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Bochvar algebras: A categorical equivalence and the generated variety

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Abstract. The proper quasivariety BCA of Bochvar algebras, which serves as the equivalent algebraic semantics of Bochvar's external logic, was introduced by Finn & Grigolia [11] and extensively studied in [8]. In this paper, we show that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs consisting of a Boolean algebra and a meet-subsemilattice (with unit) of the same. Furthermore, we provide an axiomatisation of the variety $V(\mathsf{BCA})$ generated by Bochvar algebras. Finally, we axiomatise the join of Boolean algebras and semilattices within the lattice of subvarieties of $V(\mathsf{BCA})$.

1 Introduction

In 1938, the Russian mathematician Dmitri Anatolyevich Bochvar published the influential paper "On a three-valued logical calculus and its application to the analysis of the paradoxes of the classical extended functional calculus" [3], in which he introduced a 3-valued logic aimed at resolving set-theoretic and semantic paradoxes. His proposal diverged significantly from other related approaches in at least two key aspects. First and foremost, the third, non-classical value $1/2$ was *infectious* in any sentential compound involving the standard, or *internal*, propositional connectives $\neg, \wedge, \vee$. This means that a formula would be assigned the value $1/2$ iff at least one variable within it was assigned $1/2$. This third value was interpreted as *paradoxical*. Second, the language of Bochvar's logic included *external* unary connectives J_0, J_1, J_2 (written in Finn & Grigolia's notation) that, unlike the internal connectives, could output only Boolean values.

Although the merits of Bochvar's logic as a solution to the paradoxes remain highly debatable,[1] its influence on successive developments in 3-valued logic has been significant. The internal fragment of Bochvar's logic was characterised by Urquhart [27] through the imposition of a variable inclusion strainer on the consequence relation of classical propositional logic. Building on this result, a general framework for *right variable inclusion logics* has been proposed (see [7] for a detailed account). In this context, the celebrated algebraic construction of *Płonka sums* is extended from algebras to logical matrices. Specifically, each logic L is paired with a *right*

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[1] More promising applications of Bochvar's logic concern, e.g., the treatment of fatal errors in computer science [2; 10] and the analysis of topicality in the philosophy of language [1].

variable inclusion companion L^r whose matrix models are decomposed as Płonka sums of models of L. Notably, Bochvar's internal logic serves as the right variable inclusion companion of classical logic.

Studies on Bochvar's external logic, by contrast, are comparatively scarce. Finn & Grigolia [11] provided an algebraic semantics for it with respect to the quasivariety of *Bochvar algebras*. However, their work does not employ the standard toolbox or terminology of abstract algebraic logic. Adopting a more mainstream approach, the papers [5; 8] extend Finn & Grigolia's completeness theorem to a full-fledged algebraisability result, and offers a representation of Bochvar algebras that refines the Płonka sum representations of their involutive bisemilattice reducts.[2]

In this paper, after covering the necessary preliminaries in Section 2, we further refine the representation result in [8], proving that Bochvar algebras are categorically equivalent to *Bochvar systems* — pairs consisting of a Boolean algebra and a meet-subsemilattice with unit of such (Section 3). In Section 4, we axiomatise both the variety generated by Bochvar algebras, and its proper subvariety corresponding to the varietal join of Boolean algebras and semilattices (in the appropriate similarity type).

2 Preliminaries

We assume that the reader has some familiarity with universal algebra and algebraic logic (standard references are [9] and [12], respectively). In what follows, given a class of similar algebras K, the usual class-operator symbols $\mathrm{I}(\mathsf{K})$, $\mathrm{S}(\mathsf{K})$, $\mathrm{H}(\mathsf{K})$, $\mathrm{P}(\mathsf{K})$ and $\mathrm{P_u}(\mathsf{K})$ respectively denote the closure of K under isomorphic copies, subalgebras, homomorphic images, products and ultraproducts. A class of similar algebras K is a quasivariety if $\mathsf{K} = \mathrm{ISPP_u}(K)$. When K is a finite set of finite algebras, it holds that $\mathrm{ISP}(\mathsf{K}) = \mathrm{ISPP_u}(K)$. When K is also closed under homomorphic images or, equivalently, if $\mathsf{K} = \mathrm{HSP}(\mathsf{K})$, it is said to be a *variety*. $\mathrm{V}(\mathsf{K})$ is shorthand for $\mathrm{HSP}(\mathsf{K})$. Given a quasivariety Q, by $\mathsf{Q}_{\mathrm{RSI}}$ we indicate the class of relatively subdirectly irreducible members of Q. If Q is a variety, we simply write Q_{SI}.

In this paper we will mainly deal with the algebraic language $\langle \wedge, \vee, \neg, J_2, 0, 1\rangle$, namely the expansion of the language of Boolean algebras via a unary operation symbol J_2. This is the language of Bochvar's external logic ($\mathrm{B_e}$) and external paraconsistent weak Kleene logic, the three-valued logics defined over the algebra $\mathbf{WK}^{\mathrm{e}}$ displayed in Figure 1 by taking $\{1\}$ and $\{1, 1/2\}$ as designated values, respectively.

	$\neg$
1	0
1/2	1/2
0	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

	J_2
1	1
1/2	0
0	0

Figure 1: The algebra $\mathbf{WK}^{\mathrm{e}}$.

The quasivariety generated by $\mathbf{WK}^{\mathrm{e}}$, that is, $\mathrm{ISP}(\mathbf{WK}^{\mathrm{e}})$ is the equivalent algebraic semantics of the logics $\mathrm{B^e}$ and $\mathrm{PWK^e}$ (see [5, Theorem 35] and [4, Theorem 7.1]) and is called the *quasivariety of Bochvar algebras* — BCA, in brief. Two additional unary operations can be defined on $\mathbf{WK}^{\mathrm{e}}$ from the operations J_2 and $\neg$, namely $J_0 a := J_2 \neg a$; $J_1 a := \neg(J_2 a \vee J_2 \neg a)$. Those operations were taken as primitive by Finn & Grigolia [11, pp. 233-234], where one finds the following quasi-equational basis for the quasivariety BCA.

[2] Bochvar's 3-valued algebra was employed by Halldén [14] to introduce another logic that incorporates both internal and external connectives; crucially, however, Halldén — unlike Bochvar — treated 1/2 as a designated value. The internal fragment of Halldén's logic has been thoroughly investigated in recent years under the name *paraconsistent weak Kleene logic* (see [7, Chapter 7] for a survey). By contrast, the external logic has received relatively little attention in this case as well (exceptions include [26; 4]). It is important to observe that the quasivariety of Bochvar algebras algebraises not only Bochvar's, but also Halldén's external logic. Finally, it is worth mentioning that Kleene independently introduced the internal fragments of both logics in his [17].

Definition 1. The quasivariety of Bochvar algebras can be axiomatised through the following identities and quasi-identities:

(1) $\varphi \vee \varphi \approx \varphi$;
(2) $\varphi \vee \psi \approx \psi \vee \varphi$;
(3) $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
(4) $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
(5) $\neg(\neg\varphi) \approx \varphi$;
(6) $\neg 1 \approx 0$;
(7) $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
(8) $0 \vee \varphi \approx \varphi$;
(9) $J_2 J_k \varphi \approx J_k \varphi$, for every $k \in \{0,1,2\}$;
(10) $J_0 J_k \varphi \approx \neg J_k \varphi$, for every $k \in \{0,1,2\}$;
(11) $J_1 J_k \varphi \approx 0$, for every $k \in \{0,1,2\}$;
(12) $J_k \neg\varphi \approx J_{2-k} \varphi$, for every $k \in \{0,1,2\}$;
(13) $J_i \varphi \approx \neg(J_j \varphi \vee J_k \varphi)$, for $i \neq j \neq k \neq i$;
(14) $J_k \varphi \vee \neg J_k \varphi \approx 1$, for every $k \in \{0,1,2\}$;
(15) $(J_i \varphi \vee J_k \varphi) \wedge J_i \varphi \approx J_i \varphi$, for $i, k \in \{0,1,2\}$;
(16) $\varphi \vee J_k \varphi \approx \varphi$, for $k \in \{1,2\}$;
(17) $J_0(\varphi \vee \psi) \approx J_0 \varphi \wedge J_0 \psi$;
(18) $J_2(\varphi \vee \psi) \approx (J_2 \varphi \wedge J_2 \psi) \vee (J_2 \varphi \wedge J_2 \neg\psi) \vee (J_2 \neg\varphi \wedge J_2 \psi)$;
(19) $J_0 \varphi \approx J_0 \psi \;\&\; J_1 \varphi \approx J_1 \psi \;\&\; J_2 \varphi \approx J_2 \psi \;\Rightarrow\; \varphi \approx \psi$.

The semantic properties of Bochvar algebras have been investigated in [8], where appropriate representation theorems are provided and the following simplified axiomatisation for BCA is offered:

Theorem 2 (Bonzio & Pra Baldi, [8, Theorem 3.3])**.** The following is a quasi-equational basis for BCA.

(1) $\varphi \vee \varphi \approx \varphi$;
(2) $\varphi \vee \psi \approx \psi \vee \varphi$;
(3) $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
(4) $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
(5) $\neg(\neg\varphi) \approx \varphi$;
(6) $\neg 1 \approx 0$;
(7) $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
(8) $0 \vee \varphi \approx \varphi$;
(9) $J_0 J_2 \varphi \approx \neg J_2 \varphi$;
(10) $J_2 \varphi \approx \neg(J_0 \varphi \vee J_1 \varphi)$;
(11) $J_2 \varphi \vee \neg J_2 \varphi \approx 1$;
(12) $J_2(\varphi \vee \psi) \approx (J_2 \varphi \wedge J_2 \psi) \vee (J_2 \varphi \wedge J_2 \neg\psi) \vee (J_2 \neg\varphi \wedge J_2 \psi)$;
(13) $J_0 \varphi \approx J_0 \psi \;\&\; J_2 \varphi \approx J_2 \psi \;\Rightarrow\; \varphi \approx \psi$.

