - \frac{N-1}{r}|v_r| \ge -\alpha u - (N-1)M_1 = -\alpha u - C_1. \tag{24}$$

Similarly, for w_{rr}, for some constant $M_2 > 0$, we obtain

$$\begin{aligned} w_{rr} &\le \frac{\gamma}{|\Omega|}\bar{m} - \frac{N-1}{r}w_r \le \frac{\gamma}{|\Omega|}\bar{m} + \frac{N-1}{r}|w_r| \\ &\le \frac{\gamma}{|\Omega|}\bar{m} + (N-1)M_2 = C_2, \end{aligned} \tag{25}$$

Taking into account (18), and by using (23), (24), and (25) in (22), we get

$$c_3(r,t) \le 3\chi\alpha u + C$$

with $C = (N-1)\frac{M}{C_0} + \lambda + \xi(N-1)M_2 + \chi(N-1)M_1 + 2\xi\frac{\gamma}{|\Omega|}\bar{m}$. And, since u is bounded in $[0,R]\times[0,T]$, we have $\sup_{(0,R)\times[0,T]} c_3(r,t) < \infty$. By applying a comparison principle (see [17, Proposition 52.4]), we conclude that $\varphi = u_r \le 0$. □

3. **Blow-up in L^∞.** The aim of this section is to study problem (3) in the case in which the attraction prevails over repulsion in the sense that $\chi\alpha - \xi\gamma > 0$.

If we set $V(x,t) = \chi v(x,t) - \xi w(x,t)$, we obtain

$$\begin{cases} u_t = \nabla \cdot \left(\dfrac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}\right) - \nabla\cdot(u\nabla V) + \lambda u - \mu u^k, \\ 0 = \Delta V + (\chi\alpha - \xi\gamma)u - M(t), \end{cases} \tag{26}$$

with $M(t) = \chi m_1(t) - \xi m_2(t)$. A problem similar to (26) was studied in [15] with $\chi\alpha - \xi\gamma > 0$.

In order to analyze the blow up phenomena of the solution to (3), as in [22] and [7], we introduce radial coordinates with $s = r^N \in [0,R^N]$, $t \in [0,T_{\max})$, defining the following functions:

$$U(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1}u(\rho,t)\,d\rho, \tag{27}$$

$$V(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} v(\rho,t)d\rho,$$

$$W(s,t) := \int_0^{s^{\frac{1}{N}}} \rho^{N-1} w(\rho,t)d\rho.$$

Thus, we have

$$U_s(s,t) = \frac{1}{N}u(s^{\frac{1}{N}},t) \geq 0, \quad U_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}u_r(s^{\frac{1}{N}},t), \tag{28}$$

$$V_s(s,t) = \frac{1}{N}v(s^{\frac{1}{N}},t) \geq 0, \quad V_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}v_r(s^{\frac{1}{N}},t),$$

$$W_s(s,t) = \frac{1}{N}w(s^{\frac{1}{N}},t) \geq 0, \quad W_{ss}(s,t) = \frac{1}{N^2}s^{\frac{1}{N}-1}w_r(s^{\frac{1}{N}},t).$$

From the second and third equations in (3), we deduce

$$\frac{1}{r^{N-1}}\left(r^{N-1}v_r(r,t)\right)_r = -\alpha u(r,t) + m_1(t), \tag{29}$$

$$\frac{1}{r^{N-1}}\left(r^{N-1}w_r(r,t)\right)_r = -\gamma u(r,t) + m_2(t)$$

and by integrating (29),

$$r^{N-1}v_r = -\alpha\int_0^r \rho^{N-1}u d\rho + m_1(t)\frac{r^N}{N} = -\alpha U + m_1(t)\frac{r^N}{N}, \tag{30}$$

$$r^{N-1}w_r = -\gamma\int_0^r \rho^{N-1}u d\rho + m_2(t)\frac{r^N}{N} = -\gamma U + m_2(t)\frac{r^N}{N}$$

for all $r \in (0,R)$, $t \in (0,T_{\max})$.

