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A Nyström method for Volterra-Fredholm integral equations with highly oscillatory kernel

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Abstract

In the present paper, we propose a Nyström method for a class of Volterra-Fredholm integral equations containing a fast oscillating kernel. The approximation tool consists of the $\ell-$ Iterated Boolean sums of Bernstein operators, also known as Generalized Bernstein (GB) operators, based on equally spaced nodes of the interval $[-1,1]$. The corresponding GB polynomials associated with any continuous function depend on the additional parameter ℓ, which can be suitably chosen in order to improve the rate of convergence, as the smoothness of the function increases. Hence, the low degree of approximation by the classical Bernstein polynomials or by piecewise polynomials functions, typically based on equispaced nodes, is overcome in some sense. The numerical method we propose here is stable and convergent in the space of the continuous functions equipped with the uniform norm. Error estimates are proved in Hölder-Zygmund type subspaces and some numerical tests confirm the theoretical error estimates.

Keywords:Volterra-Fredholm integral equation, Approximation by polynomials, Generalized Bernstein polynomials.
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1 Introduction

In this paper, we propose a Nyström-type method to approximate the solution of the following integral equation

$$f(y)+\mu_1\int_{-1}^{y} h(x,y)f(x)dx+\mu_2\int_{-1}^{1} e^{-\mathrm{i}\omega(x-y)}f(x)dx=g(y), \quad y\in[-1,1], \tag{1}$$

where $\mu_1,\mu_2\in\mathbb{R}$, $\omega\in\mathbb{R}^+$, $\mathrm{i}=\sqrt{-1}$, f is the unknown solution, g and $h(x,y)$ are given continuous functions defined in $[-1,1]$ and $[-1,1]\times[-1,1]$, respectively.

In a nutshell, we investigate the numerical treatment of a second kind integral equation having both a Volterra integral of the type

$$(\mathcal{V}f)(y)=\int_{-1}^{y} h(x,y)f(x)dx, \tag{2}$$

and an integral with a highly oscillator kernel $e^{-\mathrm{i}\omega(x-y)}$ for $\omega\gg 1$. By separating $e^{-\mathrm{i}\omega(x-y)}$ into the real and imaginary parts, we deal with integrals of the type

$$(\mathcal{K}^{\omega}f)(y)=\int_{-1}^{1} k_{\omega}(x,y)f(x)dx, \tag{3}$$

where $k_\omega(x,y)=\cos(\omega(x-y))$ or $k_\omega(x,y)=\sin(\omega(x-y))$.

Many models arising in the engineering area from parabolic boundary value problems and more in general in the development of evolutionary problems like epidemic diffusion, biological evolution, etc. can be described by equations of this mixed type (see e.g. [19]). Since in some applications the kernels and/or the right-hand side function are known in a discrete form, mainly on equally spaced nodes, all the efficient Nyström methods involving Gauss-Legendre or more in general product integration rules based on orthogonal polynomials, cannot be applied. In addition, Nyström methods based on piecewise polynomial approximation show saturation phenomena and low degree of approximation (see e.g. [16]). Moreover, the presence of an oscillating kernel, in many cases implies a solution having an oscillating behavior [18].

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To overcome the aforesaid drawbacks, we propose here a Nyström method involving quadrature rules obtained by means of the so called *Generalized Bernstein polynomials* $\bar{B}_{m,\ell}(f) \in \mathbf{P}_m$. The employed product integration rule is based on equally spaced knots in $[-1,1]$, and by exploiting the convergence properties of the operator $\bar{B}_{m,\ell}$, the rate of convergence of the method increases as the smoothness of the involved functions (kernel and right-hand side) increases. This improvement not only depends on the degree m, but significantly on the order ℓ of the boolean sum defining the GB polynomial $\bar{B}_{m,\ell}(f)$. Indeed, the iterated Boolean sums of Bernstein operators, for suitable choices of ℓ, accelerate the degree of convergence of the classical Bernstein operator, being $m^{-\ell}$ the saturation order of $\bar{B}_{m,\ell}$ [14]. Moreover, any function $f \in Z_{2\ell}$ can be uniformly approximated by $\bar{B}_{m,\ell}(f)$ with order $\mathcal{O}\left(m^{-\ell}\right)$ [10]. In any case, the presence of the oscillatory kernel, even though in the product rule the coefficients are "exactly" computed, introduces more difficulties, since the solution can oscillate, requiring additional effort [18]. The stability and the convergence of the method are proved in the space of the continuous functions, and error estimates are given in Hölder-Zygmund spaces.

The outline of the paper is as follows. In Section 2 are collected some preliminary results about the properties and the construction of GB polynomials employed in the numerical method and the spaces of functions mainly we are dealing with. In Section 3 the discrete operators introduced to approximate $\mathcal{K}$, $\mathcal{V}$ are presented with an estimate of convergence. Section 4 contains the main results: the approximate equation and its unique solvability, providing convergence and stability theorems. Finally, Section 5 provides some numerical experiments, for analyzing the performance of the method. To improve the readability, the proofs are given in the last Section 6.

2 Preliminaries

Throughout the paper, $\mathcal{C}$ will denote a generic positive constant having different meanings at different occurrences, and by writing $\mathcal{C} \neq \mathcal{C}(a,b,..)$ will be underlined that $\mathcal{C}$ is independent of $a,b,...$
For $m \in \mathbb{N}$, let $N_0^m = \{0,1,2,\dots,m\}$ and denote by $\mathbf{P}_m$ the space of the algebraic polynomials of degree at most m.
Following a standard notation, $C^0 := C^0([-1,1])$ denotes the space of the continuous functions on $[-1,1]$ equipped with the uniform norm

$$\|f\| := \max_{x\in[-1,1]} |f(x)|.$$

In C^0, setting $\varphi(x) := \sqrt{1-x^2}$, let us define the φ–modulus of smoothness by Ditzian and Totik [4]

$$\omega_\varphi^r(f,t) = \sup_{0<h\le t} \|\Delta_{h\varphi}^r f\|, \qquad r \in \mathbb{N}$$

where

$$\Delta_{h\varphi(x)}^r f(x) = \sum_{k=0}^r (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} f\left(x+(r-2k)\frac{h}{2}\varphi(x)\right).$$

Let us also introduce the Hölder–Zygmund space of order $\lambda > 0$

$$Z_\lambda = \left\{ f \in C^0 : \quad \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda} < \infty, \quad r > \lambda \right\}, \tag{4}$$

equipped with the norm

$$\|f\|_{Z_\lambda} := \|f\| + \sup_{t>0} \frac{\omega_\varphi^r(f,t)}{t^\lambda}, \qquad r > \lambda.$$

In the case $k \in \mathbb{N}$, we define the Sobolev space

$$W_k = \left\{ f \in C^0 : f^{(k-1)} \in \mathcal{AC}, \|f^{(k)}\varphi^k\| < \infty \right\},$$

where $\mathcal{AC}$ denotes the space of the absolutely continuous functions on $[-1,1]$. We equip the space W_k with the norm

$$\|f\|_{W_k} = \|f\| + \|f^{(k)}\varphi^k\|.$$

We mention that $W_k \subset Z_k$ and $\|f\|_{Z_k} \le \mathcal{C}\|f\|_{W_k}$.