It is not difficult to check that the congruence generated on $\mathbf{WK}^{\mathrm{e}}$ by the pair $(0,1)$ yields a quotient which is not in BCA (as it fails to satisfy the quasi-identity (13)).

In the available representation theorems for BCA, the key tool is a well-known algebraic construction introduced by Jerzy Płonka [23; 24; 22] (see also [25; 7]) and later dubbed *Płonka sum*. A *semilattice direct system* consists of a family of similar algebras $\{\mathbf{A}_i\}_{i \in I}$ with pairwise disjoint universes, such that the index set I is the universe of a lower-bounded semilattice $\langle I, \vee, i_0 \rangle$ with induced partial order $\leqslant$.[3] Moreover, whenever

[3] The similarity type of such algebras must contain at least an operation symbol of arity 2 or greater. This constraint will be kept implicit in what follows.

$i \leqslant j$, for $i, j \in I$, we have a homomorphism p_{ij} from the algebra $\mathbf{A}_i$ to the algebra $\mathbf{A}_j$. Such homomorphisms satisfy a *compatibility property*: p_{ii} is the identity, for every $i \in I$, and $p_{jk} \circ p_{ij} = p_{ik}$, for every $i \leqslant j \leqslant k$. The algebras in $\{\mathbf{A}_i\}_{i\in I}$ are called the *fibres* of the system.

Given a semilattice direct system of algebras

$$\mathbb{A} = \langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I} = \langle I, \vee, i_0\rangle, \{p_{ij} : i \leqslant_{\mathbf{I}} j\}\rangle,$$

the *Płonka sum* over it is the algebra $\mathbf{A} = \mathcal{P}_{\text{ł}}(\mathbb{A}) = \mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ (of the same similarity type as the algebras in $\{\mathbf{A}_i\}_{i\in I}$) whose universe is the disjoint union $A = \bigsqcup_{i\in I} A_i$ and where a generic n-ary term operation $g^{\mathbf{A}}$ is computed as follows:

$$g^{\mathbf{A}}(a_1, \ldots, a_n) \coloneqq g^{\mathbf{A}_k}(p_{i_1 k}(a_1), \ldots p_{i_n k}(a_n)), \tag{1}$$

where $k = i_1 \vee \cdots \vee i_n$ and $a_1 \in A_{i_1}, \ldots, a_n \in A_{i_n}$. If the similarity type contains any constant operation e, then $e^{\mathbf{A}} = e^{\mathbf{A}_{i_0}}$. The *fibres* of a Płonka sum are the fibres of its underlying semilattice direct system. A fibre $\mathbf{A}_i$ is *trivial* if its universe is a singleton. We will also say that a *fixpoint* is the universe of a trivial fibre. Notice that if a is a fixpoint, then $a = g(a)$ for every unary operation g.

For our purposes, the crucial application of Płonka sums is with respect to semilattice direct systems of Boolean algebras. The resulting algebras have a remarkable feature: they satisfy all and only the valid Boolean identities $\varphi \approx \psi$ that are *regular*, i.e., such that the same variables appear in φ and ψ (this is a consequence of properties of the Płonka sum construction, see e.g. [7, Theorem 2.3.2]).

The class of algebras obtained as Płonka sums over a semilattice direct system of Boolean algebras is a variety IBSL, whose members are known as *involutive bisemilattices* [7, Chapter 2]. Equivalently, this variety can be defined as follows [6]:

Definition 3. An *involutive bisemilattice* is an algebra $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ of type $\langle 2, 2, 1, 0, 0\rangle$ satisfying:

I1. $\varphi \vee \varphi \approx \varphi$;
I2. $\varphi \vee \psi \approx \psi \vee \varphi$;
I3. $\varphi \vee (\psi \vee \delta) \approx (\varphi \vee \psi) \vee \delta$;
I4. $\neg\neg\varphi \approx \varphi$;
I5. $\varphi \wedge \psi \approx \neg(\neg\varphi \vee \neg\psi)$;
I6. $\varphi \wedge (\neg\varphi \vee \psi) \approx \varphi \wedge \psi$;
I7. $0 \vee \varphi \approx \varphi$;
I8. $1 \approx \neg 0$.

Observe that all the identities in Definition 3 are regular. In the Płonka sum representation $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ of an involutive bisemilattice $\mathbf{A}$, given $a, b \in A$, it is possible to check that $a, b \in A_i$ for some $i \in I$ if and only if $a \wedge (a \vee b) = a$ and $b \wedge (b \vee a) = b$. Moreover, if $a \in A_i$ and $i \leqslant j$, it holds $p_{ij}(a) = a \wedge (a \vee b)$, for any $b \in A_j$.

We list some examples of involutive bisemilattices:

(1) Boolean algebras, that is, the involutive bisemilattices that satisfy one of the equivalent identities $\varphi \vee (\varphi \wedge \psi) \approx \varphi$, $\varphi \wedge \neg\varphi \approx 0$ or $\varphi \vee \neg\varphi \approx 1$; their Płonka sum representations consist of a single fibre.
(2) Semilattices with zero, that is, the involutive bisemilattices that satisfy one of the equivalent identities $\varphi \vee \psi \approx \varphi \wedge \psi$, $\neg\varphi \approx \varphi$ or $0 \approx 1$; their Płonka sum representations consist entirely of trivial fibres.
(3) The algebra $\mathbf{WK}$, namely, the J_2-free reduct of $\mathbf{WK}^{\mathrm{e}}$ in Figure 1. This algebra can be represented as the unique Płonka sum whose fibres are a 2-element Boolean algebra and a trivial one. $\mathbf{WK}$ generates the variety IBSL, in symbols $\mathrm{HSP}(\mathbf{WK}) = \mathsf{IBSL}$.

In light of the above observations, it is easy to see that each Bochvar algebra has an IBSL-reduct, because the identities in Definition 3 coincide with the identities (1)–(8) of Theorem 2. Given a Bochvar algebra $\mathbf{A}$, we will denote by $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ the Płonka decomposition of its IBSL-reduct.

The subquasivariety SIBSL of IBSL, axiomatised relative to IBSL by the quasi-identity

$$\varphi \approx \neg\varphi \;\&\; \psi \approx \neg\psi \Rightarrow \varphi \approx \psi,$$

comprises those members of IBSL with at most one fixpoint, and is generated by **WK** as a quasivariety [7, Theorem 7.1.19].[4] Since each defining quasi-identity of SGIB is valid in $\mathbf{WK}^{\mathrm{e}}$, the following proposition holds.

Proposition 4 (Bonzio & Pra Baldi, [8, Proposition 3.5])**.** Every Bochvar algebra has a SIBSL-reduct.

The connection between IBSL and Płonka sums of Boolean algebras provides some representation theorems for Bochvar algebras, as established in [8]. We recall them below, as they play a key role in the subsequent sections. Notably, in the Płonka sum representation of a Bochvar algebra all the homomorphisms are surjective. Consequently, each fibre is a quotient of the algebra $\mathbf{A}_{i_0}$; moreover, it is a quotient modulo a principal congruence.

Lemma 5. Let $\mathbf{A}$ be a Bochvar algebra with $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ the Płonka decomposition of its IBSL-reduct. Then

(1) $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ has surjective homomorphisms;
(2) for every $i \neq i_0$, $p_{i_0 i}$ is not injective;
(3) for every $a \in A$ and every $i \in I$ (with $a \in A_i$), $J_2 a \in p_{i_0 i}^{-1}(a)$ and $J_0 a \in p_{i_0 i}^{-1}(\neg a)$.

Theorem 6 (Bonzio & Pra Baldi, [8, Theorem 3.17])**.** Let $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ be an involutive bisemilattice whose Płonka sum representation is such that

(1) all homomorphisms are surjective and $p_{i_0 i}$ is not injective for every $i_0 \neq i \in I$;
(2) for each $i \in I$ there exists an element $a_i \in A_{i_0}$ such that $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ is an isomorphism, with $a_i \neq a_j$ for $i \neq j$ and, in particular, $a_j < a_i$ for each $i < j$.

Define, for every $a \in A_i$ and $i \in I$, $J_2 a = p_{i_0 i}^{-1}(a) \cap [0, a_i]$ (which is unique by (2)). Then $\mathbf{B} = \langle A, \wedge, \vee, \neg, 0, 1, J_2\rangle$, is a Bochvar algebra.

Theorem 7 (Bonzio & Pra Baldi, [8, Corollary 3.18])**.** Let $\mathbf{A} \in$ SIBSL be such that the underlying semilattice direct system of its Płonka sum representation has surjective and non-injective homomorphisms. The following are equivalent:

(1) $\mathbf{A}$ is the reduct of a Bochvar algebra;
(2) for each $i \in I$, $1/\ker_{p_{i_0 i}}$ is a principal filter, with generator $a_i \in A_{i_0}$. Moreover, if $i \neq j$ then $a_i \neq a_j$ and $a_j < a_i$ for each $i < j$;
(3) for each $i \in I$ there exists an element $a_i \in A_{i_0}$ such that $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ is an isomorphism. Moreover, if $i \neq j$ then $a_i \neq a_j$ and $a_j < a_i$ for each $i < j$.

Occasionally, if $\mathbf{A}$ is a Bochvar algebra and $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$ is the Płonka decomposition of its IBSL-reduct, the top element of the fibre $\mathbf{A}_i$ will be denoted as 1^{A_i} or as 1_i, while the bottom element of the same fibre will be denoted as 0^{A_i} or as 0_i.