The radial version of the first equation in (3) is

$$\begin{aligned} U_t(s,t) &= \int_0^{s^{\frac{1}{N}}} r^{N-1}u_t(r,t)dr = \int_0^{s^{\frac{1}{N}}} \big(r^{N-1}\frac{uu_r}{\sqrt{u^2+u_r^2}}\big)_r dr \\ &\quad - \chi\int_0^{s^{\frac{1}{N}}} \left(r^{N-1}u(r,t)v_r\right)_r dr + \xi\int_0^{s^{\frac{1}{N}}} \left(r^{N-1}u(r,t)w_r\right)_r dr \\ &\quad + \lambda\int_0^{s^{\frac{1}{N}}} r^{N-1}u(r,t)dr - \mu\int_0^{s^{\frac{1}{N}}} r^{N-1}u^k(r,t)dr \\ &= s^{1-\frac{1}{N}}\frac{uu_r}{\sqrt{u^2+u_r^2}} - \chi s^{1-\frac{1}{N}}uv_r + \xi s^{1-\frac{1}{N}}uw_r \\ &\quad + \lambda U - \mu N^{k-1}\int_0^s U_s^k(\sigma,t)d\sigma. \end{aligned}$$

By using (28) and (29), we obtain

$$U_t(s,t) = \frac{N^2 s^{2-\frac{2}{N}}U_sU_{ss}}{\sqrt{U_s^2 + N^2s^{2-\frac{2}{N}}U_{ss}^2}} - \chi N(-\alpha U + \frac{m_1(t)}{N}s)U_s \tag{31}$$

$$+\xi N(-\gamma U+\frac{m_2(t)}{N}s)U_s+\lambda U-\mu N^{k-1}\int_0^s U_s^k(\sigma,t)d\sigma$$

$$=\frac{N^2 s^{2-\frac{2}{N}}U_sU_{ss}}{\sqrt{U_s^2+N^2s^{2-\frac{2}{N}}U_{ss}^2}}+N(\chi\alpha-\xi\gamma)UU_s-[\chi m_1(t)-\xi m_2(t)]sU_s$$

$$+\lambda U-\mu N^{k-1}\int_0^s U_s^k(\sigma,t)d\sigma$$

$$\geq -Ns^{1-\frac{1}{N}}U_s+N(\chi\alpha-\xi\gamma)UU_s-\chi\frac{\alpha}{|\Omega|}\bar{m}sU_s-\mu N^{k-1}\int_0^s U_s^k(\sigma,t)d\sigma,$$

where in the last inequality we have used

$$\frac{N^2 s^{2-\frac{2}{N}}U_sU_{ss}}{\sqrt{U_s^2+N^2s^{2-\frac{2}{N}}U_{ss}^2}}\geq -Ns^{1-\frac{1}{N}}U_s$$

and (17), and we have neglected the two positive terms $\xi m_2(t)sU_s$ and λU .

In order to obtain our blow-up result, we derive a basic differential inequality describing the evolution of the following moment-like functional $y(t)$:

$$y(t):=\int_0^{R^N} s^{-a}U^b(s,t)ds,\quad t\in[0,T_{\max}) \tag{32}$$

with $a>0$ and $b\in(0,1)$.

We have the following lemma.

Lemma 3.1. *Let $\Omega=B_R(0)\subset\mathbb{R}^N$, $N\geq 3$, and $R>0$. Let u_0 satisfy (4). If $\chi\alpha-\xi\gamma>0$, then for all $a>0$, $b\in(0,1)$, the function U defined in (27) satisfies*

$$\frac{1}{b}\frac{d}{dt}\int_0^{R^N} s^{-a}U^b(s,t)ds \tag{33}$$

$$\geq -b_1\int_0^{R^N} s^{-\frac{1}{N}-a}U^bds+b_2\int_0^{R^N} s^{-a-1}U^{b+1}ds-b_3\int_0^{R^N} s^{-a}U^bds$$

$$-\frac{\mu N^{k-1}}{a-1}\int_0^{R^N} s^{1-a}U^{b-1}(s,t)U_s^k(s,t)ds-b_4$$

with

$$b_1=\frac{N}{b}\Big(a-\Big(1-\frac{1}{N}\Big)\Big), \tag{34}$$

$$b_2=a\frac{N}{b+1}(\chi\alpha-\xi\gamma),$$

$$b_3=\chi\frac{\alpha}{|\Omega|b}\bar{m}(a-1),$$

$$b_4=\frac{R^{N(1-a)-1}}{b}\Big(\frac{\bar{m}}{\omega_N}\Big)^b\Big[N+\chi\frac{\alpha}{|\Omega|}\bar{m}R\Big],$$

with $\bar{m}$ defined in (15).