Finally, for our aims, we recall the following inequality

$$\omega_\varphi^k(f,t) \le \mathcal{C} t^r \|f\|_{Z_r}, \quad \forall f \in Z_r, \quad k > r, \quad t > 0, \quad \mathcal{C} \neq \mathcal{C}(f). \tag{5}$$

2.1 Generalized Bernstein polynomials in $[-1,1]$

Let us introduce in this paragraph the essential approximation tool that we will use in our numerical method, i.e. the Generalized Bernstein polynomials [5, 13, 14, 17]. Here, we will only mention the properties we will use but an accurate description of all its features and applications can be found in the survey [17].
Let $\bar{B}_m(f)$ be the mth classical Bernstein polynomial of $f \in C^0$

$$\bar{B}_m(f,x) := \sum_{k=0}^{m} \bar{p}_{m,k}(x) f(x_k), \qquad x_k := -1 + k\frac{2}{m}, \qquad x \in [-1,1], \tag{6}$$

where

$$\bar{p}_{m,k}(x) := \binom{m}{k}\left(\frac{1+x}{2}\right)^k \left(\frac{1-x}{2}\right)^{m-k}, \qquad k = 0,1,\ldots,m.$$

Fixed $\ell \in \mathbb{N}, \ell \geq 1$, denoted by $\mathcal{I}$ the identity operator and setting

$$\bar{B}_{m,\ell} := \mathcal{I} - (\mathcal{I} - \bar{B}_m)^\ell : f \in C^0 \to C^0, \qquad \bar{B}_m^i = \bar{B}_m(\bar{B}_m^{i-1}), \qquad \bar{B}_m^0 = \mathcal{I},$$

the mth Generalized Bernstein polynomial of $f \in C^0$ takes the form

$$\bar{B}_{m,\ell}(f,x) := \sum_{j=0}^{m} \bar{p}_{m,j}^{(\ell)}(x) f(x_j), \qquad x_j := -1 + j\frac{2}{m}, \qquad x \in [-1,1], \tag{7}$$

where

$$\bar{p}_{m,j}^{(\ell)}(x) = \sum_{i=1}^{\ell} \binom{\ell}{i} (-1)^{i-1} \bar{B}_m^{i-1} \bar{p}_{m,j}(x). \tag{8}$$

Let us recall that they can be also written as [9]

$$\bar{B}_{m,\ell}(f,x) = \sum_{i=0}^{m} \left(\sum_{j=0}^{m} c_{i,j}^{(m,\ell)} f(x_j) \right) \bar{p}_{m,i}(x), \tag{9}$$

being $c_{i,j}^{(m,\ell)}$ the entry (i,j) of the matrix $C_{m,\ell} \in \mathbb{R}^{(m+1)\times(m+1)}$ defined as

$$C_{m,\ell} = I + (I-A) + \ldots + (I-A)^{\ell-1}, \quad C_{m,1} = I, \tag{10}$$

where I is the identity matrix of order $m+1$ and A is defined as

$$A := (A_{i,j}) \qquad A_{i,j} := \bar{p}_{m,j}(x_i), \qquad (i,j) \in N_0^m \times N_0^m. \tag{11}$$

Setting

$$\bar{\mathbf{p}}_m(x) = [\bar{p}_{m,0}(x), \bar{p}_{m,1}(x), \ldots, \bar{p}_{m,m}(x)], \quad \mathbf{f} = [f(x_0), f(x_1), \ldots, f(x_m)]^T,$$

$\bar{B}_{m,\ell}(f)$ can be expressed as

$$\bar{B}_{m,\ell}(f,x) = \bar{\mathbf{p}}_m(x) C_{m,\ell} \mathbf{f}. \tag{12}$$

Let us now state a relation between the polynomial $\bar{B}_{m,\ell}(f,x)$, $x \in [-1,1]$ and the generalized Bernstein polynomial $B_{m,\ell}(g,y)$, defined for $y \in [0,1]$. Denoting by

$$B_m(g,y) = \sum_{k=0}^{m} y^k (1-y)^{m-k} g\left(\frac{k}{m}\right) =: \sum_{k=0}^{m} p_{m,k}(y) g\left(\frac{k}{m}\right), \qquad y \in [0,1],$$

the m-th Bernstein polynomial on $[0,1]$, $B_{m,\ell}(g,y)$ is defined as

$$B_{m,\ell}(g,y) := \sum_{j=0}^{m} p_{m,j}^{(\ell)}(y) g\left(\frac{k}{m}\right), \tag{13}$$

where

$$p_{m,j}^{(\ell)}(y) = \sum_{i=1}^{\ell} \binom{\ell}{i} (-1)^{i-1} B_m^{i-1} p_{m,j}(y), \quad B_m^i = B_m(B_m^{i-1}), \qquad B_m^0 = \mathcal{I}. \tag{14}$$

Observing that

$$\bar{p}_{m,k}\left(2\frac{j}{m} - 1\right) = p_{m,k}\left(\frac{j}{m}\right),$$

it follows

$$\bar{p}_{m,j}^{(\ell)}(x) = p_{m,j}^{(\ell)}\left(\frac{1+x}{2}\right), \qquad x \in [-1,1].$$

As consequence, if $g(y) = f(2y-1)$, $y \in [0,1]$ one has

$$\bar{B}_{m,\ell}(f,x) = B_{m,\ell}\left(g, \frac{1+x}{2}\right). \tag{15}$$

About the convergence for the sequence $\{B_{m,\ell}(f)\}_{m,\ell}$, we recall the following result by Gonska and Zhou [10].

Theorem 2.1. *Let $\ell \in \mathbb{N}$ be fixed. Then, for all $m \in \mathbb{N}$ and for any $f \in C^0([0,1])$, setting $\varphi_1(x) = \sqrt{x(1-x)}$, we have*

$$\|f - B_{m,\ell}(f)\| \le \mathcal{C}\left\{\omega^{2\ell}_{\varphi_1}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|}{m^\ell}\right\}, \qquad \mathcal{C} \ne \mathcal{C}(m,f).$$

Moreover, for any $0 < \lambda \le 2\ell$ we obtain

$$\|f - B_{m,\ell}(f)\| = \mathcal{O}\left(\frac{1}{\sqrt{m^\lambda}}\right).$$

By Theorem 2.1 and in view of the relation established in (15) the following results holds.