3 A categorical equivalence for Bochvar algebras

Capitalising on the results that conclude the previous section, we aim to show that all the information about a Bochvar algebra $\mathbf{A}$, whose involutive bisemilattice reduct is represented as $\mathcal{P}_{\text{ł}}(\mathbf{A}_i)_{i\in I}$, is encoded in two components: *a)* the bottom fibre $\mathbf{A}_{i_0}$ of the attendant semilattice direct system, and *b)* the meet-semilattice of the principal filters of $\mathbf{A}_{i_0}$ whose associated congruences determine the other fibres. Indeed, in this section we will establish that the algebraic category of Bochvar algebras is equivalent to a category whose objects are pairs

[4] This quasivariety is introduced in [6], while [21] considers the same class in a type with no constants.

consisting of a Boolean algebra $\mathbf{B}$ and a meet-subsemilattice with unit $\mathbf{I}$ of $\mathbf{B}$. We will observe that this category is a subcategory of a *comma category* [19] based on Boolean algebras and semilattices. This observation could open avenues for further exploration, particularly by linking Bochvar algebras to well-established algebraic and categorical concepts.

Definition 8. A *Bochvar system* is a pair $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ such that $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1 \rangle$ is a Boolean algebra and $\mathbf{I} = \langle I, \wedge, 1 \rangle$ is a meet-subsemilattice with unit of $\mathbf{B}$.

Hereafter, for $b \in B$, by $[b)$ we denote the principal lattice filter generated by b in $\mathbf{B}$, and for F a filter of $\mathbf{B}$, by $\mathbf{B}/F$ we denote the quotient $\mathbf{B}/\vartheta_F$, where

$$\vartheta_F = \big\{ \langle a, b \rangle \in B^2 : (\neg a \vee b) \wedge (\neg b \vee a) \in F \big\}.$$

Our aim is to associate a Bochvar algebra to each Bochvar system. Thus, if $\mathbb{B} = \langle \mathbf{B}, \mathbf{I} \rangle$ is a Bochvar system, let $\mathbb{A}_{\mathbb{B}} = \big\langle \{\mathbf{A}_i\}_{i \in I}, \mathbf{I}^{\partial}, \{p_{ij} : i \leqslant_{\mathbf{I}^{\partial}} j\} \big\rangle$ be such that:

(1) for all $i \in I$, $\mathbf{A}_i \coloneqq \mathbf{B}/[i)$;
(2) $\mathbf{I}^{\partial}$ is the lower-bounded join-semilattice dual to $\mathbf{I}$;
(3) for all $i, j \in I$ such that $i \leqslant_{\mathbf{I}^{\partial}} j$, $p_{ij}(a/[i)) \coloneqq (a/[i))/[j)$.

Lemma 9. $\mathbb{A}_{\mathbb{B}}$ is a semilattice direct system of Boolean algebras.

Proof. It suffices to show that each p_{ij} is well-defined, that p_{ii} is the identity on $\mathbf{A}_i$ and that for $i \leqslant_{\mathbf{I}^{\partial}} j \leqslant_{\mathbf{I}^{\partial}} k$, $p_{jk} \circ p_{ij} = p_{ik}$.

Since $\mathbf{I}$ is a subsemilattice with unit of $\mathbf{B}$, whenever $i \leqslant_{\mathbf{I}^{\partial}} j$, we have that $j \leqslant_{\mathbf{B}} i$ and hence $[i) \subseteq [j)$. Therefore $p_{ij}(a/[i)) = (a/[i))/[j)$ is a congruence class in $\mathbf{A}_i = \mathbf{B}/[i)$ and thus p_{ij} is a well-defined surjective homomorphism from $\mathbf{A}_i$ to $\mathbf{A}_j$. By definition, $p_{ii}(a/[i)) = (a/[i))/[i) = a/[i)$ because the congruence that corresponds to the filter $[i)$ in $\mathbf{B}$ is the identity congruence in $\mathbf{B}/[i)$. Finally, by the general homomorphism theorem (see, e.g., [20, Theorem B2]),

$$p_{jk} \circ p_{ij}(a/[i)) = ((a/[i))/[j))/[k) = (a/[i))/[k). \qquad \square$$

Since $\mathbb{A}_{\mathbb{B}}$ is a semilattice direct system of Boolean algebras, $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}})$ is an involutive bisemilattice. We now prove that it can be expanded to a Bochvar algebra in a unique way.

Theorem 10. $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}})$ is the involutive bisemilattice reduct of a unique Bochvar algebra $\mathbf{A}_{\mathbb{B}}$.

Proof. We use the equivalence between conditions (1) and (2) of Theorem 7. First of all, we must show that $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}}) \in \mathsf{SIBSL}$. As already remarked, $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}}) \in \mathsf{IBSL}$; we must show that it has at most one trivial fibre. However, if $0^{\mathbf{B}} \in I$, $\mathbf{B}/[0^{\mathbf{B}})$ will be the unique trivial fibre in $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}})$; otherwise, $\mathcal{P}_{\mathfrak{l}}(\mathbb{A}_{\mathbb{B}})$ will have no trivial fibre, because all the algebras in $\{\mathbf{A}_i\}_{i \in I}$ are quotients of $\mathbf{B}$ modulo congruences that are strictly less than $\nabla = B \times B$.

Further, observe that for all $i \in I$,

$$\begin{aligned} \ker_{p_{1i}} &= \{\langle a, b \rangle \in B^2 : p_{1i}(a) = p_{1i}(b)\} \\ &= \{\langle a, b \rangle \in B^2 : a/[i) = b/[i)\}, \end{aligned}$$

so $1/\ker_{p_{1i}} = \{a \in B : \langle a, 1 \rangle \in \ker_{p_{1i}}\} = \{a \in B : a \in 1/[i)\} = [i)$, which is a principal filter of $\mathbf{B}$ with generator i. Moreover, if $i <_{\mathbf{I}^{\partial}} j$, then clearly $j <_{\mathbf{I}} i$ and thus $j <_{\mathbf{B}} i$, as $\mathbf{I}$ is a subsemilattice of $\mathbf{B}$.

Uniqueness follows from the observation that if $a/[i) \in \mathbf{A}_i$, then $J_2(a/[i)) = p_{1i}^{-1}(a/[i))$, which is uniquely determined by Theorem 7 (3). $\square$

After canonically associating a Bochvar algebra to a given Bochvar system, we now proceed in the opposite direction. Let $\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1 \rangle$ be a Bochvar algebra, whose involutive bisemilattice reduct decomposes as $\mathcal{P}_{\mathfrak{l}}(\mathbf{A}_i)_{i \in I}$. We define $\mathbb{B}_{\mathbf{A}} \coloneqq \langle \mathbf{A}_{i_0}, \mathbf{K} \rangle$, where $K = \{J_2^{\mathbf{A}}(1^{A_i}) : i \in I\}$, and for $J_2^{\mathbf{A}}(1^{A_i}), J_2^{\mathbf{A}}(1^{A_j}) \in K$, $J_2^{\mathbf{A}}(1^{A_i}) \leqslant_{\mathbf{K}} J_2^{\mathbf{A}}(1^{A_j})$ iff $j \leqslant_{\mathbf{I}} i$.

Theorem 11. $\mathbb{B}_{\mathbf{A}}$ is a Bochvar system.

Proof. $\mathbf{A}_{i_0}$ is a Boolean algebra. K is a subset of $B = A_{i_0}$, by Lemma 5. It contains $1^{\mathbf{B}} = 1^{\mathbf{A}_{i_0}} = J_2^{\mathbf{A}}(1^{\mathbf{A}_{i_0}})$. We now show that it is closed under meets in $\mathbf{B}$. Using Definition 1 (18) and the fact that for all $i \in I$ we have $J_2^{\mathbf{A}}(0^{A_i}) \leqslant^{\mathbf{A}} J_2^{\mathbf{A}}(1^{A_i})$,

$$\begin{aligned}
J_2^{\mathbf{A}}(1^{A_i}) \wedge^{\mathbf{B}} J_2^{\mathbf{A}}(1^{A_j}) &= J_2^{\mathbf{A}}(1^{A_i}) \wedge^{\mathbf{A}_{i_0}} J_2^{\mathbf{A}}(1^{A_j}) \\
&= (J_2^{\mathbf{A}}(1^{A_i}) \wedge^{\mathbf{A}_{i_0}} J_2^{\mathbf{A}}(1^{A_j})) \\
&\quad \vee^{\mathbf{A}} (J_2^{\mathbf{A}}(1^{A_i}) \wedge^{\mathbf{A}_{i_0}} J_2^{\mathbf{A}}(0^{A_j})) \\
&\quad \vee^{\mathbf{A}} (J_2^{\mathbf{A}}(0^{A_i}) \wedge^{\mathbf{A}_{i_0}} J_2^{\mathbf{A}}(1^{A_j})) \\
&= J_2^{\mathbf{A}}(1^{A_i} \vee^{\mathbf{A}} 1^{A_j}) \\
&= J_2^{\mathbf{A}}(1^{A_{i \vee^{\mathbf{I}} j}}).
\end{aligned}$$

□

The next goal in our agenda is showing that the previous correspondences between Bochvar algebras and Bochvar systems are mutually inverse (modulo isomorphism).

Theorem 12. If $\mathbf{A}$ is a Bochvar algebra, then $\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}$ is isomorphic to $\mathbf{A}$.