Proof. Following the steps in [15, Lemma 3.2], we obtain (33). □

Lemma 3.2. *Let us assume the hypotheses of Lemma 3.1 and*

$$\frac{1}{N-1} < b < 1, \quad 1 < a < (b+1)\Big(1 - \frac{1}{N}\Big). \tag{35}$$

(i) *If*

$$1 < k < 2, \quad \mu > 0, \tag{36}$$

then

$$\frac{1}{b}\frac{d}{dt}\int_0^{R^N} s^{-a}U^b(s,t)ds \geq \bar{c}_1\Big(\int_0^{R^N} s^{-a}U^b ds\Big)^{\frac{b+1}{b}} - \bar{c}_2\int_0^{R^N} s^{-a}U^b ds - \bar{c}_3, \tag{37}$$

for all $t \in (0, T_{\max})$*, with*

$$\bar{c}_1 = \frac{abN}{2(b+1)}(\chi\alpha - \xi\gamma)\Big(\frac{-a+b+1}{R^{N(-a+b+1)}}\Big)^{\frac{b+1}{b}}, \tag{38}$$

$$\bar{c}_2 = \chi\frac{\alpha}{|\Omega|}\bar{m}(a-1),$$

$$\bar{c}_3 = b(b_4 + \tilde{c}_1 + \tilde{c}_3),$$

where

$$\tilde{c}_1 = (\frac{N}{b}(a - (1 - \frac{1}{N})))^{b+1}\frac{1}{\delta_1^b}\frac{1}{b+1}\ \frac{R^{N[(b+1)(1-\frac{1}{N})-a]}}{(b+1)(1-\frac{1}{N}) - a},$$

$$\tilde{c}_3 = \frac{2-k}{b+1}(\frac{\mu N^{k-1}}{a-1})^{\frac{b+1}{2-k}}\delta_2^{-\frac{b+k-1}{2-k}},$$

$$\delta_1 = \frac{1}{b}\frac{aN(\chi\alpha - \xi\gamma)}{4}, \quad \delta_2 = \frac{b+1}{b+k-1}\frac{aN(\chi\alpha - \xi\gamma)}{4}.$$

(ii) *If*

$$k = 2, \quad \mu < \frac{a(a-1)}{b+1}(\chi\alpha - \xi\gamma) \tag{39}$$

then

$$\frac{1}{b}\frac{d}{dt}\int_0^{R^N} s^{-a}U^b(s,t)ds \geq \tilde{b}_1\Big(\int_0^{R^N} s^{-a}U^b ds\Big)^{\frac{b+1}{b}} - \tilde{b}_2\int_0^{R^N} s^{-a}U^b ds - \tilde{b}_3 \tag{40}$$

for all $t \in (0, T_{\max})$*, with*

$$\tilde{b}_1 = \frac{1}{2}b\Big[a\frac{N}{b+1}(\chi\alpha - \xi\gamma) - \frac{\mu N}{a-1}\Big]$$

$$\tilde{b}_2 = \chi\frac{\alpha}{|\Omega|}\bar{m}(a-1),$$

$$\tilde{b}_3 = b(b_4 + \tilde{c}_1),$$

where $\tilde{c}_1 = (\frac{N}{b}(a - (1-\frac{1}{N})))^{b+1}\frac{1}{\delta_1^b}\frac{1}{b+1}\frac{R^{N[(b+1)(1-\frac{1}{N})-a]}}{(b+1)(1-\frac{1}{N})-a}$, $\delta_1 = \frac{1}{2}\frac{b+1}{b}[a\frac{N}{b+1}(\chi\alpha - \xi\gamma) - \frac{\mu N}{a-1}]$*, and* b_4 *is as defined in* (34).