Theorem 2.2. *Let $\ell \in \mathbb{N}$ be fixed. Then, for all $m \in \mathbb{N}$ and for any $f \in C^0$, we have*

$$\|f - \bar{B}_{m,\ell}(f)\| \le \mathcal{C}\left\{\omega^{2\ell}_{\varphi}\left(f, \frac{1}{\sqrt{m}}\right) + \frac{\|f\|}{m^\ell}\right\},$$

where $\varphi(x) = \sqrt{1-x^2}$ and $\mathcal{C} \ne \mathcal{C}(m,f)$. Moreover, for any $0 < \lambda \le 2\ell$ we obtain

$$\|f - \bar{B}_{m,\ell}(f)\| = \mathcal{O}\left(\frac{1}{\sqrt{m^\lambda}}\right).$$

In particular, by (5), we deduce that for each $f \in Z_r$

$$\|f - \bar{B}_{m,\ell}(f)\| \le \mathcal{C}\left(\frac{1}{\sqrt{m^r}} + \frac{1}{m^\ell}\right)\|f\|_{Z_r}, \tag{16}$$

where $\mathcal{C} \ne \mathcal{C}(m,f)$, and with $r \le 2\ell$, we have

$$\|f - \bar{B}_{m,\ell}(f)\| \le \mathcal{C}\frac{\|f\|_{Z_r}}{\sqrt{m^r}}. \tag{17}$$

Let us highlight the role of the additional parameter $\ell > 1$. It allows to increase the approximation rate achieved by the classical Bernstein polynomials, being $m^{-\ell}$ the saturation order of $\bar{B}_{m,\ell}(f) \sim f \in Z_r$ with $r \le 2\ell$ (see for instance [14]).

Let us now provide some details about the computation of the polynomials $B_{m,\ell}(f)$, with m fixed and ℓ varying, according to the definition in (9). Starting from the matrix A defined in (11), it can be constructed by rows by using the recurrence relation

$$\begin{cases} \bar{p}_{m,k}(x) = \dfrac{(1-x)}{2}\bar{p}_{m-1,k}(x) + \dfrac{(1+x)}{2}\bar{p}_{m-1,k-1}(x), & k \in N_0^m,\ m \ge 1, \\ \bar{p}_{m,k}(x) = 0, & k \notin N_0^m, \end{cases} \tag{18}$$

requiring for each row, $\mathcal{O}(m^2)$ floating point operations (flops). Since A is centrosymmetric, i.e. $a_{ij} = a_{n+2-i,n+2-j}$ (equivalently $\mathcal{A} = \mathbf{J}\mathcal{A}\mathbf{J}$, where $\mathbf{J}$ is the reversal matrix (see [11, pp. 33-36])) it will be enough to compute at most $\frac{m+1}{2}$ its rows, and hence, its construction requires $\frac{m^3}{2}$ flops. Therefore, once A is constructed, the matrix $C_{m,\ell}$ defined in (10), requiring $(\ell-2)$ products of centrosymmetric matrices, can be computed in $\mathcal{O}((\ell-2)(m+1)^3/2)$ flops since the product of two centrosymmetric matrices of order $m+1$ is performable into $\mathcal{O}((m+1)^3/2)$ flops. Finally, we recall that a significant reduction of the computational cost is realized by constructing the sequence $\{\bar{B}_{m,2^p}(f)\}_{p\ge1}$, in light of the following relation:

$$C_{m,2^p} = C_{m,2^{p-1}} + (I-A)^{2^{p-1}} C_{m,2^{p-1}}.$$

3 A discrete approximation of the integral operators

In this section, we present the discrete operators $\mathcal{V}_m^{(\ell)}$ and $\mathcal{K}_m^{(\omega,\ell)}$, that we propose in order to approximate $\mathcal{V}$ and $\mathcal{K}^\omega$, respectively, by analyzing their convergence in C^0.

3.1 The approximation of the operator $\mathcal{V}$

Let us consider the operator $\mathcal{V}$ defined in (2) as a map from C^0 to C^0 and let us approximate it by proceeding as in [6] (see also [16]).
We introduce, for each fixed $\ell \in \mathbb{N}$, the sequence $\{\mathcal{V}_m^{(\ell)} f\}_m$ defined as

$$(\mathcal{V}_m^{(\ell)} f)(y) = \int_{-1}^{y} \bar{B}_{m,\ell}(f h(\cdot,y), x)dx = \frac{1+y}{2}\sum_{j=0}^{m} h(x_j,y) f(x_j) \sum_{i=0}^{m} c_{i,j}^{(m,\ell)} \left(\int_{-1}^{1} \bar{p}_{m,i}(\rho^{-1}(z))dz\right),$$

where $z = \rho(x) := 2\frac{1+x}{1+y} - 1$. Basically, the discrete operators $\{\mathcal{V}_m^{(\ell)} f\}_m$ have been defined by replacing the integrand function $k(x,y)f(x)$ of the original operator $\mathcal{V}$ with the GB polynomial (9).
By using the Gauss-Legendre rule of order $q := \left\lfloor \frac{m}{2} \right\rfloor + 1$ which is exact for polynomials of degree m, we have

$$\int_{-1}^{1} \bar{p}_{m,i}(\rho^{-1}(z))dz = \sum_{k=1}^{q} \lambda_{q,k}\, \bar{p}_{m,i}(\rho^{-1}(v_k))$$

where $\{\lambda_{q,k}\}_{k=1}^q$ and $\{\nu_k\}_{k=1}^q$ are the weights and the nodes of the q-point Gauss-Legendre rule, respectively. Then, we can write

$$(\mathcal{V}_m^{(\ell)} f)(y) = \sum_{j=0}^m h(x_j, y) f(x_j) \hat{Q}_j^{(\ell)}(y) \tag{19}$$

where

$$\hat{Q}_j^{(\ell)}(y) = \frac{1+y}{2} \sum_{i=0}^m c_{i,j}^{(m,\ell)} \left(\sum_{k=1}^q \lambda_{q,k} \bar{p}_{m,i}(\rho^{-1}(\nu_k)) \right).$$

The discrete operator (19) uniformly converges to the operator (2) as $m \to \infty$, for each fixed ℓ. This has been stated in [6] in a more general context. However, we report it below to make the paper self-contained.

Theorem 3.1. *For any $f \in C^0$, under the assumption $h(x,y)$ continuous in $[-1,1] \times [-1,1]$, it is*

$$\lim_m \|(\mathcal{V} - \mathcal{V}_m^{(\ell)}) f\| = 0. \tag{20}$$

Moreover, if $f \in Z_r$ and $\sup_{y \in [-1,1]} h(\cdot, y) \in Z_r$, the following error estimate holds

$$\|(\mathcal{V} - \mathcal{V}_m^{(\ell)}) f\| \le \mathcal{C} \left(\frac{1}{(\sqrt{m})^r} + \frac{1}{m^\ell} \right) \|f\|_{Z_r}, \tag{21}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Remark 1. For each $f \in Z_r$ with $0 < r \le 2\ell$ we have

$$\|(\mathcal{V} - \mathcal{V}_m^{(\ell)}) f\| \le \frac{\mathcal{C}}{\sqrt{m^r}} \|f\|_{Z_r}, \tag{22}$$

where $\mathcal{C} \neq \mathcal{C}(m, f)$.

