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ORIGINAL ARTICLE

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Deformation measures of micropolar theory in spatial and material description – A review

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Abstract In this paper we present a review of some of the foundations of micropolar continua. The focus is initially on the classical representation of fields in spatial or material description. We shall remind the reader that traditionally both are based on the notion of the indestructible material particle. We will give reasons why these traditional concepts may fail, namely if we wish to study more complex processes, such as agglomeration or crushing of matter. As a way out, we will present a suitable extension, which we call *true spatial description*. We shall also demonstrate that the classical twofold approach can lead to serious misunderstandings that may result in unnecessary scientific controversies. Further attention is paid to the various deformation measures that are encountered in the literature on micropolar materials. It will be discussed under which circumstances which one should preferably be used. In this context the kinetic equation for the microinertia tensor deserves particular attention: it was recently extended by a production term. This additional feature can be used to describe processes in matter associated with micromorphological change, for example during crushing and milling of substances. Here a continuum description in terms of material, indestructible particles is no longer possible, and true spatial notation becomes a must.

1 Micropolar continua and outline to the paper

The ultimate references on micropolar continua—solids as well as fluids—are Eringen's books [25] and [26]. A predecessor of them is the article by Eringen and Kafadar [27]. All these books favor index notation. For a more modern approach in direct notation the book by Eremeyev *et al.* [17] may serve as an excellent example.

In this paper we are aiming at a clarification of the following issues, which in fact are all interconnected:

- The spatial and the material description of continuous fields: what is their common basis, are they fully equivalent, and are there situations where the associated notion of the material particle can become insufficient?
- Which strain measures for micropolar continua are useful in (true) spatial description and which ones for material description? In which way are they related to each other? When are they different and when do they coincide?
- What is the status of the kinetic equation for the balance of microinertia? Are there circumstances under which it needs to be be extended and and, if so, how?

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In our presentation direct notation will be preferred but often the corresponding Cartesian index form will also be presented. This is done to provide clarification and ease of use for the reader. Our survey should be helpful for scientists working in solids and fluids alike. However, modern materials might require a hybrid formulation so that a presentation of both worlds in one paper seems more than justified.

2 Field descriptions revisited

In this section we will introduce the spatial and the material descriptions of fields. Although one might think that this is all well known, it turns out that in certain situations the spatial and the material description that uses the current position—also known as Eulerian description—must clearly be distinguished from each other and that the latter may need to be extended. We will explain why (true) spatial description is a much wider concept than the Eulerian (material) one. For that reason, we decided to clarify differences from the very start. We begin with a short review of the situation.

2.1 An incomplete review of the literature

It is well known that continuum theory received an immense boost from the rational spirit introduced by the school of Clifford Truesdell. Maybe the new concepts even had a slightly stronger impact in solid than in fluid mechanics. This can be suspected after reading how Truesdell explains the various possibilities for describing the fields relevant to continuous matter: the fluid mechanics perspective—the so-called Eulerian description—is just one of many. However, in order to understand such subtle issues, we must first accept Truesdell's rather mathematical points of view. In this section we will basically follow his nomenclature. However, whenever required additional information and notation from other texts will be provided.

On page 35 of [71] Truesdell considers massy bodies that consist of infinitely many *substantial points* X. These are often also referred to as *material elements* ([35], pg. 18) or *material particles* ([4], pg. 8). In short: they contain an infinitesimally small amount of massy matter.

The substantial points are moving. Truesdell calls the invertible mapping

$$\boldsymbol{x} = \boldsymbol{\chi}(X, t) \quad \Leftrightarrow \quad X = \boldsymbol{\chi}^{-1}(\boldsymbol{x}, t) \tag{2.1}$$

the *motion* of the body made of material particles ([71], pg. 35 and 86), where $\boldsymbol{x}$ is the vector to the point in space currently occupied by X at time t. A remark: if we keep in mind that the motion of a particular particle X is concerned, the simpler notation $\boldsymbol{x} = \check{\boldsymbol{x}}(t)$ can be used. It still distinguishes between value and function of position, $\boldsymbol{x}$ and $\check{\boldsymbol{x}}(t)$, respectively. It also immediately allows to see that a position is meant because the symbol $\boldsymbol{x}$ is used throughout. For certain transformations in equations, it is easier to handle than Truesdell's more detailed but also unwieldy expression shown in Eq. $(2.1)_1$.

Consequently, the velocity of a particular (fixed) material particle X is obtained by differentiation of the motion function with respect to time ([71], pg. 36):[1]

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{\chi}(X, t)}{\partial t}\right|_X =: \dot{\boldsymbol{\chi}}(X, t)\,. \tag{2.2}$$

In here, the dot on top of $\boldsymbol{\chi}$ is initially just a shorthand notation for the partial derivative in time while keeping the material particle X fixed. In more physics-based terms we could also refer to it as a *total derivative* of the motion. In general, let us assign some arbitrary physical property $\psi = F(X, t)$ to the material particle X. In words, total time derivative means the ratio of the change of this physical property divided by time interval for this particular material point, hence ([22], pg. 69, [71], pg. 97),

$$\frac{\mathrm{d}F(X, t)}{\mathrm{d}t} = \left.\frac{\partial F(X, t)}{\partial t}\right|_X =: \dot{F}(X, t) \quad \Rightarrow \quad \boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\chi}(X, t)}{\mathrm{d}t} \equiv \frac{\mathrm{d}\check{\boldsymbol{x}}(t)}{\mathrm{d}t}. \tag{2.3}$$

Moreover, each material particle follows its own path $\boldsymbol{\chi}(X, t) = \check{\boldsymbol{x}}(t)$: the motion it undergoes signifies nothing else but its current 3D spatial position $\boldsymbol{x}$ at the current time t. It is now granted that each material

[1] The symbol =: means that the expression on the right is defined by what is written on the left. Truesdell uses it for example on pg. 97 of [71].

particle is uniquely identifiable at time $t = 0$ by its position vector $\boldsymbol{X}$. Truesdell now relates the particle X to $\boldsymbol{X}$ through the mapping $\boldsymbol{\kappa}$ ([71], pg. 94):

$$\boldsymbol{X} = \boldsymbol{\kappa}\,(X)\,. \tag{2.4}$$

The motion is now rewritten using a new function $\boldsymbol{\chi}_\kappa$, which is bijective, *i.e.*, uniquely invertible:[2]

$$\boldsymbol{x} = \boldsymbol{\chi}\left(\boldsymbol{\kappa}^{-1}\,(\boldsymbol{X}, t)\,, t\right) \equiv \boldsymbol{\chi}_\kappa\,(\boldsymbol{X}, t) \quad \Leftrightarrow \quad \boldsymbol{X} = \boldsymbol{\chi}_\kappa^{-1}(\boldsymbol{x}, t). \tag{2.5}$$

A critical summary is in order.

- Note that on pg. 5 of [69] or on pg. 35 of [71] the material particle is initially called X, whatever X may be. By doing so, it remains vague how this very particle can technically be identified, maybe by some color or a running number? In [69] Truesdell introduces the identifying (reference) position $\boldsymbol{X}$ of the particle X on the same page. He calls the Cartesian coordinates of $\boldsymbol{X}$ its *intrinsic coordinates*. However, in this publication he does *not* talk about the concept of a *reference position* at all. Indeed, this is much different in [71], where $\boldsymbol{X}$ is introduced much later than the somewhat vague identifier X, namely on pg. 94. We will return to this issue shortly.
- Being a continuum concept the size of a material particle is infinitely small in the sense of calculus. However, in order to capture reality its "size" should be chosen such that homogenized quantities obtained by averaging the corresponding properties of its micro-constituents (corpuscles like atoms, molecules, grains, crystals, *etc*.) represent convenient substitutes for their counterparts on the microlevel. In other words: the continuum hypothesis must be fulfilled and the material particle must be representative.
- To quote Truesdell from [71], pp. 36: "This statement [namely Eq. (2.1)], which asserts that two distinct substantial points never come to occupy the same place at the same time, is sometimes called the Axiom of Impenetrability."
- Consequently, from the axiom, we must also conclude that material points are indestructible and cannot exchange matter. This limits the applicability of this concept to certain physical processes. For example, the crushing and agglomeration of matter relevant to soil mechanics are impossible to describe. A similar comment applies to mixtures of substances. In this case, exchange of matter in form of various species is required—the mixing process—which may or may not lead to chemical reactions. As we see later, in the so-called rational mixture theories of the school of Truesdell all the mixing and all the chemistry happens within one infinitesimally small material particle. But what if we wish or even need to exchange matter from adjacent regions? This remains unanswered by the aforementioned mixture theories and illustrates the limits of the material particle concept. As we shall see, in rational mixture theories it even leads to conceptual problems in context with the idea of a single reference configuration.

We are now ready to list and comment on Truesdell's four methods of describing fields of the continuum. We can find them on page 10 of [69] and on page 95 of [71]. They are referred to as the *substantial*[3], the *referential*, the *spatial*, and the *relative* description, which due to the hypotheses of smoothness of the continuum, are all equivalent, as Truesdell remarks initially. In other words, one can be derived from the other by purely mathematical transformations. Before we start explaining, it should be pointed out that *all of them* are based on the concept of a material particle, in particular what Truesdell calls the spatial description method. Let us give a short verbal summary of each, before we will present the spatial (how fluid mechanicians understand it) and the referential ones in more mathematical detail:

- The *substantial description* uses directly the labels X of the substantial points and the time parameter t. Truesdell envisions the designation X, which is continuous, to be an extension of discrete mass point labels, which he calls X_q with the mass M_q, and q is a (in the limit continuous) running index. Most certainly, such a (continuous) tagging is not so easy to do in practice. Truesdell seems to be aware of this dilemma when he says "With the substantial description for continua, strictly interpreted, few analytical tools are at hand."
- The *referential description* shows a way out of this trap: the position vectors of all material points in the body at a fixed time are used to label the material points. One reference position is particularly important, the one at time $t = 0$ leading to the positions $\boldsymbol{X}$ for the material point X. It was used without further explanation in Eq. (2.5). Now, Truesdell refers to it as the *Lagrangian description* and says it tracks back to Euler without providing reference. At this point it should already be mentioned that the Lagrangian

[2] Later we shall depart from Truesdell's choice of symbols for the functions and use tildes and hats instead.

[3] called *material* in [69]

description is the most practical one to use for describing the deformation of solids. However, this does not mean that it is forbidden to be used for fluids, although it might lead to some clumsy computation. Moreover, as was mentioned by Lurie, it is not necessary to regard the reference position as corresponding to some fixed instant, but this assumption is often convenient ([55], pg. 2). Let us note that in Lagrangian description the velocity must be written following Truesdell's notation as follows:

$$\boldsymbol{v} = \left.\frac{\partial \boldsymbol{\chi}_\kappa(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}} = \frac{\mathrm{d}\boldsymbol{\chi}_\kappa(\boldsymbol{X}, t)}{\mathrm{d}t}. \tag{2.6}$$

We will "streamline" this notation later.

- For Truesdell in *spatial description* the place $\boldsymbol{x}$ and the time t are taken as independent variables in the arguments of functions. He mentions that this choice of variables is also known as *Eulerian description* in the world of fluids. It is "spatial" in the sense that we can describe properties of matter in every point in space $\boldsymbol{x}$ for a certain moment of time t.
  Let us return to the arbitrary physical property ψ and write ([71], pg. 97):

$$\psi = F(X, t) = F\left(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t\right) =: f(\boldsymbol{x}, t). \tag{2.7}$$

The last representation means that the physical quantity is now considered as a function of the field of *all* space points $\boldsymbol{x}$ and *all* times t. Indeed, material particles X are located on the positions $\boldsymbol{x}$. However, there is no emphasis on a special particle X, and $\boldsymbol{x}$ and t are *independent* as Truesdell says so aptly. Now, time derivatives of a property concern a particular particle X. Hence, in order to specify a total time derivative of the property ψ of the particle X in spatial description, we must rewrite the arguments of the last term in Eq. (2.7) and then apply the chain rule:

$$\psi = f(\boldsymbol{x}, t) = f\left(\boldsymbol{\chi}(X, t), t\right) \quad \Rightarrow \quad \frac{\mathrm{d}f}{\mathrm{d}t} = \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_t \cdot \frac{\mathrm{d}\boldsymbol{\chi}(X, t)}{\mathrm{d}t} + \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_t, \tag{2.8}$$

where Eq. $(2.3)_2$ was used and the velocity $\boldsymbol{v}$ was brought in. Alternatively we could have emphasize the position symbol $\boldsymbol{x}$ in the last relation slightly more:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_t \cdot \frac{\mathrm{d}\check{\boldsymbol{x}}(t)}{\mathrm{d}t} + \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v} \cdot \left.\frac{\partial f}{\partial \boldsymbol{x}}\right|_t. \tag{2.9}$$

If, in particular, we choose $f\left(\boldsymbol{\chi}(X, t), t\right) = \boldsymbol{\chi}(X, t) = \check{\boldsymbol{x}}(t) \equiv \boldsymbol{x}$, then the partial derivative with respect to time drops out because of the independence of $\boldsymbol{x}$ and t, and, by evaluating the last relation, the equation for the velocity becomes an identity:

$$\frac{\mathrm{d}\boldsymbol{\chi}(X, t)}{\mathrm{d}t} = \boldsymbol{0} + \boldsymbol{v} \cdot \boldsymbol{1} = \boldsymbol{v}. \tag{2.10}$$

This already indicates the special role of velocity in spatial description, where it is the primary kinematic quantity. The motion does no longer have this prominent role, as it did in the case of Lagrangian description shown in Eq. (2.1) or (2.5). Truesdell turns the last relation around ([71], pg. 97) to obtain:

$$\boldsymbol{v} = \frac{\mathrm{d}\boldsymbol{\chi}(X, t)}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{\chi}\left(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), t\right)}{\mathrm{d}t} =: \dot{\boldsymbol{x}}(\boldsymbol{x}, t). \tag{2.11}$$

The dot on top of $\boldsymbol{x}$ is a shorthand notation for the total derivative $\frac{\mathrm{d}}{\mathrm{d}t}$.

- For Truesdell the *relative description* is obtained by choosing the current location $\boldsymbol{x}$ from Eq. (2.5) at time t as reference for identifying the material particle X. He explains this concept on pp. 109 of [71]: two different times and the corresponding positions of a material particle are considered. One time is the current time, t, and the other, τ, is arbitrary, and relative to t. It can be past or future. By exploiting Eq. (2.1) we have:

$$\boldsymbol{x} = \boldsymbol{\chi}(X, t), \quad \boldsymbol{\xi} = \boldsymbol{\chi}(X, \tau). \tag{2.12}$$

We now eliminate X in the second relation:

$$\boldsymbol{\xi} = \boldsymbol{\chi}\left(\boldsymbol{\chi}^{-1}(\boldsymbol{x}, t), \tau\right) \equiv \boldsymbol{\chi}_t(\boldsymbol{x}, \tau). \tag{2.13}$$

The new function $\boldsymbol{\chi}_t$ is called *relative transplacement* by Truesdell. This is an interesting coinage, which deserves some explanation and to be placed in context with other notions more common in continuum theory, specifically with what is called *displacement*. Recall that $\boldsymbol{\xi}$ is the position of the material particle at some other time τ then the present time t, and $\boldsymbol{x}$ its position at time t that is chosen as reference. Traditionally, the displacement of a particle, defined as $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$, is the difference in position between the current and some (previous or future) reference location, $\boldsymbol{x}$ and $\boldsymbol{X}$, respectively. Hence we may want to call

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{\xi} \equiv \boldsymbol{\chi}_t(\boldsymbol{x}, t) - \boldsymbol{\chi}_t(\boldsymbol{x}, \tau) \tag{2.14}$$

the *displacement* of the material particle from time τ to time t, in order to establish some connection between the two notions. However, notice that the original idea of a displacement has been switched, in the sense that "current" is now "reference."

In the same context Truesdell says explicitly "Sometimes we shall wish to calculate the relative transplacement when the motion is given to us only through the spatial description of the velocity field." A very important statement, indeed, because it turns out that this is exactly the situation we face in fluid mechanics, where the velocity—and not the motion—is the primary field. He is building a bridge between the world of spatial description (fluid mechanics perception) and the notion of the motion of a material particle by combining Eq. $(2.12)_2$ and the function $\dot{\boldsymbol{x}}$ from Eq. (2.11):

$$\frac{\partial \boldsymbol{\xi}(X, \tau)}{\partial \tau} = \dot{\boldsymbol{x}}(\boldsymbol{\xi}, \tau)\,. \tag{2.15}$$

In this differential equation the right hand side—the velocity function—is known and the relative position $\boldsymbol{\xi}$ is the unknown to be obtained by integration. To this end, an initial condition is required:

$$\boldsymbol{\xi}\big|_{t=\tau} = \boldsymbol{\chi}(X, \tau = t) = \boldsymbol{\chi}_t(\boldsymbol{x}, t) \equiv \boldsymbol{x}. \tag{2.16}$$

A remark: Truesdell's use of a partial derivative with respect to time in Eq. (2.15) makes the equation look like a partial differential equation. This is deceiving. The equation is solved for a fixed material particle X. In this sense, it is an ordinary differential equation, and providing boundary conditions is unnecessary.

At the end of these explanations a few historical notes are helpful:

- The original reference to consult in context with Euler's introduction of the Lagrangian description is [28]. On pg. 275 we read "On doit aussi supposer que l'état du fluide dans un certain tems soit connu, & que je nommerai l'état primitif du fluide: cet état étant quasi arbitraire, il faut premièrement connoitre la disposition des particules, dont le fluide est composé, & le mouvement qui leur aura été imprimé, à moins que dans l'état primitif le fluide n'ait été en repos."[4] We may interpret the *état primitif* as (some special) reference state.
- The fixed position of Eulerian description—the fixed grid point—is introduced on pg. 279 of the same reference, where Euler says: "& si l'on met constantes les coordonnées x, y, & z, & qu'on considère le seul tems t comme variable, on trouvera le mouvement, non d'un certain élément du fluide, mais de tous les éléments, qui passeront successivement par le même point Z, où l'on connoitra à chaque tems le mouvement de cet élément du fluide, qui se trouvera alors dans le point Z."[5] In other words: Euler introduces a virtual space grid, through which matter moves. This is an important feature of true spatial description, which Truesdell does not mention explicitly and does not use either.
- On the other hand, Euler also talks about finding the pathways of fluid particles, for example on pg. 278: "X. Mais voyons aussi quel chemin décrira l'élément du fluide, qui est á présent en Z, pendant le tems infiniment petit dt; ou à quel point il se trouvera un inftant aprés. Or, si nous exprimons l'espace par le produit de la vitesse & du tems, l'élément du fluide, qui est à présent en Z, sera porté dans la direction ZP par l'espace $= u\,dt$, dans la direction ZQ par l'espace $= v\,\mathrm{d}t$, & dans la direction ZR par l'espace $= w\,dt$. Donc, si nous posons:

$$\mathrm{ZP} = u\,dt\ ,\quad \mathrm{ZQ} = v\,dt\ ,\quad \&\quad \mathrm{ZR} = w\,dt\ , \tag{2.17}$$

[4] We must also assume that the state of the fluid at a certain point in time is known, & I will refer to as the primitive state of the fluid: since this state is virtually arbitrary, we must first know the arrangement of the particles that make up the fluid, & the motion that has been imparted to them, unless the fluid was at rest in its primitive state.

[5] & If we set the coordinates x, y, & z as constants, & and consider only time t as a variable, we will find the motion, not of a certain element of the fluid, but of all the elements, which will successively pass through the same point Z, where we will know at each time the motion of this element of the fluid, which will then be at point Z.

& qu'on acheve de ces trois côtés le parallelepipede, l'angle opposé 'á Z marquera le point, oú l'élément du fluide en question se trouvera après le tems dt, & la diagonale de ce parallelepipede, qui est $= dt\sqrt{(uu + vv + ww)}$, donnera le vrai chemin décrit ...."[6] We may interpret the "élément" as a material particle.

- Finally, it is worthwhile to point out again that in spatial description the primary quantity is not the displacement of a fluid particle (we could even call it the relative transplacement or simply motion), but the velocity field. Euler explains this on pg. 278: "Soient donc u, v, & w les vitesses de ce mouvement décomposé selon les trois directions ZP, ZQ, & ZR, & il est clair que ces trois quantités doivent aussi être considérées comme des fonctions des quatre variables x, y, z, & t. Car ayant trouvé la nature de ces fonctions, si l'on met le tems t constant, on connoîtra par la variabilité des coordonnées x, y, & z, les trois vitesses u, v, & w, & partant le vray mouvement dont chaque élément du fluide est porté dans l'instant présent."[7]

After this first encounter and introduction to the convoluted situation we are facing, we shall, first, concentrate on a proper definition of what we call *true spatial description* because, in contrast to Truesdell's spatial description, it can but does not have to do with the motion of material particles. Second, we shall present unambiguously the material point of view, which we refer to as *Lagrangian material* and *Eulerian material description*. These two always make use of the concept of an indestructible material point. To let it sink in, once more: we present true spatial and Eulerian material description *separately* in order to point out that the former is a much wider concept than the latter because *a priori* it is *not* restricted to the description of indestructible material points.