Proof. By following the steps of the proof of [15, Lemma 3.3], we obtain (37) and (40). □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1. Both cases $1 < k < 2$ and $k = 2$ of Lemma 3.2 lead to a differential inequality of the type

$$y'(t) \geq d_1 y^{\beta}(t) - d_2 y(t) - d_3, \quad \beta > 2, \tag{41}$$

with $y(t)$ defined in (32) and $\beta = 1 + \frac{1}{b}$.

By following the steps in the proof of Theorem 1.1 in [15], we first prove the case $1 < k < 2$ (with $d_i = b\bar{c}_i,\ i = 1, 2, 3$), and then we prove the case $k = 2$ (with $d_i = b\tilde{b}_i,\ i = 1, 2, 3$).

For clarity, we show the main steps of the proofs.

In the case $1 < k < 2$, by integrating (41) over the time interval $[0, t]$ for $0 < t < T_{\max}$, we obtain

$$y(t) \geq y_0 + d_1 \int_0^t y^{\beta}(\tau)d\tau - d_2 \int_0^t y(\tau)d\tau - d_3 t, \tag{42}$$

with $y_0 = \int_0^{R^N} s^{-a} U_0^b ds$ and $U_0 = U_0(s) = \int_0^{s^{\frac{1}{N}}} \rho^{N-1} u_0(\rho)d\rho$. By using the Young inequality, for $\epsilon \in (0, \frac{d_1}{d_2})$, we can write $y(t) \leq \epsilon y^{\beta} + c(\epsilon)$. This, along with (42), entails

$$y(t) \geq y_0 + D_1 \int_0^t y^{\beta}(\tau)d\tau - D_2 t \tag{43}$$

with $D_1 = d_1 - d_2\epsilon > 0$ and $D_2 = d_3 + d_2 c(\epsilon)$. Then, we introduce the auxiliary function $z(t) := y_0 - \left(\frac{D_2}{D_1}\right)^{\frac{1}{\beta}} + D_1 \int_0^t y^{\beta} d\tau - D_2 t$, which satisfies

$$z'(t) \geq D_1 z^{\beta}(t),$$

$$z(0) = y_0 - \left(\frac{D_2}{D_1}\right)^{\frac{1}{\beta}} > 0,$$

where we can find positive radially decreasing initial data $u_0 \in C^2(\overline{\Omega})$ satisfying (8) as well as $y_0 - \left(\frac{D_2}{D_1}\right)^{\frac{1}{\beta}} > 0$ (see the steps in the proof of Theorem 1.1 in [15] pg. 12). By the comparison principle, we obtain that $z(t)$ is positive, and if $t \to T_{\max}$, then $z(t) \to +\infty$ with $T_{\max} \leq \frac{z_0^{1-\beta}}{(\beta-1)D_1}$. The proof now is an immediate consequence of the blow-up criterion given in Lemma 2.1.

In a similar manner, we obtain the proof when $k = 2$, where if $a \to (b+1)(1-\frac{1}{N})$ and $b \to 1$, then $\frac{a(a-1)}{b+1}(\chi\alpha - \xi\gamma) \to (1-\frac{2}{N})(1-\frac{1}{N})(\chi\alpha - \xi\gamma)$. Thus, if $\mu < (1-\frac{2}{N})(1-\frac{1}{N})(\chi\alpha - \xi\gamma)$ (i.e. if (7) holds), then we can take a, b satisfying (35) and (39), and we repeat the steps in the proof of the case $1 < k < 2$ with $d_i = b\tilde{b}_i,\ i = 1, 2, 3$. □

4. Blow-up in L^p.

In this section, our goal is to prove the blow up in L^p-norm of the classical solution of (3). To this end, starting from the hypotheses of the blow-up in L^{∞}-norm, we proceed by contradiction by using the following lemma.