Remark 2. The error estimate (21) was already proved in [6, Theorem 3.1] under the same hypothesis. Here, in addition we prove the convergence (20) by assuming that the functions f and h are continuous.

3.2 The approximation of the operator $\mathcal{K}^\omega$

Let us now focus on the operator $\mathcal{K}^\omega : C^0 \to C^0$ defined in (3). In order to approximate it, we propose to use the product integration rule developed in [7]. This is a quadrature scheme based on the approximation of the single function f by the polynomial $\bar{B}_{m,\ell}(f)$ and on the accurate computation of the resulting coefficients.
To be more precise, we write

$$\begin{aligned}(\mathcal{K}^\omega f)(y) &= \int_{-1}^1 k_\omega(x,y) \bar{B}_{m,\ell}(f,x) dx + \int_{-1}^1 k_\omega(x,y)[f(x) - \bar{B}_{m,\ell}(f,x)] dx \\ &=: (\mathcal{I}_m^\omega f)(y) + (\mathcal{R}_m f)(y). \end{aligned} \tag{23}$$

To avoid the consequences of the fast oscillations for ω large, let us partition the interval $[-1,1]$ into $N = \lfloor \frac{\omega}{\pi} \rfloor + 1$ intervals i.e.

$$[-1, 1] = \bigcup_{s=1}^N [x_{s-1}, x_s], \quad x_0 = -1, \quad x_s = -1 + s\frac{2}{N},$$

and write

$$(\mathcal{I}_m^\omega f)(y) = \frac{1}{N} \sum_{s=1}^N \int_{-1}^1 k_\omega(\gamma_s^{-1}(z), y) \bar{B}_{m,\ell}(f(\gamma_s^{-1}), z) \, dz,$$

where $z = \gamma_s(x) := N(x - x_{\iota-1}) - 1$.
Hence, by approximating each integral by the m-point Gauss-Legendre rule, we define the operators $\mathcal{K}_m^{(\omega,\ell)}$

$$(\mathcal{I}_m^\omega f)(y) \sim (\mathcal{K}_m^{(\omega,\ell)} f)(y) = \frac{1}{N} \sum_{s=1}^N \sum_{k=1}^m k_\omega(\gamma_s^{-1}(z_k), y) \bar{B}_{m,\ell}(f(\gamma_s^{-1}), z_k)) \lambda_{m,k}, \tag{24}$$

where $\{\lambda_{m,k}\}_{k=1}^m$, $\{z_k\}_{k=1}^m$ denote the weights and the nodes of the m-point Gauss-Legendre rule, respectively.
By making explicit the expression of $\bar{B}_{m,\ell} f(\gamma_s^{-1}(z_k))$, we have the final product formula

$$(\mathcal{K}_m^{(\omega,\ell)} f)(y) = \sum_{j=0}^m f(x_j) \hat{T}_j^{(\omega,\ell)}(y), \tag{25}$$

where

$$\hat{T}_j^{(\omega,\ell)}(y) = \frac{1}{N} \sum_{i=0}^m c_{i,j}^{(m,\ell)} \left(\sum_{s=1}^N \sum_{k=1}^m \lambda_{m,k} \bar{p}_{m,i} \left(\gamma_s^{-1}(z_k) \right) k_\omega(\gamma_s^{-1}(z_k), y) \right) =: \frac{1}{N} \sum_{i=0}^m c_{i,j}^{(m,\ell)} q_i(y). \tag{26}$$

As proved in [7], rule (25) is stable in C^0, i.e. for each fixed $\omega \in \mathbf{R}^+$ and $\ell \ge 1$

$$\sup_{y \in [-1,1]} \sup_m |(\mathcal{K}_m^{(\omega,\ell)} f)(y)| \le \mathcal{C} \|f\|_\infty, \qquad \forall f \in C^0.$$

In the following theorem it is stated the convergence of the rule, in the case when $f \in C^0$.

Theorem 3.2. *For any $f \in C^0$, and $\forall \ell \geq 1$, under the assumption $\frac{\pi(m+\omega)}{\omega(2m-1)} < 1$, it is*

$$\lim_m \|(\mathcal{K} - \mathcal{K}_m^{(\omega,\ell)})f\| = 0. \tag{27}$$

Moreover, under the assumption $f \in Z_r$ with $0 < r \leq 2\ell$ we have

$$\|(\mathcal{K} - \mathcal{K}_m^{(\omega,\ell)})f\| \leq \mathcal{C}\left(\frac{\|f\|_{Z_r}}{(\sqrt{m})^r} + \|f\|\, m^{\frac{3}{2}}\left(\frac{\pi}{\omega}\cdot\frac{m+\omega}{2m-1}\right)^m\right), \tag{28}$$

where $\mathcal{C} \neq \mathcal{C}(m, f, \omega)$ and $\mathcal{C} = \mathcal{C}(\ell)$.

Remark 3. The convergence of formula (25) was already proved in [7, Theorem 3.2] by providing an error estimate under the assumption $f \in W_r$. Here, we prove the convergence (27) by assuming that f is continuous and furnish an estimate of the error in the case $f \in Z_r$, with $0 < r \leq 2\ell$.

4 The Nyström method

Let us consider equation (1) that, according to the introduced notation, can be rewritten as

$$(\mathcal{I} + \mu_1\mathcal{V} + \mu_2\mathcal{K}^\omega)f = g, \tag{29}$$

where $\mathcal{I}$ is the identity operator. Let us note that if $\ker\{\mathcal{I} + \mu_1\mathcal{V} + \mu_2\mathcal{K}^\omega\} = \{0\}$ equation (29) admits a unique solution f^* in C^0, for any $g \in C^0$.
Then, in order to approximate it, we propose a Nyström-type method based on the quadrature rules (19) and (25).
Consider the following finite dimensional equation

$$(\mathcal{I} + \mu_1\mathcal{V}_m^{(\ell)} + \mu_2\mathcal{K}_m^{(\omega,\ell)})f_m^{(\ell)} = g, \tag{30}$$

where $f_m^{(\ell)}$ is the unknown, and $\mathcal{V}_m^{(\ell)}$ and $\mathcal{K}_m^{(\omega,\ell)}$ are defined in (19) and (25), respectively.
By collocating (30) at the points $x_i = -1 + i\frac{2}{m}$, for $i = 0, \dots, m$, we get the linear system

$$\sum_{j=0}^{m}\left[\delta_{ij} + \mu_1 h(x_j, x_i)\hat{Q}_j^{(\ell)}(x_i) + \mu_2\hat{T}_j^{(\omega,\ell)}(x_i)\right]a_j = g(x_i), \qquad i = 0, \dots, m \tag{31}$$

where $\{a_j = f_m^{(\ell)}(x_j)\}_{j=0}^m$ are the unknowns. The above system can be written also in a matrix form as