2.2 True spatial description

An *inertial frame of reference* is used, in order to capture space in a continuous manner by introducing a three-dimensional grid and within it an open volume v,[8] as indicated in Fig. 1 for the planar case. For simplicity we will assume that the grid does not move nor change. Note that it also does not have to be orthogonal. It has only been drawn like this in the figure for the sake of convenience. The important point is that grid cells are created and that, in principle, they can be infinitesimally small, because we expect perfect continuity of physical space. Therefore, we have the option of dividing it up as finely as we like, at least in principle, from a mathematical point of view.

The spatial cells are continuously distributed, they are completely immaterial and identifiable by vectors $\boldsymbol{r}$. The letter $\boldsymbol{r}$ was chosen to distinguish it from the current position vector $\boldsymbol{x} = \check{\boldsymbol{x}}(t)$ of a material particle used in Eq. (2.5) or (later) in Section 2.3. $\boldsymbol{r}$ is called радиус-вектор (radius vector) in the Russian literature [55] as well as in some Western literature [68], [77]. In contrast to $\boldsymbol{x}$ the position $\boldsymbol{r}$ does not move at all. However, if one so wishes, there might be an indirect time dependence because the observer O might use a time-dependent coordinate system. Then we must write $\boldsymbol{r} = r_i(t)\boldsymbol{e}_i(t)$, and the time dependencies in the components $r_i(t)$ and in the base vectors $\boldsymbol{e}_i(t)$, which, for simplicity, have been chosen as orthonormal Cartesian, cancel each other so that, in effect, $\boldsymbol{r}$ stays put.

The matter in this point in space is now examined by the so-called *observational point* [9] [36] $\check{\boldsymbol{r}}(t)$, which is positioned on $\boldsymbol{r}$ at time t. This new notion needs further explanations:

- In a way the observational point $\check{\boldsymbol{r}}(t)$ is an analogue of the moving position of the material particle $\check{\boldsymbol{x}}(t)$, as evident from the way of writing. Both can move.
- Unlike the set of *all* space points $\boldsymbol{r}$, the observational point is *not* a field, it is just a function.

[6] X. Let us consider what path will be described by a fluid element now at Z during the infinitely small time dt; or the point at which it will be an instant later. If we express the distance as the product of velocity & time, a fluid element currently at Z will travel a distance $u\,dt$ in the direction ZP, a distance $v\,dt$ in the direction ZQ, & a distance $w\,dt$ in the direction ZR. Therefore, if we set $\mathrm{ZP} = u\,dt$, $\mathrm{ZQ} = v\,dt$, & $\mathrm{ZR} = w\,dt$, and from these three sides complete the construction of the parallelepiped, then the corner opposite the point Z will represent the point at which the fluid element in question will be after the time dt, \$ the diagonal of the parallelepiped, which is equal to $= dt\sqrt{(uu + vv + ww)}$, will give the true path described ....

[7] Let u, v, and w be the velocities of this motion decomposed according to the three directions ZP, ZQ, and ZR, and it is clear that these three quantities must also be considered as functions of the four variables x, y, z, and t. For, having found the nature of these functions, if we set time t constant, we will know, from the variability of the coordinates x, y, and z, the three velocities u, v, and w, and thus the true motion in which each element of the fluid is carried at the present moment.

[8] in the literature this is also referred as the control volume concept [32], pg. 125 and 168

[9] точка наблюдения in the original [81], pg. 118

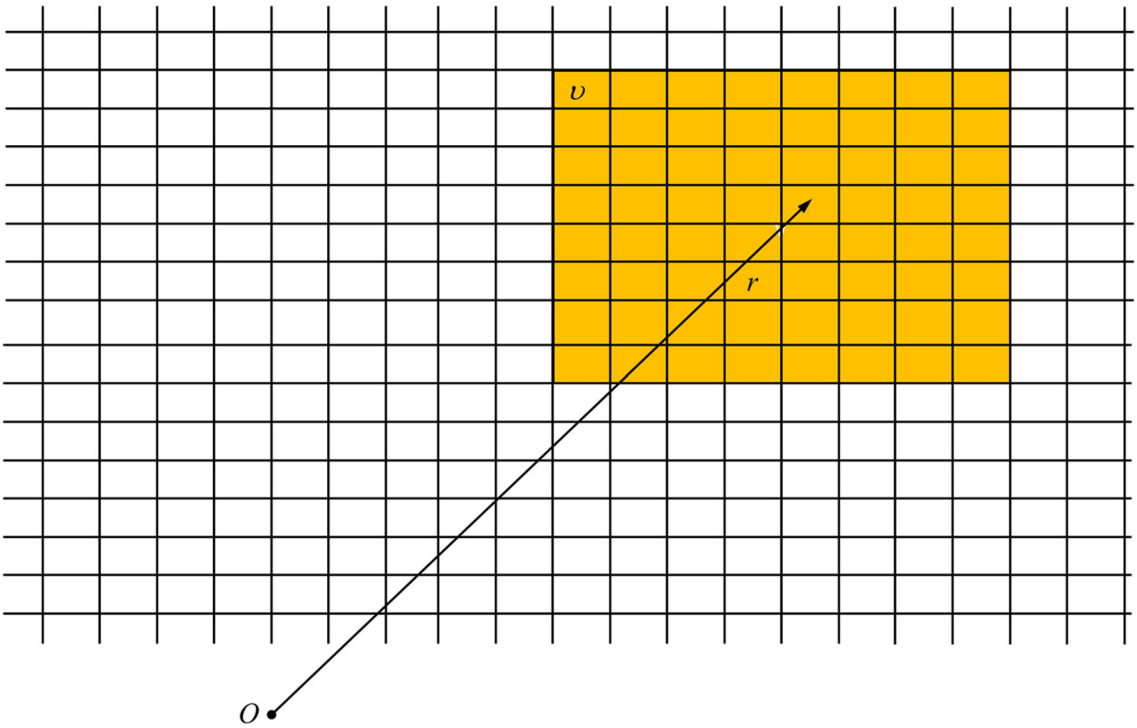

Fig. 1 The concept of an open stationary volume v for spatial description.

- However, this function can be chosen freely, whereas the movement $\check{\boldsymbol{x}}(t)$ of the material particle is to be determined according to the laws of physics.
- Moreover, the observational point is completely immaterial, and the material point is not. Nevertheless, by means of the observational point we shall examine properties of the matter currently located at $\boldsymbol{r}$.
- The observational point can move with the matter, *or* it can stay at $\boldsymbol{r}$, *or* it can move to another cell.
- An illustrating summary: with the observational point we can "measure" the velocity of matter and other physical quantities. The situation is similar to measuring temperature in a moving medium. The thermometer can also move independently within the medium. The result of the measurement is a function of time that shows the temperature value at the point of observation. However, it cannot say anything about the temperature (field) in points that are close by. Therefore, it is not a field.

Now, we still have to clarify where exactly $\check{\boldsymbol{r}}(t)$ is located in relation to a grid cell. As shown in the figure, it is conceivable that it lies in the geometric center of a single grid cell, in other words, it belongs to this very grid cell at the immovable space point $\boldsymbol{r}$ alone. But it would also be possible to place it in a node of the mesh. Then it obviously belongs to several cells. However, it should be kept in mind that for the applicability of calculus the cell size should approach zero. This means that in the *mathematical* continuum limit the differences become obsolete again. In this context, also recall the *physical* continuum hypothesis, which in a continuum theory must be fulfilled, and which is problematized in [42].

We now consider a field quantity ψ[10] related to a physical quantity of the matter[11] within the cell at the current time. In other words, we relate it to the observational point of the grid cell and write,

$$\psi = \overline{\psi}\left(\check{\boldsymbol{r}}(t), t\right), \tag{2.18}$$

or also,

$$\psi = \overline{\psi}\left(\boldsymbol{r}, t\right). \tag{2.19}$$

[10] It could also be of vectorial or tensorial nature, $\boldsymbol{\lambda}$ or $\boldsymbol{\Lambda}$, respectively.

[11] Here we are only concerned with the true spatial description of matter. However, true spatial description can do much more, cracks and voids can be simulated, and also electromagnetic fields that exist in vacuum. An example of how true spatial description in hybrid formulation with Eulerian material description can be used for describing porous media can be found in [13].

The two notations deserve a comment: (2.18) focuses on one observational point, which is potentially moving, similarly as in Eq. (2.11), where (after the definition symbol) the motion $\boldsymbol{x} = \boldsymbol{\chi}(X, t) = \check{\boldsymbol{x}}(t)$ of a material point X, as stated in Eq. $(2.1)_1$, was emphasized. The second notation, Eq. (2.19), emphasizes the field $\boldsymbol{r}$ of grid points. It would be misleading to interpret them as non-moving observational points, because there is only ONE observational point.[12] Moreover, for clarity's sake, let it be said that ψ is the field *value* at the grid cell position $\boldsymbol{r}$ at time t of the field property in question, and $\overline{\psi}$ is the field *function* used to calculate this very value. One should also realize that the function has to stay the same, independently of whether the location $\boldsymbol{r}$ or the observational point $\check{\boldsymbol{r}} = (t)$ at the location is used in its first argument.

At this time, it should already be mentioned that in this paper functions in true spatial description are easily identifiable by a bar, in contrast to functions in Lagrangian and Eulerian material description, which are introduced in the next section and which will carry a tilde and a hat, respectively. In fact, the notation employed in Eq. (2.19) is the typical notation of fluid mechanics, where true spatial description is preferred, see, for example, [10], Section 2.1, or [75], pg. 14. The second reference makes it very clear by emphasizing that a fixed point in space is meant. It is therefore not surprising that scientists from the solid mechanics field might get confused. In both fluid mechanics references it is referred to as Eulerian specification but this is *not necessarily* the Eulerian material description Truesdell had in mind (for his point of view see Section 2.1). We proceed to explain.

To begin with, Batchelor makes a most interesting statement on pg. 71 of [10], which shows that true spatial description is not limited to massy material particles: "The first, usually called the Eulerian type, is like the specification of an electromagnetic field in that the flow quantities are defined as functions of position in space ($\mathbf{x}$) and time (t)." This being said, we must also give credit to Truesdell, who explicitly says on pg. 19 of [71]: "thus excluding tacitly from our discourse any bodies that are not massy." In other words, that what was not promised you cannot expect to get.

What about the issue that the matter at the point $\boldsymbol{r}$ of a grid cell at time t might get accumulated from different locations, as illustrated in Fig. 2? Consequently, let us ask: can the concept of Truesdell's material particle be extended such that it can be "destroyed?" In other words, is it allowed for it to be open to receiving or emitting matter from/to different sites? It is fair to say that Batchelor did not have such phenomena in mind when he introduced spatial description. In fact, on pg. 72 of his book he talks about a *pathline*, which is just another word for the motion of the indestructible material particle described by Eq. (2.5). He also says quite clearly on pg. 71 "Nevertheless, the notion of material volumes, surfaces and lines which consist always of the same [*sic*!] fluid particles and move with them is indispensable, and will often be employed within the framework of an Eulerian specification of the flow field." However, also note that Batchelor uses the word "often." So the question arises, which exceptions there are.

The first example where the concept of an indestructible material particle becomes problematic are rational mixture theories formulated in the Truesdellian way by postulating his three famous metaphysical principles ([70], pg. 83):

1. All properties of the mixture must be mathematical consequences of properties of the constituents.
2. So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.
3. The motion of the mixture is governed by the same equations as in a single body.

The single body mentioned in the third principle consists of indestructible material particles. All constituents of the mixture are trapped within a material particle. This is all the mixing there is, and all the chemistry occurs within an isolated particle. In fact, this is a strong limitation, because further exchange of matter and the associated mixing cannot happen: material particles are isolated and do not allow to mix. True spatial description does not have this constraint from the very start: as illustrated in Fig. 2 the matter in one cell at a certain time can stem from various locations. Indeed, in true spatial description the chemical reaction will also take place in a single grid cell, but the material required for it does not have one origin only. Let us in some kind of figurative language refer to the matter in a cell at time t as "fluid particle." In general, this matter stems from other "fluid particles" located in other grid cells at time $t - \Delta t$. In contrast to Truesdell's material particles these "particles" can exchange matter.

This conundrum and further consequences were detected and addressed a decade ago in several articles. Wilmański [76] refers to the work on Bowen (for example [12]), which builds upon Truesdell's ideas on

[12] The situation is similar to Eulerian material description: With the motion we follow ONE material particle $\boldsymbol{X}$. Its position $\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)$ is not the same as the space position $\boldsymbol{x}$, which is the set of ALL matter filled space. In true spatial description we have $\check{\boldsymbol{r}}(t)$ (ONE observational point) and the set of ALL spatial points $\boldsymbol{r}$.

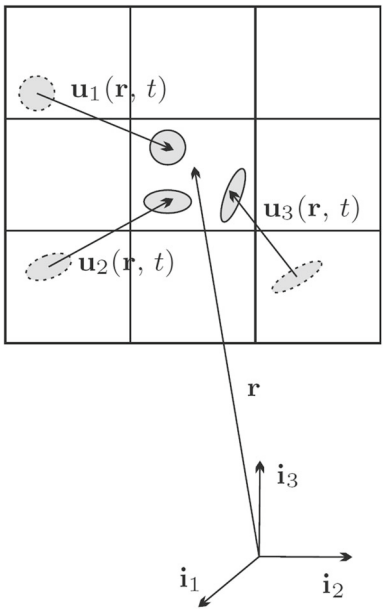

Fig. 2 Transport of matter into a grid location $\boldsymbol{r}$ from different locations (adapted from [42], $\boldsymbol{u}_i$ are displacement vectors of corpuscles at the microlevel).

mixtures. One of Wilmański's points of criticism has to do with the notion of the material particle and the components it contains. He objects to the idea that each component in Bowen's treatment requires its own reference body. He says on pg. 100: "It is even worse when we choose the same reference configuration, say, $\mathcal{B}_0$, for all components as this yields particles of different components to lie very far apart in any current configuration. Then, in the Eulerian description [Truesdell's understanding of spatial description described in Section 2.1], particles interact with each other on long distances which means that the theory should be nonlocal." Klika [46] discusses and comments on the work of Wilmański as follows: "Briefly, the mixture cannot be described correctly in material description, because every constituent has its properties defined on different bodies ..." More recently, the idea that matter enters a spatial cell from various locations has led numerics to the formulation of the so-called material point method [79]. In short: the puristic idea of an indestructible material point must give way to the needs of capturing the deformation of more complex matter.

The second example where the material point concept reaches its limits comes from soil mechanics, specifically from the attempt to study agglomeration or crushing of matter. For example Liu *et al.* [54] leave the grounds of traditional continuum mechanics and use Discrete Element Methods (DEM) to examine the breakage between discrete corpuscles in a representative volume element. DEM was also used to study agglomerates of seedbeds in [9]. Authors of this paper tried to extend the traditional concepts of continuum mechanics and proposed a simulation for crushing matter by using a true spatial formulation and a production term for the microinertia ([30], [29]). The extended kinetic equation that was used for the microinertia will be discussed shortly.

We shall now *define* the substantial (also known as material[13]) derivative of a field quantity pertinent to true spatial description. As the name already suggests, the *substantial time derivative* refers to the *rate of change of some physical property ψ of the matter* due to its movement. The matter is under scrutiny by the observational point $\check{\boldsymbol{r}}(t)$ located at the grid point $\boldsymbol{r}$ at that moment of time t. On the other hand, matter moves with the velocity $\boldsymbol{v} = \overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t), t\big)$ through space as recorded by the inertial observer O using his observational point $\check{\boldsymbol{r}}(t)$ at the space grid position $\boldsymbol{r}$ at time t. Hence, matter that was originally located at $\boldsymbol{r}$ is transported to a new location at the position $\boldsymbol{r} + \boldsymbol{v}\Delta t$ as shown in Fig. 3. So to repeat, with the substantial time derivative we are detecting property change by following the matter as indicated by the black arrow. In contrast, the *total time derivative* refers to when we are *not* following the matter but the observational point $\check{\boldsymbol{r}}(t)$ instead.[14] We

[13] This is another dangerous adjective to use, because it could remind us of the indestructible material particle. But this is not meant here. What is meant is that matter=material, currently at one position, is moved to some other place.

[14] In Eulerian material description we are following the material point. Hence, *a priori*, substantial and total derivative must coincide.

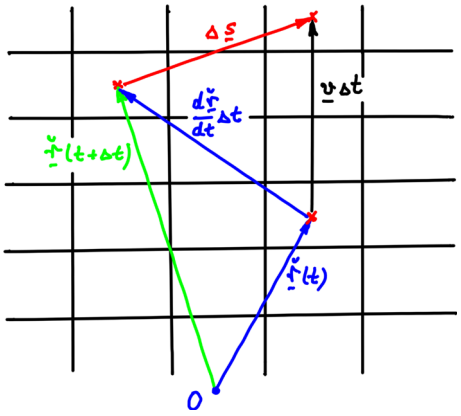

Fig. 3 Illustrating the meaning of a moving observational point.

will get to it a little later. Let us focus on the grid location $\boldsymbol{r}$ first and, therefore, define the substantial derivative as:

$$\frac{\delta\overline{\psi}(\boldsymbol{r},t)}{\delta t} := \lim_{\Delta t\to 0}\frac{\overline{\psi}\big(\boldsymbol{r}+\boldsymbol{v}\Delta t, t+\Delta t\big)-\overline{\psi}\big(\boldsymbol{r},t\big)}{\Delta t}, \tag{2.20}$$

at least for the moment.

One must realize that in this scenario the observational point $\check{\boldsymbol{r}}(t)$ stays put at position $\boldsymbol{r}$. Moreover, although this expression looks like a derivative from a mathematical point of view, it is *not*, since the right-hand side contains the vector $\boldsymbol{v}\Delta t$, which is not part of the function $\overline{\psi}\,(\boldsymbol{r},t)$. It is based on physics arguments because it makes use of the velocity of the matter.

On the other hand, we may now let the observational point move as indicated by the green and blue pathways in Fig. 3. In order to get to the cell where the matter would be at time $t+\Delta t$ we need to close the green or the blue paths by the red vector $\Delta\boldsymbol{s}$ as shown in the figure. Then, instead of Eq. (2.20), we must write:

$$\frac{\delta\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\delta t} = \lim_{\Delta t\to 0}\frac{\overline{\psi}\big(\check{\boldsymbol{r}}(t+\Delta t)+\Delta\boldsymbol{s}, t+\Delta t\big)-\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\Delta t}. \tag{2.21}$$

Again: although this expression also looks like a derivative from a mathematical point of view, it is *not*, since the right-hand side contains the vector $\Delta\boldsymbol{s}$, which is not an argument of the function $\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)$. It is based on physics due to the relocation vector $\Delta\boldsymbol{s}$ required because the observer changes his observational point within the time interval Δt from $\check{\boldsymbol{r}}(t)$ to $\check{\boldsymbol{r}}(t+\Delta t)$ (also see Zhilin [81], pg. 118). Note that now we have used $\check{\boldsymbol{r}}(t)$ as argument in the substantial derivative. This was done mainly for didactical reasons, because now the emphasis is on the observational point. However, mathematically it does not matter, and we have $\frac{\partial\check{\boldsymbol{r}}(t)}{\partial\boldsymbol{r}} = \mathbf{1}$: both are on top of each other.

