ON THE STRUCTURE OF BOCHVAR ALGEBRAS

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Abstract. Bochvar algebras consist of the quasivariety BCA playing the role of equivalent algebraic semantics for Bochvar (external) logic, a logical formalism introduced by Bochvar [4] in the realm of (weak) Kleene logics. In this paper, we provide an algebraic investigation of the structure of Bochvar algebras. In particular, we prove a representation theorem based on Płonka sums and investigate the lattice of subquasivarieties, showing that Bochvar (external) logic has only one proper extension (apart from classical logic), algebraized by the subquasivariety NBCA of BCA. Furthermore, we address the problem of (passive) structural completeness ((P)SC) for each of them, showing that NBCA is SC, while BCA is not even PSC. Finally, we prove that both BCA and NBCA enjoy the amalgamation property (AP).

§1. Introduction. The recent years have seen a renaissance of interests and studies around weak Kleene logics, logical formalisms that were considered, in the past, not particularly attractive in the panorama of three-valued logics, due to reputed "odd" behavior of the third-value. The late (re)discovery of weak Kleene logic regards, almost exclusively, *internal* rather than *external* logics: the latter, in essence, consisting of linguistic expansions of the former. More precisely, here, for external Kleene logics we understand the external version of Bochvar logic (introduced by Bochvar himself [4]) and of Paraconsistent weak Kleene logic (introduced by Segerberg [35]).

The idea of considering the external connectives, thus enriching the (internal) logical vocabulary, is originally due to Russian logician D. Bochvar [4]. His aim was, from the one side, to adopt a non-classical base to get rid of set-theoretic and semantic paradoxes (by interpreting them to $1/2$) and, from the other, to preserve the expressiveness of classical logic. Although his attempt failed in reaching the former purpose (as it can be shown that paradoxes resurface [37]), the work of Bochvar has left us with a logic extremely rich in expressivity and whose potential has yet to be discovered and applied in its full capacity. Indeed, external weak Kleene logics have the advantage of limiting the infectious behavior of the third value—the feature making them apparently little attractive—which is confined to internal formulas only, and to recover all the consequences of classical logic in the purely external part of the language (this holding true for Bochvar external logic only). We believe that these features may turn out to

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be very useful in computer science and AI, providing new tools for modeling errors, concurrence, and debugging.

From a mathematical viewpoint, internal weak Kleene logics show quite a weak connection with respect to their algebraic counterparts, as they are examples of the so-called non-protoalgebraic logics. On the other hand, a recent work [6] has shown that Bochvar external logic is algebraizable with the quasivariety of Bochvar algebras (introduced in [17]) as its equivalent algebraic semantics. This observation gives a justified starting point for a deeper, intertwined, investigation of Bochvar external logic and Bochvar algebras, which is the main scope of the present work.

The flourishing trend of algebraic research around weak Kleene logics has strongly connected them with the algebraic theory of Płonka sums (see, e.g., [9]). Recently, the tools offered by Płonka sums have fruitfully been extended to the structural analysis of residuated structures, establishing a natural connection with substructural logics (see [19, 22]). In line with this trend, we will further extend the application of the method. Not surprisingly, since Bochvar external logic is a linguistic expansion of Bochvar logic, we will show that the construction of the Płonka sum will play an important role in characterizing the structure of Bochvar algebras.

The paper is organized into five sections: in Section 2, we present Bochvar external logic as the logic induced by a single matrix, and we recall the axiomatization due to Finn and Grigolia. In Section 3, we first introduce the quasivariety of Bochvar algebras and prove some basic facts, including that any Bochvar algebra has an involutive bisemilattice reduct. We then proceed by describing the structure of Bochvar algebras: the main result is a representation theorem in terms of Płonka sums of Boolean algebras plus some additional operations. Section 4 is concerned with the study of the lattice of subquasivarieties of the quasivariety of Bochvar algebras, which is dually isomorphic to the lattice of extension of Bochvar external logic. We show that there are only three nontrivial quasivarieties of Bochvar algebras and we address the problem of (passive) structural completeness for each of them. Finally, in Section 5, we show that every quasivariety of Bochvar algebras has the amalgamation property. We conclude the paper with Appendix A, where we provide a new (quasi)equational basis for the quasivariety of Bochvar algebras. The proposed axiomatization significantly simplifies the traditional one introduced by Finn and Grigolia [17].

§2. Bochvar external logic. Kleene's three-valued logics—introduced by Kleene in his *Introduction to Metamathematics* [23]—are traditionally divided into two families, depending on the meaning given to the connectives: *strong Kleene* logics—counting strong Kleene and the logic of paradox—and *weak Kleene* logics, namely Bochvar logic [4] and paraconsistent weak Kleene logic (sometimes referred to as Hallden's logic [21]). Kleene logics are traditionally defined over the (algebraic) language of classical logic. However, the intent of one of the first developers of these formalisms, D. Bochvar, was to work within an enriched language allowing to express all classical "two-valued" formulas—which he referred to as *external formulas*—beside the genuinely "three-valued" ones.