$$G_{m+1}^{(\ell)}\mathbf{a}_{m+1} = \mathbf{b}_{m+1}, \qquad \text{with} \qquad \mathbf{b}_{m+1} = [g(x_0), \dots, g(x_m)]^T, \qquad \mathbf{a}_{m+1} = [a_0, \dots, a_m]^T$$

and

$$G_{m+1}^{(\ell)} = I + \mu_1 Q_{m+1}^{(\ell)} + \mu_2 T_{m+1}^{(\ell)}, \qquad \mathcal{Q}_{m+1} = \left[Q_{m+1}^{(\ell)}\right]_{i,j} = \left[h(x_j, x_i)\hat{Q}_j^{(\ell)}(x_i)\right], \qquad \mathcal{T}_{m+1} = \left[T_{m+1}^{(\ell)}\right]_{i,j} = \left[\hat{T}_j^{(\omega,\ell)}(x_i)\right], \tag{32}$$

being I the identity matrix of order $m+1$.
This is a square linear system of order $m+1$ whose unique solution $\mathbf{a}_{m+1}^* = [a_0^*, \dots, a_m^*]^T$ provides the Nyström interpolant

$$f_m^{(\ell)}(y) = g(y) - \sum_{j=0}^{m}\left(\mu_1 h(x_j, y)\hat{Q}_j^{(\ell)}(y) + \mu_2\hat{T}_j^{(\omega,\ell)}(y)\right)a_j^*.$$

Next theorem states conditions assuring the method is stable and convergent.

Theorem 4.1. *Assume that $\ker\{\mathcal{I} + \mu_1\mathcal{V} + \mu_2\mathcal{K}^\omega\} = \{0\}$ and consider the functional equation (30) with ℓ fixed. Then, for m sufficiently large the operators $(\mathcal{I} + \mu_1\mathcal{V}_m^{(\ell)} + \mu_2\mathcal{K}_m^{(\omega,\ell)})$ are invertible and their inverse are uniformly bounded w.r.t. m on C^0. Moreover, denoted by f^* the unique solution of (29), if $g \in Z_r$, under the assumption*

$$(1+y)\sup_{x\in[-1,1]}|k(x,y)| \in Z_r, \tag{33}$$

for $0 < r \leq 2\ell$, one has

$$\|f^* - f_m^{(\ell)}\| \leq \mathcal{C}\frac{1}{(\sqrt{m})^r}\|f^*\|_{Z_r}, \tag{34}$$

where $\mathcal{C} \neq \mathcal{C}(m, f^*)$.

Remark 4. The approximate solution $f_m^{(\ell)}$ depends on the degree m, related to the dimension of the final linear system, and on the additional parameter ℓ chosen accordingly to the smoothness of the solution, in turn depending on the regularity of the involved known functions $h(x,y), g(x)$. Consequently, it makes sense to increase the parameter ℓ in order to obtain the maximum rate of convergence with the minimal system dimension.

When the solution $f \in Z_r$, the choice of the parameter ℓ is guided by the error estimate (34), since $0 < r \le 2\ell$. In case $f \in Z_r$ with $r > 2\ell$, or f is an analytical function, the error behaves as $F(m,\ell) = m^{-\ell}$ and improves for increasing values of both m and ℓ. To accelerate convergence it is convenient to increase the value of ℓ only if $\ell < \ell^* := m \log m$, and above this threshold it is more useful to increase m. Indeed, the following inequality

$$-m^{-\ell} \log m = \frac{\partial F}{\partial \ell} < \frac{\partial F}{\partial m} = -\ell m^{-\ell-1} < 0 \tag{35}$$

holds if $\ell < m \log m$. This behavior is not surprising, since $\{\bar{B}_{m,\ell}(f)\}_{\ell\to\infty}$ uniformly converges to the Lagrange polynomial $L_m(f) \in \mathbf{P}_m$, interpolating f at equidistant nodes, whose Lebesgue constants diverge exponentially (see for instance [3, 15]). However, these considerations do not take into account the fact that the solution of the integral equation (1) may oscillate, due to the presence of the oscillatory kernel. In these cases the threshold ℓ^* can be slightly larger and our advice is to increase ℓ to overcome these difficulties.

Finally, about the computational cost required to construct the matrix of the final linear system (31), we mention that the most effort is due to the matrix $\mathcal{T}_{m+1}$ defined in (32) and related to the discrete operator $\mathcal{K}_m^{(\omega,\ell)}$ given in (24) and (25). In fact, once the matrix $C_{m,\ell}$ has been constructed, the evaluation of its entries $\hat{T}_j^{(\omega,\ell)}(y)$ given in (26) requires those of

$$S := \{\bar{B}_{m,\ell}(f(\gamma_s^{-1}), z_k)\}_{\substack{k=1,2,\dots m,\\ s=1,2,\dots,N}}.$$

Fixed s and k, in view of (12) $\bar{B}_{m,\ell}(f(\gamma_s^{-1}), z_k)$ requires the evaluation of the vector $\bar{\mathbf{p}}(z_k)$ performable in m^2 flops, and $(m+1)$ evaluations of the composite function $f(\gamma_s^{-1})$. Overall, S needs $\mathcal{O}\left(N \times m^3\right)$ flops and $N \times (m+1)$ function evaluations where $N = \left\lfloor \frac{\omega}{\pi} \right\rfloor + 1$.

5 Numerical Tests

In this section we carry out five examples to show the performance of our method and confirm the theoretical estimates established in Section 4. In the first three examples, the exact solution f^* is known and to show the accuracy of the proposed method we compute, for fixed values of the parameter ℓ and ω, the relative errors

$$\epsilon_m^{(\ell,\omega)}(f) = \frac{\|f_m^{(\ell)} - f^*\|_\infty}{\|f^*\|_\infty},$$

where the infinity norm is taken on the equispaced points $y_i \in [-1,1]$ for $i = 1,\dots,10^3$. In the last two examples, we do not know the exact solution of equations. Therefore, we compute for fixed values of the parameter ℓ and ω, the relative errors

$$\xi_m^{(\ell,\omega)}(f) = \frac{\|f_{2m}^{(\ell)} - f_m^{(\ell)}\|_\infty}{\|f_{2m}^{(\ell)}\|_\infty},$$

where, also here, the infinity norm is taken on the equispaced points $y_i \in [-1,1]$ for $i = 1,\dots,10^3$. Moreover, in Figure 1 we illustrate the approximated solutions of these last two examples.

In all the experiments we show the well conditioning of the final linear system we solve, by reporting the condition number in infinity norm of the matrix $G_{m+1}^{(\ell)}$ defined in (31).

All the computations were carried out in Matlab R2021b in double precision on an Intel Core i7-2600 system (8 cores), under the Debian GNU/Linux operating system.