Now, following the rules of vector addition, we find with Fig. 3:

$$\Delta\boldsymbol{s} = \left(\overline{\boldsymbol{v}}\,(\check{\boldsymbol{r}}(t),t) - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\Delta t. \tag{2.22}$$

In order to evaluate the limit in (2.21), we first perform a Taylor expansion in the first argument:

$$\overline{\psi}\big(\check{\boldsymbol{r}}(t+\Delta t)+\Delta\boldsymbol{s}, t+\Delta t\big) = \overline{\psi}\big(\check{\boldsymbol{r}}(t+\Delta t), t+\Delta t\big) + \Delta\boldsymbol{s}\cdot\frac{\partial\overline{\psi}\big(\boldsymbol{r},t+\Delta t\big)}{\partial\boldsymbol{r}}. \tag{2.23}$$

We now insert the equations into each other in order to identify the total derivative $\frac{\mathrm{d}}{\mathrm{d}t}$. Recall that the total derivative is supposed to indicate the *change of property* ψ per Δt *as perceived by the observational point*. In

this spirit we write:

$$\begin{aligned}\frac{\delta\overline{\psi}(\boldsymbol{r},t)}{\delta t} &= \lim_{\Delta t\to 0}\frac{\overline{\psi}\big(\check{\boldsymbol{r}}(t+\Delta t),t+\Delta t\big)-\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\Delta t}\\ &\quad+\left(\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)-\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\Big(\lim_{\Delta t\to 0}\overline{\psi}\big(\boldsymbol{r},t+\Delta t\big)\Big)\\ &= \frac{\mathrm{d}\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\mathrm{d}t}+\left(\overline{\boldsymbol{v}}\,\big(\check{\boldsymbol{r}}(t),t\big)-\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\overline{\psi}\big(\boldsymbol{r},t\big),\end{aligned}\tag{2.24}$$

with the definition for the total derivative in spatial description, and the nabla operator with respect to the spatial grid (and not with respect to the moving observational point, of which there is only one, and a gradient makes no sense):[15]

$$\frac{\mathrm{d}\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\mathrm{d}t}:=\lim_{\Delta t\to 0}\frac{\overline{\psi}\big(\check{\boldsymbol{r}}(t+\Delta t),t+\Delta t\big)-\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\Delta t}\,,\quad(\cdot)\,\nabla_{\boldsymbol{r}}=\frac{\partial(\cdot)}{\partial\boldsymbol{r}}.\tag{2.25}$$

Note that we wrote the final expression for the substantial derivative in Eq. (2.24) slightly differently than in [81], pg. 117, [82], pg. 355, or [7], pg. 548: although it is mathematically possible to use the observational point $\check{\boldsymbol{r}}(t)$, we put $\boldsymbol{r}$ in the argument of the substantial derivative, thus emphasizing that it focuses on the change of the properties of the matter currently in the grid cell. In the case of the total derivative we did it the other way round, because the total derivative emphasizes the change of property due to movement of the observational point. And, finally, we applied the gradient to the property written in terms of the spatial field $\overline{\psi}\big(\boldsymbol{r},t\big)$.

In any case, the result (2.24) shows that, in general, the substantial ($\delta/\delta t$) and the total derivative ($\mathrm{d}/\mathrm{d}t$) of fields in true spatial description are *not* equal. A *relative velocity* term combines them.[16] Even more accurately, we must say that this is the difference between the velocity $\boldsymbol{v}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)$ of the matter we previously referred to as fluid particle and the velocity $\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}$ of the observational point $\check{\boldsymbol{r}}(t)$, which is currently at the grid position $\boldsymbol{r}$. In this context note that, in general, the latter velocity is due to *two* time dependencies: the observational point has time-dependent coordinates $\check{r}_i(t)$ and the observer uses a time dependent base $\boldsymbol{e}_i(t)$, such that $\check{\boldsymbol{r}}(t)=\check{r}_i(t)\boldsymbol{e}_i(t)$, all for simplicity written in orthonormal Cartesian form. In contrast to $\boldsymbol{r}$ the two time dependencies in $\check{r}_i(t)$ and $\boldsymbol{e}_i(t)$ do *not* cancel each other. If finally the observational point is forced to move instantaneously with the matter, *i.e.*, $\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)$, then substantial and total time derivative coincide:[17]

$$\frac{\delta\overline{\psi}(\boldsymbol{r},t)}{\delta t}=\frac{\mathrm{d}\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\mathrm{d}t}.\tag{2.26}$$

After this "derivation" several remarks are in order:

- A word of caution regarding the meaning of the velocity function $\overline{\boldsymbol{v}}\,\big(\check{\boldsymbol{r}}(t),t\big)$: first of all, it is completely different from the functions $\dot{\boldsymbol{\chi}}$ and $\dot{\boldsymbol{x}}$ used by Truesdell in Eqs. (2.2) and (2.11). The latter were relevant in the case of an indestructible material point, whereas the velocity $\boldsymbol{v}=\overline{\boldsymbol{v}}\,\big(\check{\boldsymbol{r}}(t),t\big)$ concerns the *current state of movement* of the "fluid particle" mentioned above. How, in principle, could we determine the value $\boldsymbol{v}$ of this function? To this end, we would have to homogenize the velocities of each microparticle constituting the matter at the grid point $\boldsymbol{r}$ at time t. This way we bring the notion of velocity to the continuum scale. Also recall that in true spatial description these microconstituents could originate from different grid cells and could move into different ones later. This is impossible in context with material particles. A material particle always keeps its microconstituents. Therefore, referring to $\overline{\boldsymbol{v}}\,\big(\check{\boldsymbol{r}}(t),t\big)$ as a *material* velocity now transcends the velocity of *material* particles: the velocity of the *matter* of the "fluid particle" is meant, and it would be more appropriate to speak of *the velocity of the local matter*. Of course, if it keeps its microconstituents during the movement the "fluid particle" is a material point and we return to the traditional situation.

[15] For scalar functions ψ the distinction between nablas applied to the left or to the right is irrelevant.

[16] In this context the notion of objectivity under Euclidean observer transformations should at least be mentioned.

[17] Note already now that in Eulerian material description, where the concept of a material particle is used, there cannot be a difference between substantial and total derivative, because the motion of the material particle takes over the role of the moving observational point.

- Let us temporarily put the emphasis on the field of all grid points $\boldsymbol{r}$ and use the notation $\frac{\mathrm{d}\overline{\psi}(\boldsymbol{r},t)}{\mathrm{d}t}$ for the total derivative of the quantity ψ. Recall again that $\nabla_{\boldsymbol{r}}$ is a gradient in space around a grid cell point $\boldsymbol{r}$ at which the observational point $\check{\boldsymbol{r}}(t)$ is located at time t. Then, of course, we can form a gradient of this quantity, $\nabla_{\boldsymbol{r}}\left(\frac{\mathrm{d}\overline{\psi}(\boldsymbol{r},t)}{\mathrm{d}t}\right)$.
- Now, let us consider the gradient of a physical quantity in a grid cell point, $\nabla_{\boldsymbol{r}}\overline{\psi}\,(\boldsymbol{r},t)$, and take its total time derivative, $\frac{\mathrm{d}\nabla_{\boldsymbol{r}}\overline{\psi}(\boldsymbol{r},t)}{\mathrm{d}t}$. It is the same as $\nabla_{\boldsymbol{r}}\left(\frac{\mathrm{d}\overline{\psi}(\boldsymbol{r},t)}{\mathrm{d}t}\right)$, because in both cases we are referring to the same space point $\boldsymbol{r}$ with the same neighborhood. This explains Zhilin's comment [81], pg. 119 regarding the interchangability of what he calls the operator of total time derivative [18] (straight d's) and the non-interchangability of the material (=substantial) time derivatives (deltas) in true spatial description:

$$\nabla_{\boldsymbol{r}}\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\boldsymbol{r}}\ ,\quad \nabla_{\boldsymbol{r}}\frac{\delta}{\delta t}\neq\frac{\delta}{\delta t}\nabla_{\boldsymbol{r}}. \tag{2.27}$$

  Moreover, by pure algebra, we find that

$$\nabla_{\boldsymbol{r}}\left(\frac{\delta\overline{\psi}\,(\boldsymbol{r},t)}{\delta t}\right)=\frac{\delta}{\delta t}\left(\nabla_{\boldsymbol{r}}\overline{\psi}\,(\boldsymbol{r},t)\right)+\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\,(\boldsymbol{r},t)\right)\cdot\left(\nabla_{\boldsymbol{r}}\overline{\psi}\,(\boldsymbol{r},t)\right). \tag{2.28}$$

  During its derivation it was observed that $\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}$ is no field.
- Let us further assume that the coordinate system used by the observer O from Fig. 3 to describe $\check{\boldsymbol{r}}(t)$ is *not* time dependent. As it can be concluded from what was said previously, by this we mean that $\check{\boldsymbol{r}}(t)=\check{r}_i(t)\boldsymbol{e}_i$ and *not* $\check{\boldsymbol{r}}(t)=\check{r}_i(t)\boldsymbol{e}_i(t)$, where $\boldsymbol{e}_i$ refers (for simplicity) to an orthonormal Cartesian base of O. Hence, we have to put on the right hand side of $(2.25)_1$ $\overline{\psi}\big(\check{r}_1(t),\check{r}_2(t),\check{r}_3(t),t\big)\equiv\overline{\psi}\big(\check{r}_i(t),t\big)$. Otherwise the $\boldsymbol{e}_i(t)$ would enlarge the functional dependence with respect to time. Then the total derivative in Eq. (2.25) can be made explicit by using the chain rule as follows,

$$\begin{aligned}\frac{\mathrm{d}\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\mathrm{d}t}&=\left.\frac{\partial\overline{\psi}\big(\check{r}_i(t),t\big)}{\partial t}\right|_{r_i}+\left.\frac{\partial\overline{\psi}\big(\check{r}_i(t),t\big)}{\partial r_i}\right|_{t}\frac{\mathrm{d}\check{r}_i(t)}{\mathrm{d}t}=\\&\left.\frac{\partial\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\partial t}\right|_{\boldsymbol{r}}+\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\cdot\nabla_{\boldsymbol{r}}\overline{\psi}\big(\boldsymbol{r},t\big).\end{aligned} \tag{2.29}$$

  Only in this special case we may say that $\frac{\partial}{\partial r_i}\big|_t$ (or simply $\nabla_{\boldsymbol{r}}=\boldsymbol{e}_i\frac{\partial}{\partial r_i}$) can be exchanged with the partial time derivative $\frac{\partial}{\partial t}\big|_{r_i}$. Moreover, by inserting this in Eq. (2.24) we obtain in this special case:

$$\frac{\delta\overline{\psi}\big(\boldsymbol{r},t\big)}{\delta t}=\left.\frac{\partial\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)}{\partial t}\right|_{\boldsymbol{r}}+\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)\cdot\nabla_{\boldsymbol{r}}\overline{\psi}\big(\boldsymbol{r},t\big). \tag{2.30}$$

  In [7] it is said "The last expression just coincides with the commonly used material derivative ... ." Superficially speaking, this looks as if it were so, but the devil is in the details: in true spatial description the last line Eq. (2.24) shows how the substantial derivative looks in its full generality.
- As an example for the application of the last equation we choose $\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)=\check{\boldsymbol{r}}(t)=\check{r}_i(t)\boldsymbol{e}_i\equiv r_i\boldsymbol{e}_i$. Note that there is no *explicit* time dependence in this case.[19] Hence Eq. (2.30) yields:

$$\frac{\delta\check{\boldsymbol{r}}(t)}{\delta t}=\boldsymbol{0}+\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)\cdot\nabla_{\boldsymbol{r}}\boldsymbol{r}\equiv\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)\cdot\boldsymbol{1}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big), \tag{2.31}$$

  as could be expected. In fact, we must stress that the last relation is an identity and not an equation defining the velocity.

[18] оператор полного дифференцирования по времени

[19] What we refer to is the second t in $\overline{\psi}\big(\check{\boldsymbol{r}}(t),t\big)$.

- Zhilin in [81], pg. 118 gives Eq. (2.24) yet another twist. He says: "If the observation point is motionless relative to the reference frame, then expression (3.3) [=(2.24)] simplifies and takes the form" [20]

$$\frac{\delta\overline{\psi}(\boldsymbol{r},t)}{\delta t}=\frac{\mathrm{d}\overline{\psi}(\boldsymbol{r},t)}{\mathrm{d}t}+\overline{\boldsymbol{v}}(\boldsymbol{r},t)\cdot\nabla_{\boldsymbol{r}}\overline{\psi}(\boldsymbol{r},t). \tag{2.32}$$

  Because of the motionlessness we have put in Eq. (2.24) $\check{\boldsymbol{r}}\equiv\boldsymbol{r}$. Note that Zhilin uses straight d's for the time differentiation and *not* partial ones. However, if we accept this, then in view of Eq. (2.30) total and partial time differentiation *are the same* for a motionless observational point and if the coordinate system of the observer O is not time dependent.

  Let us summarize important issues:

- We will use the nabla operator in Eq. (2.23) for the gradient between grid cells of the field $\overline{\psi}$ in true spatial description, such that $(\cdot)\,\nabla_{\boldsymbol{r}}=\frac{\partial(\cdot)}{\partial\boldsymbol{r}}$. It must be clearly distinguished from the gradient between material points $\boldsymbol{X}$ to be used later in Eq. (2.52), which is denoted by $\nabla_{\boldsymbol{X}}$. And finally, there is the gradient between material points in current configuration in Eq. (2.51) denoted by $\nabla_{\boldsymbol{x}}$.
- Note that, for easy identification, we will always write $\frac{\delta\psi}{\delta t}$ for the substantial time derivative of fields in true spatial description. As we shall see later there is no difference between a total $\frac{\mathrm{d}\psi}{\mathrm{d}t}$ and substantial time derivative $\frac{\delta\psi}{\delta t}$ in Eulerian material description. But in true spatial description there is: Eq. (2.24). At this point it is appropriate to cite Zhilin [81], pg. 117: "В литературе ... понятие материальной производной считаеця строгим и общепринятым. Однако мы не придерживаемся этой точки зрения, ибо фактически в литературе не только оцуцтвует ясное определение материальной производной, но, кроме того, явно (а чаще неявно) смешиваются материальное и пространственное описания."[21]

We will now turn to the question as to whether it is reasonable to introduce the notion of a reference configuration in true spatial description and, if so, under which circumstances. In fact, the answer to this question has led to some confusion if not irritation—in particular in context with materials the treatment of which requires higher continuum theories: the infamous Ericksen-Leslie-Parodi and Eringen-Lee controversy. The controversy is described in several articles. For support of the Eringen-Lee side we cite [49], [50], [20], [21], and for the Ericksen-Leslie-Parodi perspective see [66], [67], [52]. For an article of a mediator one may consult Rymarz [65]. Regarding the introduction of a reference placement in spatial description we find the following statements in the aforementioned papers:

- Lee and Eringen in [50], pg. 1321: "... did not catch [= Ericksen *et al.*] the concept of the *reference state* which we tried to convey through our papers ..."
- Leslie in [52], pg. 22: "The approach of Lee and Eringen, however, is conceptually quite different, since it invokes the notion of a reference configuration, which in turn leads to displacement gradients appearing explicitly in the constitutive relations. As Shahinpoor [...] points out, this aspect of their constitutive theory gives rise to an inconsistency between theory and experiment for steady simple shear flow."
- Rymarz in [65], pg. 12: "... according to the concept of fluidity, the reference configuration is the present one."

In all of these statements—even though they come from two different sides—there is some truth. Let us start by reasserting that the true spatial description of fields is a formulation in the "here and now:" we look at matter in what the solid mechanics community would call its *current placement* or *current configuration*. True spatial description is completely a *current* one.

To be more specific: both schools presented an approach for handling the flow of nematic crystals (see, for example, [51], [26], Chapter 12). Both use the concept of an indestructible material particle. In [51], pg. 266 Leslie chooses a relative description as in Eq. (2.13) to describe its motion. Hence the reference position $\boldsymbol{X}$ at time $t=0$ does not explicitly occur. On the other hand, Eringen mentions on pg. 143 of [26] the identifier $\boldsymbol{X}$ explicitly. We are tempted to say that he is a bit more on the solids side. However, when it comes to solving

[20] Если точка наблюдения неподвижна относительно системы отсчета, то выражение (3.3) упрощается и принимает вид; his choice of symbols has been adjusted to our nomenclature

[21] In the literature ..., the notion of material derivative is considered to be strict and generally accepted. However, we do not adhere to this point of view, because in fact the literature not only lacks a clear definition of the material derivative, but, in addition, explicitly (and more often implicitly) mixes material and spatial descriptions.

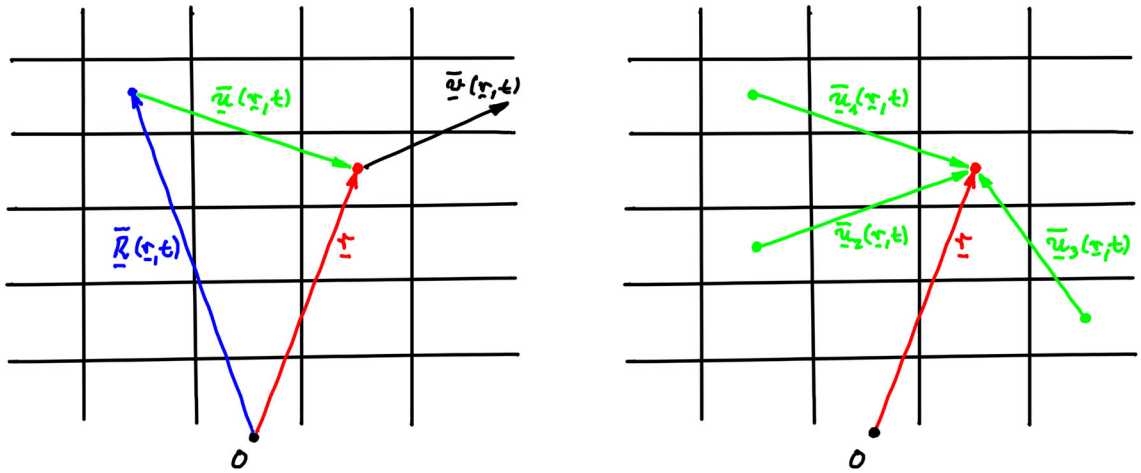

Fig. 4 The displacement vector and the notion of reference vector in true spatial description.

concrete problems, which are studied by Leslie on pp. 276 and by Eringen on pp. 184, the results are written in what Truesdell calls spatial form $(2.8)_1$ by both authors: they are both thinking in terms of a material particle X at the position $\boldsymbol{x} = \check{\boldsymbol{x}}(t)$ with its motion according to Eq. $(2.1)_1$. They are definitely *not* thinking in terms of matter that—wherever it may come from—assembles at time t in a grid cell at point $\boldsymbol{r}$ according to Eq. (2.19). Hence the approach is perfectly correct but limited to matter that does neither decompose nor reassemble.

Now let us turn back to the question as to whether this leaves room for a reference configuration in true spatial configuration? The answer is positive, but its introduction requires a trick. Recall that the velocity of matter, $\boldsymbol{v} = \overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t), t\big)$, appeared first in Eq. (2.22) as a "mathematical vehicle" for transporting matter in space. This was absolutely justified because velocity is the *primary kinematic quantity* in true spatial description as emphasized above. However, in order to introduce *strain measures* for describing the deformation of (micropolar) matter, we need to talk about its relative displacement. Therefore, we define a displacement vector $\boldsymbol{u}$ as shown in Fig. 4 (left), analogously as in material description ([34], Section 2.2), by pure *formal* mathematical reasoning:

$$\boldsymbol{u} = \overline{\boldsymbol{u}}\,(\boldsymbol{r}, t) = \boldsymbol{r} - \overline{\boldsymbol{R}}\,(\boldsymbol{r}, t)\,. \tag{2.33}$$

A few remarks on the time-dependent reference vector $\boldsymbol{R} = \overline{\boldsymbol{R}}\,(\boldsymbol{r}, t)$ in Eq. (2.33) are in order:

- The reference vector assigns the matter currently located at the spatial grid point $\boldsymbol{r}$ mathematically a formal origin. Since the matter in this point will be exchanged as time goes on, $\overline{\boldsymbol{R}}\,(\boldsymbol{r}, t)$ is different at each moment of time: it is explicitly time dependent.
- Recall that in true spatial description the matter currently at position $\boldsymbol{r}$ can be obtained by mix of microparticles from different locations. This is indicated in Fig. 4 (right). Hence, in some cases, it does not correspond to the actual displacement of a "single piece" of matter. Therefore, in general, the reference vector $\overline{\boldsymbol{R}}\,(\boldsymbol{r}, t)$ is a formally introduced characteristic, indeed.
- Let us now assume that the matter at the cell $\boldsymbol{r}$ at time t can be considered to be a material indestructible particle. Then both the reference $\boldsymbol{R}$ as well as the displacement $\boldsymbol{u}$ from Eq. (2.33) are not formal nor fictitious any more. They correspond to the real displacement and the real reference vector of a particular particle. But this makes the reference vector still time dependent because we consider different material particles at different moments of time at position $\boldsymbol{r}$. By combining some of the notation from Eq. (2.4) with the notation of true spatial description we may want to write

$$\overline{\boldsymbol{u}}\,(\boldsymbol{r}, t) = \boldsymbol{r} - \overline{\boldsymbol{X}}\,(\boldsymbol{r}, t)\,. \tag{2.34}$$

Actually, this is not quite the expression for the displacement familiar from solid mechanics (for example [34], Section 2.2),

$$\boldsymbol{u}\,(X, t) = \boldsymbol{x} - \boldsymbol{X} = \boldsymbol{\chi}\,(X, t) - \boldsymbol{\kappa}\,(X)\,, \tag{2.35}$$

although both look similar. This is because solid mechanics prefers the Lagrangian description of Eqs. $(2.1)_1$ and (2.4), as indicated in the second part of the last relation, and not a Eulerian representation.