Proof. Let $\mathbf{A} = \langle A, \wedge, \vee, \neg, J_2, 0, 1\rangle$ be a Bochvar algebra, whose involutive bisemilattice reduct decomposes as $\mathcal{P}_{\mathrm{ł}}(\mathbf{A}_i)_{i\in I}$. Recall that $\mathbb{B}_{\mathbf{A}} = \langle \mathbf{A}_{i_0}, \mathbf{K}\rangle$, where $K = \{J_2^{\mathbf{A}}(1^{A_i}) : i \in I\}$. However, as a set of indices, we may as well use its bijective copy I, replacing when convenient each $J_2^{\mathbf{A}}(1^{A_i})$ by i. Thus, taking into account the fact that the order of $\mathbf{K}$ dualises the order of $\mathbf{I}$, the Płonka sum representation of the involutive bisemilattice reduct of $\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}$ is $\left\langle \{\mathbf{A}_i^*\}_{i\in I}, \mathbf{I}, \{p_{ij}^* : i \leqslant_{\mathbf{I}} j\}\right\rangle$, where $\mathbf{A}_i^* = \mathbf{A}_{i_0}/[i) = \mathbf{A}_{i_0}/[J_2^{\mathbf{A}}(1^{A_i}))$ and for all $a \in A_{i_0}$,

$$p_{ij}^*(a/[J_2^{\mathbf{A}}(1^{A_i}))) = (a/[J_2^{\mathbf{A}}(1^{A_i})))/[J_2^{\mathbf{A}}(1^{A_j})).$$

To prove that $\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}$ and $\mathbf{A}$ have isomorphic involutive bisemilattice reducts, it suffices to show that for all $i, j \in I$, $\mathbf{A}_i^* \cong \mathbf{A}_i$ and, up to isomorphism, $p_{ij}^* = p_{ij}$. However, using Theorem 7 (3), $f(a) \coloneqq J_2^{\mathbf{A}} a/[i)$ is an isomorphism between $\mathbf{A}_i$ and $\mathbf{A}_{i_0}/(1/\mathrm{ker}_{p_{1i}})$ for every $i \in I$. So $\mathbf{A}_i \cong \mathbf{A}_{i_0}/(1/\mathrm{ker}_{p_{1i}}) = \mathbf{A}_{i_0}/[i) = \mathbf{A}_i^*$. Moreover, let $a \in A_i$. Identifying a with $f(a)$,

$$\begin{aligned}
p_{ij}(J_2^{\mathbf{A}} a/[i)) &= (J_2^{\mathbf{A}} a/[i))/[j) \\
&= p_{ij}^*(J_2^{\mathbf{A}} a/[i)).
\end{aligned}$$

To round off our proof, it remains to be established that the term operations realised by J_2 in both algebras coincide. Indeed, applying Theorem 6 and working again up to isomorphism:

$$J_2^{\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}} a = J_2^{\mathbf{A}_{\mathbb{B}_{\mathbf{A}}}}(J_2^{\mathbf{A}} a/[i)) = p_{i_0 i}^{-1}(J_2^{\mathbf{A}} a/[i)) = J_2^{\mathbf{A}}(J_2^{\mathbf{A}} a/[i)) = J_2^{\mathbf{A}} a.$$

□

As implied by the definition of homomorphism to be found below, two Bochvar systems $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1\rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2\rangle$ are said to be isomorphic when there exists an isomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in I_2$ whenever $i \in I_1$.

Theorem 13. If $\mathbb{B} = \langle \mathbf{B}, \mathbf{I}\rangle$ is a Bochvar system, then $\mathbb{B}$ is isomorphic to $\mathbb{B}_{\mathbf{A}_{\mathbb{B}}}$.

Proof. Recall that to $\mathbb{B} = \langle \mathbf{B}, \mathbf{I}\rangle$ is associated a unique Bochvar algebra $\mathbf{A}_{\mathbb{B}}$ over the semilattice direct system of Boolean algebras $\mathbb{A}_{\mathbb{B}} = \left\langle \{\mathbf{A}_i\}_{i\in I}, \mathbf{I}^{\partial}, \{p_{ij} : i \leqslant_{\mathbf{I}^{\partial}} j\}\right\rangle$, defined before Lemma 9, so that $\mathbb{B}_{\mathbf{A}_{\mathbb{B}}} = \langle \mathbf{A}_{1^{\mathbf{B}}}, \mathbf{K}\rangle$, where $K = \{J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_i}}) : i \in I\}$ and $J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_i}}) \leqslant_{\mathbf{K}} J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_j}})$ iff $j \leqslant_{\mathbf{I}^{\partial}} i$ iff $i \leqslant_{\mathbf{I}} j$.

Firstly, we must show that $\mathbf{B} \cong \mathbf{A}_{1^{\mathbf{B}}}$, which is clear since $\mathbf{A}_{1^{\mathbf{B}}} = \mathbf{B}/[1^{\mathbf{B}}) \cong \mathbf{B}$. The other requirement is obvious. Actually, we even have $\mathbf{K} \cong \mathbf{I}$. Indeed, let $f\colon I \to K$ be defined by $f(i) = J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_i}})$. f is bijective by Theorem 7. Moreover,

$$
\begin{aligned}
f(i \wedge^{\mathbf{I}} j) &= f(i \vee^{\mathbf{I}^{\partial}} j) \\
&= J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_{i\vee j}}}) \\
&= J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_i}}) \wedge^{\mathbf{K}} J_2^{\mathbf{A}_{\mathbb{B}}}(1^{\mathbf{A}_{\mathbb{B}_j}}) \\
&= f(i) \wedge^{\mathbf{K}} f(j).
\end{aligned}
$$

□

In the following, $\mathfrak{B}$ will denote the algebraic category of Bochvar algebras. We now define a category $\mathfrak{S}$ whose objects are Bochvar systems. If $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1 \rangle$ and $\mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2 \rangle$ are objects in $\mathfrak{S}$, a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$ is a homomorphism g from $\mathbf{B}_1$ to $\mathbf{B}_2$ such that $g(i) \in I_2$ for every $i \in I_1$. Observe that any such g is also a homomorphism from $\mathbf{I}_1$ to $\mathbf{I}_2$.

Let us briefly recall the notion of a *comma category.* If $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ are categories and S, T are functors, respectively from $\mathfrak{A}$ to $\mathfrak{C}$ and from $\mathfrak{B}$ to $\mathfrak{C}$, called the *source* and the *target* functor, the comma category $S \downarrow T$ is defined as follows:

(1) Its objects are triples $\langle A, B, f \rangle$, with A an object in $\mathfrak{A}$, B an object in $\mathfrak{B}$ and $f\colon S(A) \to T(B)$ a morphism in $\mathfrak{C}$.
(2) The morphisms from $\langle A, B, f \rangle$ to $\langle A', B', f' \rangle$ are pairs $\langle g, h \rangle$ where $g\colon A \to A'$ is a morphism in $\mathfrak{A}$, $h\colon B \to B'$ is a morphism in $\mathfrak{B}$ and the following diagram commutes:

$$
\begin{tikzcd}
S(A) \arrow[r, "S(g)"] \arrow[d, "f"] & S(A') \arrow[d, "f'"] \\
T(B) \arrow[r, "T(h)"] & T(B').
\end{tikzcd}
$$

Now, consider the comma category $S \downarrow T$, where the source functor S is the identity functor from the algebraic category $\mathfrak{Sem}$ of meet semilattices with unit to itself, and the target functor T is the forgetful functor from the algebraic category $\mathfrak{Bool}$ of Boolean algebras to $\mathfrak{Sem}$. $\mathfrak{S}$ can be viewed as a full subcategory of $S \downarrow T$, whose objects are those $\langle A, B, f \rangle$ such that f is injective (since A must be a subreduct of B) and whose morphisms are the $S \downarrow T$-morphisms between such objects. Therefore, $\mathfrak{S}$ is well-defined.

We now define the map Γ as follows:

(1) If $\mathbf{A}$ is an object in $\mathfrak{B}$, let $\Gamma(\mathbf{A}) \coloneqq \mathbb{B}_{\mathbf{A}}$;
(2) If $f\colon \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism in $\mathfrak{B}$, let $\Gamma(f)$ be the restriction of f to $\mathbf{A}_{1_{i_0}}$.

Similarly, we define the map Ξ as follows:

(1) If $\mathbb{B}$ is an object in $\mathfrak{S}$, let $\Xi(\mathbb{B}) \coloneqq \mathbf{A}_{\mathbb{B}}$;
(2) If $g\colon \mathbb{B}_1 \to \mathbb{B}_2$ is a morphism in $\mathfrak{S}$, let $\Xi(g)$ be defined as follows: $\Xi(g)(a/[i)) \coloneqq g(a)/[g(i))$.

Lemma 14. Γ is a functor from $\mathfrak{B}$ to $\mathfrak{S}$.

Proof. If $\mathbf{A}$ is an object in $\mathfrak{B}$, $\Gamma(\mathbf{A})$ is an object in $\mathfrak{S}$ by Theorem 11. Next, we take care of morphisms. Let $\mathbf{A}_1, \mathbf{A}_2$ be Bochvar algebras (with the associated Płonka representations of their involutive bisemilattice reducts), let f be a homomorphism from $\mathbf{A}_1$ to $\mathbf{A}_2$, and let $a \in A_{1_{i_0}}$. Then $\Gamma(f)(a) = f \restriction A_{1_{i_0}}(a) = f(a) \in A_2$. However, if $a \in A_{1_{i_0}}$, $a \vee^{\mathbf{A}_1} \neg^{\mathbf{A}_1} a = 1^{\mathbf{A}_1}$, and thus $f(a) \vee^{\mathbf{A}_2} \neg^{\mathbf{A}_2} f(a) = f(a \vee^{\mathbf{A}_1} \neg^{\mathbf{A}_1} a) = f(1^{\mathbf{A}_1}) = 1^{\mathbf{A}_2}$ and hence $f(a) \in A_{2_{i_0}}$. Since f is a homomorphism of Bochvar algebras, its restriction to $\mathbf{A}_{1_{i_0}}$ is a homomorphism of Boolean algebras. Moreover, for $i \in I_1$, $\Gamma(f)(J_2^{\mathbf{A}_1}(1^{\mathbf{A}_{1_i}})) = f(J_2^{\mathbf{A}_1}(1^{\mathbf{A}_{1_i}})) = J_2^{\mathbf{A}_2}(f(1^{\mathbf{A}_{1_i}})) = J_2^{\mathbf{A}_2}(1^{\mathbf{A}_{2_{f^*(i)}}})$, where f^* is the homomorphism from $\mathbf{I}_1$ to $\mathbf{I}_2$ induced by f, viewed as a homomorphism of involutive bisemilattices. □

Lemma 15. Ξ is a functor from $\mathfrak{S}$ to $\mathfrak{B}$.