Lemma 4.1. *Let $\Omega = B_R(0) \subset \mathbb{R}^N$, $N \geq 1$, and $R > 0$. Assume that u_0 is nonnegative and $u_0 \in C^2(\overline{\Omega})$. Let (u, v, w) be a classical solution of system (3). Then, if there exists $C > 0$ such that for some $p_0 > N$,*

$$\|u(\cdot, t)\|_{L^{p_0}(\Omega)} \leq C \quad \textit{for all } t \in (0, T_{\max}),$$

then, for some $\hat{C} > 0$,

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le \hat{C} \quad \textit{for any } t \in (0, T_{\max}).$$

Proof. Following the steps in [14, Lemma 3.1], we can consider the $L^p(\Omega)$-norm, $p > 1$, of $u(x,t)$ of (3) to obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx = &-\int_\Omega \nabla u^{p-1}\frac{u\nabla u}{\sqrt{u^2+|\nabla u|^2}}dx + \chi(p-1)\int_\Omega u^{p-1}\nabla u \nabla v dx\\
&- \xi(p-1)\int_\Omega u^{p-1}\nabla u\nabla w dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx\\
= &-(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx - \chi\frac{p-1}{p}m_1(t)\int_\Omega u^p dx\\
&+ (\chi\alpha - \xi\gamma)\frac{p-1}{p}\int_\Omega u^{p+1}dx + \xi\frac{p-1}{p}m_2(t)\int_\Omega u^p dx\\
&+ \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx.
\end{aligned}$$

Neglecting the second and last negative terms and applying (18) leads us to

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le &-(p-1)\int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx + (\chi\alpha-\xi\gamma)\frac{p-1}{p}\int_\Omega u^{p+1}dx\\
&+ \Big(\lambda + \xi\frac{\gamma}{|\Omega|}\bar{m}\Big)\int_\Omega u^p dx.
\end{aligned}$$

Thanks to the inequality proved in [1, Lemma 6.1],

$$\int_\Omega u^{p-1}|\nabla u| dx \le \int_\Omega \frac{u^{p-1}|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}}dx + \int_\Omega u^p dx$$

we can deduce that

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx \le &-(p-1)\int_\Omega u^{p-1}|\nabla u| dx + (p-1)\int_\Omega u^p dx\\
&+ (\chi\alpha-\xi\gamma)\frac{p-1}{p}\int_\Omega u^{p+1}dx + (\lambda + \frac{\gamma}{|\Omega|}\bar{m})\int_\Omega u^p dx\\
\le &-\frac{p-1}{p}\int_\Omega |\nabla u^p| dx + \mathcal{A}\int_\Omega u^p dx + \mathcal{B}\int_\Omega u^{p+1}dx
\end{aligned}$$

for all $t \in (0, T_{\max})$, with

$$\begin{cases}\mathcal{A} := p - 1 + \lambda + \frac{\gamma}{|\Omega|}\bar{m},\\ \mathcal{B} := (\chi\alpha - \xi\gamma)\frac{p-1}{p}.\end{cases}$$

As a particular case of the proof of Lemma 3.1 in [14], we obtain

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le c_1 p^{1+N}\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^2 + c_2 p\Big(\int_\Omega u^{\frac{p}{2}}dx\Big)^{2\frac{p+1}{p}}.$$

Defining

$$p := p_j = p_0 2^j$$

for nonnegative integers j and introducing

$$M_j := \sup_{t\in(0,T)} \int_\Omega u^{p_j}\,dx$$

with $j \geq 1$, and for a fixed $T \in (0, T_{\max})$, we have

$$\int_\Omega u^{p_j}\,dx = M_j \leq \max\left\{ \int_\Omega u_0^{p_j}\,dx,\ c_1 p_j^{1+N} M_{j-1}^2 + c_2 p_j M_{j-1}^{2^{\frac{p_j+1}{p_j}}} \right\} \text{ for all } j \geq 1.$$

By induction and a limiting procedure as in [19, p. 714], we obtain

$$\limsup_{j\to\infty} M_j^{\frac{1}{p_j}} \leq c_3$$

for some $c_3 > 0$. In view of the definition of M_j, it follows that

$$\sup_{t\in(0,T)} \|u(\cdot,t)\|_{L^\infty(\Omega)} \leq c_3.$$

Hence, we obtain that if $\|u(\cdot,t)\|_{L^{p_0}(\Omega)}$ is bounded for some $p_0 > N$, then $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. □

Proof of Theorem 1.2. Taking into account Lemma 4.1, if there exists some $p_0 > N$ such that $\|u(\cdot,t)\|_{L^{p_0}(\Omega)} \leq C$, then we also have that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. This is a contradiction since Theorem 1.1 holds. Therefore, $\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^p(\Omega)} = \infty$ for all $p > N$. □

5. **Global existence and boundedness.** The purpose of the present section is to study the global existence and boundedness of the solution (u, v, w) of (3). The following lemma is the core of this argument.