5.1 Example 1

Let us test the accuracy of the method on the following equation:

$$f(y) + \frac{1}{2}\int_{-1}^{y} (e^{2(x-y)} - x) f(x)dx + \frac{1}{3}\int_{-1}^{1} \sin(\omega(x-y)) f(x)dx = g(y), \quad y \in [-1,1],$$

where $g(y) = \cos(y) + \frac{1}{2}H_1(y) + \frac{1}{3}H_2(y)$, with

$$H_1(y) = \cos(1) + \sin(1) - \frac{3}{5}\cos(y) + \left(\frac{1}{5} - y\right)\sin(y) + \frac{1}{5}e^{-2(1+y)}(\sin(1) - 2\cos(1)),$$

$$H_2(y) = \frac{2}{\omega^2 - 1}\sin(\omega y)(\cos(\omega)\sin(1) - \omega\cos(1)\sin(\omega)).$$

The solution is the smooth function $f^*(y) = \cos y$. Table 1 displays a fast convergence of the approximate solution to the exact one, by virtue of the smoothness of the known functions, as well as good condition numbers, for each fixed value of $\omega = 10, 10^2, 10^3$. In Table 2 we show the relative errors for the fixed value of $m = 64$ and increasing values of ℓ. The results highlight that if $\ell \ge 2^6$ we do not get any improvement in the accuracy.

Table 1: Numerical results for Example 5.1 with $\ell = 2^6$

m	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$
4	10	1.62e-02	2.80e+00	10^2	1.69e-02	2.77e+00	10^3	1.66e-02	2.75e+00
8		1.15e-04	2.60e+00		1.16e-04	2.26e+00		1.16e-04	2.26e+00
16		4.26e-08	2.52e+00		4.16e-08	1.98e+00		4.16e-08	1.97e+00
32		1.09e-12	2.60e+00		1.06e-12	2.01e+00		1.06e-12	1.99e+00
64		2.36e-15	2.64e+00		2.47e-15	2.06e+00		2.95e-15	2.00e+00

Table 2: Numerical results for Example 5.1 with $m = 64$

ℓ	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$
2	10	1.39e-04	2.50e+00	10^2	1.38e-04	2.01e+00	10^3	1.38e-04	1.99e+00
2^2		1.83e-06	2.59e+00		1.82e-06	2.02e+00		1.82e-06	1.99e+00
2^3		2.71e-09	2.62e+00		2.66e-09	2.02e+00		2.66e-09	2.00e+00
2^4		1.91e-12	2.63e+00		1.93e-12	2.03e+00		1.93e-12	2.00e+00
2^5		2.87e-15	2.63e+00		2.71e-15	2.04e+00		3.61e-15	2.00e+00
2^6		2.36e-15	2.64e+00		2.47e-15	2.06e+00		2.95e-15	2.00e+00
2^7		4.66e-15	2.65e+00		5.27e-15	2.10e+00		6.64e-15	2.00e+00
2^8		1.28e-14	2.67e+00		8.19e-15	2.19e+00		8.35e-15	2.00e+00

5.2 Example 2

Let us apply our method to the following equation

$$f(y) + \frac{1}{5}\int_{-1}^{y}(x^2+y^2)f(x)dx + \frac{1}{7}\int_{-1}^{1}\cos(\omega(x-y))f(x)dx = e^{y+1} + \frac{1}{5}I_1(y) + \frac{I_2(y)}{7(1+\omega^2)},$$

where

$$I_1(y) = 2e^{y+1}(y^2-y+1) - y^2 - 5$$
$$I_2(y) = e^2(\omega\sin(\omega(1-y)) + \cos(\omega(1-y)) + \omega\sin(\omega(1+y)) - \cos(\omega(1+y)).$$

The exact solution is $f^*(y) = e^{y+1}$. Table 3 contains the errors and the condition number of G_{m+1}^{ℓ} for increasing values of m. Again, we can deduce that the method is convergent and leads to a well-conditioning system. Moreover, even if the wavenumber ω is large, the errors go quickly to zero as the kernel $h(x,y)$ of the operator $\mathcal{V}$ and the right-hand side g are smooth.

Table 3: Numerical results for Example 5.2 with $\ell = 2^6$

m	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\mathrm{cond}\left(G_{m+1}^{(\ell)}\right)$
4	10	1.69e-03	2.08e+00	10^2	2.35e-03	2.05e+00	10^3	2.04e-03	2.06e+00
8		2.96e-06	2.16e+00		2.78e-06	2.14e+00		2.78e-06	2.15e+00
16		2.77e-10	2.24e+00		3.17e-10	2.19e+00		3.16e-10	2.20e+00
32		3.86e-15	2.25e+00		4.04e-15	2.21e+00		6.58e-15	2.22e+00
64		1.44e-15	2.26e+00		1.75e-15	2.22e+00		3.50e-15	2.23e+00

5.3 Example 3

Let us consider the following equation

$$f(y) + \frac{1}{20}\int_{-1}^{y}\sinh(x+y)f(x)dx + \frac{1}{30}\int_{-1}^{1}\cos(\omega(x-y))f(x)dx = g(y), \quad y \in [-1,1],$$

where the right-hand side g is chosen so that the exact solution is the oscillating function $f^*(y) = \cos(17y)$. In Table 4 we report the results. With respect to the first two examples, we need to increase m and ℓ since the solution is oscillating. Nevertheless, this increase of m and ℓ does not have any impact on the well-conditioning of the linear systems.

Table 4: Numerical results for Example 5.3 with $\ell = 2^{12}$

m	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\epsilon_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$
32	10	5.35e-01	1.69e+00	10^2	3.69e-01	1.49e+00	10^3	2.70e-01	1.54e+00
64		3.35e-05	1.52e+00		2.07e-05	1.46e+00		1.72e-05	1.44e+00
128		1.26e-11	1.41e+00		1.38e-11	1.39e+00		1.40e-11	1.37e+00
256		4.24e-12	1.35e+00		2.72e-12	1.35e+00		2.90e-12	1.33e+00

5.4 Example 4

Let us now consider an equation whose exact solution is not known

$$f(y) + \frac{1}{4}\int_{-1}^{y} e^{x+y} f(x)dx + \frac{1}{\pi}\int_{-1}^{1} \cos(\omega(x-y))f(x)dx = \left|y - \frac{1}{2}\right|^{\frac{5}{2}}, \quad y \in [-1,1].$$

Here, the two kernels are smooth function but the right-hand side $g \in Z_{5/2}$. Hence, according to Theorem 4.1 we expect a convergence of order $\mathcal{O}(m^{-5/4})$. The numerical results contained in Table 5 shows a faster convergence with respect to the theoretical expectation. Moreover, it exhibits the well-conditioning of the linear system (31).

