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Adaptive sliding mode boundary control of a perturbed diffusion process

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Summary
This paper proposes a sliding-mode-based adaptive boundary control law for stabilizing a class of uncertain diffusion processes affected by a matched disturbance. The matched disturbance is assumed to be uniformly bounded along with its time derivative, whereas the corresponding upper bounding constants are not known. This motivates the use of adaptive control strategies. In addition, the spatially-varying diffusion coefficient is also uncertain. To achieve asymptotic stability of the plant origin in the L_2-sense in the presence of the disturbance, a discontinuous boundary feedback law is proposed where the gain of the discontinuous control term is adjusted according to a gradient-based adaptation law. A constructive Lyapunov analysis supports the stability properties of the considered closed-loop system, yielding sufficient convergence conditions in terms of suitable inequalities involving the controllers' tuning parameters. Simulation results are presented to corroborate the theoretical findings.

KEYWORDS
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1 | INTRODUCTION

Many industrial processes are governed by partial differential equations (PDEs). Particularly, various classes of diffusion PDEs have been derived to characterize the dynamics of relevant engineering processes such as distillation processes,[1] tubular reactors,[2,3] Lithium-ion batteries,[4] and many others.

The majority of engineering processes modeled by PDEs can be controlled through manipulable signals acting at the boundary of the spatial domain. The most popular and powerful approach presently available in the literature to deal with the boundary control of PDEs is the so-called *backstepping*.[5–8]

In its original formulation, the backstepping approach requires the plant model to be perfectly known. However, physical system models are often affected by uncertainties, including both parametric uncertainty, that is, the imperfect knowledge of the model parameters, as well as the possible presence of exogenous disturbance signals.

To cope with parametric uncertainties, the backstepping method has been correspondingly revisited and *adaptive* backstepping boundary controllers have been developed for various classes of parabolic and non-parabolic PDEs with unknown parameters. An overview of the early results in the area of adaptive backstepping can be found in Reference 9, whereas more recent related contributions are, for example, References 10–13. In this recent literature not only uncertain destabilizing system parameters (such as the diffusion and reaction coefficients) are successfully managed but, in addition, an uncertain boundary input delay is also taken into account.

The rejection of exogenous matched disturbances with an uncertain shape goes beyond the capabilities of the backstepping method. In contrast, it can be achieved by resorting to discontinuous boundary control laws designed according to the sliding-mode paradigm (see e.g., References 14–17). Presently, the discontinuous control synthesis in the infinite-dimensional setting is well documented[3,18–20] and it is generally shown to retain the main robustness features as those possessed by its finite-dimensional counterparts. To asymptotically stabilize open-loop unstable reaction-diffusion processes while also asymptotically rejecting a uniformly bounded matched disturbance (with an upper bound known in advance), backstepping can profitably be combined with the sliding mode control approach as done, for example, in Reference 21.

Conventional sliding-mode-based controllers are generally insensitive to matched disturbances provided that these exogenous signals and/or their time derivatives are uniformly bounded and provided that the corresponding bounding constants are known in advance in such a way that the gain parameters of the discontinuous controller can be tuned accordingly.

In the present paper, we aim to provide the asymptotic rejection of a matched boundary disturbance with an *unknown upper bound*, which demands adaptive sliding-mode boundary control techniques. In References 22–24 similar disturbance rejection problems were addressed by sliding mode control techniques in the finite-dimensional setting where an ODE models the process to be controlled. In the above references, two distinct types of adaptation mechanisms were considered. In References 22,23 the authors suggest a gradient-based and monodirectional adaptation law where the adaptive control gain can only increase over time until it asymptotically converges to a constant value. This method provides the asymptotic rejection of the bounded disturbance. In Reference 24 the adaptation method is bidirectional in that the adaptive gain can both increase and decrease over time, which is an advantage in implementation as it demands less control effort and may reduce the chattering phenomenon. This approach, however, can only provide a finite, although arbitrarily large, level of disturbance attenuation.

In Reference 25 a sliding mode-based adaptive boundary control design was investigated for a class of distributed-parameter systems (governed by flexible string or Euler-Bernoulli beam PDEs, as opposed to the diffusion PDE considered in the present work) subject to an exogenous disturbance whose magnitude is uniformly bounded by an unknown constant. In Reference 26 the more complicated scenario of a coupled PDE/ODE system modeling the vibrations occurring in a spacecraft was tackled within the same methodological framework. Both papers[25,26] considered the previously mentioned bidirectional adaptation method, yielding, as in the finite-dimensional setting of Reference 24, an arbitrarily large level of disturbance attenuation in that the norm of the closed-loop trajectories is only guaranteed to reach a vicinity of zero.

It should be also noted that a gradient-based adaptive unit-vector controller was suggested in Reference 27 to reject a distributed disturbance acting on the entire spatial domain of a diffusion PDE and having an unknown upper bound to its L_2-norm. However, in Reference 27 a *distributed* controller was involved, which is less relevant from the practical viewpoint compared to the boundary controller under investigation in the present work.