Lemma 5.1. *Assume that* (10) *holds. Then, for any* $p > 1$*, there exists* $C > 0$ *such that the solution of* (3) *satisfies*

$$\int_\Omega u^p dx \leq C. \tag{44}$$

Proof. By multiplying with u^{p-1} the first equation of (3), integrating by parts over Ω, and using the second and third equations of (3), we obtain

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx &= -(p-1)\int_\Omega \nabla(u^{p-1})\frac{|\nabla u|}{\sqrt{u^2+|\nabla u|^2}} - \chi\frac{p-1}{p}\int_\Omega u^p \Delta v dx \\
&\quad + \xi\int_\Omega u^p \Delta w dx + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx \\
&= -(p-1)\int_\Omega u^{p-1}\frac{|\nabla u|^2}{\sqrt{u^2+|\nabla u|^2}} - \frac{p-1}{p}(\xi\gamma-\chi\alpha)\int_\Omega u^{p+1}dx \\
&\quad - \chi m_1(t)\frac{p-1}{p}\int_\Omega u^p dx + \xi m_2(t)\frac{p-1}{p}\int_\Omega u^p dx \\
&\quad + \lambda\int_\Omega u^p dx - \mu\int_\Omega u^{p+k-1}dx.
\end{aligned}$$

By neglecting the first and the third negative terms on the right-hand side of this identity, we get

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p dx \le& -(p-1)(\xi\gamma-\chi\alpha)\int_\Omega u^{p+1}dx \\ &+[\lambda p+\xi m_2(t)(p-1)]\int_\Omega u^p dx-\mu p\int_\Omega u^{p+k-1}dx \\ \le& -(p-1)(\xi\gamma-\chi\alpha)\int_\Omega u^{p+1}dx \\ &+\left[\lambda p+\xi\gamma\frac{\bar{m}}{|\Omega|}(p-1)\right]\int_\Omega u^p dx-\mu p\int_\Omega u^{p+k-1}dx \\ =& J_1+J_2+J_3,\end{aligned} \tag{45}$$

where in J_2 we have used the estimate (18).

The rest of the proof is divided into two cases.

5.1. **Boundedness: repulsion-dominant case.** In this section, we focus our attention on the case where the repulsion dominates over the attraction in the sense that $\chi\alpha-\xi\gamma<0$.

In this case, neglecting the negative term J_3 and adding $\int_\Omega u^p dx$ in (45), we have

$$\frac{d}{dt}\int_\Omega u^p dx+\int_\Omega u^p dx \le -(p-1)(\xi\gamma-\chi\alpha)\int_\Omega u^{p+1}dx+c_1\int_\Omega u^p dx \tag{46}$$

with $c_1=1+\lambda p+\xi\gamma\frac{\bar{m}}{|\Omega|}(p-1)$.

We estimate the second term on the right-hand side of (46) by the Hölder and the Young inequalities to get

$$\begin{aligned}c_1\int_\Omega u^p dx \le& \left(\epsilon_1\int_\Omega u^{p+1}dx\right)^{\frac{p}{p+1}}\left(c_1^{p+1}\frac{1}{\epsilon_1^p}|\Omega|\right)^{\frac{1}{p+1}} \\ \le& \epsilon_1\frac{p}{p+1}\int_\Omega u^{p+1}dx+c_2,\end{aligned} \tag{47}$$

with $c_2=\frac{c_1^{p+1}}{p+1}\frac{1}{\epsilon_1^p}|\Omega|$. Since $\chi\alpha-\xi\gamma<0$, by choosing $\epsilon_1=\frac{p^2-1}{p}(\xi\gamma-\chi\alpha)>0$ and replacing (47) on the right-hand side of (46), it follows that

$$\frac{d}{dt}\int_\Omega u^p dx+\int_\Omega u^p dx\le c_2,$$

and hence an ODE comparison argument yields (44) with $C=\max\{\int_\Omega u_0^p dx, c_2\}$. This proves Lemma 5.1 in the repulsion-dominant case $\chi\alpha-\xi\gamma<0$. □

We now are ready to prove Case 1 of Theorem 1.3.