We now investigate the consequences when we express the velocity of matter in true material description $\overline{\boldsymbol{v}}(\boldsymbol{r},t)$ in terms of a substantial time derivative of another field, namely the displacement from Eq. (2.33). To this end, we define, again formally,

$$\frac{\delta\overline{\boldsymbol{u}}\big(\boldsymbol{r},t\big)}{\delta t}=\overline{\boldsymbol{v}}\,(\boldsymbol{r},t)\,. \tag{2.36}$$

One must realize that the displacement vector $\boldsymbol{u}$ is *not* the fundamental kinematic characteristic in true spatial description. Rather the velocity $\boldsymbol{v}$ is, which was emphasized before. In fact, we must look at Eq. (2.36) as a partial differential equation for $\boldsymbol{u}$ provided the velocity field $\boldsymbol{v}$ is known and given. In order to work out the details, we put the observational point $\check{\boldsymbol{r}}(t)$ on top of the grid point $\boldsymbol{r}$ at time t and rewrite Eq. (2.36) as follows:

$$\frac{\delta\overline{\boldsymbol{u}}\big(\check{\boldsymbol{r}}(t),t\big)}{\delta t}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big). \tag{2.37}$$

Now let us imagine that the observational point is motionless so that $\check{\boldsymbol{r}}=\boldsymbol{r}$ and, in addition, that the observer O does not use a time dependent coordinate system. Then Eq. (2.30) can be applied and we can directly rewrite (2.37) into a partial differential equation for $\overline{\boldsymbol{u}}\,(\boldsymbol{r},t)$ for given $\overline{\boldsymbol{v}}\,(\boldsymbol{r},t)$,

$$\frac{\partial\overline{\boldsymbol{u}}\big(\boldsymbol{r},t\big)}{\partial t}+\overline{\boldsymbol{v}}\big(\boldsymbol{r},t\big)\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\big(\boldsymbol{r},t\big)=\overline{\boldsymbol{v}}\big(\boldsymbol{r},t\big). \tag{2.38}$$

Once $\overline{\boldsymbol{u}}\,(\boldsymbol{r},t)$ has been determined from this equation, it can serve, if required, to determine $\overline{\boldsymbol{R}}\,(\boldsymbol{r},t)$ from

$$\overline{\boldsymbol{R}}\,(\boldsymbol{r},t)=\boldsymbol{r}-\overline{\boldsymbol{u}}\,(\boldsymbol{r},t)\,. \tag{2.39}$$

It seems fair to compare the relation (2.38) with the *ordinary* differential equation for the relative transplacement $\boldsymbol{\xi}$ obtained in (2.15). First of all, we notice that we must also provide initial conditions, namely,

$$\overline{\boldsymbol{u}}\,(\boldsymbol{r},t=0)=\boldsymbol{0}. \tag{2.40}$$

This condition agrees with Eq. (2.16) if we observe Eq. (2.14). But in contrast to the differential equation for the relative transplacement, it is not enough because we are now facing a partial differential equation, such that boundary conditions are required as well. The boundary conditions have to be specified according to the specific boundary-value problem. But we leave it with this remark because in true spacial description no one really needs to calculate the displacement itself: only gradients of displacement are important, and these do not require boundary conditions.

In order to familiarize ourselves with the possibilities the observational point has to offer, let us assume that it now moves instantaneously with the matter at position $\boldsymbol{r}$, such that $\frac{\delta\check{\boldsymbol{r}}(t)}{\delta t}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)$. Then, if we take the substantial time derivative of Eq. (2.39) and observe (2.31) and (2.37), we obtain:

$$\frac{\delta\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)}{\delta t}=\frac{\delta\check{\boldsymbol{r}}(t)}{\delta t}-\frac{\delta\overline{\boldsymbol{u}}\big(\check{\boldsymbol{r}}(t),t\big)}{\delta t}=\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)-\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t),t\big)=\boldsymbol{0}. \tag{2.41}$$

Thus, by looking at Eq. (2.24) for the general form of the substantial derivative and taking into account that the observational point is moving with the matter we must conclude that the total derivative of the reference vector $\boldsymbol{R}$ vanishes:[22]

$$\frac{\mathrm{d}\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)}{\mathrm{d}t}=\boldsymbol{0}. \tag{2.42}$$

We conclude that $\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)$ does not change in time under these assumptions:

$$\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)=\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(0),0\big). \tag{2.43}$$

On the other hand, we can also execute the total time derivative in Eq. (2.42):

$$\left.\frac{\partial\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)}{\partial\boldsymbol{r}}\right|_t\frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}+\left.\frac{\partial\overline{\boldsymbol{R}}\big(\check{\boldsymbol{r}}(t),t\big)}{\partial t}\right|_{\boldsymbol{r}}=\boldsymbol{0}\quad\Rightarrow\quad\frac{\partial\overline{\boldsymbol{R}}\big(\boldsymbol{r},t\big)}{\partial t}+\overline{\boldsymbol{v}}\,(\boldsymbol{r},t)\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{R}}\big(\boldsymbol{r},t\big)=\boldsymbol{0}. \tag{2.44}$$

In the last step it was observed that the observational point moves with the velocity of matter (which is assumed to be known) and that it is located at the grid point $\boldsymbol{r}$. This partial differential equation is identically satisfied because of the solution (2.43) that was obtained before.

[22] also compare Eq. (2.26)

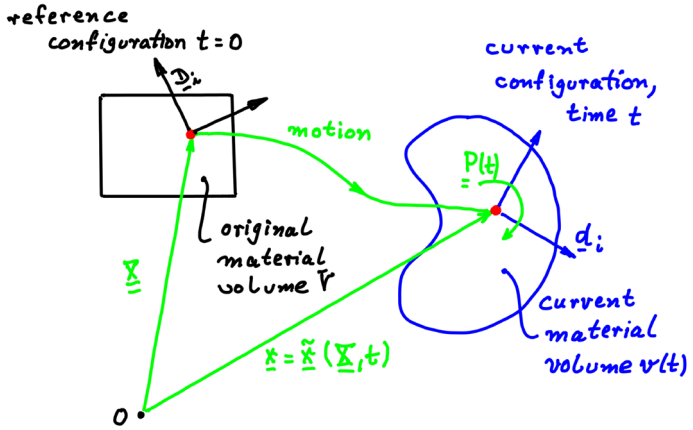

Fig. 5 The material (Lagrangian) method of description.

2.3 Descriptions based on the notion of material points

In this section we will again talk about Truesdellian material points as we did already in Section 2.1. Besides now putting the emphasis on *micropolar material points*, we will introduce a notation for the fields that is line with the nomenclature used for true spatial description, which was a bar over the field symbol. Recall that Truesdell uses different letters for different functions in different field representations. For example, $\boldsymbol{\chi}$ refers to the motion of a material particle X, Eq. $(2.1)_1$, and $\boldsymbol{\kappa}$ applies when the material particle is assigned a reference position vector $\boldsymbol{X}$, and so on. Instead, we will keep the letter for a physical quantity ψ and use bars, hats, and tildes to distinguish the various field descriptions. This has the benefit of being immediately discernable. We proceed to explain.

Consider Fig. 5: a macroscopic region in space, called $v(t)$, contains the same amount of matter at all times t. Alternatively we may say that it is composed of the same (infinitely many) material points at all times. This is referred to as the *current configuration* or *current placement*. Initially, at time t_0 (which may be equal to 0), the matter is contained in the so-called *reference configuration* or *reference placement*. We refer to this volume as $v(t_0)$ or V for short. The translational kinematics of material points (for ordinary as well as for micropolar material) in the reference configuration can be identified uniquely by their *reference position vector* $\boldsymbol{X}$. In this aspect, there is no difference to nonpolar material points. Also recall that by definition the matter assigned to a material particle can neither get lost nor can matter be added.

A remark: the material volumes, V and $v(t)$, could be obtained by the method of section (free body principle). In this case, they would be surrounded by other matter. Of course, their surfaces, ∂V and $\partial v(t)$, are impenetrable for matter in order to guarantee that the volumes always consist of the same material particles. However, it is also conceivable that there is no matter surrounding them and that they float amebalike through space.

In order to establish a consistent scheme, we now rewrite Eqs. (2.1) and (2.4): each material particle follows its own path, it will undergo a *motion*. Its current position $\boldsymbol{x}$ at the current time t is identifiable by using the motion function $\tilde{\boldsymbol{x}}$, which is bijective:

$$\boldsymbol{x} = \tilde{\boldsymbol{x}}(\boldsymbol{X}, t) \quad \Leftrightarrow \quad \boldsymbol{X} = \hat{\boldsymbol{X}}(\boldsymbol{x}, t). \tag{2.45}$$

The first relation in this equation we shall refer to as *Lagrangian material field description* of the motion field. We add the adjective "material" in order to emphasize that an indestructible material particle is considered. Note that we distinguish carefully between the function and the value of the function by adding a tilde. Also note the hat on the inverted function. This new symbol is necessary because of the new function it represents.

As in Eq. (2.2) we will now define the translational velocity of a material particle, for obvious reasons also known as *material velocity*,[23]

$$\boldsymbol{v} = \tilde{\boldsymbol{v}}(\boldsymbol{X}, t) = \left.\frac{\partial \tilde{\boldsymbol{x}}(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}} =: \frac{\mathrm{d}\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)}{\mathrm{d}t} \equiv \frac{\mathrm{d}\check{\boldsymbol{x}}(t)}{\mathrm{d}t}, \tag{2.46}$$

[23] A micropolar material particle has also a *rotational* velocity. We will talk about it later.

because the reference position of the material particle $\boldsymbol{X}$ is fixed, it is a constant for each particle. In material description the motion $\boldsymbol{x} = \tilde{\boldsymbol{x}}(X, t)$ is the primary quantity and the material velocity $\tilde{\boldsymbol{v}}(\boldsymbol{X}, t)$ is secondary—it is derived from it. Recall that in contrast the velocity $\overline{\boldsymbol{v}}(\boldsymbol{r}, t)$ is the primary variable in true spatial description. There is—strictly speaking—no motion function in true spatial description, although in fluid mechanics there exists the notion of the *pathline* of a fluid particle. If we wish, Eq. (2.45) describes a pathline—it enters fluid mechanics, which focuses on (true) spatial description, through the backdoor by using a concept from the world of material description. Also note that the definition of the material velocity in Eq. (2.46) is based on a Lagrangian material field description. This is emphasized by the use of the tilde. Note that, in principle, the straight d's in (2.46) require us to evaluate the total differential of a multivalued function as follows:

$$\frac{\mathrm{d}\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)}{\mathrm{d}t} = \left.\frac{\partial\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)}{\partial\boldsymbol{X}}\right|_t \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} + \left.\frac{\partial\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)}{\partial t}\right|_{\boldsymbol{X}}. \tag{2.47}$$

and $\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} \equiv \boldsymbol{0}$ because right now we are focusing on a single material particle.

However, there is a catch to obtain such a simple result, where straight d and partial time derivatives are interchangeable: a potential stumbling stone appears when we switch to what is known as *Eulerian description* of fields (see Section 2.1 for how it is commonly introduced in continuum mechanics). The switch consists in replacing the identifier $\boldsymbol{X}$ in the arguments of a field in Lagrangian description by the motion $\tilde{\boldsymbol{x}}(\boldsymbol{X}, t)$, which is possible because of the bijective property shown in $(2.45)_1$. In order to emphasize that it also requires us to use the notion of an indestructible material particle we will refer to it as *Eulerian material description*. This is indeed unusual because in many books (for example, [44], pg. 12 being a very recent reference), only the Lagrangian description is called "material." In this spirit we write:

$$\psi = \tilde{\psi}(\boldsymbol{X}, t) = \hat{\psi}\big(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t\big) = \hat{\psi}\big(\check{\boldsymbol{x}}(t), t\big) = \hat{\psi}\big(\boldsymbol{x}, t\big). \tag{2.48}$$

Note the resemblance of Eqs. $(2.48)_{3,4}$ to Eqs. (2.18), (2.19), respectively. In order to distinguish the various field formulations as clearly as possible, we used $\check{\boldsymbol{r}}(t)$ for the location of the (potentially moving) observational point—a known to-be-prescribed function with respect to the observer O—in Eq. (2.18), whereas $\check{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(\boldsymbol{X}, t)$ in Eq. $(2.48)_3$ is the Lagrange field of the motion, which is unknown and to-be-calculated. The argument $\check{\boldsymbol{x}}(t)$ could only be used because by writing $\hat{\psi}\big(\check{\boldsymbol{x}}(t), t\big)$ we are still looking at a single material particle. In contrast to that the form $\hat{\psi}\big(\boldsymbol{x}, t\big)$ in Eq. $(2.48)_4$ means that we are now considering the field of space points $\boldsymbol{x}$ at time t, all populated by different material particles identified by $\boldsymbol{X}$. However, the latter is a dangerous way of writing, which nevertheless can be found in textbooks. The *caveat* to be made is this: as it becomes evident from Eq. (2.19) in Section 2.2 it can easily be confused with the true spatial description of fields. If we wish to be on the safe side we should always put a hat on top of the corresponding function for distinction (in contrast to the bar used for true spatial fields) and (even better) note all the arguments. However, that this is not standard practice is a different story. Let us point out that a clear distinction can be helpful if hybrid problems are considered, for example, a fluid passing through some elastic skeleton [13].

In summary: the world of *material description of fields* comprises both—the Lagrangian as well as the Eulerian material formulation.

One more remark on the meaning of the symbol ψ in Eq. (2.48). It is the value of the field in question for that particular material point at time t. The value stays the same, no matter as to whether a Eulerian material or Lagrangian material formulation is used. How about the value ψ in Eqs. (2.18) or (2.19) in true spatial description? To this end, let us assume once more that all microparticles of the matter currently at $\boldsymbol{r}$ stay together. Then such a "fluid particle" is a material point. Of course, the value ψ shown in (2.18) or (2.19) is the same as the value ψ in Eq. (2.48). However, if the microparticles constituting the matter in $\boldsymbol{r}$ at time t stem from various locations, the material description shown in Eq. (2.48) makes no sense to begin with and only a true spatial representation can be used.

Let us now consider the meaning of the substantial time derivative in the framework of material particles. To this end, the expression (2.24) for the substantial time derivative in true spatial description will guide us. First note that in the case of material particles the observational point follows the motion of the material particle, $\check{\boldsymbol{r}}(t) \to \check{\boldsymbol{x}}(t) = \tilde{\boldsymbol{x}}(\boldsymbol{X}, t)$. Second, the velocity is the material velocity, $\overline{\boldsymbol{v}}\big(\check{\boldsymbol{r}}(t), t\big) \to \hat{\boldsymbol{v}}\big(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t\big)$ from Eq. (2.46), after replacing the reference position $\boldsymbol{X}$ by the motion according to Eq. $(2.45)_1$. Hence, the relative

velocity term of Eq. (2.24) will vanish.[24] We obtain the simple expression

$$\frac{\delta\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\delta t}=\frac{\mathrm{d}\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\mathrm{d}t}. \tag{2.49}$$

This shows that in material description there is no difference between substantial (δ) and total derivative (d). Now recall that the total derivative of a physical property in true spatial description, $\frac{\mathrm{d}\overline{\psi}(\check{\boldsymbol{r}}(t),t)}{\mathrm{d}t}$, meant the ratio of the change of some physical property related to the observational point during its movement to a different grid position divided by the corresponding time increment, namely the first term after the first equal sign in Eq. (2.24). We now write in full analogy,

$$\begin{aligned}\frac{\mathrm{d}\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\mathrm{d}t}&=\lim_{\Delta t\to 0}\frac{\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t+\Delta t),t+\Delta t\right)-\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\Delta t}\equiv\\ &\lim_{\Delta t\to 0}\frac{\tilde{\psi}\left(\boldsymbol{X},t+\Delta t\right)-\tilde{\psi}\left(\boldsymbol{X},t\right)}{\Delta t}=\left.\frac{\partial\tilde{\psi}(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}},\end{aligned} \tag{2.50}$$

where in the last step we switched from a Eulerian material to a Lagrangian material field description. Therefore, in Lagrangian material description the substantial, the total, and the partial time derivative are interchangeable. This is very different from true spatial description.

Strictly following mathematical rules, we can alternatively write for scalar functions ψ

$$\begin{aligned}&\frac{\mathrm{d}\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\mathrm{d}t}=\left.\frac{\partial\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\partial\boldsymbol{x}}\right|_{t}\cdot\frac{\mathrm{d}\tilde{\boldsymbol{x}}(\boldsymbol{X},t)}{\mathrm{d}t}+\left.\frac{\partial\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\partial t}\right|_{\boldsymbol{x}}=\\ &\left.\frac{\partial\hat{\psi}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)}{\partial t}\right|_{\boldsymbol{x}}+\hat{\boldsymbol{v}}\left(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\right)\cdot\nabla_{\boldsymbol{x}}\hat{\psi}\left(\boldsymbol{x},t\right)=\frac{\partial\psi}{\partial t}+\boldsymbol{v}\cdot\nabla\psi\ .\end{aligned} \tag{2.51}$$

Note that this is the analogue to Eq. (2.29) in true spatial description and that we have assumed a non-moving coordinate system for O so that $\tilde{\boldsymbol{x}}(\boldsymbol{X},t)=\tilde{x}_i(\boldsymbol{X},t)\boldsymbol{e}_i$. Moreover, after the last equal sign we have written the result how it is typically presented in textbooks, for example [53], pg. 15. This was done on purpose to remind the reader of the slightly non-diligent attitude that can often be found. Most likely, the authors will say that it is clear from the context what is meant. It should be pointed out that the nabla operator $\nabla_{\boldsymbol{x}}$ examines, at a fixed time t, gradients of field quantities between space points $\boldsymbol{x}$, on which material particles are located, *i.e.*, in Cartesian notation, $\nabla_{\boldsymbol{x}}\hat{\psi}\left(\boldsymbol{x},t\right)=\boldsymbol{e}_i(t)\frac{\partial\hat{\psi}(\boldsymbol{x},t)}{\partial x_i}$. Alternatively, we can say that $\nabla_{\boldsymbol{x}}$ is a gradient of the properties between moving material particles.

A remark on the inherent time-dependencies: note that we allowed time-dependent coordinates for the observer, $\boldsymbol{x}=x_i(t)\boldsymbol{e}_i(t)$. Hence, the time dependencies of the coordinates $x_i(t)$ and of the base $\boldsymbol{e}_i(t)$ eliminate each other for a fixed $\boldsymbol{x}$. This is in agreement with the fact that $\nabla_{\boldsymbol{x}}$ is an objective quantity. Moreover, the situation is such that we are screening the neighborhood around a *fixed* point $\boldsymbol{x}$ at one moment in time. On the other hand, there is a material point at the position $\boldsymbol{x}$ which moves, $\boldsymbol{x}=\tilde{\boldsymbol{x}}(\boldsymbol{X},t)$. The material points surrounding it also move. The gradient examines this neighborhood: if material points around $\boldsymbol{x}$ are neighbors at time t, they can be apart in the future or in the past, and at $\boldsymbol{x}$ different material point have arrived. Then $\nabla_{\boldsymbol{x}}$ is screening a "new" neighborhood, and in this sense the gradient $\nabla_{\boldsymbol{x}}$ is time dependent. This is very different from the gradient in reference placement $\nabla_{\boldsymbol{X}}$, which always examines the same neighborhood.

At this point a remark is in order: In this paper we prefer to use *left gradients* $\nabla(\cdot)$, as it is customary in the Russian and (some of the) German literature. However, this preference can not always be sustained: Sometimes the custom of using *right gradients* $(\cdot)\nabla$ must be followed unless we want to get in conflict with firmly established nomenclature, as for example in the case of the deformation gradient, which is a function in Lagrangian material description ([47], [11], pg. 96),[25]

$$\widetilde{\boldsymbol{F}}=\frac{\partial\tilde{\boldsymbol{x}}}{\partial\boldsymbol{X}}=\frac{\partial\tilde{x}_i}{\partial X_J}\boldsymbol{i}_i\boldsymbol{I}_J=(\nabla_{\boldsymbol{X}}\tilde{\boldsymbol{x}})^{\mathsf{T}}=\tilde{\boldsymbol{x}}\nabla_{\boldsymbol{X}}\ ,\quad(\cdot)\nabla_{\boldsymbol{X}}=\frac{\partial(\cdot)}{\partial\boldsymbol{X}}\ ,\quad\widetilde{F}_{iJ}=\frac{\partial\tilde{x}_i}{\partial X_J}. \tag{2.52}$$

[24] In principle the gradient operation with respect to the grid cell points $\boldsymbol{r}$ must also be replaced by the gradient with respect to space points $\boldsymbol{x}$ on which material particles are located, $\boldsymbol{x}$. Hence, $\nabla_{\boldsymbol{r}}\to\nabla_{\boldsymbol{x}}$. But the corresponding term from Eq. (2.24) drops out anyway.