In the present paper, the system under consideration is a diffusion process with a spatially-varying uncertain diffusion coefficient. The manipulable control is of Neumann type and it is applied through one of the boundaries of the spatial domain. At the same controlled boundary, a matched disturbance is affecting the system's dynamics. The only measured signal to be used for feedback is the state at the controlled boundary. The magnitudes of the disturbance and of its time derivative are supposed to be uniformly bounded but the corresponding upper bounding constants are both *unknown*.

For this class of systems, a sliding-mode-based adaptive boundary feedback is considered. The proposed boundary control law has two components: a proportional term with constant gain, and a discontinuous switching term having an adaptive time-varying gain that is adjusted on-line. The adaptation algorithm is gradient-based and monodirectional as that used in References 22,23.

The application of the gradient-based adaptation method in the framework of the boundary control of distributed-parameter systems raises nontrivial challenges in developing the appropriate Lyapunov-based stability analysis. A constructive Lyapunov-based convergence proof is developed which yields the appropriate tuning conditions for the controller parameters and demonstrates the global asymptotic stability of the plant origin in the L_2-sense. Some terms of the Lyapunov functions employed in the convergence analysis feature a rather unconventional form, and their finding represents one of the major challenges overcome in the present work.

The paper is structured as follows. Section 2 describes the system under consideration and states the control problem at hand. In Section 3 the proposed adaptive boundary controller is presented and the well-posedness and stability properties of the closed-loop system are investigated via Lyapunov analysis. Simulation results corroborating the demonstrated properties of the closed-loop system are discussed in Section 4 and concluding remarks and perspectives for future research activities are given in Section 5.

1.1 Notation

The notation used throughout the paper is fairly standard. $H^{\ell}(0,1)$, with $\ell = 0, 1, 2, \ldots,$ denotes the Sobolev space of absolutely continuous scalar functions $f(\zeta)$ on $(0,1)$ with square integrable derivatives $f^{(i)}(\zeta)$ up to the order ℓ and the H^{ℓ}-norm

$$\|f(\cdot)\|_{H^{\ell}} = \sqrt{\int_0^1 \sum_{i=0}^{\ell} [f^{(i)}(\zeta)]^2 \, \mathrm{d}\zeta}. \tag{1}$$

We shall also utilize the standard notations $H^0(0,1) = L_2(0,1)$ and

$$\|f(\cdot)\|_{H^0} = \|f(\cdot)\|_{L_2}. \tag{2}$$

The function $\text{sign}\{\cdot\}$ represents the multi-valued function $\text{sign}\{z\} : \mathbb{R} \to [-1,1]$ such that

$$\text{sign}\{z\} \in \begin{cases} \{1\} & \text{for } z > 0 \\ [-1,1] & \text{for } z = 0 \\ \{-1\} & \text{for } z < 0 \end{cases}. \tag{3}$$

2 PROBLEM FORMULATION

Consider the space- and time-varying scalar field $z(x,t)$, evolving in the space $L_2(0,1)$, with the spatial variable $x \in [0,1]$ and time variable $t \geq 0$. Let it be governed by the perturbed boundary-value problem (BVP)

$$z_t(x,t) = [\theta(x) z_x(x,t)]_x, \tag{4a}$$

$$z_x(0,t) = -[u(t) + \psi(t)], \tag{4b}$$

$$z_x(1,t) = 0 \tag{4c}$$

of Neumann type, where $\theta(x)$ is the spatially-varying diffusion coefficient, and with initial condition

$$z(x,0) = z_0(x) \in L_2(0,1). \tag{5}$$

The variable $u(t)$ is the manipulable boundary control signal, applied through the Neumann-type boundary condition (4b). The matched boundary disturbance $\psi(t)$ and its time derivative are of class L_∞ and assumed to be uniformly bounded according to the following

Assumption 1. There are unknown constants Φ and Φ_d such that

$$|\psi(t)| \leq \Phi, \tag{6}$$

$$|\dot{\psi}(t)| \leq \Phi_\mathrm{d}. \tag{7}$$

Since the two constants Φ and Φ_d involved in the inequalities (6) and (7) are supposed to be unknown, an adaptive and robust control strategy is needed in order to achieve disturbance rejection.

The spatially-varying system's diffusivity coefficient $\theta(x)$, which is also unknown, is assumed to be of class C^1 and positive everywhere. The following assumption on lower and upper bounds to the function $\theta(x)$ is thus made.

Assumption 2. There exist unknown constants θ_m and θ_M such that

$$0 < \theta_m \le \theta(x) \le \theta_M \quad \forall x \in [0,1]. \tag{8}$$

The control goal is that of designing the boundary control signal $u(t)$ to guarantee the asymptotic stability of the underlying uncertain and perturbed closed-loop system. The meaning of the boundary-value problem (4a)–(5) in the closed-loop is specified in the sequel.