Proof. If $\mathbb{B}$ is an object in $\mathfrak{S}$, $\Xi(\mathbb{B})$ is an object in $\mathfrak{B}$ by Theorem 10. Let $\mathbb{B}_1 = \langle \mathbf{B}_1, \mathbf{I}_1 \rangle, \mathbb{B}_2 = \langle \mathbf{B}_2, \mathbf{I}_2 \rangle$ be Bochvar systems, let g be a morphism from $\mathbb{B}_1$ to $\mathbb{B}_2$, and let $a/[i] \in A_{\mathbb{B}_1}$, where $a \in B_1$, $i \in I_1$. Then $g(a) \in B_2$, $g(i) \in I_2$, and thus $\Xi(g)(a/[i]) = g(a)/[g(i)]$ is well-defined. We have to show that it is a homomorphism of Bochvar algebras. We check just the clauses for $\wedge$ and J_2. To keep the notation reasonably unencumbered, we will not add superscripts to the semilattice joins of $\mathbf{I}_1$ and $\mathbf{I}_2$, relying on the context to disambiguate.

$$\begin{aligned}
\Xi(g)(a/[i] \wedge^{\mathbf{B}_1} b/[j]) &= \Xi(g)(p_{i,i\vee j}(a/[i]) \wedge^{\mathbf{B}_{1_{i\vee j}}} p_{j,i\vee j}(b/[j])) \\
&= \Xi(g)((a/[i])/[i\vee j) \wedge^{\mathbf{B}_{1_{i\vee j}}} (b/[j))/[i\vee j)) \\
&= \Xi(g)(a/[i\vee j) \wedge^{\mathbf{B}_{1_{i\vee j}}} b/[i\vee j)) \\
&= \Xi(g)(a \wedge^{\mathbf{B}_1} b/[i\vee j)) \\
&= g(a \wedge^{\mathbf{B}_1} b)/[g(i\vee j)) \\
&= (g(a) \wedge^{\mathbf{B}_2} g(b))/[g(i)\vee g(j)) \\
&= g(a)/[g(i)\vee g(j)) \wedge^{\mathbf{B}_{2_{g(i)\vee g(j)}}} g(b)/[g(i)\vee g(j)) \\
&= g(a)/[g(i)) \wedge^{\mathbf{B}_2} g(b)/[g(j)) \\
&= \Xi(g)(a/[i)) \wedge^{\mathbf{B}_2} \Xi(g)(b/[j)).
\end{aligned}$$

$$\begin{aligned}
\Xi(g)(J_2^{\mathbf{B}_1}(a/[i))) &= \Xi(g)(p_{1,i}^{-1}(a/[i))) \\
&= \Xi(g)(a/\{1^{\mathbf{B}_1}\}) \\
&= g(a)/[g(1^{\mathbf{B}_1})) \\
&= g(a)/\{1^{\mathbf{B}_2}\} \\
&= p_{1,g(i)}^{-1}(g(a)/[g(i))) \\
&= J_2^{\mathbf{B}_2}(g(a)/[g(i))) \\
&= J_2^{\mathbf{B}_2}(\Xi(g)(a/[i))).
\end{aligned}$$

□

Theorem 16. The functors Γ and Ξ induce a categorical equivalence between the categories $\mathfrak{B}$ and $\mathfrak{S}$.

Proof. This results follows from Theorems 12 and 13 and Lemmas 14 and 15 if we can prove that for any morphism $f\colon \mathbf{A}_1 \to \mathbf{A}_2$ in $\mathfrak{B}$ and any morphism $g\colon \mathbb{B}_1 \to \mathbb{B}_2$ in $\mathfrak{S}$, $\Xi(\Gamma(f)) = f$ and $\Gamma(\Xi(g)) = g$. We have to show the commutativity of the following diagram:

$$\begin{tikzcd}
\mathbf{A}_1 \arrow[r, "f"] \arrow[d, "\Xi\Gamma"] & \mathbf{A}_2 \arrow[d, "\Xi\Gamma"] \\
\Xi(\Gamma(\mathbf{A}_1)) \arrow[r, "\Xi\Gamma(f)"] & \Xi(\Gamma(\mathbf{A}_2)).
\end{tikzcd}$$

However, for $a \in \mathbf{A}_{1_i}$,

$$\begin{aligned}
\Xi(\Gamma(f))(\Xi(\Gamma(a))) &= \Xi(\Gamma(f))(J_2 a/[i)) \\
&= f \restriction \mathbf{A}_{1_{i_0}}(J_2 a)/[f \restriction \mathbf{A}_{1_{i_0}}(i)) \\
&= J_2 f(a)/[f(i)) \\
&= \Xi(\Gamma(f(a))).
\end{aligned}$$

The commutativity of the diagram

$$\begin{tikzcd}
\mathbb{B}_1 \arrow[r, "g"] \arrow[d, "\Gamma\Xi"] & \mathbb{B}_2 \arrow[d, "\Gamma\Xi"] \\
\Gamma(\Xi(\mathbb{B}_1)) \arrow[r, "\Gamma\Xi(g)"] & \Gamma(\Xi(\mathbb{B}_2))
\end{tikzcd}$$

is shown similarly. □

4 On the variety generated by Bochvar algebras

In this section we investigate the variety generated by the proper quasivariety BCA. For a start, in Section 4.1 we provide an equational basis for this variety and characterise the Płonka sum representations of its members. Like IBSL, $\mathrm{V}(\mathsf{BCA})$ contains as subvarieties term-equivalent counterparts BA of Boolean algebras and SL of semilattices with zero — in the former subvariety the operation symbol J_2 is interpreted as the identity function, while in the latter it is interpreted as the constant function yielding 0 (or, equivalently, 1) for each argument. In Section 4.2 we axiomatise the join of these (independent) varieties in the lattice of subvarieties of $\mathrm{V}(\mathsf{BCA})$, which coincides with the class of isomorphic copies of the product of a Boolean algebra and a semilattice.

4.1 Axiomatisation and basic properties

Definition 17. The class K is defined as the variety axiomatised by the following identities:

$\mathbf{K_1}$ $\varphi \vee \varphi \approx \varphi$;
$\mathbf{K_2}$ $\varphi \vee \psi \approx \psi \vee \varphi$;
$\mathbf{K_3}$ $\varphi \vee (\psi \vee \delta) \approx (\varphi \vee \psi) \vee \delta$;
$\mathbf{K_4}$ $\neg\neg\varphi \approx \varphi$;
$\mathbf{K_5}$ $\varphi \wedge \psi \approx \neg(\neg\varphi \vee \neg\psi)$;
$\mathbf{K_6}$ $\varphi \wedge (\neg\varphi \vee \psi) \approx \varphi \wedge \psi$;
$\mathbf{K_7}$ $0 \vee \varphi \approx \varphi$;
$\mathbf{K_8}$ $1 \approx \neg 0$;
$\mathbf{K_9}$ $J_2\varphi \vee \neg J_2\varphi \approx 1$;
$\mathbf{K_{10}}$ $\varphi \vee J_2\psi \approx \varphi \vee J_2(\varphi \vee \psi)$;
$\mathbf{K_{11}}$ $\varphi \wedge J_2\varphi \approx \varphi$;
$\mathbf{K_{12}}$ $J_2(\varphi \wedge \neg\varphi) \approx 0$.

We begin with a simple arithmetical lemma.

Lemma 18. In K, the following holds:

(1) $\varphi \vee J_2\varphi \approx \varphi$;
(2) $\varphi \approx J_2\varphi \vee (\varphi \wedge \neg\varphi)$;
(3) $J_2J_2\varphi \approx J_2\varphi$;
(4) $\varphi \vee \neg J_2\psi \approx \varphi \vee \neg J_2(\varphi \vee \psi)$.

Proof. (1). Let $\mathbf{A} \in \mathsf{K}$. Observe that $J_2 0 = 0$. Indeed, using $\mathbf{K_4}$, $\mathbf{K_5}$, $\mathbf{K_8}$, and $\mathbf{K_{12}}$, $0 = J_2(0 \wedge \neg 0) = J_2(0 \wedge 1) = J_2 0$. From this we have, for any $a \in A$, that $a = a \vee 0 = a \vee J_2 0 = a \vee J_2(0 \vee a) = a \vee J_2 a$ by $\mathbf{K_7}$, $\mathbf{K_{10}}$.

(2). Using $\mathbf{K_{11}}$, (1), and involutive bisemilattice properties, we obtain

$$J_2a \vee (a \wedge \neg a) = (J_2a \vee a) \wedge (J_2a \vee \neg a) = a \wedge (J_2a \vee \neg a) = (a \wedge J_2a) \vee (a \wedge \neg a) = a \vee (a \wedge \neg a) = a.$$

(3). Using $\mathbf{K_9}$, (2), and involutive bisemilattice properties, $J_2a = J_2J_2a \vee (J_2a \wedge \neg J_2a) = J_2J_2a$.