[25] The base $\boldsymbol{I}_I$ of the reference configuration shall coincide with the current base $\boldsymbol{i}_i$ as in [25], Section 1.1. Here it was written like that only to point out where the capital and small indices in $\widetilde{F}_{iJ}$ come from.

Let us rewrite Eq. (2.51) in a simplified notation, in order to interpret each contribution easily:

$$\frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} = \frac{\partial\hat{\psi}}{\partial t} + \hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}}\hat{\psi}. \tag{2.53}$$

We may say the following: The term $\frac{\partial\hat{\psi}}{\partial t}$ characterizes the temporal change of the physical property ψ of the material particle at its current position, whilst the second term, $\hat{\boldsymbol{v}} \cdot \nabla_{\boldsymbol{x}}\hat{\psi}$, indicates a temporal change of the property $\hat{\psi}$ due to convective transport, enabled through the velocity $\hat{\boldsymbol{v}}$ of the material point. It is therefore customary to refer to the combination as *substantial* and also—most appropriate in the context of material particles—as *material time derivative* of the field property ψ. In order to comply with the notation established in Eq. (2.24) in true spatial description, we could now also introduce the δ symbol. But as we can see, this is not really necessary: the total and the substantial time derivative in Eulerian material description are the same, because the material point and the observational point are identical in material description. Alternatively, we may say that in Eulerian material description the role of the observational point in true spatial description is taken over by the motion of the material particle. Finally, note the fundamental difference to Eq. (2.24), with its relative velocity $\overline{\boldsymbol{v}} - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}$ instead of $\hat{\boldsymbol{v}}$, and where $\overline{\boldsymbol{v}}$ is *not* the velocity of a material point but that of the matter at grid location $\boldsymbol{r}$. However, if the matter consists of material points, then it is the material point velocity.

Mind that Eqs. (2.51) or (2.53) are only applicable to Eulerian material description, $\hat{\psi}$, and not to the Lagrangian one, $\tilde{\psi}$. Also, note that the equation looks a lot like (2.30). Nevertheless beware: in (2.53) the symbols mean $\psi = \hat{\psi}\big(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t\big)$, $\boldsymbol{v} = \hat{\boldsymbol{v}}\big(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t\big)$, $\nabla_{\boldsymbol{x}}(\cdot)$ and *not* $\psi = \overline{\psi}\big(\tilde{\boldsymbol{r}}(t), t\big)$, $\boldsymbol{v} = \overline{\boldsymbol{v}}\big(\tilde{\boldsymbol{r}}(t), t\big)$, $\nabla_{\boldsymbol{r}}(\cdot)$, respectively.

To repeat, in the world of material particles, or in other words, in Eulerian material description of fields, the material time derivative is not discernible from the total time derivative, which is a purely mathematical construct. In summary: we write for the substantial time derivative

$$\frac{\partial\tilde{\psi}}{\partial t} \quad \text{in Lagrangian material} \quad \text{and} \quad \frac{\mathrm{d}\hat{\psi}}{\mathrm{d}t} \quad \text{in Eulerian material description.} \tag{2.54}$$

In context with Eq. (2.53), we should remind ourselves that, due to the way the material velocity was introduced in Eq. (2.46), it is originally a field written in Lagrangian notation, $\boldsymbol{v} = \tilde{\boldsymbol{v}}(\boldsymbol{X}, t)$. However, all fields can also be rewritten in Eulerian material notation. Hence, $\boldsymbol{v} = \hat{\boldsymbol{v}}(\boldsymbol{x}, t)$ following Eq. (2.48). Thus, all the fields and all the equations they are used in can be rewritten in terms of Lagrange or Eulerian material description. Moreover, it is obvious that the partial time derivative in Eq. (2.53) can be exchanged with $\nabla_{\boldsymbol{x}}(\cdot) \equiv \frac{\partial(\cdot)}{\partial\boldsymbol{x}}$, if the observer O does not use time-dependent coordinates. Also recall the footnote from before: in all equations the scalar quantity ψ can simply be replaced by vectorial or tensorial quantities $\boldsymbol{\lambda}$ and $\boldsymbol{\Lambda}$, respectively, but nabla must be applied to these functions correctly, namely from the right.

In the same context, also note the substantial time derivatives in Nanson's formulae, which are frequently used in material field formulation ([34], pg. 74):

$$\mathrm{d}v = \det\boldsymbol{F}\ \mathrm{d}V\ , \quad \frac{\mathrm{d}\det\boldsymbol{F}}{\mathrm{d}t} = \nabla\cdot\boldsymbol{v}\ \det\boldsymbol{F} \quad \Leftrightarrow \quad \frac{\mathrm{d}\det\boldsymbol{F}}{\mathrm{d}t} = \frac{\partial v_k}{\partial x_k}\det\boldsymbol{F}. \tag{2.55}$$

Nanson's formulae are also known as Piola's transform or Piola's identity, see *e.g.*, [14], [55]. Note that the material velocity appears in these formulae. What about the volume elements of the grid cells in spatial description from Fig. 2.2? Recall that we assumed that mesh does not change. However, if it did, this would lead us to the so-called ALE (Arbitrary Lagrangian Eulerian) formulation [33]. There the intention is to readjust the mesh over time for numerical reasons. Such meshes also have a velocity, but it has nothing to do with the velocity of material points. Equations similar in form to (2.55) exist helping to deal with such a situation mathematically. However, we shall not explain the details in this paper.

This means that the shape and size of the material volume element will change over time just like the current material volume $v(t)$ differs in shape and size from the reference volume $V = v(t = 0)$.

Let us finally examine what happens if we choose the motion $\boldsymbol{x} = \hat{\boldsymbol{x}}(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t) = \hat{\boldsymbol{x}}(\check{\boldsymbol{x}}(t), t) \equiv \check{\boldsymbol{x}}(t)$ for $\hat{\psi}$ in Eq. (2.51). Then we compute

$$\frac{\mathrm{d}\hat{\boldsymbol{x}}}{\mathrm{d}t} = \left.\frac{\partial\hat{\boldsymbol{x}}}{\partial t}\right|_{\boldsymbol{x}} + \left.\frac{\partial\hat{\boldsymbol{x}}}{\partial\boldsymbol{x}}\right|_{t} \cdot \frac{\mathrm{d}\check{\boldsymbol{x}}}{\mathrm{d}t} = \boldsymbol{0} + \boldsymbol{1}\cdot\boldsymbol{v} = \boldsymbol{v} \tag{2.56}$$

while observing Eq. (2.46) in the last step, thus going around full circle.

2.4 The deformation measure of classical continua in true spatial and material notation

Eq. (2.33) will form our basis to define a deformation measure for classical and micropolar continua at current time t in true spatial representation. Following [81], pg. 123 or [42], pg. 1774 we apply $\nabla_{\boldsymbol{r}}$ to the displacement and define:[26]

$$\overline{\boldsymbol{g}}\left(\boldsymbol{r},t\right) := \mathbf{1} - \nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\left(\boldsymbol{r},t\right) \quad \Leftrightarrow \quad \overline{g}_{ij} = \delta_{ij} - \frac{\partial \overline{u}_j}{\partial r_i}. \tag{2.57}$$

From (2.33) we conclude that $\overline{\boldsymbol{g}}\left(\boldsymbol{r},t\right) = \nabla_{\boldsymbol{r}}\overline{\boldsymbol{R}}\left(\boldsymbol{r},t\right)$. Thus, the symbol $\overline{\boldsymbol{g}}$ corresponds to the the gradient of the formally introduced reference vector. Let us dwell upon that a little more. Starting from Eq. (2.36), observing the general relation (2.24) for the spatial derivative, and then specializing to a non-moving observational point in the grid cell $\boldsymbol{r}$ we may write:

$$\frac{\mathrm{d}\overline{\boldsymbol{u}}}{\mathrm{d}t} = \overline{\boldsymbol{v}} \cdot \left(\mathbf{1} - \nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) = \overline{\boldsymbol{v}} \cdot \overline{\boldsymbol{g}}. \tag{2.58}$$

In words: the temporal change of displacement in the observational point[27] can be expressed by the quantity $\overline{\boldsymbol{g}}$. It also describes the *difference between the reference position vectors of matter in neighboring grid cells*. In order to see this we write:

$$\mathrm{d}\overline{\boldsymbol{R}}(\boldsymbol{r},t) = \left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{R}}\right)^{\mathsf{T}} \cdot \mathrm{d}\boldsymbol{r} = \overline{\boldsymbol{g}}^{\mathsf{T}} \cdot \mathrm{d}\boldsymbol{r}. \tag{2.59}$$

This relation shows what the line segment between two fluid particles was like that is currently given by $\mathrm{d}\boldsymbol{r}$. Moreover, note that, because of the way Eq. (2.33) was written, $\overline{\boldsymbol{g}}$ is a quantity of current time—a veritable basis for measuring measure in true spatial description. Moreover, note that in [42] $\overline{\boldsymbol{g}}$ is referred to as *spatial* deformation gradient tensor. This notion is, in fact, dichotomous. The only quantity universally declared as deformation gradient tensor in continuum mechanics is $\widetilde{\boldsymbol{F}}$, and written in Lagrangian material description, mentioned in Eq. (2.52). In order to appreciate the meaning of both quantities, and to understand the difference, let us reintroduce the displacement vector from (2.35) consistently in Lagrangian material notation:

$$\boldsymbol{u} = \tilde{\boldsymbol{u}}\left(\boldsymbol{X},t\right) = \tilde{\boldsymbol{x}}\left(\boldsymbol{X},t\right) - \boldsymbol{X}, \tag{2.60}$$

Recall that unlike the displacement vector in true spatial description, Eq. (2.33), this $\boldsymbol{u}$ *not* a formally introduced quantity. It has a physical meaning, because it describes the displacement of a material particle from its reference to its current location in space. We use it for introducing the deformation gradient $\widetilde{\boldsymbol{F}}$:

$$\widetilde{\boldsymbol{F}} = \frac{\partial \tilde{\boldsymbol{x}}}{\partial \boldsymbol{X}} \equiv \tilde{\boldsymbol{x}}\nabla_{\boldsymbol{X}} = \tilde{\boldsymbol{u}}\nabla_{\boldsymbol{X}} + \mathbf{1} \quad \Leftrightarrow \quad \widetilde{F}_{ij} = \frac{\partial \tilde{x}_i}{\partial X_J} = \frac{\partial \tilde{u}_i}{\partial X_J} + \delta_{iJ}, \tag{2.61}$$

where Eqs. (2.52), (2.60) were observed and right gradient notation has been used. The intuitive meaning of this quantity is that it can be used to compute how the *distance between material particles* changes:

$$\mathrm{d}\boldsymbol{x} = \frac{\partial \tilde{\boldsymbol{x}}(\boldsymbol{X},t)}{\partial \boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{X} = \tilde{\boldsymbol{x}}\nabla_{\boldsymbol{X}} \cdot \mathrm{d}\boldsymbol{X} = \widetilde{\boldsymbol{F}} \cdot \mathrm{d}\boldsymbol{X}. \tag{2.62}$$

This should be compared to Eq. (2.59) in order to appreciate analogies but still realize the fundamental differences. Obviously, the inverse of the deformation gradient is given by:

$$\widehat{F}^{-1}_{Ij} = \frac{\partial \widehat{X}_I}{\partial x_j} = \frac{\partial\left(x_I - \hat{u}_I\right)}{\partial x_j} = \delta_{Ij} - \frac{\partial \hat{u}_I}{\partial x_j} \mathrel{\widehat{=}} \overline{g}_{jI}, \tag{2.63}$$

where $\mathrel{\widehat{=}}$ is supposed to mean "corresponds to something like" because the arguments of the corresponding functions are completely different, namely $\hat{\boldsymbol{F}}^{-\mathsf{T}}\left(\hat{\boldsymbol{X}}(\boldsymbol{x},t),t\right)$ and $\overline{\boldsymbol{g}}(\boldsymbol{r},t)$.

Alternatively to the previous equation, we may write:

$$\overline{g}_{iJ} \mathrel{\widehat{=}} \widehat{F}^{-\mathsf{T}}_{iJ} \quad \Leftrightarrow \quad \overline{\boldsymbol{g}} = \left(\frac{\partial \overline{\boldsymbol{R}}}{\partial \boldsymbol{r}}\right)^{\mathsf{T}} = \nabla_{\boldsymbol{r}}\overline{\boldsymbol{R}} \;\; \widehat{=} \;\; \widehat{\boldsymbol{F}}^{-\mathsf{T}} = \left(\frac{\partial \hat{\boldsymbol{X}}}{\partial \boldsymbol{x}}\right)^{\mathsf{T}} = \nabla_{\boldsymbol{x}}\hat{\boldsymbol{X}}, \tag{2.64}$$

[26] At this point recall once more that true spatial description is also capable of handling empty space as encountered in cracks, voids, or pores. As an example of what to do with deformation measures in the case of porous media see [13].

[27] Recall that this is what the total derivative means.

and, if we wish, we could say that $\overline{\boldsymbol{g}}$ corresponds to the inverse transposed deformation gradient tensor $\widehat{\boldsymbol{F}}^{-\mathsf{T}}$. Let us assume in true spatial description that the matter at $\boldsymbol{r}$ at time t can be considered to be a material point; then—of course—the values of the two functions are the same. But, the functions themselves are different and not only because they have different arguments. And, very important, they change differently in time:

$$\frac{\mathrm{d}\hat{\boldsymbol{F}}^{-\mathsf{T}}}{\mathrm{d}t} = \frac{\mathrm{d}\nabla_{\boldsymbol{x}}\hat{\boldsymbol{X}}\left(\boldsymbol{x},t\right)}{\mathrm{d}t} \quad \text{and} \quad \frac{\mathrm{d}\overline{\boldsymbol{g}}}{\mathrm{d}t} = \frac{\mathrm{d}\overline{\boldsymbol{R}}(\boldsymbol{r},t)}{\mathrm{d}t}\nabla_{\boldsymbol{r}}. \tag{2.65}$$

Moreover, for the gradient and for the divergence of the velocity of matter we may write in true spatial notation:[28]

$$\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}} = -\frac{\delta\overline{\boldsymbol{g}}}{\delta t}\cdot\overline{\boldsymbol{g}}^{-1}\,, \quad \nabla_{\boldsymbol{r}}\cdot\overline{\boldsymbol{v}} = -\frac{\delta\overline{\boldsymbol{g}}}{\delta t}\cdot\cdot\,\overline{\boldsymbol{g}}^{-1}. \tag{2.66}$$

For proof we start from the general definition of the substantial time derivative in true spatial description from Eq. (2.24) and apply it to the displacement $\overline{\boldsymbol{u}}$. Furthermore we observe Eqs. (2.24) and (2.36) to obtain:

$$\frac{\delta\overline{\boldsymbol{u}}}{\delta t} = \frac{\mathrm{d}\overline{\boldsymbol{u}}}{\mathrm{d}t} + \left(\overline{\boldsymbol{v}} - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}} \equiv \overline{\boldsymbol{v}}. \tag{2.67}$$

To this relation we apply $\nabla_{\boldsymbol{r}}$ and observe the interchangability relation $(2.27)_1$:

$$\begin{aligned}
\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}} = \nabla_{\boldsymbol{r}}\left(\frac{\mathrm{d}\overline{\boldsymbol{u}}}{\mathrm{d}t}\right) + \nabla_{\boldsymbol{r}}\left(\overline{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\cdot\nabla_{\boldsymbol{r}}\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}} = \\
\frac{\mathrm{d}\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\mathrm{d}t} + \left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\right)\cdot\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) + \overline{\boldsymbol{v}}\cdot\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\cdot\nabla_{\boldsymbol{r}}\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}} \\
\frac{\mathrm{d}\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\mathrm{d}t} + \left(\overline{\boldsymbol{v}} - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) + \left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\right)\cdot\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) = \frac{\delta\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\delta t} + \left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\right)\cdot\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right),
\end{aligned} \tag{2.68}$$

and therefore with Eq. $(2.57)_2$:

$$\frac{\delta\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\delta t} = \nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\cdot\left(\mathbf{1}-\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right) \equiv \nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}\cdot\overline{\boldsymbol{g}} \quad \Rightarrow \quad \nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}} = \frac{\delta\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\delta t}\cdot\overline{\boldsymbol{g}}^{-1}. \tag{2.69}$$

On the other hand, by application of $\frac{\delta}{\delta t}$ to Eq. $(2.57)_2$ we obtain:

$$\frac{\delta\left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{u}}\right)}{\delta t} = -\frac{\delta\overline{\boldsymbol{g}}}{\delta t}, \tag{2.70}$$

which in combination with Eq. (2.69) concludes the proof of Eq. $(2.66)_1$. Eq. $(2.66)_2$ follows by calculating the trace of $(2.66)_1$.

A remark for those interested in the original literature: Zhilin [81] claims on pg. 123 that Eqs. (2.66) can be found in the book by Palmov [61]. However, this citation is misleading. It is true that a quantity $\hat{\boldsymbol{g}} = \left(\nabla_{\boldsymbol{x}}\hat{\boldsymbol{X}}\right)\cdot\left(\hat{\boldsymbol{X}}\nabla_{\boldsymbol{x}}\right)$,[29] the Almansi strain—in Eulerian material notation—is introduced in Eq. (1.27) of that book. We realize immediately that this is not quite the object $\overline{\boldsymbol{g}}$ from Eq. (2.64). If we wish, we could call our $\overline{\boldsymbol{g}}$ the root of this strain measure—if the matter in $\boldsymbol{r}$ at time t can be considered as a material point, in other words, if it is possible to convert true spatial description into a Eulerian material one. For the material (=substantial) derivative of his $\hat{\boldsymbol{g}}$, indicated by a dot, Palmov writes in essence on pg. 19:

$$\dot{\hat{\boldsymbol{g}}} = -\left(\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}}\right)\cdot\hat{\boldsymbol{g}} - \hat{\boldsymbol{g}}\cdot\left(\nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}}\right). \tag{2.71}$$

Surely this is not our Eq. (2.66) unless it is transformed and specialized further. In short (2.66) does not appear in Palmov's book.

[28] The inner scalar product between two tensors second rank is defined by $\boldsymbol{A}\cdot\cdot\boldsymbol{B} = A_{ij}B_{ji}$. The equivalent to Eq. $(2.66)_2$ in Eulerian material description is shown in $(2.55)_2$.

[29] In true spatial description this would correspond to some $\overline{\boldsymbol{g}} = \left(\nabla_{\boldsymbol{r}}\overline{\boldsymbol{R}}\right)\cdot\left(\overline{\boldsymbol{R}}\nabla_{\boldsymbol{r}}\right)$, by using the notation from Eq. (2.64), which Palmov does *not* really advocate. In fact, he speaks of "the Eulerian approach" (pg. 18), which indicates that he is thinking in terms of a Eulerian material formulation.

2.5 Microangular velocity at the current time (material points perspective)

Consider Fig. 5 once again. First, let us recall, following Kafadar and Eringen [45], that "physically every material point of a polar medium is *phenomenologically equivalent to a rigid body*. As such, a material particle of this continuum may translate according to a motion as" $(2.45)_1$ "and may undergo a rotation which is most easily described by a proper orthogonal tensor" $\boldsymbol{P}$, such that

$$\boldsymbol{P}\cdot\boldsymbol{P}^{\mathsf{T}}=\mathbf{1}=\boldsymbol{P}^{\mathsf{T}}\cdot\boldsymbol{P}\ ,\quad \det\boldsymbol{P}=1. \tag{2.72}$$

In classical rigid body mechanics, rotation is defined with respect to a rotating reference frame fixed in the body. Thus, in addition to the reference position vector $\boldsymbol{X}$ we introduce triads of orthogonal base vectors $\widetilde{\boldsymbol{D}}_i(\boldsymbol{X},t=t_0)=:\boldsymbol{D}_i(\boldsymbol{X})$ at the reference time t_0. These triads may be arbitrarily oriented and vary from point to point, or, alternatively, we may assign the same reference state to all material particles, since our main goal is to characterize the relative evolution of rotation. In the latter case, the $\boldsymbol{X}$ in $\boldsymbol{D}_i(\boldsymbol{X})$ becomes superfluous. Furthermore, we may postulate that the reference directors coincide with the coordinate system's base vectors, so that particle rotations are measured with respect to this fixed system.