3 | CONTROLLER SYNTHESIS

In Reference 16 a similar control goal is attained by a control law consisting of a proportional and a discontinuous part, where the discontinuous control component is needed to achieve robustness against disturbances. Notice that in Reference 16 it was addressed the simplified scenario where the magnitude of the matched disturbance is uniformly bounded by an a priori known constant.

To deal with the unknown bounds for the magnitude of the matched disturbance $\psi(t)$ and of its time derivative, as specified in the Assumption 1, in the present paper an extension to this control law is made by allowing the gain of the discontinuous part to be time-varying, that is,

$$u(t) = -kz(0,t) - M(t)\text{sign}\{z(0,t)\}. \tag{9a}$$

The adaptive switching gain $M(t)$ evolves according to the adaptation law

$$\dot{M}(t) = \gamma|z(0,t)|, \tag{9b}$$

where the initial value $M(0) = M_0$ satisfies

$$M_0 \ge 0. \tag{10}$$

With a positive adaptation gain

$$\gamma > 0 \tag{11}$$

the right-hand side of (9b) is nonnegative whence it easily turns out that (9b)–(11) describe a *monodirectional* adaptation law, that is, the gain $M(t)$ is monotonically increasing. The growth rate of $M(t)$ depends on the parameter γ and the magnitude of the state at the boundary $x = 0$. Hence, adaptation stops when $|z(0,t)| = 0$. Since the control law (9a) feeds back $z(0,t)$, the overall dynamic controller (9a), (9b) only requires information of the state at the controlled boundary $x = 0$.

There are three controller parameters to be tuned, namely the initial value M_0 of the adaptive switching gain $M(t)$, the adaptation gain γ and the proportional gain k. It is later shown that the latter one is required to satisfy the following inequality:

$$k \ge \max\left\{\frac{1}{2}, \gamma\right\}. \tag{12}$$

3.1 | Well posedness of the closed loop system

The proposed control input (9a) undergoes discontinuities in the state subspace $z(0,t) = 0$. Similar to Reference 16, definition 1, the meaning of the closed-loop system (4a)–(5), driven by (9a)–(12) with the multi-valued input (3), is adopted in the weak sense beyond the discontinuity manifold $z(0,t)=0$, otherwise, it is viewed in the Filippov sense. In addition to Reference 16, the interested reader may also refer to Reference 20 for more details on weak and Filippov (sliding mode) solutions in the PDE setting.

Since the above closed-loop system is of class C^1 beyond its discontinuity manifold, it possesses a unique local weak solution once initialized with $z_0(x)$ such that $z_0(0)\neq 0$,[28] theorem 23.2. If a sliding mode occurs on the discontinuity manifold $z(0,t)=0$ then it is governed by the same PDE (4a) subject to the mixed-type boundary conditions

$$z(0,t)=0,\ z_x(1,t)=0, \tag{13}$$

since the Neumann-type boundary condition (4c) remains in force. In fact, the latter boundary condition is necessary for the sliding mode to exist in the closed-loop system in question, and it results in the boundary-value problem (4a), (13) which is well-recognized to be well-posed.

3.2 | Main result

The Lyapunov-based convergence analysis of the closed-loop system is presented in the following Theorem 1, which represents the main result of the present paper.

Theorem 1. *Consider PDE (4a) with boundary conditions (4b), (4c), initial condition (5), and let Assumptions 1 and 2be fulfilled. Let the system be controlled by the adaptive boundary control law (9a), (9b) and let the control parameters be tuned according to (10), (11) and (12). Then, the zero solution $z^*(x,t)=0$ is globally asymptotically stable in the $L_2(0,1)$-sense, that is,*

$$\lim_{t\to\infty}\|z(\cdot,t)-z^*(\cdot,t)\|_{L_2}=0 \quad \forall z_0(x). \tag{14}$$

Proof. The structure of the proof is divided into the following four steps: (step 1) It is preliminarily shown by Lyapunov analysis that the L_2-norm $\|z(\cdot,t)\|_{L_2}$ of the potential closed-loop trajectories and the adaptive switching gain $M(t)$ are both uniformly bounded. (step 2) Based on the uniform boundedness, the system is concluded to be forward complete in the sense that its solutions globally exist for all $t\geq 0$. By virtue of these results, a more elaborated Lyapunov analysis is subsequently developed (step 3) to show the uniform boundedness of $\int_0^1\theta(x)z_x^2(x,t)\,\mathrm{d}x$ and $|z(0,t)|$.

Finally, owing to the above boundedness results, we further develop (in step 4) the Lyapunov analysis eventually enabling the application of *Barbalat's* Lemma[29] to demonstrate the global asymptotic closed-loop stability in the $L_2(0,1)$-sense.