(4). By De Morgan's laws and the dual of $\mathbf{K_6}$ it holds that $\varphi \vee \neg(\varphi \vee \psi) = \varphi \vee \neg\psi$. This, together with $\mathbf{K_{10}}$ yields

$$\begin{aligned}\varphi \vee \neg J_2\psi &= \varphi \vee \neg(\varphi \vee J_2\psi)\\ &= \varphi \vee \neg(\varphi \vee J_2(\varphi \vee \psi))\\ &= \varphi \vee \neg J_2(\varphi \vee \psi).\end{aligned}$$

□

Recall that by $\mathbf{WK}^{\mathrm{e}}$ we denote the 3-element Bochvar algebra of Figure 1, whose IBSL-reduct is $\mathbf{WK}$.

Lemma 19. Let $\mathbf{A} \in \mathsf{K}$. Then,

(1) $\mathbf{A}$ has an IBSL-reduct;
(2) The Płonka representation of $\mathbf{A}$ has surjective homomorphisms;
(3) If the lowest fibre $\mathbf{A}_0$ in the Płonka representation of $\mathbf{A}$ is a 2-element Boolean algebra, then there is a unique way to turn the IBSL-reduct of $\mathbf{A}$ into a K-algebra. This is done by defining the J_2 operation as follows, for every $a \in A$:

$$J_2 a = \begin{cases} 1 & \text{if } a = 1_i \text{ and } i \in I^+, \\ 0 & \text{otherwise,} \end{cases} \tag{J-def}$$

where $I^+ = \{i \in I \; : \; |A_i| > 1\}$.

Proof. (1). This is obvious as $\mathbf{K_1}$–$\mathbf{K_8}$ axiomatise IBSL (see Definition 3).

(2). By (1), $\mathbf{A}$ admits a Płonka sum representation; let $\langle I, \leqslant\rangle$ be its underlying semilattice. For $i, j \in I$, suppose $i \leqslant j$ and let $a \in A_j$. By $\mathbf{K_9}$ we have that $J_2 a \in A_0$. Moreover, Lemma 18 (1) and $\mathbf{K_{11}}$ entail $J_2 a \wedge (J_2 a \vee a) = a$, i.e., $p_{0j}(J_2 a) = a$. By the composition property of Płonka homomorphisms, $a = p_{0j}(J_2 a) = p_{ij}(p_{0i}(J_2 a))$, thus a is the image under p_{ij} of $p_{0i}(J_2 a)$. This shows the claim.

(3). Let $\mathbf{A}$ be such that the universe of the bottom fibre is $A_0 = \{0, 1\}$. Since every homomorphism p_{ij} in the Płonka sum represetation of $\mathbf{A}$ is surjective, every fibre $\mathbf{A}_i$ is such that $|A_i| \leqslant 2$. If $|A_i| = 1$ and $a \in A_i$, then $a = a \wedge \neg a$ and $J_2 a = J_2(a \wedge \neg a) = 0$, by $\mathbf{K_{12}}$. If $|A_i| = 2$, then $J_2 0_i = J_2(0_i \wedge 1_i) = 0$ (again by $\mathbf{K_{12}}$). On the other hand, suppose by contradiction that $J_2 1_i = 0$, hence, by $\mathbf{K_{11}}$, $1_i = 1_i \wedge J_2 1_i = 1_i \wedge 0 = 1_i \wedge p_{0i}(0) = 1_i \wedge 0_i = 0_i$, in contradiction with the fact that $|A_i| = 2$, i.e., $0_i \neq 1_i$. □

Lemma 20. Let $\mathbf{A} \in \mathsf{K}$ be such that $\mathbf{A}_0$ is a 2-element Boolean algebra. Assume $a \in A_i, b \in A_j$ and $\mathbf{A}_i, \mathbf{A}_j$ are non-trivial. Then $\mathbf{A}_{i \vee j}$ is non-trivial.

Proof. Assume $a \in A_i$, $b \in A_j$, and that $\mathbf{A}_i$, $\mathbf{A}_j$ are non-trivial. Suppose, towards a contradiction, that $\mathbf{A}_{i\vee j}$ is trivial. Then $a \vee b \in A_{i \vee j}$, which by (J-def) in Lemma 19 entails $J_2(a \vee b) = 0 = J_2(0_i \vee 1_j)$. By $\mathbf{K_{10}}$, however, we have that $0_i = 0_i \vee J_2(0_i \vee 1_j) = 0_i \vee J_2 1_{i\vee j} = 1_i$, a contradiction. □

Theorem 21. The variety generated by BCA is K.

Proof. Since $\mathrm{V}(\mathsf{BCA}) = \mathrm{H}(\mathsf{BCA}) = \mathrm{HSP}(\mathbf{WK}^{\mathrm{e}}) = \mathrm{V}(\mathbf{WK}^{\mathrm{e}})$ and $\mathbf{K_1}$–$\mathbf{K_{12}}$ hold in $\mathbf{WK}^{\mathrm{e}}$, clearly $\mathrm{V}(\mathsf{BCA}) \subseteq \mathsf{K}$. We will obtain the converse inclusion by showing that $\mathsf{K}_{\mathrm{SI}} \subseteq \mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$, where $\mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$ consists of the trivial algebra, the 2-element semilattice, the 2-element Boolean algebra and $\mathbf{WK}^{\mathrm{e}}$. Our strategy is an adaption of the results contained in [16], and it proceeds as follows. Let $\mathbf{A} \in \mathsf{K}_{\mathrm{SI}}$. By Lemma 19, $\mathbf{A}$ has an IBSL-reduct with surjective homomorphisms and underlying semilattice I. Let 0 be the least element of I. For $a \in A_0$ define the relation

$$\Theta_a \coloneqq \{\langle b, c\rangle \in \mathbf{A}^2 : a \vee b = a \vee c,\, a \vee \neg b = a \vee \neg c\}.$$

The relation Θ_a is an IBSL congruence [6, Lemma 24]. Moreover, suppose $\langle b, c\rangle \in \Theta_a$. Then:

$$a \vee J_2 b = a \vee J_2(a \vee b) = a \vee J_2(a \vee c) = a \vee J_2 c,$$

where the first and the last equalities hold by $\mathbf{K_{10}}$, while the second one holds because of the assumption $\langle b, c\rangle \in \Theta_a$. Similarly, using Lemma 18 (4), we obtain

$$a \vee \neg J_2 b = a \vee \neg J_2(a \vee b) = a \vee \neg J_2(a \vee c) = a \vee \neg J_2 c,$$

thus Θ_a is compatible with J_2 and it is a congruence on $\mathbf{A}$.

We claim that $\Theta_a = \Delta$ if and only if $a = 0$. The right-to-left direction of this claim is obvious. Conversely, assume $\Theta_a = \Delta$. For any $b \in A$ we have $a \vee b = a \vee (a \vee b)$ and

$$a \vee \neg(a \vee b) = a \vee (\neg a \wedge \neg b) = (a \vee \neg a) \wedge (a \vee \neg b) = a \vee \neg b,$$

where the above equivalences are justified by the fact that De Morgan's laws and the distributivity laws hold in IBSL. Hence, $b = a \vee b$ for every $b \in A$, which is true if and only if $a = 0$. This proves the claim.

We now show that $\mathbf{A}_0$ is either a trivial algebra or the 2-element Boolean algebra. For $a \in A_0$, let $\langle b, c\rangle \in \Theta_a \cap \Theta_{\neg a}$. From the definition of this congruence it follows that $a \vee b = a \vee c$ and $a \wedge b = a \wedge c$. Thus

$$\begin{aligned} b \wedge (b \vee a) &= b \wedge (c \vee a) \\ &= (b \wedge c) \vee (b \wedge a) \\ &= (b \wedge c) \vee (c \wedge a) \\ &= c \wedge (c \vee a). \end{aligned}$$

Since $a \in A_0$, for any $i \in I$ and $d \in A_i$ we have that $d \wedge (d \vee a) = d$. So, the above equivalences entail that $b = c$, namely $\Theta_a \cap \Theta_{\neg a} = \Delta$. Since $\mathbf{A}$ is subdirectly irreducible, $\Theta_a = \Delta$ or $\Theta_{\neg a} = \Delta$ which by the above claim entails $a = 0$ or $\neg a = 0$ (i.e., $a = 1$). Therefore, we have shown that $\mathbf{A}_0$ is either trivial or a 2-element Boolean algebra. If it is a trivial algebra, then $\mathbf{A}$ is a semilattice by Lemma 19 (2), thus $\mathbf{A}$ is the 2-element semilattice, i.e., the unique subdirectly irreducible semilattice.

So, suppose that $\mathbf{A}_0$ is a 2-element Boolean algebra. Notice that for every $i \in I$, $\mathbf{A}_i$ is either trivial or a 2-element Boolean algebra, because each homomorphism is surjective. Suppose, towards a contradiction, that $\mathbf{A} \notin \mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$. We distinguish two cases, depending on the presence of non-trivial fibres in the Płonka representation of $\mathbf{A}$.

Let us begin with the case in which there is at most one trivial fibre. Our assumption that $\mathbf{A} \notin \mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$, together with (J-def), ensures that $\mathbf{A}_0$ cannot be the only non-trivial fibre, for otherwise $\mathbf{A} \in \mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$. Define a relation Θ' as follows:

$$\langle a, b\rangle \in \Theta' \iff \begin{cases} a = b \text{ or} \\ \exists i, j \in I^+,\ a = 1_i,\ b = 1_j \text{ or} \\ \exists i, j \in I^+,\ a = 0_i,\ b = 0_j, \end{cases}$$

where $I^+ = \{i \in I : \mathbf{A}_i \text{ is non-trivial}\}$.