At the current time t, a material point occupies a new position $\boldsymbol{x}=\tilde{\boldsymbol{x}}(\boldsymbol{X},t)$ and rotates such that the directors attached to the particle are given in Lagrange and Eulerian material notation by (also see [17], pg. 11)

$$\tilde{\boldsymbol{d}}_i\,(\boldsymbol{X},t)=\widetilde{\boldsymbol{P}}(\boldsymbol{X},t)\cdot\boldsymbol{D}_i(\boldsymbol{X})\ ,\quad \hat{\boldsymbol{d}}_i\,(\boldsymbol{x},t)=\widehat{\boldsymbol{P}}(\boldsymbol{x},t)\cdot\widehat{\boldsymbol{D}}_i(\boldsymbol{x},t=t_0). \tag{2.73}$$

A few remarks: it was already mentioned that we focus on *micropolar* materials. However, the notions we are introducing were originally intended to cover a wider class of "extended continua" as outlined in the work by Eringen, *e.g.*, [25], also see the comparison on various extended theories in [74]. For example, $\boldsymbol{P}$ is denoted by $\boldsymbol{\chi}$ in Eringen's work and volume conservation, *i.e.*, $\det\boldsymbol{P}=1$, is not necessarily required. In fact, all of Eq. (2.72) does not hold in this more general case. In Eringen's work it is allowed that $\boldsymbol{P}$ covers the whole orthogonal group. Consequently, then $\det\boldsymbol{P}$ may change its sign under the mirror reflection transformation, which is important for material symmetry analysis [18].

The microrotation tensor is now used to define the *microgyration tensor* (also known as *left spin tensor*) of micropolar theory,[30] by means of substantial time derivatives:

$$\widetilde{\boldsymbol{S}}=\frac{\mathrm{d}\widetilde{\boldsymbol{P}}}{\mathrm{d}t}\cdot\widetilde{\boldsymbol{P}}^{\mathsf{T}}\quad\Leftrightarrow\quad\widetilde{S}_{ij}=\frac{\mathrm{d}\widetilde{P}_{iM}}{\mathrm{d}t}\widetilde{P}_{jM}. \tag{2.74}$$

Because of (2.72) the microgyration tensor is antisymmetric,

$$\widetilde{\boldsymbol{S}}=-\widetilde{\boldsymbol{S}}^{\mathsf{T}}, \tag{2.75}$$

and we may therefore introduce the so-called (left) *microangular velocity vector* $\tilde{\boldsymbol{\omega}}$, which is an axial quantity, as follows:[31]

$$\widetilde{\boldsymbol{S}}=\mathbf{1}\times\tilde{\boldsymbol{\omega}}=\tilde{\boldsymbol{\omega}}\times\mathbf{1}\quad\Leftrightarrow\quad\widetilde{S}_{ij}=-\epsilon_{ijk}\tilde{\omega}_k \tag{2.76}$$

or

$$\tilde{\boldsymbol{\omega}}=-\frac{1}{2}\widetilde{\boldsymbol{S}}_{\times}\equiv-\frac{1}{2}\left[\frac{\mathrm{d}\widetilde{\boldsymbol{P}}}{\mathrm{d}t}\cdot\widetilde{\boldsymbol{P}}^{\mathsf{T}}\right]_{\times}\quad\Leftrightarrow\quad\tilde{\omega}_i=-\frac{1}{2}\epsilon_{ijk}\widetilde{S}_{jk}. \tag{2.77}$$

By combining Eqs. (2.74) and (2.76) we arrive at what Zhilin [81], pg. 121 calls the *Poisson relation*:

$$\frac{\mathrm{d}\widetilde{\boldsymbol{P}}}{\mathrm{d}t}=\tilde{\boldsymbol{\omega}}\times\widetilde{\boldsymbol{P}}. \tag{2.78}$$

A few remarks are due at this point:

[30] Denoted by $\boldsymbol{v}$ in Eringen [25], pp. 24.

[31] The vector invariant of a tensor second rank is given by: $\boldsymbol{A}=A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\ \Rightarrow\ \boldsymbol{A}_{\times}=A_{ij}\boldsymbol{e}_i\times\boldsymbol{e}_j$. It was introduced by Gibbs [77].

- Up to now we have written all fields pertinent to micropolar theory in Lagrangian material description. Of course, this is not a must. Keeping the bijectivity of Eq. (2.45) in mind we could (for example) write the microangular velocity in Eulerian material formulation,

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}\big(\tilde{\boldsymbol{x}}\,(\boldsymbol{X},t)\,,t\big) = \hat{\boldsymbol{\omega}}\,(\boldsymbol{x},t)\,. \tag{2.79}$$

  In fact, we may say that we have two reference quantities for micropolar materials, the material particle identifier X (or reference position vector $\boldsymbol{X}$) and the reference directors $\boldsymbol{D}_i$. However, $\boldsymbol{\omega}$ always belongs to the current configuration.
- In material description of micropolar materials the microrotation tensor $\widetilde{\boldsymbol{P}}$ is the primary quantity and the microangular velocity $\tilde{\boldsymbol{\omega}}$ is secondary—it is derived from it as shown in Eq. (2.77).
- It is important to realize: The microangular velocity field $\tilde{\boldsymbol{\omega}}$ is not "visible" on the continuum level. It is *not* a "vortex," as one might think. Visible vortices are known from fluid mechanics. They are characterized by the *vorticity*, which is nothing else but the antisymmetric velocity gradient $\widehat{\boldsymbol{W}}$:

$$\widehat{\boldsymbol{W}} = \frac{1}{2}\big(\hat{\boldsymbol{v}}\nabla_{\boldsymbol{x}} - \nabla_{\boldsymbol{x}}\hat{\boldsymbol{v}}\big) \quad \Leftrightarrow \quad \widehat{W}_{ij} = \frac{1}{2}\left(\frac{\partial \hat{v}_i}{\partial x_j} - \frac{\partial \hat{v}_j}{\partial x_i}\right), \tag{2.80}$$

  from which the corresponding vector invariant $\hat{\boldsymbol{w}}$, the *vorticity vector*, follows:

$$\hat{\boldsymbol{w}} = -\frac{1}{2}\widehat{\boldsymbol{W}}_{\times} \quad \Leftrightarrow \quad \hat{w}_i = -\frac{1}{2}\epsilon_{ijk}\widehat{W}_{jk} = \frac{1}{2}\epsilon_{ijk}\frac{\partial \hat{v}_k}{\partial x_j}. \tag{2.81}$$

  In contrast to $\widehat{\boldsymbol{W}}$ or $\hat{\boldsymbol{w}}$, the microgyration tensor $\widehat{\boldsymbol{S}}$ and the microangular velocity $\hat{\boldsymbol{\omega}}$ are intrinsic quantities of the continuum that characterize the capability of matter for internal rotation due to additional internal degrees of freedom.
- In this section we will use a mix of symbols from different sources: The microgyration tensor second rank, $\boldsymbol{\nu}$, is used mostly in Eringen's work. In fact, his starting point are the conceptually more general micromorphic continua, which are then eventually specialized to micropolar materials (see pg. 27 of [25], where the microgyration vector—our microangular velocity $\boldsymbol{\omega}$—is introduced, which is (confusingly) also called $\boldsymbol{\nu}$ by Eringen).

Let us now talk about how to interpret and choose the directors and about the role of the microrotation tensor in a more figurative, non-formal way of speech. To this end, we first turn to Eringen's microinertia tensor $\boldsymbol{J} = \tilde{\boldsymbol{J}}(\boldsymbol{X},t) = \hat{\boldsymbol{J}}\big(\tilde{\boldsymbol{x}}(\boldsymbol{X},t),t\big) = \hat{\boldsymbol{J}}\,(\boldsymbol{x},t)$, which was finally presented here in Eulerian material notation.

Being a symmetric second-rank tensor and assuming that it is not degenerate, the inertia tensor of a material particle in the reference configuration,

$$\widetilde{\boldsymbol{J}}\,(\boldsymbol{X},t=t_0) =: \boldsymbol{J}_0(\boldsymbol{X}), \tag{2.82}$$

can be diagonalized in its eigenbase $\boldsymbol{E}_i\,(\boldsymbol{X})$, $i = 1,2,3$ as follows:

$$\boldsymbol{J}_0(\boldsymbol{X}) = J_1(\boldsymbol{X})\boldsymbol{E}_1(\boldsymbol{X})\boldsymbol{E}_1(\boldsymbol{X}) + J_2(\boldsymbol{X})\boldsymbol{E}_2(\boldsymbol{X})\boldsymbol{E}_2(\boldsymbol{X}) + J_3(\boldsymbol{X})\boldsymbol{E}_3(\boldsymbol{X})\boldsymbol{E}_3(\boldsymbol{X}). \tag{2.83}$$

If desired, we may take the reference directors to coincide with the principal axes of the particle, *i.e.*, $\boldsymbol{D}_k(\boldsymbol{X}) = \boldsymbol{E}_k(\boldsymbol{X})$. Since every particle in a polar medium is equivalent to a rigid body, the inertia of individual microparticles does not change under deformation of the continuum. Therefore, we can write

$$\widehat{\boldsymbol{J}}(\boldsymbol{x},t) = \widehat{\boldsymbol{P}}(\boldsymbol{x},t)\cdot\widehat{\boldsymbol{J}}_0\,(\boldsymbol{x},t=t_0)\cdot\widehat{\boldsymbol{P}}^{\mathsf{T}}(\boldsymbol{x},t). \tag{2.84}$$

It follows that the principal values remain unchanged, while the principal axes rotate together with the particle. If the reference directors are chosen to coincide with the principal axes of inertia, then

$$\widehat{\boldsymbol{J}}(\boldsymbol{x},t) = \widehat{J}_1(\boldsymbol{x},t=t_0)\boldsymbol{d}_1\boldsymbol{d}_1 + \widehat{J}_2(\boldsymbol{x},t=t_0)\boldsymbol{d}_2\boldsymbol{d}_2 + \hat{J}_3(\boldsymbol{x},t=t_0)\boldsymbol{d}_3\boldsymbol{d}_3, \tag{2.85}$$

where the directors in the current configuration are given by

$$\boldsymbol{d}_i = \hat{\boldsymbol{d}}_i(\boldsymbol{x},t) = \widehat{\boldsymbol{P}}(\boldsymbol{x},t)\cdot\widehat{\boldsymbol{E}}_i\,(\boldsymbol{x},t=t_0) = \widehat{\boldsymbol{P}}(\boldsymbol{x},t)\cdot\widehat{\boldsymbol{D}}_i\,(\boldsymbol{x},t=t_0)\,. \tag{2.86}$$

But what happens if the eigenvalues degenerate, or—in the extreme case—if the microparticles are spherical? In that situation, the eigenvectors of the microinertia tensor cannot be used to identify the reference

directors $\boldsymbol{D}_i(\boldsymbol{X})$. We are then left with no alternative but to assign them formally, as discussed earlier in this section.

Also notice that since the reference inertia tensor always remains the same it follows from (2.84), while observing (2.78)

$$\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} \cdot \boldsymbol{J}_0 \cdot \boldsymbol{P}^{\mathsf{T}} + \boldsymbol{P} \cdot \boldsymbol{J}_0 \cdot \frac{\mathrm{d}\boldsymbol{P}^{\mathsf{T}}}{\mathrm{d}t} = \boldsymbol{\omega} \times \boldsymbol{J} - \boldsymbol{J} \times \boldsymbol{\omega}. \tag{2.87}$$

The kinetic equation in this form can be found in [25], pg. 42, or [26], pg. 112. Let us summarize by saying that it is conceptually useful to characterize the directors as eigenvectors of the microinertia tensor, but it is not an absolute must and in the degenerate case impossible. From a formal perspective the microrotational tensor is the most important ingredient in order to characterize internal rotational degrees of freedom of micropolar matter in material description. However, the directors are required to capture the relative advancement of microrotation.

In the same context, let us talk about another issue that is different in the theories of the two schools mentioned near the end of Section 2.2. The Ericksen-Leslie school models nematic crystals by using the concept of a single director, which is a non-deformable 1D digit on the continuum scale. This has the effect that the balance of spin degenerates to what is called the director balance, [56], pg. 354. Eringen, however, uses a more general rigid microstructure described by the tensor of microinertia, which is a 3D continuum field. For this, he makes use of the kinetic equation (2.87) and the microinertia tensor is part of a complete spin balance. In this sense, Eringen is more general. But his microinertia tensor does not change its form. It is a rigidly rotating object (see Eq. (2.84) and the relations before) just like the 1D digit of Ericksen-Leslie. So, it is perfectly legitimate to use the concept of an indestructible material point. At least this is so in the beginning of Eringen's work. Later, in [23] and [26], pg. 118, he changes his initial point of view. He considers the situation of a viscous fluid that can stick gradually to suspensions. Hence, the microinertia tensor will change and a production term is required. However, this renders the concept of a material particle problematic. Eringen does not mention this at all because, more or less without being aware of it, he is using a true spatial description for solving problems already. We will investigate this even more in the following section.

2.6 Microangular velocity and its gradient (true spatial perspective)

We start by noting that Eringen's formulation mentioned at the end of the previous section is also completely within the world of material particles, either in Lagrangian or in material Eulerian formulation. However, in this paper we also want to emphasize true spatial description and introduce the quantities relevant for micropolar theory accordingly. In this context, the following papers are worthwhile consulting. For the nonlinear theory see: [81], Section 3.2.1: [31], Sections 2/3, [42], Sections 2/5, [38], Section 4.4, [43], pg. 5, [40], Section 2.1, [41], Section 2.1. The linear theory is outlined in [37], Section 2, [39], Section 8.2.2.

Recall that in true spatial description of non-polar matter the velocity $\boldsymbol{v} = \overline{\boldsymbol{v}}(\boldsymbol{r}, t)$ was the primary quantity, which had to be assumed as known. The motion, or rather the displacement $\boldsymbol{u} = \overline{\boldsymbol{u}}(\boldsymbol{r}, t)$, were secondary. In fact, the latter followed from Eq. (2.36), which could be transformed into a partial differential equations, namely (2.38).

In the case of micropolar matter in true spatial description it is similar. The starting point is the equivalent of Eq. (2.78), but in true spatial notation,

$$\frac{\delta \overline{\boldsymbol{P}}}{\delta t} = \overline{\boldsymbol{\omega}} \times \overline{\boldsymbol{P}} \tag{2.88}$$

with

$$\overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} = \mathbf{1} = \overline{\boldsymbol{P}}^{\mathsf{T}} \cdot \overline{\boldsymbol{P}}\,, \quad \det \overline{\boldsymbol{P}} = 1. \tag{2.89}$$

In the first relation, the microangular velocity field $\overline{\boldsymbol{\omega}}(\boldsymbol{r}, t)$ must be known, so that the microrotation tensor $\overline{\boldsymbol{P}}(\boldsymbol{r}, t)$ can be calculated from it, provided an initial condition is assigned, for example, $\overline{\boldsymbol{P}}(\boldsymbol{r}, t = 0) = \mathbf{1}$. We conclude that the microrotation tensor is no longer the primary kinematic quantity for internal rotational degrees of freedom. Rather the microangular velocity is. Hence, similar to $\overline{\boldsymbol{R}}(\boldsymbol{r}, t)$ or $\overline{\boldsymbol{u}}(\boldsymbol{r}, t)$, the microrotational tensor becomes a formal, derivable quantity. Only if the fluid particle remains undisturbed $\overline{\boldsymbol{P}}(\boldsymbol{r}, t)$ characterizes its actual rotation.

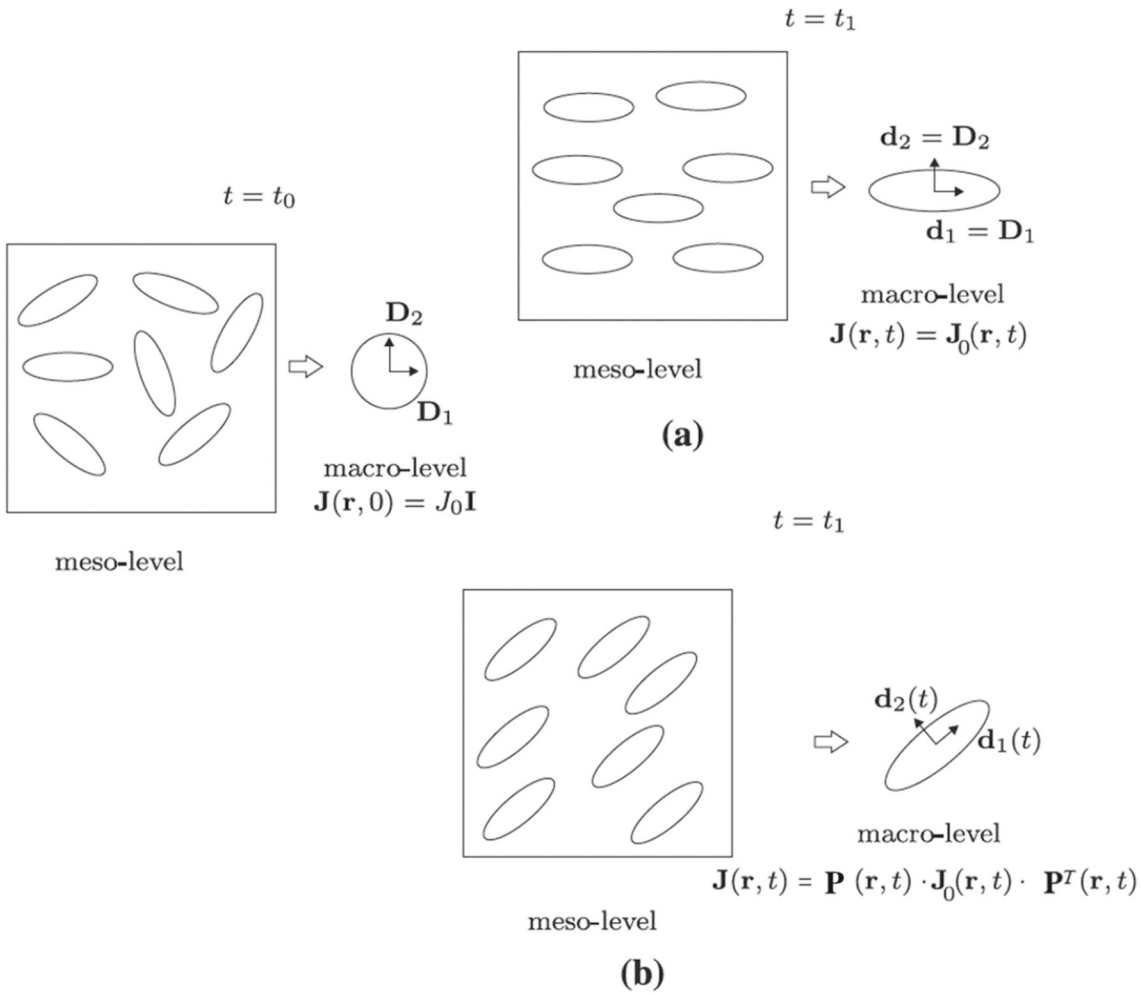

Fig. 6 Particle alignment caused by an external magnetic field.

For completeness we also note the analogues of Eqs. (2.74) and (2.77) in true spatial notation:

$$\overline{\boldsymbol{S}} = \frac{\delta \overline{\boldsymbol{P}}}{\delta t} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}}, \quad \overline{\boldsymbol{\omega}} = -\frac{1}{2}\overline{\boldsymbol{S}}_{\times} = -\frac{1}{2}\left[\frac{\delta \overline{\boldsymbol{P}}}{\delta t} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}}\right]_{\times}. \tag{2.90}$$

Note that the microrotational tensor $\overline{\boldsymbol{P}}(\boldsymbol{r}, t = 0)$ describes only relative rotation between positions. Therefore, in order to determine the actual orientation of a fluid particle, either current or reference directors must be introduced. The initial condition of Eq. (2.88) also depends on this choice. Which option is preferable depends on the problem under consideration.

If the fluid particles are indestructible, the situation is more or less analogous to the material description. For example, one may treat the reference directors as reference frames for describing rotation and postulate that all reference directors are aligned with the coordinate system. Alternatively, the directors may be associated with the principal axes of inertia of a polar particle. Following the logic of spatial description, which focuses on the present configuration, it is more natural to introduce the current directors as the primary ones and determine the reference directors via the rotation tensor. However, if the initial orientation of the particles is known (*e.g.*, all were initially aligned in the same direction), then it is more reasonable to introduce reference directors and to determine the current ones by using the angular velocity and the corresponding rotation tensors. In both cases, it should be emphasized that, as with the displacement vector, the directors themselves are not of primary interest; only their gradients, or in most modern theories, the relative rotations of the particles, are important.

As repeatedly noted, in general, a spatial description allows modeling of more complex situations, where the material composing a fluid particle may have arrived at the observation point from different locations. In such cases, associating reference directors with the principal axes of inertia becomes problematic. Moreover, the very concept of the inertia tensor in this context requires rethinking, and the notion of an inertia tensor in the reference state becomes ambiguous.