Step 1. Uniform boundedness of $\|z(\cdot,t)\|_{L_2}$ and $M(t)$

For the closed-loop system in question, consider the Lyapunov function candidate

$$V(t)=V_1(t)+V_2(t), \tag{15}$$

where

$$V_1(t)=\frac{1}{2}\int_0^1 z^2(x,t)\,\mathrm{d}x=\frac{1}{2}\|z(\cdot,t)\|_{L_2}^2 \tag{16}$$

and

$$V_2(t)=\frac{\theta(0)}{2\gamma}(M(t)-\Phi)^2. \tag{17}$$

Strictly speaking $V_1(t)$ is a functional but for simplicity it is referred to as a function. Furthermore, $V_1(z(\cdot,t))=V_1(t)$ is written for this and, analogously, for other functions.

According to (4a), and performing integration by parts, the time derivative of $V_1(t)$ along potential weak solutions of the closed-loop system beyond the discontinuity manifold $z(0,t)=0$ is manipulated as

$$\dot{V}_1(t) = \int_0^1 z(x,t)z_t(x,t)\,\mathrm{d}x = \int_0^1 z(x,t)[\theta(x)z_x(x,t)]_x\,\mathrm{d}x = \theta(x)z_x(x,t)z(x,t)|_0^1 - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x = \theta(1)z_x(1,t)z(1,t) - \theta(0)z_x(0,t)z(0,t) - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x. \tag{18}$$

By substituting the boundary conditions (4b), (4c) and the control law (9a) into the right-hand side of (18) one ends up with

$$\dot{V}_1(t) = -\theta(0)kz^2(0,t) - \theta(0)M(t)|z(0,t)| + \theta(0)\psi(t)z(0,t) - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x. \tag{19}$$

The time derivative of $V_2(t)$ is obtained by virtue of (9b), this yields

$$\dot{V}_2(t) = \theta(0)M(t)|z(0,t)| - \theta(0)\Phi|z(0,t)|. \tag{20}$$

Combining (19) and (20), performing further manipulations and taking advantage of (6) yields

$$\dot{V}(t) = \dot{V}_1(t) + \dot{V}_2(t) = -\theta(0)kz^2(0,t) + \theta(0)\psi(t)z(0,t) - \theta(0)\Phi|z(0,t)| - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x \le -\theta(0)kz^2(0,t) - \theta(0)[\Phi - |\psi(t)|]|z(0,t)| - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x \le 0. \tag{21}$$

Similar manipulations of the Lyapunov derivative along the potential sliding modes, governed by (4a), (13), result in $\dot{V}(t) \le 0$ as well. Indeed, specifying the right-hand side of (18) according to (13) yields

$$\dot{V}_1(t) = \theta(1)z_x(1,t)z(1,t) - \theta(0)z_x(0,t)z(0,t) - \int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x = -\int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x \tag{22}$$

whereas in light of (13) the right-hand side of (20) becomes identically zero. Thus, along the potential sliding modes, governed by (4a), (13) the Lyapunov derivative $\dot{V}(t)$ becomes

$$\dot{V}(t) = -\int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x \le 0. \tag{23}$$

Inequalities (21) and (23) imply that along potential weak solutions of the closed-loop system relation $\dot{V}(t) \le 0$ holds both beyond and along the discontinuity manifold $z(0,t)=0$. Hence, $V(t)$ is nonincreasing, which means that

$$0 \le V(t) \le V(0) \quad \forall t \ge 0. \tag{24}$$

From (24) and (15)–(17), and taking into account that

$$V_1(t) \le V(t) \le V(0), \tag{25}$$

$$V_2(t) \le V(t) \le V(0), \tag{26}$$

one straightforwardly derives the following uniform upper bounds for $\|z(\cdot,t)\|_{L_2}$ and $M(t)$

$$\|z(\cdot,t)\|_{L_2} \le \sqrt{2V(0)}, \tag{27}$$

$$M(t) \le \Gamma := \Phi + \sqrt{\frac{2\gamma}{\theta(0)}V(0)}. \tag{28}$$

Step 2. Forward completeness of the closed-loop system The closed-loop trajectories are uniformly bounded in the state space $L_2(0,1)\times\mathbb{R}$ because of the *a priori* estimate (24) of the Lyapunov function candidate (15)–(17). Hence, along with the Lyapunov function candidate, an arbitrary local solution of the closed-loop system admits an *a priori* estimate in the state space $L_2(0,1)\times\mathbb{R}$, too. Thus, such a solution is continuously extendible to the right for all $t\geq 0$ because otherwise, its norm would escape to infinity in finite time which is impossible due to (24).