Furthermore, let us now define $\Theta^\star$ as:

$$\langle a, b\rangle \in \Theta^\star \iff \begin{cases} a = b \text{ or} \\ \exists i \in I,\ a, b \in A_i. \end{cases}$$

Clearly both are equivalence relations. We now show that Θ' is a congruence on $\mathbf{A}$. The compatibility with $\neg$ is obvious once we recall that whenever $a, b \in A_i$ then also $\neg a, \neg b \in A_i$. Suppose $\langle a, b\rangle \in \Theta'$ and $\langle c, d\rangle \in \Theta'$. Without loss of generality we may assume $a \neq b$ and $a = 1_i$, $b = 1_j$, $c = 0_k$, $d = 0_z$. Therefore $\mathbf{A}_i$, $\mathbf{A}_j$ are non-trivial, by definition of Θ'. If one among $\mathbf{A}_k$ and $\mathbf{A}_z$ is trivial, then $c = d$ and $k = z$, which will be the top element of I. Therefore, since $\mathbf{A}$ has at most one trivial fibre, $\langle a \vee c, b \vee d\rangle \in \Theta$, as desired. Otherwise, $\mathbf{A}_k, \mathbf{A}_z$ are non-trivial and $c \neq d$. By Lemma 20, $\mathbf{A}_{i\vee k}$ and $\mathbf{A}_{j\vee z}$ are non-trivial, thus $a \vee c = 1_{i\vee k}$ and $b \vee d = 1_{j\vee z}$, namely $\langle a \vee c, b \vee d\rangle \in \Theta'$. Suppose now $\langle a, b\rangle \in \Theta'$ so, without loss of generality $a = 1_i, b = 1_j$ for some $i, j \in I^+$. By (J-def) in Lemma 19 we obtain $J_2 a = 1 = J_2 b$, thus Θ' is compatible with J_2 and it is a congruence on $\mathbf{A}$.

The relation $\Theta^\star$ is an involutive bisemilattice congruence (see e.g. [7, Theorem 2.3.11]) and $J_2 a \in A_0$ for each $a \in A$, so $\Theta^\star$ is compatible with J_2. Recall that, as already noticed, there are additional non-trivial fibres other than $\mathbf{A}_0$; let $\mathbf{A}_q$ be one of them, distinct from $\mathbf{A}_0$. Thus, $\langle 1, 1_q\rangle \in \Theta' \neq \Delta \neq \Theta^\star$, because $\langle 0, 1\rangle \in \Theta^\star$. If we show $\Theta' \cap \Theta^\star = \Delta$, we would then obtain the desired contradiction with the fact that $\mathbf{A}$ is subdirectly

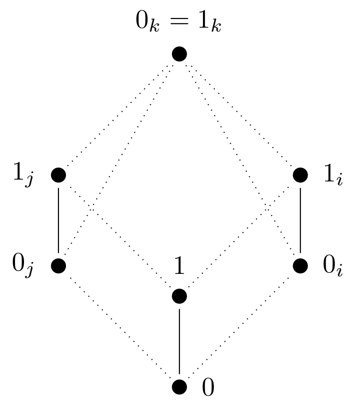

Figure 2: A forbidden configuration

irreducible. To this end, notice that for distinct elements a, b, we have that $\langle a, b\rangle \in \Theta \cap \Theta^\star$ if and only if either $a = 1_i$, $b = 1_j$ and $i = j$, or $a = 0_i$, $b = 0_j$ and $i = j$. In both cases, this is true if and only if $a = b$, as desired.

We now consider the case in which there are two or more trivial fibres. Define the relation Θ as:

$$\langle a, b\rangle \in \Theta \iff \begin{cases} a = b \text{ or} \\ a = \neg a,\ b = \neg b. \end{cases}$$

This is an equivalence relation. We show that it is a congruence on $\mathbf{A}$. If $\langle a, b\rangle \in \Theta$, $a = \neg a$, $b = \neg b$, so $\langle \neg a, \neg b\rangle \in \Theta$. Moreover $J_2 a = 0 = J_2 b$, by Lemma 19 (3), i.e., $\langle J_2 a, J_2 b\rangle \in \Theta$. If $\langle a, b\rangle \in \Theta$, $\langle c, d\rangle \in \Theta$, then $a \vee c$, $b \vee d$ must be fixpoints. This proves that Θ is a congruence. Notice that $\Theta^\star$ is a congruence on $\mathbf{A}$ also in this case, for the very same argument used previously. Observe also that $\Theta \neq \Delta \neq \Theta^\star$, because $\langle 0, 1\rangle \in \Theta^\star$, while Θ collapses (at least) the two trivial fibres. We proceed to show that $\Theta \cap \Theta^\star = \Delta$. We have that $\langle a, b\rangle \in \Theta \cap \Theta^\star$ if and only if $a = \neg a$, $b = \neg b$ and $a, b \in A_i$ for some $i \in I$. However, this is true if and only if $a = b$, again a contradiction.

Therefore, we conclude $\mathbf{A} \in \mathrm{HS}(\mathbf{WK}^{\mathrm{e}})$. This proves $K \subseteq \mathrm{V}(\mathsf{BCA})$ and, finally, $\mathsf{K} = \mathrm{V}(\mathsf{BCA})$. $\square$

Remark 22. Observe that Lemma 20 allows us to build a SIBSL with surjective homomorphisms that cannot be equipped with a $\mathrm{V}(\mathsf{BCA})$ structure. This is the case of the algebra displayed in Figure 2, let us call it $\mathbf{A}$. This algebra is a SIBSL whose underlying semilattice I consists of four elements: the bottom element, i, j and $k = i \vee j$. Notice that i and j are incomparable elements in I. The corresponding fibres $\mathbf{A}_0$, $\mathbf{A}_i$ and $\mathbf{A}_j$ are 2-element Boolean algebras, while $\mathbf{A}_k$ is a trivial algebra. Dotted lines represent Płonka homomorphisms. Suppose now that $\mathbf{A}$ can be equipped with an appropriate operation J_2 satisfying $\mathbf{K_1}$–$\mathbf{K_{12}}$. By Lemma 19, we have that $J_2(0_i) = 0 = J_2(1_j \vee 0_i) = J_2(0_k)$. By $\mathbf{K_{10}}$, however, we have that $0_i = 0_i \vee J_2(0_i \vee 1_j) = 0_i \vee J_2 1_j = 1_i$, a contradiction. This shows that any IBSL having $\mathbf{A}$ as a subalgebra cannot be turned into a $\mathrm{V}(\mathsf{BCA})$ algebra.

By Theorem 21, we are in a position to help ourselves to all the equational properties of Bochvar algebras when carrying out computations concerning members of $\mathsf{K} = \mathrm{V}(\mathsf{KBCA})$. For example, in what follows, the axioms of Definition 1 will be freely employed when necessary.

Remark 23. As previously noted, all semilattices with zero can be viewed as members of $\mathrm{V}(\mathsf{BCA})$, such that all primitive operation symbols other than $\vee$ and 0 are either redundant, or realise the identity, or realise constant operations. For this reason $\mathrm{V}(\mathsf{BCA})$ does not satisfy any congruence identities in the type of lattices.

4.2 The varietal join of Boolean algebras and semilattices

In this subsection we show that BA and SL, whose relative equational bases w.r.t. $\mathrm{V}(\mathsf{BCA})$ are given by the identities $J_2x \approx x$ and $J_2x \approx 1$ respectively, are *independent* subvarieties of $\mathrm{V}(\mathsf{BCA})$ and, therefore, that their varietal join $\mathsf{BA} \vee \mathsf{SL}$ comprises precisely the isomorphic images of direct products of a Boolean algebra and a semilattice. Although this result directly provides an axiomatisation of $\mathsf{BA} \vee \mathsf{SL}$, using the algorithms given in, for instance, [15] or [18], we derive a much simpler equational basis by explicitly describing the direct product decomposition.

Recall that two subvarieties V_1, V_2 of a variety V of type τ are said to be *independent* if there exists a binary term $\varphi(x, y)$ of type τ such that $\mathsf{V}_1 \vDash \varphi(x, y) \approx x$ and $\mathsf{V}_2 \vDash \varphi(x, y) \approx y$. Recall, moreover, that the *direct product* of two similar varieties V_1, V_2 is defined as

$$\mathsf{V}_1 \times \mathsf{V}_2 \coloneqq I(\{\mathbf{A}_1 \times \mathbf{A}_2 : \mathbf{A}_1 \in \mathsf{V}_1,\ \mathbf{A}_2 \in \mathsf{V}_2\}).$$

The following classical theorem about independent varieties is due to Grätzer, Lakser & Płonka [13]:

Theorem 24. Let V_1, V_2 be independent subvarieties of an arbitrary variety V. Then V_1, V_2 are such that their join $\mathsf{V}_1 \vee \mathsf{V}_2$ (in the lattice of subvarieties of V) is their direct product $\mathsf{V}_1 \times \mathsf{V}_2$.

We aim at showing that the varieties BA and SL are independent. Before that, we confirm that it is legitimate to identify these classes with Boolean algebras and semilattices, respectively.

Lemma 25. Let $\mathbf{A} \in \mathrm{V}(\mathsf{BCA})$. Then:

(1) $\mathbf{A} \models J_2x \approx x$ iff the involutive bisemilattice reduct of $\mathbf{A}$ is a Boolean algebra;
(2) $\mathbf{A} \models J_2x \approx 1$ iff the involutive bisemilattice reduct of $\mathbf{A}$ is a semilattice.

Proof. (1). If the involutive bisemilattice reduct of $\mathbf{A}$ is a Boolean algebra, its Płonka sum representation has a single fibre, whence by $\mathbf{K_{11}}$ and Lemma 18 (1), for all $a \in A$ we have that $J_2a = J_2a \vee (J_2a \wedge a) = a$. Conversely, if $\mathbf{A} \models J_2x \approx x$, then for all $a \in A$ we have that $a \wedge \neg a = J_2(a \wedge \neg a) = 0$, hence the involutive bisemilattice reduct of $\mathbf{A}$ is a Boolean algebra.