Table 5: Numerical results for Example 5.4 with $\ell = 2^8$

m	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$
4	10	1.60e-01	3.83e+00	10^2	1.81e-02	3.83e+00	10^3	1.24e-02	3.84e+00
8		3.49e-02	6.02e+00		1.24e-02	4.74e+00		1.56e-03	4.75e+00
16		3.20e-03	5.54e+00		2.47e-03	4.84e+00		5.04e-04	4.85e+00
32		9.36e-06	5.12e+00		1.51e-04	4.92e+00		1.04e-04	4.93e+00
64		1.81e-06	5.12e+00		4.10e-05	4.97e+00		4.32e-05	4.97e+00
128		3.37e-07	5.14e+00		8.72e-06	5.10e+00		1.75e-05	5.00e+00
256		5.70e-08	5.16e+00		5.78e-06	5.07e+00		1.28e-05	5.02e+00

5.5 Example 5

Let us apply our method to the following equation

$$f(y) + \frac{1}{2\pi}\int_{-1}^{y} \log(x+y+4)f(x)dx + \frac{1}{9}\int_{-1}^{1} \sin(\omega(x-y))f(x)dx = y\sin(25y), \quad y \in [-1,1],$$

whose solution is not known analytically. In this case, even if the known functions are smooth, due to the oscillations of the kernel $\sin(\omega(x-y))$ and of the right-hand side term $y\sin(25y)$, it is necessary to increase the values of m and ℓ to get better results. The machine precision is not achieved but we suppose that this behavior can be justified by the oscillating nature of the unknown solution $f^*(y)$. Indeed, it has been proved that the solution of integral equations of the type (1) inherits the highly oscillating nature of the involved kernels; see, for instance [2, 18].

Table 6: Numerical results for Example 5.5 with $\ell = 2^{12}$

m	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$	ω	$\xi_m^{(\ell,\omega)}(f)$	$\text{cond}\left(G_{m+1}^{(\ell)}\right)$
32	10	2.74e-01	2.87e+00	10^2	2.98e-01	3.02e+00	10^3	1.21e-01	3.17e+00
64		2.52e-03	2.54e+00		2.03e-03	2.64e+00		2.99e-03	2.68e+00
128		2.02e-08	2.36e+00		1.96e-04	2.48e+00		7.10e-05	2.40e+00
256		2.73e-13	2.24e+00		5.36e-06	2.32e+00		7.22e-06	2.26e+00

6 The proofs

Proof of Theorem 2.2. In view of (15), setting $g(y) := f(2y-1)$ with $y = \frac{1+x}{2}$, we have

$$\max_{x\in[-1,1]} |f(x) - \bar{B}_{m,\ell}(f,x)| = \max_{y\in[0,1]} |g(y) - B_{m,\ell}(g,y)|,$$

and consequently Theorem 2.1 yields

$$\max_{y\in[0,1]} |g(y) - B_{m,\ell}(g,y)| \leq \mathcal{C}\omega_{\varphi_1}^{2\ell}\left(g, \frac{1}{\sqrt{m}}\right) + \frac{1}{m^\ell}\max_{y\in[0,1]}|g(y)|.$$

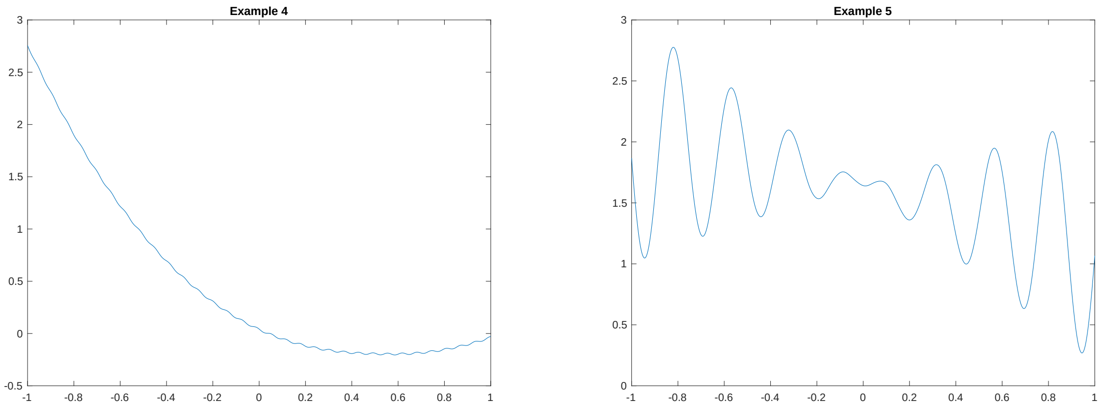

Figure 1: Plot of the solutions $f_{32}^{(2^8)}$ and $f_{256}^{(2^{12})}$ of Examples 4 and 5, respectively, with $\omega = 100$.

Then the statement of the theorem follows once the following inequality is proved

$$\omega_{\varphi_1}^r(g,t) \sim \mathcal{C}\omega_\varphi^r(f,t). \tag{36}$$

To this end, we recall [4]

$$\omega_{\varphi_1}^r(g,t) \sim K_{r,\varphi_1}(g,t^r) := \inf_{h\in H}\left\{\|h-g\|_{[0,1]} + t^r\|h^{(r)}\varphi_1^r\|_{[0,1]}\right\},$$

where $H = \{h:\ h^{(r-1)} \in AC(0,1)\}$ with $AC(a,b)$ the set of the absolutely continuous function in every closed subset $[c,d] \subset (a,b)$. Noting that the following equalities hold true

$$\sup_{y\in[0,1]}|h(y)-g(y)| = \sup_{x\in[-1,1]}\left|\tilde{h}(x)-f(x)\right|, \qquad \text{with} \qquad \tilde{h}(x) = h\left(\frac{1+x}{2}\right),$$
$$\sup_{y\in[0,1]}|h^{(r)}(y)\varphi_1^r(y)| = \sup_{x\in[-1,1]}\left|\tilde{h}^{(r)}(x)\varphi^r(x)\right|,$$

it follows

$$\|h-g\|_{[0,1]} + t^r\|h^{(r)}\varphi_1^r\|_{[0,1]} = \|\tilde{h}-f\| + t^r\|\tilde{h}^{(r)}\varphi_r\|.$$

Therefore, since the space $H \equiv \tilde{H} := \{\tilde{h}:\ h^{(r-1)} \in AC(-1,1)\}$, taking the infimum on $H \equiv \tilde{H}$ it follows

$$\omega_{\varphi_1}^r(g,t) \sim K_{r,\varphi_1}(g,t^r) = K_{r,\varphi}(f,t^r) \sim \omega_\varphi^r(f,t)$$

from which (36) is deduced. □

Proof of Theorem 3.1. We start from

$$|(\mathcal{V}f)(y) - (\mathcal{V}_m^{(\ell)}f)(y)| \le \int_{-1}^{1} |f(x)h(x,y) - \bar{B}_{m,\ell}(fh(\cdot,y),x)|dx.$$