Following Ivanova and Vilchevskaya [42], we introduce the macroscopic inertia tensor as the average moment of inertia of all microparticles i within the cell located at $\boldsymbol{r}$ at time t:

$$\overline{\boldsymbol{J}}\,(\boldsymbol{r},t) = \frac{1}{N}\sum_{i=1}^{N}\boldsymbol{J}_i. \tag{2.91}$$

In view of Eqs. (2.82) and (2.84) from Eulerian material description, we now assume that the macroscopic tensor of inertia obeys similar relations in spatial description:

$$\overline{\boldsymbol{J}}(\boldsymbol{r},t) = \overline{\boldsymbol{P}}(\boldsymbol{r},t)\cdot\check{\boldsymbol{J}}(\boldsymbol{r},t)\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}(\boldsymbol{r},t)\ , \quad \check{\boldsymbol{J}}(\boldsymbol{r},0) = \check{\boldsymbol{J}}_0(\boldsymbol{r}). \tag{2.92}$$

However, unlike in the material description, the reference tensor $\check{\boldsymbol{J}}(\boldsymbol{r},t)$ is a function of the current position vector and time. To clarify the meaning of $\check{\boldsymbol{J}}(\boldsymbol{r},t)$ let us consider a simple example without translational velocity: imagine metal filings homogeneously distributed throughout the medium (see Fig. 6 on the left). We choose this configuration as both the reference and initial configuration of a dynamic problem. Then the specific density of the inertia tensor in the reference configuration $\boldsymbol{J}(\boldsymbol{r},0) \equiv \check{\boldsymbol{J}}(\boldsymbol{r},0) = J_0\boldsymbol{1}$ is a spherical tensor. If an external static magnetic field is applied in horizontal direction, the metal filings align with the field direction, and the medium becomes transversely isotropic (Fig. 6a). The orientation of each particle is characterized by a microrotation tensor at the microlevel. At the continuum level, the initially random distribution becomes ordered, leading to a change in the macroscopic inertia tensor. Note that the macroscopic microangular velocity is zero; hence, following Eq. (2.88), the microrotation tensor is the identity tensor, $\overline{\boldsymbol{P}}\,(\boldsymbol{r},t) = \boldsymbol{1}$ and, therefore, Eq. (2.92) yields $\overline{\boldsymbol{J}}\,(\boldsymbol{r},t) = \check{\boldsymbol{J}}\,(\boldsymbol{r},t)$. Thus, in this case, the tensor $\check{\boldsymbol{J}}(\boldsymbol{r},t)$ changes due to macroscopic variations in the moments of inertia of the particles rather than rigid-body rotation. It follows that the current and reference directors coincide, $\boldsymbol{d}_i = \boldsymbol{D}_i$, and one may either arbitrarily assign the reference directors (with no guarantee that they will align with the principal axes of $\check{\boldsymbol{J}}(\boldsymbol{r},t)$) or simply adopt the principal axes as the current, and therefore also the reference, directors (as shown in Fig. 6a).

Now suppose that the external magnetic field rotates with some angular velocity. This causes the filings to follow and to rotate coherently. As a result, their average microangular velocity becomes nonzero, and the tensor of inertia $\check{\boldsymbol{J}}(\boldsymbol{r},t)$ either rotates relative to the reference directors $\boldsymbol{D}_k$ (Fig. 6b) or, conversely, the reference directors $\boldsymbol{D}_i$ rotate relatively to the current ones $\boldsymbol{d}_i\,(t)$ according to Eq. $(2.92)_1$.

Let us now compute the material derivative of Eq. $(2.92)_1$ while observing (2.88):

$$\frac{\delta\boldsymbol{J}}{\delta t} = \frac{\delta\boldsymbol{P}}{\delta t}\cdot\check{\boldsymbol{J}}\cdot\boldsymbol{P}^{\mathsf{T}} + \boldsymbol{P}\cdot\check{\boldsymbol{J}}\cdot\frac{\delta\boldsymbol{P}^{\mathsf{T}}}{\delta t} + \boldsymbol{P}\cdot\frac{\delta\check{\boldsymbol{J}}}{\delta t}\cdot\boldsymbol{P}^{\mathsf{T}} = \boldsymbol{\omega}\times\boldsymbol{J} - \boldsymbol{J}\times\boldsymbol{\omega} + \boldsymbol{\chi}, \tag{2.93}$$

where the terms with angular velocity describe the rigid-body rotation of the inertia tensor, while $\boldsymbol{\chi} = \boldsymbol{P}\cdot\check{\boldsymbol{\chi}}\cdot\boldsymbol{P}^{\mathsf{T}}$ with $\check{\boldsymbol{\chi}} = \frac{\delta\check{\boldsymbol{J}}}{\delta t}$ characterizes changes in the inertia tensor due to internal structural transformations, such as consolidation, fragmentation, or mutual reorientation of microparticles. We may call this term a *production of microinertia*. Various examples for the use of the production term in true spatial description can be found in [59], [30], [57], [58], [72], [29], [73]. One must realize that such a term does not exist when indestructible material particles are involved. Therefore, in material description—Lagrangian or Eulerian—this term cannot occur *a priori*.

Finally, let us note that in Eq. (2.92) the tensor $\check{\boldsymbol{J}}(\boldsymbol{r},t)$ is expressed in tensorial form without specifying a coordinate system. Thus, the same coordinate system can be used to describe both rotational and translational motion. In this case, the inertia tensors generally have six components, and it is more natural to use reference directors as reference frames for characterizing rotations. Alternatively, one can adopt the principal axes of each fluid particle as the current directors and define the reference directors accordingly.

Analogously to Eq. (2.88) we now introduce another measure, $\boldsymbol{\theta} = \overline{\boldsymbol{\theta}}\,(\boldsymbol{r},t)$, based on spatial and not on time differentiation:

$$\nabla_{\boldsymbol{r}}\overline{\boldsymbol{P}} = \overline{\boldsymbol{\theta}}\times\overline{\boldsymbol{P}} \quad\Rightarrow\quad \overline{\theta}_{ij} = \frac{1}{2}\epsilon_{jkl}\overline{P}_{kM}\frac{\partial\overline{P}_{lM}}{\partial r_i}. \tag{2.94}$$

Note that, in Eulerian material description, we can introduce:

$$\nabla_{\boldsymbol{x}} \widehat{\boldsymbol{P}} = \hat{\boldsymbol{\theta}} \times \hat{\boldsymbol{P}} \quad \Rightarrow \quad \hat{\theta}_{ij} = \frac{1}{2} \epsilon_{jkl} \hat{P}_{kM} \frac{\partial \hat{P}_{lM}}{\partial x_i}. \tag{2.95}$$

The quantity $\boldsymbol{\theta}$ is known as the *wryness tensor second rank*.[32]

Next, in analogy to Eq. (2.66)—the gradient of the material velocity—we investigate the gradient of the microangular velocity. We shall prove that:

$$\nabla_{\boldsymbol{r}} \overline{\boldsymbol{\omega}} = \frac{\delta \overline{\boldsymbol{\theta}}}{\delta t} + \overline{\boldsymbol{\theta}} \times \overline{\boldsymbol{\omega}} + (\nabla_{\boldsymbol{r}} \overline{\boldsymbol{v}}) \cdot \overline{\boldsymbol{\theta}}. \tag{2.96}$$

The proof is complex and the advice given in [81] (pp. 121) or in [31] (Section 3) sometimes does not provide sufficient detail and is formulated for arbitrary curvilinear coordinate systems so that co-/contravariant notation becomes necessary. This introduces another complication. The additional notes in [81], Appendix D.1.2 continue using curvilinear coordinates and quantities of the reference configuration—namely the right microangular velocity vector $\boldsymbol{\Omega}$—that will be discussed later in Eq. (3.5). We therefore decided to present the proof here in a slightly different way.

We first investigate a derivate of Eq. (2.88) ([81], pg. 121) by redefining this relation in terms of total derivatives in time and using an auxiliary field $\overline{\boldsymbol{\omega}}'$ for adjustment:

$$\frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t} = \overline{\boldsymbol{\omega}}' \times \overline{\boldsymbol{P}} \quad \Leftrightarrow \quad \overline{\boldsymbol{\omega}}' = -\frac{1}{2} \left[\frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} \right]_{\times}. \tag{2.97}$$

Moreover, in analogy to this relation—but now for space derivatives—we introduce three auxiliary *vectors* $\boldsymbol{\phi}_i = \overline{\boldsymbol{\phi}}_i(\boldsymbol{r}, t)$, $i = 1, 2, 3$ by:

$$\frac{\partial \overline{\boldsymbol{P}}}{\partial r_i} = \overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{P}} \quad \Leftrightarrow \quad \overline{\boldsymbol{\phi}}_i = -\frac{1}{2} \left[\frac{\partial \overline{\boldsymbol{P}}}{\partial r_i} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} \right]_{\times}. \tag{2.98}$$

Using all of this and because of $\left(\overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{P}}\right)^{\mathsf{T}} = -\overline{\boldsymbol{P}}^{\mathsf{T}} \times \overline{\boldsymbol{\phi}}_i$ and the interchangebility of total time derivative and gradient operation, Eq. $(2.27)_1$, we arrive at:

$$\begin{aligned}
\frac{\partial \overline{\boldsymbol{\omega}}'}{\partial r_i} &= -\frac{1}{2} \left[\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\partial \overline{\boldsymbol{P}}}{\partial r_i} \right) \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} + \frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t} \cdot \left(\frac{\partial \overline{\boldsymbol{P}}}{\partial r_i} \right)^{\mathsf{T}} \right]_{\times} = \\
&\quad -\frac{1}{2} \left[\left(\frac{\mathrm{d}}{\mathrm{d}t} \left(\overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{P}} \right) \right) \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} + \frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t} \cdot \left(\overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{P}} \right)^{\mathsf{T}} \right]_{\times} = \\
&\quad -\frac{1}{2} \left[\left(\frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} \right) \times \overline{\boldsymbol{P}} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} + \overline{\boldsymbol{\phi}}_i \times \frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t} \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} - \left(\overline{\boldsymbol{\omega}}' \times \overline{\boldsymbol{P}} \right) \cdot \overline{\boldsymbol{P}}^{\mathsf{T}} \times \overline{\boldsymbol{\phi}}_i \right]_{\times} = \\
&\quad -\frac{1}{2} \left[\frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} \times \mathbf{1} + \overline{\boldsymbol{\phi}}_i \times \left(\overline{\boldsymbol{\omega}}' \times \mathbf{1} \right) - \left(\overline{\boldsymbol{\omega}}' \times \mathbf{1} \right) \times \overline{\boldsymbol{\phi}}_i \right]_{\times}.
\end{aligned} \tag{2.99}$$

Now various auxiliary formulae hold:

$$-\frac{1}{2} \left[\frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} \times \mathbf{1} \right]_{\times} = \frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t}, \quad -\left[\overline{\boldsymbol{\phi}}_i \times \left(\overline{\boldsymbol{\omega}}' \times \mathbf{1} \right) \right]_{\times} = \overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{\omega}}', \quad \left[\left(\overline{\boldsymbol{\omega}}' \times \mathbf{1} \right) \times \overline{\boldsymbol{\phi}}_i \right]_{\times} = \overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{\omega}}'. \tag{2.100}$$

Therefore,

$$\frac{\partial \overline{\boldsymbol{\omega}}'}{\partial r_i} = \frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} + \overline{\boldsymbol{\phi}}_i \times \overline{\boldsymbol{\omega}}'. \tag{2.101}$$

[32] It should be noted that the tensor second rank $\hat{\boldsymbol{\theta}}$ is called $\boldsymbol{\gamma}^{\mathsf{T}}$ by Eringen [25], pg. 17. It must not be confused with Eringen's tensor of third rank $\boldsymbol{\gamma}$ (also in Eulerian material form), which will be introduced later in Eq. (3.19).

We now turn to the microangular velocity in context with the substantial derivative in the definition shown in Eq. (2.90) and combine it with the definition of the auxiliary vectors $\boldsymbol{\phi}_i$ in Eq. (2.98). In preparation of the final step in the proof we write:

$$\begin{aligned}\overline{\boldsymbol{\omega}} &= -\frac{1}{2}\left[\frac{\delta\overline{\boldsymbol{P}}}{\delta t}\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}\right]_{\times} = -\frac{1}{2}\left[\frac{\mathrm{d}\overline{\boldsymbol{P}}}{\mathrm{d}t}\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}\right]_{\times} - \frac{1}{2}\left(\left[\left(\overline{\boldsymbol{v}} - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{P}}\right]\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}\right)_{\times} = \\ &\overline{\boldsymbol{\omega}}' - \frac{1}{2}\left(\left[\left(\overline{\boldsymbol{v}} - \frac{\mathrm{d}\check{\boldsymbol{r}}(t)}{\mathrm{d}t}\right)\cdot\nabla_{\boldsymbol{r}}\overline{\boldsymbol{P}}\right]\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}\right)_{\times} = \\ &\overline{\boldsymbol{\omega}}' - \frac{1}{2}\left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\left[\frac{\partial\overline{\boldsymbol{P}}}{\partial r_j}\cdot\overline{\boldsymbol{P}}^{\mathsf{T}}\right]_{\times} = \overline{\boldsymbol{\omega}}' + \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\overline{\boldsymbol{\phi}}_j,\end{aligned} \tag{2.102}$$

and from (2.101) by using the same transformation as for the last expression of Eq. (2.102):

$$\frac{\partial\overline{\boldsymbol{\omega}}'}{\partial r_i} = \frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} + \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\omega}} - \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\phi}}_j. \tag{2.103}$$

Hence, because $\frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}$ is no field,

$$\begin{aligned}\frac{\partial\overline{\boldsymbol{\omega}}}{\partial r_i} &= \frac{\partial\overline{\boldsymbol{\omega}}'}{\partial r_i} + \frac{\partial\overline{v}_j}{\partial r_i}\overline{\boldsymbol{\phi}}_j + \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\frac{\partial\overline{\boldsymbol{\phi}}_j}{\partial r_i} = \\ &\frac{\mathrm{d}\overline{\boldsymbol{\phi}}_i}{\mathrm{d}t} + \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\omega}} - \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\phi}}_j + \frac{\partial\overline{v}_j}{\partial r_i}\overline{\boldsymbol{\phi}}_j + \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\frac{\partial\overline{\boldsymbol{\phi}}_j}{\partial r_i} = \frac{\delta\overline{\boldsymbol{\phi}}_i}{\delta t} - \\ &\left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\frac{\partial\overline{\boldsymbol{\phi}}_i}{\partial r_j} + \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\omega}} - \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\phi}}_j + \frac{\partial\overline{v}_j}{\partial r_i}\overline{\boldsymbol{\phi}}_j + \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\frac{\partial\overline{\boldsymbol{\phi}}_j}{\partial r_i} = \\ &\frac{\delta\overline{\boldsymbol{\phi}}_i}{\delta t} + \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\omega}} + \frac{\partial\overline{v}_j}{\partial r_i}\overline{\boldsymbol{\phi}}_j + \left(\overline{v}_j - \frac{\mathrm{d}\check{r}_j(t)}{\mathrm{d}t}\right)\left\{\left[\frac{\partial\overline{\boldsymbol{\phi}}_j}{\partial r_i} - \frac{\partial\overline{\boldsymbol{\phi}}_i}{\partial r_j}\right] - \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\phi}}_j\right\} = \\ &\frac{\delta\overline{\boldsymbol{\phi}}_i}{\delta t} + \overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\omega}} + \left(\frac{\partial}{\partial r_i}\overline{\boldsymbol{v}}\right)\cdot\boldsymbol{e}_j\overline{\boldsymbol{\phi}}_j.\end{aligned} \tag{2.104}$$

We finally multiply this by the Cartesian unit vector $\boldsymbol{e}_i$ from the left to obtain Eq. (2.96) if we define $\overline{\boldsymbol{\theta}} = \boldsymbol{e}_k\overline{\boldsymbol{\phi}}_k$.

For completeness, we remark that an auxiliary formula was required in the last step of Eq. (2.104), which is known as Cartan's first structural equation:

$$\overline{\boldsymbol{\phi}}_i\times\overline{\boldsymbol{\phi}}_j = \frac{\partial\overline{\boldsymbol{\phi}}_j}{\partial r_i} - \frac{\partial\overline{\boldsymbol{\phi}}_i}{\partial r_j}. \tag{2.105}$$

Its proof can be found in [81], pp. 467.

3 The various strain measures of micropolar theory

3.1 Introductory remarks

In this section we will discuss in great detail deformation measures for micropolar matter from Eringen's and Eremeyev's work. There are many different ones, but they are all based in the world of material description, specifically Lagrangian and Eulerian material description. We will compare them to the deformation measures of true spatial description, of which there are not that many. In preparation, we wish to remind the reader of the following:

- Pietraszkiewicz and Eremeyev's work (for example [62], [63], [64], [17], [19]) focuses on solids and uses Lagrangian material notation. Eringen's work is relevant to solids and fluids (for example his monographs [25], [26]) and uses either Lagrangian or Eulerian material description. Both rely on the notions of material volume, indestructible material particle, reference and current placements. For example Sections 1.1/1.2 in [25] or 1.1 in [27] mention explicitly the concept of a "material point." Similarly in [17] we find these notions on pg.11, in [62] on pg. 776, in [64] on pg. 80, and in [16] on pg. 2. The term "material body" appears on pg. 15 of [17]. On the other hand, the terms "spatial frame" or just "spatial" can also be found in Eringen's work, namely in [25], *e.g.*, in 1.1 and in [27], *e.g.* 2.1. However, what is meant here is the spatial description that was described in Section 2.1, with an indestructible material particle passing by at the space position $\boldsymbol{x}$ at time t: Eringen thinks in terms of the Eulerian material description of fields at the current time.
- The differences between material Eulerian and a true spatial description were explained in Sections 2.2 and 2.3. Both are easy to confuse, since their mathematical formulation looks very much alike (see Eqns. $(2.48)_{(3,4)}$ *vs.* (2.18), (2.19)). But from a conceptionally point-of-view they have a completely different meaning. It is fair to say that the Lagrangian material description is the preferred way of analyzing problems of solid mechanics, whereas in the world of fluids without microstructural change a spatial description in terms of a Eulerian material description is to be preferred. Let us point out that, in principle, either description could be used to solve solid or fluid problems, although the mathematical solution might become cumbersome if the formulation of choice is not used. However, if it comes to problems with microstructural change, such that agglomeration or crushing of matter is an issue, the indestructible particle loses its meaning, and the concept of true spatial description is the only way out.
- Nevertheless, even the differences between material Eulerian and a true spatial description can lead to serious misunderstandings and scholarly disputes, for example, in context with the concept of reference configuration. Here, we have already mentioned the famous controversy of Eringen-Lee-Parodi *vs.* Ericksen-Leslie on nematic liquid crystals. In the context of this section the following issues are particularly important: (a) Eringen eventually concedes that nematic crystals require a framework pertinent to fluid mechanics. For example in [21] he says: "In contrast to our previous work, the approach here is novel in that the theory is hydrodynamic in character at the start, and no need arises for the use of the concept of the natural state hypothesis which seems to have caused some confusion in its interpretations (cf. Shahinpoor [1975], Lee and Eringen [1975])." But objectively speaking he still starts from (b) strain measures of the solids world pertinent to the reference configuration which he then transfers to what he calls "spatial deformation measures" (Section 2 of [21]). In fact, his new approach is only slightly better than his original one in [49], where deformation measures of the reference configuration and their application to what he calls "concept of the reference state" [50] are paramount. In fact, in his later work [24] we see the solid mechanics idea of reference and current placement and its application to liquid crystal flow resurrecting again.
- Regarding the St. Petersburg schools the following can be said. The work of Aero *et al.* ([3], [2]) does not stress the notion "spatial description" and its corresponding deformation measures particularly, but simply refer to it as "asymmetric hydrodynamics." It is fair to say that they do not speak of micropolar theory in these articles and that they refer to Ericksen-Leslie's work a lot. Eringen is not mentioned there. However, in contrast to Ericksen and Leslie they have a complete spin balance and their microinertia tensor is not the one for a digit. The work of Aero and colleagues also covers solids, but only for small deformations. Hence, the machine of nonlinear continuum mechanics with reference and current placement is not required [1], [60]. They do rely on the concept of an indestructible material particle, though. However, in the work of Zhilin and disciples true spatial description and the corresponding nonlinear deformation measures are brought to a climax: It starts in [81], Section 2.3 and carries on in recent work *e.g.*, [42],[38], [43], [40], [41].

3.2 Deformation measures for micropolar solids – Material description

In this section we shall review the various deformation measures from the literature that uses the concept of an indestructible material particle. Both formulations are possible, Lagrangian as well as Eulerian material notation, as outlined in Section 2.3. We will not use tildes and hats for the two ways of field description nor arguments, unless it is required for a better understanding.