(2). If the involutive bisemilattice reduct of $\mathbf{A}$ is a semilattice, then $\mathbf{A} \models x \approx \neg x$. Then by $\mathbf{K_9}$ for all $a \in A$ we have that $1 = J_2a \vee \neg J_2a = J_2a \vee J_2a = J_2a$. Conversely, if $\mathbf{A} \models J_2x \approx 1$, then for all $a \in A$ we have that $1 = J_2(a \wedge \neg a) = 0$ by $\mathbf{K_{12}}$, which means that hence the involutive bisemilattice reduct of $\mathbf{A}$ is a semilattice with zero. ☐

Theorem 26. BA and SL are independent subvarieties of $\mathrm{V}(\mathsf{BCA})$.

Proof. Using Lemma 25, it is easy to check that the term $\varphi(x, y) \coloneqq J_2x \vee (J_2x \wedge y)$ witnesses independence for BA and SL. ☐

Corollary 27. $\mathsf{BA} \vee \mathsf{SL} = \mathsf{BA} \times \mathsf{SL}$.

Proof. This follows from Theorem 24 and Theorem 26. ☐

Given an algebra $\mathbf{A}$ in $\mathrm{V}(\mathsf{BCA})$, we single out two notable subsets of its universe: its *open* elements and its *dense* elements. In the Płonka sum decomposition of its underlying involutive bisemilattice, the former correspond to the elements in the bottom fibre, while the latter correspond to the bottom elements in each fibre:

$$O(\mathbf{A}) \coloneqq \{a \in A : J_2a = a\};$$
$$D(\mathbf{A}) \coloneqq \{a \in A : J_2a = 0\}.$$

Lemma 28. Let $\mathbf{A} \in \mathsf{V}(\mathsf{BCA})$, and let $a \in A$.

(1) $a \in O(\mathbf{A})$ if and only if there exists $b \in A$ such that $a = J_2 b$;
(2) $a \in D(\mathbf{A})$ if and only if there exists $b \in A$ such that $a = b \wedge \neg b$;
(3) $O(\mathbf{A})$ is the universe of a Boolean subalgebra $\mathbf{O}(\mathbf{A})$ of $\mathbf{A}$;
(4) $D(\mathbf{A})$ is the universe of a semilattice $\mathbf{D}(\mathbf{A})$ which is isomorphic to a quotient of $\mathbf{A}$.

Proof. (1). The left-to-right direction is immediate. For the converse direction, by Lemma 18 (3) we have that $a = J_2 b = J_2 J_2 b = J_2 a$.

(2). If $a \in D(\mathbf{A})$, then by Lemma 18 (2), $a = J_2 a \vee (a \wedge \neg a) = 0 \vee (a \wedge \neg a) = a \wedge \neg a$. Conversely, suppose that $a = b \wedge \neg b$. Then by $\mathbf{K_{12}}$, $J_2 a = J_2(b \wedge \neg b) = 0$.

(3). Clearly $J_2 1 = 1$. Let $J_2 a = a$. Then using Definition 1 (10), $J_2 \neg a = J_0 a = J_0 J_2 a = \neg J_2 a = \neg a$. If $J_2 a = a, J_2 b = b$, then (having already proved that $O(\mathbf{A})$ is closed under $\neg$) by Definition 1 (18) we have that $J_2(a \vee b) = (J_2 a \wedge J_2 b) \vee (J_2 a \wedge J_2 \neg b) \vee (J_2 \neg a \wedge J_2 b) = (a \wedge b) \vee (a \wedge \neg b) \vee (\neg a \wedge b) = a \vee b$. Observe that the last equality holds because we are applying a J_2-free regular identity that is valid in all Boolean algebras. Finally, if $J_2 a = a$, then $J_2 J_2 a = J_2 a$. Thus, $O(\mathbf{A})$ is the universe of a subalgebra $\mathbf{O}(\mathbf{A})$ of $\mathbf{A}$. This subalgebra is Boolean by Lemma 25 (1).

(4). It is well-known (see e.g. [7, Chapter 2]) that the relation $\vartheta = \{\langle a, b\rangle : a \wedge \neg a = b \wedge \neg b\}$ is a congruence of the involutive bisemilattice reduct of $\mathbf{A}$. Moreover, if $a \mathrel{\vartheta} b$, then $J_2 a \mathrel{\vartheta} J_2 b$, since by $\mathbf{K_4}$, $\mathbf{K_5}$, $\mathbf{K_8}$, $\mathbf{K_9}$, $J_2 a \wedge \neg J_2 a = 0 = J_2 b \wedge \neg J_2 b$. So ϑ is a congruence of $\mathbf{A}$. On the other hand, it is easy to see that $D(\mathbf{A})$ contains 0 and is closed w.r.t. J_2. To show that it is closed w.r.t. joins, we use (2). If $a = x \wedge \neg x, b = y \wedge \neg y$, then, using a J_2-free regular identity valid in all Boolean algebras, $a \vee b = (x \wedge \neg x) \vee (y \wedge \neg y) = (x \vee y) \wedge \neg(x \vee y)$. Letting $\neg a = a$ for all $a \in D(\mathbf{A})$, the resulting algebra is a semilattice which is isomorphic to $\mathbf{A}/\vartheta$ via the map $\varphi(a/\vartheta) = a \wedge \neg a$. ☐

Let V be the subvariety of $\mathsf{V}(\mathsf{BCA})$ axiomatised relative to $\mathsf{V}(\mathsf{BCA})$ by the single identity $J_2 \neg x \approx \neg J_2 x$. Clearly, $\mathsf{BA}, \mathsf{SL} \subseteq \mathsf{V}$. Also, $\mathbf{WK}^{\mathrm{e}} \notin \mathsf{V}$. We will show that V coincides with the varietal join $\mathsf{BA} \vee \mathsf{SL}$.

Theorem 29. Every $\mathbf{A} \in \mathsf{V}$ is embeddable into the direct product of some $\mathbf{B} \in \mathsf{BA}$ and some $\mathbf{C} \in \mathsf{SL}$.

Proof. We will embed $\mathbf{A}$ into $\mathbf{O}(\mathbf{A}) \times \mathbf{D}(\mathbf{A})$, whence the result follows from Lemmas 25 and 28 (3)–(4). Let $a \in A$. We define $\varphi(a) \coloneqq \langle J_2 a, a \wedge \neg a\rangle$. By Lemma 28 (1)–(2), φ is well-defined. Moreover, φ is injective. Indeed, let $\varphi(a) = \varphi(b)$. Then $J_2 a = J_2 b$ and $a \wedge \neg a = b \wedge \neg b$, whence by Lemma 18 (2), $a = J_2 a \vee (a \wedge \neg a) = J_2 b \vee (b \wedge \neg b) = b$.

We show that φ preserves the operations. Let $\mathbf{B} \coloneqq \mathbf{O}(\mathbf{A})$, $\mathbf{C} \coloneqq \mathbf{D}(\mathbf{A})$ and $a, b \in A$.

(1) $\varphi(1^{\mathbf{A}}) = \langle J_2 1, 1 \wedge \neg 1\rangle = \langle 1^{\mathbf{B}}, 1^{\mathbf{C}}\rangle = 1^{\mathbf{B}\times\mathbf{C}}$.
(2) $\varphi(\neg^{\mathbf{A}} a) = \langle J_2 \neg a, \neg a \wedge \neg\neg a\rangle = \langle \neg^{\mathbf{B}} J_2 a, a \wedge \neg a\rangle = \neg^{\mathbf{B}\times\mathbf{C}}\langle J_2 a, a \wedge \neg a\rangle = \neg^{\mathbf{B}\times\mathbf{C}}\varphi(a)$.
(3) $$\begin{aligned}\varphi(a \wedge^{\mathbf{A}} b) &= \langle J_2(a \wedge b), (a \wedge b) \wedge \neg(a \wedge b)\rangle \\ &= \langle J_2 a \wedge^{\mathbf{B}} J_2 b, (a \wedge \neg a) \wedge^{\mathbf{C}} (b \wedge \neg b)\rangle \\ &= \langle J_2 a, a \wedge \neg a\rangle \wedge^{\mathbf{B}\times\mathbf{C}} \langle J_2 b, b \wedge \neg b\rangle = \varphi(a) \wedge^{\mathbf{B}\times\mathbf{C}} \varphi(b).\end{aligned}$$
(4) $\varphi(J_2^{\mathbf{A}} a) = \langle J_2 J_2 a, J_2 a \wedge \neg J_2 a\rangle = \langle J_2^{\mathbf{B}} J_2 a, J_2^{\mathbf{C}}(a \wedge \neg a)\rangle = J_2^{\mathbf{B}\times\mathbf{C}}\langle J_2 a, a \wedge \neg a\rangle = J_2^{\mathbf{B}\times\mathbf{C}}\varphi(a)$.

Observe that we have crucially utilised the identity $J_2 \neg x \approx \neg J_2 x$, as well as some J_2-free regular identities valid in all Boolean algebras. ☐

Corollary 30. The variety V coincides with the varietal join $\mathsf{BA} \vee \mathsf{SL}$ in the lattice of subvarieties of $\mathsf{V}(\mathsf{BCA})$.

Proof. Using Corollary 27 and Theorem 29, together with the fact that all the identities axiomatising V are valid both in Boolean algebras and in semilattices, we obtain the following chain of inclusions:

$$\mathsf{V} \subseteq \mathrm{ISP}(\mathsf{BA} \cup \mathsf{SL}) \subseteq \mathsf{BA} \vee \mathsf{SL} = \mathsf{BA} \times \mathsf{SL} \subseteq \mathsf{V}.$$ ☐

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