Step 3. Uniform boundedness of $\int_0^1\theta(x)z_x^2(x,t)\,\mathrm{d}x$ *and* $|z(0,t)|$

The new Lyapunov candidate function

$$W(t)=\sum_{i=1}^{7}V_i(t) \tag{29}$$

is introduced with

$$V_3(t)=\frac{1}{2}\int_0^1\theta(x)z_x^2(x,t)\,\mathrm{d}x, \tag{30}$$

$$V_4(t)=\theta(0)M(t)|z(0,t)|, \tag{31}$$

$$V_5(t)=\theta(0)\Phi|z(0,t)|-\theta(0)\psi(t)z(0,t), \tag{32}$$

$$V_6(t)=\theta(0)\frac{\Phi_\mathrm{d}}{\gamma}(\Gamma-M(t)), \tag{33}$$

$$V_7(t)=\frac{\theta(0)}{2}\left(\sqrt{k}|z(0,t)|-\frac{1}{\sqrt{k}}\Phi\right)^2. \tag{34}$$

The non-negativeness of $V_5(t)$ derives from the inequality

$$|\theta(0)\psi z(0,t)|\leq\theta(0)\Phi|z(0,t)|, \tag{35}$$

whereas the non-negativeness of $V_6(t)$ is due to the previously proven relation (28), namely the existence of the uniform upper bound Γ for the adaptive gain $M(t)$.

The time derivatives of the newly introduced functions (30)–(34) along the closed-loop system's solutions are now calculated, starting with $\dot V_3(t)$ for which integration by parts is applied. This yields

$$\begin{aligned}\dot V_3(t)&=\int_0^1\theta(x)z_x(x,t)z_{xt}(x,t)\,\mathrm{d}x\\&=z_t(x,t)\theta(x)z_x(x,t)\Big|_0^1-\int_0^1 z_t(x,t)\underbrace{[\theta(x)z_x(x,t)]_x}_{z_t(x,t)}\,\mathrm{d}x.\end{aligned} \tag{36}$$

Substituting the boundary conditions (4b), (4c) and the boundary control law (9a) into (36) gives

$$\begin{aligned}\dot V_3(t)=&-\theta(0)kz(0,t)z_t(0,t)-\theta(0)M(t)z_t(0,t)\mathrm{sign}\{z(0,t)\}+\\&\theta(0)z_t(0,t)\psi(t)-\|z_t(\cdot,t)\|_{L_2}^2.\end{aligned} \tag{37}$$

Evaluating the time derivatives of the remaining functions $V_4(t)$ to $V_7(t)$ yields

$$\begin{aligned}\dot V_4(t)&=\theta(0)\dot M(t)|z(0,t)|+\theta(0)M(t)\frac{\mathrm{d}}{\mathrm{d}t}|z(0,t)|=\\&=\theta(0)\gamma z^2(0,t)+\theta(0)M(t)z_t(0,t)\mathrm{sign}\{z(0,t)\},\end{aligned} \tag{38}$$

$$\dot V_5(t)=\theta(0)\Phi z_t(0,t)\mathrm{sign}\{z(0,t)\}-\theta(0)\dot\psi(t)z(0,t)-\theta(0)\psi(t)z_t(0,t), \tag{39}$$

$$\dot{V}_6(t) = -\theta(0)\Phi_{\mathrm{d}}|z(0,t)|, \tag{40}$$

$$\dot{V}_7(t) = \theta(0)kz_t(0,t)z(0,t) - \theta(0)\Phi z_t(0,t)\mathrm{sign}\{z(0,t)\}. \tag{41}$$

Combining (21) and (37)–(41), and reordering, leads to

$$\begin{aligned}\dot{W}(t) = \sum_{i=1}^{7}\dot{V}_i(t) = -\int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x - \|z_t(\cdot,t)\|_{L_2}^2 + \\ + (\gamma - k)\theta(0)z^2(0,t) + (\psi(t) - \dot{\psi}(t))\theta(0)z(0,t) - (\Phi + \Phi_{\mathrm{d}})\theta(0)|z(0,t)|.\end{aligned} \tag{42}$$

The sign-indefinite term $(\psi(t) - \dot{\psi}(t))\theta(0)z(0,t)$ in the right-hand side of (42) can be estimated as

$$|(\psi(t) - \dot{\psi}(t))\theta(0)z(0,t)| \leq (\Phi + \Phi_{\mathrm{d}})\theta(0)|z(0,t)| \tag{43}$$

by virtue of (6) and (7). Thus, by virtue of (43) and (11) one can further manipulate the right-hand side of (42) to get

$$\dot{W}(t) \leq 0. \tag{44}$$

The latter inequality implies that $W(t)$ is nonincreasing, which means that

$$0 \leq W(t) \leq W(0) \quad \forall t \geq 0. \tag{45}$$

From (45), and taking into account that

$$V_3(t) \leq W(t) \leq W(0), \tag{46}$$

$$V_7(t) \leq W(t) \leq W(0), \tag{47}$$

one can derive uniform upper bounds for $\int_0^1 \theta(x)z_x^2\,\mathrm{d}x$ and $|z(0,t)|$, given by

$$\int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x \leq 2W(0), \tag{48}$$

$$|z(0,t)| \leq Z := \frac{1}{k}\Phi + \sqrt{\frac{2}{k\theta(0)}W(0)}. \tag{49}$$