From the mechanics perspective, the primary kinematic variables for micropolar solids are (large) displacements $\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X}$ (*cf.*, Eq. (2.5)) and the (large) microrotation vector $\boldsymbol{\varphi} = \varphi \boldsymbol{m}$. The latter is related to

the microrotation tensor $\boldsymbol{P}$ by the Euler-Rodrigues formula:

$$\boldsymbol{P} = (1 - \cos\varphi)\, \boldsymbol{m}\boldsymbol{m} + \cos\varphi\, \mathbf{1} + \sin\varphi\, \boldsymbol{m} \times \mathbf{1}, \tag{3.1}$$

where $\boldsymbol{m}$ is the (space- and time-dependent) unit vector about which the angle φ is turned. Confusion may arise in context with the direction of $\boldsymbol{m}$. Here we follow Eringen [25], pg. 9[33], Zhilin [81], pg. 94, or Pietraszkiewicz and Eremeyev [63], [64], whereas Eremeyev *et al.* [17], pg. 88 use the opposite direction. Various vectors of rotations within the framework of micropolar continua were discussed in [63]. It should also be noted that Eringen uses the symbol $\boldsymbol{\chi}$, pg. 12, and Eremeyev *et al.*, pg. 11, the symbol $\boldsymbol{H}$ for the microrotation tensor. Recall Eq. (2.73) for which provide an interpretation: intuitively speaking, $\boldsymbol{P}$ is a proper rotation tensor (det $\boldsymbol{P} = +1$) and turns the orthogonal unit director triad of the reference configuration, $\boldsymbol{D}_k$, onto the current one, $\boldsymbol{d}_k$:[34]

$$\boldsymbol{d}_i = \boldsymbol{P} \cdot \boldsymbol{D}_i \quad \Leftrightarrow \quad \boldsymbol{P} = \boldsymbol{d}_k \boldsymbol{D}_k. \tag{3.2}$$

The last equation already indicates that $\boldsymbol{P}$ is an object of two worlds: the $\boldsymbol{D}_i = \boldsymbol{D}_i(\boldsymbol{X})$ (in Lagrangian material description) represent the reference configuration and the $\boldsymbol{d}_i = \hat{\boldsymbol{d}}_i(\tilde{\boldsymbol{x}}(\boldsymbol{X}, t), t)$ (in Eulerian material description) the current one. Hence $\boldsymbol{P}$'s first leg is in the current and its second in the reference placement. We can see this explicitly if we turn to the index notation for $\boldsymbol{P}$, in order to achieve conformity with Eringen's notation [25], Section 1.4. We transform:

$$\boldsymbol{P} = \boldsymbol{d}_k \cdot \mathbf{1}\boldsymbol{D}_k \cdot \mathbf{1} = \boldsymbol{d}_k \cdot (\boldsymbol{i}_i \boldsymbol{i}_i)\, \boldsymbol{D}_k \cdot (\boldsymbol{I}_I \boldsymbol{I}_I) = (\boldsymbol{d}_k \cdot \boldsymbol{i}_i)\, (\boldsymbol{D}_k \cdot \boldsymbol{I}_I)\, \boldsymbol{i}_k \boldsymbol{I}_I, \tag{3.3}$$

and write $\boldsymbol{P} = P_{iI} \boldsymbol{i}_k \boldsymbol{I}_I$ after identifying:[35]

$$P_{iI} = (\boldsymbol{d}_k \cdot \boldsymbol{i}_i)\, (\boldsymbol{D}_k \cdot \boldsymbol{I}_I). \tag{3.4}$$

We emphasize: the use of capital indices allows to identify the occurrence of the reference configuration in a tensor easily. In fact, this convention was used by Eringen throughout his text [25]. Now, in hindsight, the use of capital indices in Eq. (2.74) and of small letters for the left spin tensor $\boldsymbol{S}$ and in Eq. (2.76) for the left microangular velocity vector $\boldsymbol{\omega}$ becomes clear.[36] We conclude that $\boldsymbol{S}$ is an object of the current placement and so is $\boldsymbol{\omega}$.[37] Its defining equation (2.76) is also used in works of Eremeyev and coauthors, see [16]. However, in Eremeyev *et al.* [17], pg. 12 the *right angular velocity* (or microgyration vector) is introduced by

$$\boldsymbol{\Omega} = -\frac{1}{2}\left[\boldsymbol{P}^{\mathsf{T}} \cdot \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t}\right]_{\times} \quad \Leftrightarrow \quad \Omega_I = -\frac{1}{2}\epsilon_{IJK}\frac{\mathrm{d}P_{mJ}}{\mathrm{d}t}P_{mK}, \tag{3.5}$$

which means there they are using the *right spin tensor*, which is

$$\boldsymbol{S}^{\mathrm{r}} = \boldsymbol{P}^{\mathsf{T}} \cdot \frac{\mathrm{d}\boldsymbol{P}}{\mathrm{d}t} \quad \Leftrightarrow \quad S^{\mathrm{r}}_{IJ} = P_{mI}\frac{\mathrm{d}P_{mJ}}{\mathrm{d}t}. \tag{3.6}$$

Clearly, both objects are completely in the reference configuration. In [17] it is not discussed why this choice was made. However, in Zhilin [80], pp. 198 it is argued at length[38] that the use of a right angular velocity must be viewed in context with the concept of the reference state. It should be pointed out that this strict distinction between left and right microangular velocities is sometimes not mentioned, for example, [17], pg. 12.

Moreover, from their definitions it is really easy to see that both spin tensors are related by:

$$\boldsymbol{S} = \boldsymbol{P} \cdot \boldsymbol{S}^{\mathrm{r}} \cdot \boldsymbol{P}^{\mathsf{T}} \quad \Leftrightarrow \quad S_{ij} = P_{iI} P_{jJ} S^{\mathrm{r}}_{IJ}. \tag{3.7}$$

In fact, as a mnemonic, these relations obey formally the same transformation laws for objective tensors of second rank.

[33] $\boldsymbol{m} \times \mathbf{1} = -\epsilon_{klm} m_m \boldsymbol{e}_k \otimes \boldsymbol{e}_l$ in order to pick up the index form in [25]

[34] Throughout this paper Einstein's summation convention applies when needed.

[35] To repeat a previous footnote: the base $\boldsymbol{I}_I$ of the reference configuration can coincide with the current base $\boldsymbol{i}_i$. Here it was written like that to point out where the capital and small indices in $\widetilde{F}_{iJ}$ come from.

[36] For the notions "left and right spin tensors" see [80]. pg. 197; Eringen [25], pp. 24 uses the symbol $\boldsymbol{v}$ for the left spin tensor and calls it microgyration tensor, the right spin tensor does not occur.

[37] Also denoted by $\boldsymbol{v}$ in Eringen's work and called microrotation vector.

[38] To be precise: The differences are discussed in context with rigid body movement but not for micropolar media.

According to Eringen [25], pg. 14 the strain measure involving only the displacement $\boldsymbol{u}$ (or, if one so wishes, the deformation at the macrolevel) coincides with the strain measures of the classical continuum, namely the *right Cauchy-Green tensor*, another quantity fully in the reference placement:

$$\boldsymbol{C} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{F} \quad \Leftrightarrow \quad C_{IJ} = F_{kI} F_{kJ}, \tag{3.8}$$

where the deformation gradient is defined in the usual manner as a right gradient with respect to the reference configuration $\boldsymbol{X}$:

$$\boldsymbol{F} = \boldsymbol{x} \nabla_{\boldsymbol{X}} \equiv (\nabla_{\boldsymbol{X}} \boldsymbol{x})^{\mathsf{T}} \quad \Leftrightarrow \quad F_{iJ} = \frac{\partial x_i}{\partial X_J} \equiv \frac{\partial u_i}{\partial X_J} + \delta_{iJ}. \tag{3.9}$$

Clearly the deformation gradient has legs in both worlds. However, recall that there is an object corresponding to Eq. (3.8) exclusively in the current placement, namely the *Piola* or *Finger strain tensor*[39]

$$\boldsymbol{f} = \boldsymbol{F}^{-\mathsf{T}} \cdot \boldsymbol{F}^{-1} \quad \Leftrightarrow \quad f_{ij} = F^{-1}_{Mi} F^{-1}_{Mj} \equiv \frac{\partial X_M}{\partial x_i} \frac{\partial X_M}{\partial x_j}. \tag{3.10}$$

We introduce it in addition to $\boldsymbol{C}$ because of similar strain measures to be found in Eringen [25], which connect the macro- and microdeformations, both fully in the reference or in the current placement. We turn to them now.

For the micropolar continuum two additional strain measures are relevant, the so-called deformation tensor [25], also known as *Lagrangian stretch tensor* [62], $\mathfrak{C}$:

$$\mathfrak{C} = \boldsymbol{F}^{\mathsf{T}} \cdot \boldsymbol{P} \quad \Leftrightarrow \quad \mathfrak{C}_{IJ} = F_{kI} P_{kJ}. \tag{3.11}$$

A comparison with Eq. (3.8) shows that it is analogously defined to $\boldsymbol{C}$, if we only replace the deformation gradient $\boldsymbol{F} \rightarrow \boldsymbol{P}$. This also explains the choice of symbol. In addition one may say that $\mathfrak{C}$ combines macro- and microdeformation, due to the involvement of $\boldsymbol{F}$ *and* $\boldsymbol{P}$. It should also be pointed out that there are many surrogate deformation measures for $\mathfrak{C}$. In this context Table 1 in [62] is worthwhile consulting. Moreover, in Eremeyev *et al.* [17] we also find what they call the *Lagrangian stretch tensor* for which the unfortunate[40] symbol $\boldsymbol{E}$ was chosen:

$$\boldsymbol{E} = \boldsymbol{P}^{\mathsf{T}} \boldsymbol{F} - \boldsymbol{1} \equiv \mathfrak{C}^{\mathsf{T}} - \boldsymbol{1} \quad \Leftrightarrow \quad E_{IJ} = P_{mI} F_{mJ} - \delta_{IJ}, \tag{3.12}$$

also completely based in the reference configuration.

At this point, a remark is in order: we want to call a *deformation measure* a quantity that turns into a unit tensor if evaluated in the reference configuration. In this sense, the stretch tensor does not comply because it would then become the zero tensor.

Finally, a strain measure analogous to $\mathfrak{C}$ but in the current placement and in material Eulerian description was also defined in Eringen [25], pg. 15:

$$\mathfrak{c} = \boldsymbol{F}^{-\mathsf{T}} \cdot \boldsymbol{P}^{\mathsf{T}} \quad \Leftrightarrow \quad \mathfrak{c}_{ij} = F^{-1}_{Mi} P_{jM}, \tag{3.13}$$

which corresponds to $\boldsymbol{f}^{\mathsf{T}}$ if we replace in there $\boldsymbol{F}^{-1} \rightarrow \boldsymbol{P}^{-1} \equiv \boldsymbol{P}^{\mathsf{T}}$. It should also be mentioned that Eremeyev *et al.* [17], pg. 40 favor for the current Eulerian material description:

$$\boldsymbol{e} = \boldsymbol{P} \cdot \boldsymbol{E} \cdot \boldsymbol{F}^{-1} = \boldsymbol{1} - \boldsymbol{P}\boldsymbol{F}^{-1} = \boldsymbol{1} - \mathfrak{c}^{\mathsf{T}}. \tag{3.14}$$

In context with the deformation of fluid matter the velocity gradient, the symmetric part of which is also known as strain rate tensor, is of great importance. It is a deformation measure in the current configuration where all fields are in Eulerian material description, for example, $\boldsymbol{F} = \hat{\boldsymbol{F}}\left(\tilde{\boldsymbol{x}}\left(\boldsymbol{x}, t\right), t\right)$:

$$\nabla_{\boldsymbol{x}} \boldsymbol{v} = \boldsymbol{F}^{-\mathsf{T}} \cdot \frac{\mathrm{d}\boldsymbol{F}^{\mathsf{T}}}{\mathrm{d}t} \equiv -\frac{\mathrm{d}\boldsymbol{F}^{-\mathsf{T}}}{\mathrm{d}t} \cdot \boldsymbol{F}^{\mathsf{T}} \quad \Leftrightarrow \quad \frac{\partial v_i}{\partial x_j} = \frac{\mathrm{d}F_{iM}}{\mathrm{d}t} F^{-1}_{Mj}. \tag{3.15}$$

[39] There is also the right Cauchy-Green tensor, $\boldsymbol{B} = \boldsymbol{F} \cdot \boldsymbol{F}^{\mathsf{T}} = F_{iM} F^{-1}_{jM} \boldsymbol{i}_i \otimes \boldsymbol{i}_j$, but it is not so useful in comparison with Eringen's deformation measure for micropolar media shown in Eq. (3.13). Note that $\boldsymbol{f}$ is the inverse of $\boldsymbol{B}$.

[40] By tradition $\boldsymbol{E}$ is reserved for the Green-St.Venant strain tensor of conventional continua, $\boldsymbol{E} = \frac{1}{2}\left(\boldsymbol{C} - \boldsymbol{1}\right)$.

The second deformation measure pertinent to micropolar continua is the so-called *wryness tensor*. It is originally a tensor of third order and concerns exclusively microquantities, namely the microrotation and its gradient:

$$\boldsymbol{\Gamma} = \boldsymbol{P}^{\mathsf{T}} \cdot (\boldsymbol{P}\nabla_{\boldsymbol{X}}) \quad \Leftrightarrow \quad \Gamma_{KLM} = P_{mK}\frac{\partial P_{mL}}{\partial X_M}. \tag{3.16}$$

Because of $\boldsymbol{P}^{\mathsf{T}} \cdot \boldsymbol{P} = \mathbf{1}$ it is antisymmetric with respect to indices K and L or in the wording of direct notation with respect to its first two tensor places:

$$\Gamma_{KLM} = -\Gamma_{LKM}. \tag{3.17}$$

Hence it has only 9 independent components and can be mapped onto a fully populated tensor of second rank, $\boldsymbol{\Gamma}$, its tensor invariant with respect to its first two tensor places (see Eringen [25], pg. 16[41]), which is also an object fully within the reference configuration:

$$\boldsymbol{K} = \boldsymbol{\Gamma}_{\times} \equiv -\frac{1}{2}\big[\boldsymbol{P}^{\mathsf{T}} \cdot (\boldsymbol{P}\nabla_{\boldsymbol{X}})\big]_{\times} \quad \Leftrightarrow \quad K_{IJ} = \frac{1}{2}\epsilon_{IKL}\frac{\partial P_{mK}}{\partial X_J}P_{mL}. \tag{3.18}$$

Eringen also defines a wryness tensor completely based in the current placement as a material Eulerian tensor field of third rank:

$$\boldsymbol{\gamma} = \boldsymbol{P} \cdot \big(\boldsymbol{P}^{\mathsf{T}}\nabla_{\boldsymbol{x}}\big) \quad \Leftrightarrow \quad \gamma_{klm} = P_{kM}\frac{\partial P_{lM}}{\partial x_m}. \tag{3.19}$$

Because of $\boldsymbol{P} \cdot \boldsymbol{P}^{\mathsf{T}} = \mathbf{1}$ this quantity is antisymmetric with respect to k and l,

$$\gamma_{klm} = -\gamma_{lkm}, \tag{3.20}$$

and it can be replaced by the second rank wryness tensor $\boldsymbol{\theta}$ from Eq. (2.94) as follows:

$$\boldsymbol{\theta}^{\mathsf{T}} = \boldsymbol{\gamma}_{\times} \equiv -\frac{1}{2}\big[\boldsymbol{P} \cdot \big(\boldsymbol{P}^{\mathsf{T}}\nabla_{\boldsymbol{x}}\big)\big]_{\times} \quad \Leftrightarrow \quad \theta_{ij} = \frac{1}{2}\epsilon_{jkl}P_{kM}\frac{\partial P_{lM}}{\partial x_i}. \tag{3.21}$$

It is noteworthy that Eremeyev *et al.* [17], pg. 40 use in the current configuration instead:

$$\boldsymbol{k} = \boldsymbol{P} \cdot \boldsymbol{K} \cdot \boldsymbol{F}^{-1} = \boldsymbol{P} \cdot \left\{-\frac{1}{2}\big[\boldsymbol{P}^{\mathsf{T}} \cdot (\boldsymbol{P}\nabla_{\boldsymbol{x}})\big]_{\times}\right\} \quad \Leftrightarrow \quad k_{ij} = -\frac{1}{2}P_{iI}\epsilon_{IKL}P_{lK}\frac{\partial P_{lL}}{\partial x_j}. \tag{3.22}$$

3.3 Deformation measures for micropolar solids – True spatial description

Several fundamental quantities for deformation measures of micropolar continua in true spatial formulation have been formally defined already, namely the (formally introduced) displacement $\boldsymbol{u} = \overline{\boldsymbol{u}}(\boldsymbol{r}, t)$, the spatial deformation gradient $\boldsymbol{g} = \overline{\boldsymbol{g}}(\boldsymbol{r}, t)$), and the wryness tensor $\boldsymbol{\theta} = \overline{\boldsymbol{\theta}}(\boldsymbol{r}, t)$ in Eqs. (2.33), (2.57), and (2.94), respectively. The latter is also a formally introduced quantity, because the microrotation tensor $\overline{\boldsymbol{P}}$ (obtainable through solution of Eq. (2.88)) and its gradient are used in its defining equation (2.94): we have $\overline{\boldsymbol{P}} = \overline{\boldsymbol{d}}_k\overline{\boldsymbol{D}}_k$, and the reference director triad $\overline{\boldsymbol{D}}_k$ could be obtained in true spatial description from the current ones through the rotation tensor $\overline{\boldsymbol{P}}$, as explained in Section 2.6. These quantities are then used to define deformation measures for micropolar media in true spatial description, the strain rate, $\nabla_{\boldsymbol{r}}\overline{\boldsymbol{v}}$, shown in Eq. (2.66) and the gradient of angular velocity, $\nabla_{\boldsymbol{r}}\overline{\boldsymbol{\omega}}$, from Eq. (2.96).

Note that the formulation is in terms of *large deformations*. For further reference of this nonlinear theory we cite [81], [31], [42], [38], [43], [40], [41]. The linearized theory is outlined in [37] and [39].

Also note the use of small indices in the index notation of all aforementioned quantities. They are all from the "here and now." A remark: In Eq. $(2.94)_2$ we have also used capital indices M. These are to be taken with a grain of salt: it was mentioned that the use of reference quantities—here the base vectors $\boldsymbol{I}_M$—is purely formal. All fields in these equations are functions of the true spatial coordinates $\boldsymbol{r}$ and t.

Now, in the case, where the moving observational point can be chosen as the motion of a material particle, both functions for $\overline{\boldsymbol{g}}$ and $\hat{\boldsymbol{g}}$ would be the same. The same applies to the wryness tensor $\bar{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\theta}}$, respectively. In this case we are shifting to Truesdell's relative description. Then $\dot{\boldsymbol{x}}$ is $\mathring{\boldsymbol{r}}$.

Suffice it to say that, in principle, with these quantities we could also define various other deformation measures analogous to those described in Section 3.2 for the case of material description. However, so far these were not used in the literature just cited.

[41] We use the symbol $\boldsymbol{K}$ proposed by Eremeyev *et al.* [17] and avoid Eringen's choice $\boldsymbol{\Gamma}$, which would lead to confusion.

3.4 Deformation measures for micropolar solids – On relative description

Similar to Truesdell's relative description [71] of simple materials, the relative description is also possible for micropolar media such as viscoelastic fluids and other inelastic media, see [5], [78], where various types of constitutive equations based on the strain rate tensors were discussed.

4 Conclusions

In this paper the following was achieved:

- The notion of *true spatial description* was introduced and carefully distinguished from what is usually called spatial description or Eulerian description. Examples from the literature for its use were provided. The various deformation measures for micropolar media in material description for current and reference placement were reviewed and set against deformation measures useful in a true spatial description.
- The presented results could be useful for describing the interactions of highly deformed materials, such as elastomeric foams, with gases and fluids. The application of micropolar elasticity to foams is well-documented in the literature (see, for example, [17], [48]).
- Other applications include particle alignment in an electric or magnetic field [42], [73] or fiber suspensions, where fibers align within a fluid and rotary inertia plays an important role [6].
- Furthermore, some lattice or origami-type materials require enhanced continuum models, such as micropolar elasticity (see, for example, [8], [15]). Similar applications exist for thin-walled structures modeled within the micropolar approach. In fact, a shell with six parameters, also known as a six-parametric shell, is kinematically equivalent to a two-dimensional micropolar continuum [17]. The Eulerian description of Kirchhoff-Love nonlinear shells was developed in [83], [84] and extended to micropolar shells in [85].
- Finally, our results may also be useful for modeling materials with movable nonmaterial surfaces, such as phase boundaries, shock waves, and other moving defects.
- We believe that the introduced description and corresponding deformation measures could also be extended to higher-order micropolar measures and other generalized media, such as micromorphic, strain-gradient, and nonlocal elasticity.

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