In view of the continuity of f and h, by Theorem 2.2, we get

$$|(\mathcal{V}f)(y) - (\mathcal{V}_m^{(\ell)}f)(y)| \le \mathcal{C}\left(\omega_\varphi^{2\ell}\left(fh(\cdot,y);\frac{1}{\sqrt{m}}\right) + \frac{\|fh(\cdot,y)\|}{m^\ell}\right) \to 0 \qquad \forall y \in [-1,1],$$

by which (20) follows. Moreover, for $f \in Z_r$, by (16) we deduce

$$|(\mathcal{V}f)(y) - (\mathcal{V}_m^{(\ell)}f)(y)| \le \mathcal{C}\left(\frac{\mathcal{C}}{\sqrt{m^r}} + \frac{1}{m^\ell}\right)\|fh(\cdot,y)\|_{Z_r},$$

and under the assumption $\sup_{y\in[-1,1]} h(\cdot,y) \in Z_r$, we obtain inequality (21). □

Proof of Theorem 3.2. We start from (23) to obtain

$$\begin{aligned}|\mathcal{K}^\omega f(y) - \mathcal{K}_m^{(\omega,\ell)}f(y)| &\le |\mathcal{K}^\omega f(y) - \mathcal{I}_m^\omega f(y)| + |\mathcal{I}_m^\omega f(y) - \mathcal{K}_m^{(\omega,\ell)}f(y)|\\ &=: I_1(y) + I_2(y).\end{aligned} \tag{37}$$

Theorem 2.2 yields

$$I_1(y) \le \mathcal{C}\left(\omega_\varphi^{2\ell}\left(f;\frac{1}{\sqrt{m}}\right)+\frac{\|f\|}{m^\ell}\right)\int_{-1}^{1}|k_\omega(x,y)|dx \to 0,$$

whereas for the second term $I_2(y)$, we start writing

$$\begin{aligned} I_2(y) &= \frac{1}{N}\left|\sum_{s=1}^{N}\int_{-1}^{1}k_\omega(\gamma_s^{-1}(z),y)\bar{B}_{m,\ell}(f(\gamma_s^{-1}),z)dz-\sum_{k=1}^{m}k_\omega(\gamma_s^{-1}(z_k),y)\bar{B}_{m,\ell}(f(\gamma_s^{-1}),z_k))\lambda_{m,k}\right| \\ &= \frac{1}{N}\left|\sum_{j=0}^{m}f(x_j)\sum_{i=0}^{m}c_{i,j}^{(m,\ell)}\sum_{s=1}^{N}\varepsilon_m^{i,s}(y)\right|, \end{aligned} \tag{38}$$

with

$$\varepsilon_m^{i,s}(y)=\int_{-1}^{1}\bar{p}_{m,i}(\gamma_s^{-1}(z))k_\omega(\gamma_s^{-1}(z),y)dz-\sum_{k=1}^{m}\lambda_{m,k}\bar{p}_{m,i}(\gamma_s^{-1}(z_k))k_\omega(\gamma_s^{-1}(z_k),y).$$

Now, the term $|\varepsilon_m^{i,s}(y)|$ can be estimated as done in the proof of [7, Theorem 3.2], i.e.

$$|\varepsilon_m^{i,s}(y)| \le \mathcal{C}\sqrt{m}\left(\frac{1}{N}\right)^m\left(\frac{m+\omega}{2m-1}\right)^m.$$

Hence, by combining the above estimate and (38), we have

$$I_2(y) \le \|f\|\frac{1}{N}\sum_{j=0}^{m}\sum_{i=0}^{m}|c_{i,j}^{(m,\ell)}|\sum_{s=1}^{N}|\varepsilon_m^{i,s}(y)| \le \mathcal{C}\|f\|\sum_{i=0}^{m}\|C_{m,\ell}\|_\infty\sqrt{m}\left(\frac{1}{N}\right)^m\left(\frac{m+\omega}{2m-1}\right)^m. \tag{39}$$

Then, taking into account that $\|C_{m,\ell}\|_\infty \le 2^{\ell-1} \le \mathcal{C}$ with $\mathcal{C} \ne \mathcal{C}(m)$, and using the inequality $\frac{1}{N} \le \frac{\pi}{\omega}$, we have

$$I_2(y) \le \mathcal{C}\|f\|m^{\frac{3}{2}}\left(\frac{\pi}{\omega}\right)^m\left(\frac{m+\omega}{2m-1}\right)^m.$$

Consequently, (27) follows by combining last inequality and (38) with (37). Finally, (28) can be deduced by (16). □

In order to prove Theorem 4.1, we recall the following lemma proved in [8] in a more general context.

Lemma 6.1. *Under the assumption*

$$(1+y)\sup_{x\in[-1,1]}|h(x,y)| \in Z_r, \tag{40}$$

then $\mathcal{V}f \in Z_r$*, for any* $f \in C^0$.

Proof of Theorem 4.1. In view of Theorems 3.1 and 3.2, both the quadrature formulas (19) and (25) used in the Nyström method are convergent and then the sequences $\{\mathcal{V}_m^{(\ell)}\}_m$ and $\{\mathcal{K}_m^{(\omega,\ell)}\}_m$ are collectively compact [12, Theorem 12.8]. Then for each fixed ℓ [1, p. 114] and from [1, Theorem 4.1.1 p. 106] the operators

$$(\mathcal{I}+\mu_1\mathcal{V}_m^{(\ell)}+\mu_2(\mathcal{K}_m^{(\omega,\ell)})^{-1} : C^0 \to C^0,$$

exist and are uniformly bounded with respect to m. Under the assumption on g and h, the solution f^* of (29) belongs to C^0. Then, by virtue of [1, Theorem 4.1.1 p. 106] we can conclude that for any $\ell > 0$

$$\lim_m \|f^*-f_m^{(\ell)}\| = 0. \tag{41}$$

Now, to obtain the error estimate we observe that under the assumption (40), $\mathcal{V}f \in Z_r$, and being also $\mathcal{K}f \in Z_r$, for any $f \in C^0$, as a consequence, $f^* \in Z_r$, for any $g \in Z_r$. Then, by estimates (22) and (28), for $0 < r \le 2\ell$, the following error estimate holds

$$\|f^*-f_m^{(\ell)}\| \le \mu_1\|(\mathcal{V}-\mathcal{V}_m^{(\ell)})f^*\|+\mu_2\|(\mathcal{K}-\mathcal{K}_m^{(\omega,\ell)})f^*\| \le \mathcal{C}\frac{\|f^*\|_{Z_r}}{(\sqrt{m})^r}. \tag{42}$$

□

7 Conclusions

The Nyström method presented here for mixed Volterra-Fredholm integral equations can represent a valid alternative to the common methods based on equally spaced nodes, as those achieved by piecewise polynomial approximation, or classical Bernstein polynomials. The method depends on an additional parameter ℓ that can be suitably modulated to improve the rate of convergence, as the smoothness of the involved functions increases. The presence of the kernel k_ω, highly oscillating for "large" values of ω is "well handled" by means of a product rule based on a subdivision technique, relaxing high frequencies. The Nyström method is stable and convergent in C^0 and, when the solution belongs to an Hölder-Zygmund space of order r, the order of convergence is $\mathcal{O}\left(m^{-r/2}\right)$, choosing $\ell > \frac{r}{2}$, as well as the proposed numerical tests confirm.

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