Step 4. Asymptotic state stability

We are now in a position to exploit, after some preliminary manipulations, Barbalat's Lemma in order to show global asymptotic stability in the $L_2(0,1)$-sense. It follows from (21) that

$$\dot{V}(t) \leq -\int_0^1 \theta(x)z_x^2(x,t)\,\mathrm{d}x - \theta(0)kz^2(0,t). \tag{50}$$

By virtue of Assumption 2, the right-hand side of (50) can be further manipulated as follows

$$\dot{V}(t) \leq -\int_0^1 \theta_{\mathrm{m}}z_x^2(x,t)\,\mathrm{d}x - \theta_{\mathrm{m}}kz^2(0,t) = -\theta_{\mathrm{m}}\|z_x(\cdot,t)\|_{L_2}^2 - \theta_{\mathrm{m}}kz^2(0,t). \tag{51}$$

Using the Poincaré inequality (see, e.g., Reference 5)

$$\|z(\cdot,t)\|_{L_2}^2 \leq 2z^2(0,t) + 4\|z_x(\cdot,t)\|_{L_2}^2 \tag{52}$$

and rearranging it to

$$\|z_x(\cdot,t)\|_{L_2}^2 \geq \frac{1}{4}\|z(\cdot,t)\|_{L_2}^2 - \frac{1}{2}z^2(0,t) \tag{53}$$

one further manipulates the right-hand side of (51) as follows

$$\begin{aligned}\dot{V}(t) &\leq -\frac{\theta_\mathrm{m}}{4}\|z(\cdot,t)\|_{L_2}^2 + \frac{\theta_\mathrm{m}}{2}z^2(0,t) - \theta_\mathrm{m}kz^2(0,t)\\ &= -\frac{\theta_\mathrm{m}}{2}V_1(t) - \theta_\mathrm{m}\left(k-\frac{1}{2}\right)z^2(0,t).\end{aligned} \tag{54}$$

By virtue of (12) one obtains that

$$\dot{V}(t) \leq -\frac{\theta_\mathrm{m}}{2}V_1(t). \tag{55}$$

Integrating both sides of (55) from $t = 0$ to infinity yields

$$\lim_{t\to\infty}\{V(0) - V(t)\} \geq \frac{\theta_\mathrm{m}}{2}\int_0^\infty V_1(t)\,\mathrm{d}\tau. \tag{56}$$

By virtue of (24) it follows that

$$\lim_{t\to\infty}\{V(0) - V(t)\} \in [0, V(0)] \tag{57}$$

which, considered together with (56), shows that the integral term in the right-hand side of (56) exists. To apply Barbalat's Lemma to derive that $V_1(t)$ asymptotically converges to zero, it remains to show that $V_1(t)$ is uniformly continuous. The differentiability of $V_1(t)$ and uniform boundedness of $\dot{V}_1(t)$ are sufficient for this. The expression of $\dot{V}_1(t)$ was previously derived in (19).

By virtue of (8), (28), (48), and (49) it can be concluded that all terms appearing in the right-hand side of (19) are uniformly bounded, and so $\dot{V}_1(t)$ turns out to be uniformly bounded as well. This implies, according to Barbalat's Lemma, that

$$\lim_{t\to\infty}V_1(t) = \lim_{t\to\infty}\frac{1}{2}\|z(\cdot,t)\|_{L_2}^2 = 0 \tag{58}$$

which concludes the proof. ■

Remark 1. It can be verified that if the homogeneous Neumann boundary condition (4c) is replaced by the Dirichlet boundary condition $z(1,t) = 0$ then the same control law (9a), (9b) can still be adopted yielding the same convergence properties stated in Theorem 1, with the only difference that the less restrictive parameter condition $k \geq \gamma$ applies instead of (12).

Remark 2. The discontinuity manifold $z(0,t) = 0$ of the closed-loop dynamics can be considered as the sliding surface of the control system under investigation. Although the sliding surface is usually reached in finite time when the conventional (i.e., non-adaptive) sliding mode control is applied in the finite-dimensional setting, this is generally not the case when adaptive sliding mode control is employed. As a matter of fact, in References 22–24, as well as in the infinite-dimensional setting of References 25,26, the underlying sliding surface is only guaranteed to be reached asymptotically. It should be stressed that the asymptotic result established in (14) only involves the L_2-norm of the state and does not necessarily imply that $z(0,t)$ approaches zero. Establishing a pointwise convergence result would require additional and more involved stability analysis carried out in the Sobolev space $H^1(0,1)$ rather than in the space $L_2(0,1)$.