Proof of Case 1 (repulsion-dominant case) of Theorem 1.3. Lemma 5.1 implies

$$\|u(\cdot,t)\|_{L^p(\Omega)}\le\max\{\|u_0\|_{L^p(\Omega)}, c_2^{\frac{1}{p}}\}.$$

Letting $p\to\infty$, we obtain

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le\max\{\|u_0\|_{L^\infty(\Omega)}, c_3\} \tag{48}$$

with $c_3 = \frac{\lambda+\xi\gamma\frac{\bar{m}}{|\Omega|}}{\xi\gamma-\chi\alpha}$. In fact, we note that

$$c_2^{\frac{1}{p}} = \Big(\frac{c_1^{p+1}}{p+1}\frac{1}{\epsilon_1^p}|\Omega|\Big)^{\frac{1}{p}} = \frac{c_1}{\epsilon_1}\Big(\frac{c_1|\Omega|}{p+1}\Big)^{\frac{1}{p}}$$

$$= \frac{1+\lambda p+\xi\gamma\frac{\bar{m}}{|\Omega|}(p-1)}{\frac{p^2-1}{p}(\xi\gamma-\chi\alpha)}\Big(\frac{1+\lambda p+\xi\gamma\frac{\bar{m}}{|\Omega|}(p-1)|\Omega|}{p+1}\Big)^{\frac{1}{p}}$$

$$= \frac{\frac{1}{p}+\lambda+\xi\gamma\frac{\bar{m}}{|\Omega|}(1-\frac{1}{p})}{(1-\frac{1}{p^2})(\xi\gamma-\chi\alpha)}\Big(\frac{\frac{1}{p}+\lambda+\xi\gamma\frac{\bar{m}}{|\Omega|}(1-\frac{1}{p})|\Omega|}{1+\frac{1}{p}}\Big)^{\frac{1}{p}} \to \frac{\lambda+\xi\gamma\frac{\bar{m}}{|\Omega|}}{\xi\gamma-\chi\alpha} = c_3$$

as $p \to \infty$. Hence, from (48), we get $\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. □

5.2. **Boundedness: attraction-dominant case.** In this section, we study the behavior of the solution of (3) when the attraction prevails over the repulsion and $k \geq 2$. For simplicity's sake, we rewrite (45) taking into account that $\chi\alpha - \xi\gamma > 0$

$$\frac{d}{dt}\int_\Omega u^p dx \leq (p-1)(\chi\alpha-\xi\gamma)\int_\Omega u^{p+1}dx + [\lambda p + \xi\gamma\frac{\bar{m}}{|\Omega|}(p-1)]\int_\Omega u^p dx - \mu p\int_\Omega u^{p+k-1}dx. \tag{49}$$

Proof of Case 2 (attraction-dominant case) of Theorem 1.3. Following the steps in [15, Section 6], we consider the cases $k > 2$ and $k = 2$ separately, and we estimate the terms on the right-hand side of (49) to obtain Lemma 5.1 and the desired result in the attraction-dominant case. In particular, if $k = 2$, if $\mu > (\frac{N-1}{N})(\chi\alpha - \xi\gamma)$, we can take p close to N such that $p > N$ and $\mu > (\frac{p-1}{p})(\chi\alpha - \xi\gamma)$. With this fixed value of $p > N$, we obtain $\|u(\cdot,t)\|_{L^p(\Omega)} \leq C$. Taking into account Lemma 4.1, we see that $\|u(\cdot,t)\|_{L^\infty(\Omega)}$ is bounded. This proves that $\limsup_{t\nearrow T_{\max}} \|u(\cdot,t)\|_{L^\infty(\Omega)} < \infty$. Thanks to Lemma 2.1, we arrive at the conclusion. □

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