4 | SIMULATION RESULTS

The properties of the proposed control scheme are investigated by numerical simulations. The diffusivity function is chosen to be linearly increasing from $x = 0$ to $x = 1$, that is,

$$\theta(x) = 0.05 + 0.1x. \quad (59)$$

The matched disturbance $\psi(t)$ is selected as a sinusoidal signal that is activated at $t = 2$, that is,

$$\psi(t) = \begin{cases} 0 & \text{for } 0 \le t < 2 \\ 20\sin(\pi(t-2)) & \text{for } t \ge 2 \end{cases}. \quad (60)$$

Note that Assumptions 1 and 2 are fulfilled. The initial condition is set to

$$z_0(x) = 0.2(1 - \cos(\pi x)), \quad (61)$$

which is compatible to the boundary conditions (4b), (4c).

In order to numerically solve the closed-loop Boundary-Value Problem, the semi-discretization approach is followed where the spatial domain $x \in [0, 1]$ is partitioned into $n = 100$ uniformly spaced solution nodes. The resulting 100th-order finite-dimensional system is then discretized by the forward Euler method with a step size of $T_s = 10^{-5}$.

A preliminary simulation (TEST 1) is carried out where just the proportional part of the control law (9a) is utilized, that is, $M(t) = 0 \forall t \ge 0$, with a proportional gain of $k = 40$. The time evolution of $\|z(\cdot, t)\|_{L_2}$ in TEST 1 is depicted in Figure 1, where it can be seen that the disturbance (60) causes a steady state deviation from zero as well as permanent oscillations. This demonstrates the need for the sliding mode control component in the control law to achieve disturbance rejection.

In the next TEST 2, the controller is implemented in its complete form, that is, also including the discontinuous component neglected in the previous simulation. The adaptation gain γ is selected in accordance with (11) and (12) as $\gamma = 40$, whereas the initial value for the adaptive switching gain is chosen as $M(0) = 0$. Figure 2 depicts the spatiotemporal profile of $z(x, t)$ whereas Figure 3 shows the corresponding L_2-norm time evolution, that asymptotically vanishes revealing that the control goal is achieved. The sinusoidal disturbance acting at $x = 0$ from $t = 2$ on is suppressed by the sliding-mode base boundary control signal acting at the same spatial location.

The time evolution of the adaptive gain $M(t)$ is displayed in the upper plot of Figure 4 alongside the magnitude of the disturbance (60). At $t \ge 2$ the matched disturbance (60) becomes non-zero, yielding an increase of the adaptive gain $M(t)$ to compensate for its effect. As expected, the adaptive gain $M(t)$ eventually settles to a constant value. Additionally the boundary control signal $u(t)$, composed of a proportional and a discontinuous part, is displayed in the bottom plot of the same Figure 4. Finally, Figure 5 displays the boundary state $z(0, t)$. It is interesting to observe that during the convergence transient the switching frequency of the discontinuous control component fluctuates, being very large in the time intervals when the adaptive gain $M(t)$ is dominating the magnitude of the disturbance $|\psi(t)|$ and, as a result, the boundary state $z(0, t)$ is constrained to evolve into a small vicinity of zero. As shown in Figure 5, the convergence process of approaching the discontinuity manifold $z(0, t) = 0$ takes place asymptotically (see also the Remark 2).

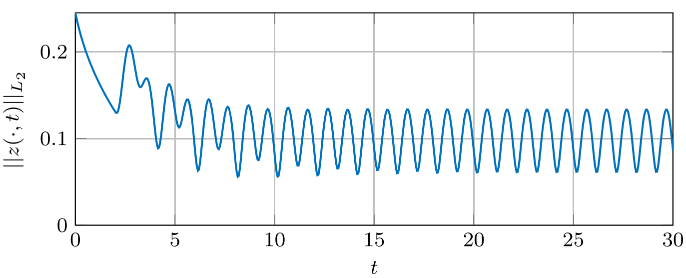

FIGURE 1 Time evolution of $\|z(\cdot, t)\|_{L_2}$ in the TEST 1.

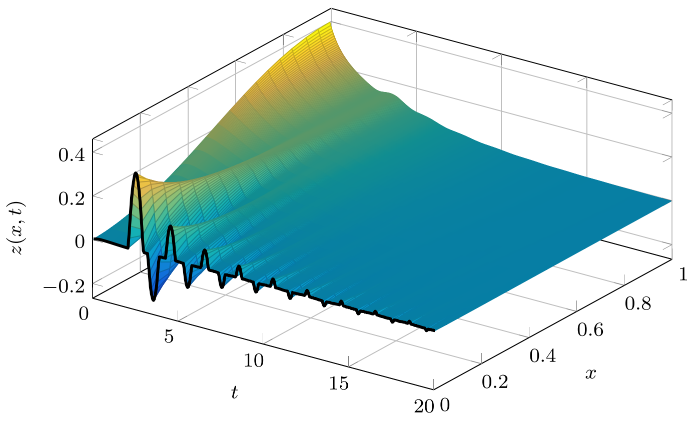

FIGURE 2 Spatiotemporal profile of $z(x,t)$ in the TEST 2.

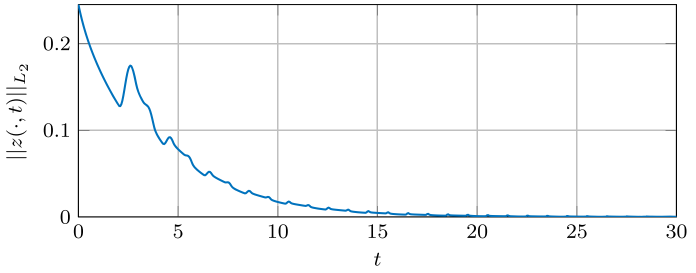

FIGURE 3 The state norm $\|z(\cdot,t)\|_{L_2}$ in the TEST 2.

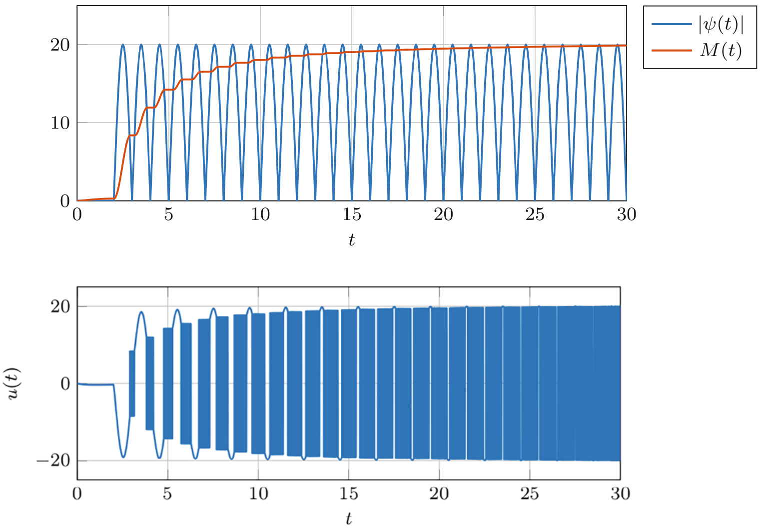

FIGURE 4 Top plot: The adaptive gain $M(t)$ and the disturbance magnitude $|\psi(t)|$ in TEST 2. Bottom plot: the boundary control $u(t)$ in TEST 2.

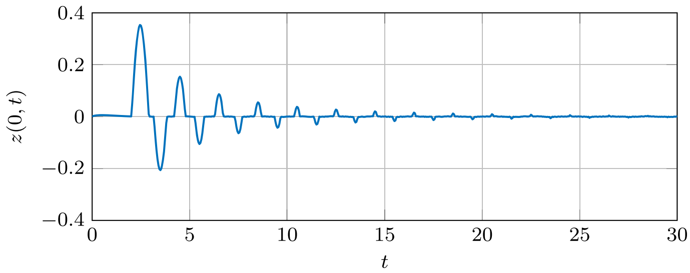

FIGURE 5 The boundary state $z(0,t)$ in TEST 2.

5 | CONCLUSIONS

A sliding-mode-based adaptive boundary control law has been proposed for stabilizing a class of uncertain diffusion processes while simultaneously rejecting a matched disturbance uniformly bounded along with its time derivative. The corresponding upper bounding constants are not known in advance, motivating the use of an adaptive control scheme. The proposed controller features an intuitive structure and requires a measurement of the state at the boundary only. Despite its straightforward structure, a rather unconventional Lyapunov function had to be devised to prove the stability properties of the closed-loop system.

Simulation studies were carried out to demonstrate the closed-loop performance. They revealed that $M_0 = 0$ is a practical choice for the initial value of the adaptive switching gain and led to no over-estimation of Φ in those experiments.

There are multiple possible directions for future research activities. It would be of interest to extend the problem to the more general scenario where multiple collocated actuators and matched disturbances, possibly acting inside the spatial domain instead of just at one of its boundaries, are present. Another interesting problem is to assess the effect of unmatched disturbances as it was done in Reference 16 in the non-adaptive case. The experimental validation of the proposed technique using laboratory prototypes, and its performance comparison with the non-adaptive version of the algorithm, also represent attractive and interesting topics to address. Lastly, future research will also investigate the potential use in the present scenario of *bidirectional* adaptation algorithms similar to those used in References 25,26 which also allow a potential transient decrease of the adaptive switching gain.

AUTHOR CONTRIBUTIONS

Paul Mayr: Investigation, software, visualization, writing – original draft. **Yury Orlov**: Investigation, writing – original draft. **Alessandro Pisano**: Conceptualization, methodology, project administration, writing – original draft. **Stefan Koch**: Writing – review and editing. **Markus Reichhartinger**: Writing – review and editing.

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CONFLICT OF INTEREST STATEMENT

The authors declare no potential conflict of interests.

DATA AVAILABILITY STATEMENT

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

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