The result of this choice is the language $\mathcal{L}$: $\langle \neg, \vee, \wedge, J_0, J_1, J_2, 0, 1\rangle$ (of type $(1, 2, 2, 1, 1, 1, 0, 0)$), which is obtained by enriching the classical language by three unary connectives J_0, J_1, J_2 (and the constants $0, 1$). The language $\mathcal{L}$ can be referred to as *external language*, in contrast with the traditional language upon which Kleene

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	$\neg$
1	0
1/2	1/2
0	1

$\vee$	0	1/2	1
0	0	1/2	1
1/2	1/2	1/2	1/2
1	1	1/2	1

$\wedge$	0	1/2	1
0	0	1/2	0
1/2	1/2	1/2	1/2
1	0	1/2	1

φ	$J_0\varphi$
1	0
1/2	0
0	1

φ	$J_1\varphi$
1	0
1/2	1
0	0

φ	$J_2\varphi$
1	1
1/2	0
0	0

Figure 1. The algebra $\mathbf{WK}^e$.

logics are defined. Let **Fm** refer to the formula algebra over the language $\mathcal{L}$, and to *Fm* as its universe.

The intended algebraic interpretation of the language $\mathcal{L}$ is traditionally given via the three-elements algebra $\mathbf{WK}^e = \langle\{0, 1, 1/2\}, \neg, \vee, \wedge, J_0, J_1, J_2, 0, 1\rangle$ displayed in Figure 1.

The value 1/2 is traditionally read as "meaningless" (see, e.g., [15, 36]) due to its infectious behavior. It is immediate to check that the $\vee, \wedge$-reduct of $\mathbf{WK}^e$ is not a lattice (it is an involutive bisemilattice), as it fails to satisfy absorption, hence the operations $\vee$ and $\wedge$ induce two (different) partial orders. In the following, we will refer to $\leq$ as the one induced by $\vee$ (i.e., $x \leq y$ iff $x \vee y = y$). With reference to such order, it holds $0 < 1 < 1/2$.

The language $\mathcal{L}$ allows to define the so-called *external* formulas (see Definition 2.3), namely those that are evaluated into $\{0, 1\}$ *only* (which is the universe of a Boolean subalgebra of $\mathbf{WK}^e$), for any homomorphism $h\colon \mathbf{Fm} \to \mathbf{WK}^e$ ($J_k\varphi$, for any $\varphi \in Fm$ and $k \in \{0, 1, 2\}$, are examples of external formulas).

Definition 2.1. *Bochvar* external *logic* B^e *is the logic induced by the matrix* $\langle\mathbf{WK}^e, \{1\}\rangle$.

In words, B^e is the logic with the only distinguished value 1.[1] B^e is a linguistic expansion of Bochvar logic B, which is defined by the matrix $\langle\mathbf{WK}, \{1\}\rangle$, where **WK** is the J_k-free reduct of $\mathbf{WK}^e$. Since B^e is defined by a finite set of finite matrices, it is a finitary logic, in the sense that $\Gamma \vdash_{\mathrm{B}^e} \varphi$ entails $\Delta \vdash_{\mathrm{B}^e} \varphi$ for some finite $\Delta \subseteq \Gamma$ (the relation among B^e and other three-valued logics can be found in [12]).

The following technicalities are needed to introduce a Hilbert-style axiomatization of B^e.

Definition 2.2. *An occurrence of a variable* x *in a formula* φ *is* open *if it does not fall under the scope of* J_k, *for every* $k \in \{0, 1, 2\}$. *A variable* x *in* φ *is* covered *if all of its occurrences are not open, namely if every occurrence of* x *in* φ *falls under the scope of* J_k, *for some* $k \in \{0, 1, 2\}$.

The intuition behind the notion of external formulas is made precise by the following.

Definition 2.3. *A formula* $\varphi \in Fm$ *is called* external *if all its variables are covered.*

[1] The different choice (on the same formula algebra) of the truth set $\{1, 1/2\}$ defines the logic H_0 studied by Segerberg [35].

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A Hilbert-style axiomatization of B^e has been introduced by Finn and Grigolia [17]. In order to present it, let

$$\varphi \equiv \psi := \bigwedge_{i=0}^{2} J_i\varphi \leftrightarrow J_i\psi,$$

and α, β, γ denote external formulas.
Axioms

(A1) $(\varphi \vee \varphi) \equiv \varphi$;
(A2) $(\varphi \vee \psi) \equiv (\psi \vee \varphi)$;
(A3) $((\varphi \vee \psi) \vee \chi) \equiv (\varphi \vee (\psi \vee \chi))$;
(A4) $(\varphi \wedge (\psi \vee \chi) \equiv ((\varphi \wedge \psi) \vee (\varphi \wedge \chi))$;
(A5) $\neg(\neg\varphi) \equiv \varphi$;
(A6) $\neg 1 \equiv 0$;
(A7) $\neg(\varphi \vee \psi) \equiv (\neg\varphi \wedge \neg\psi)$;
(A8) $0 \vee \varphi \equiv \varphi$;
(A9) $J_2\alpha \equiv \alpha$;
(A10) $J_0\alpha \equiv \neg\alpha$;
(A11) $J_1\alpha \equiv 0$;
(A12) $J_i\neg\varphi \equiv J_{2-i}\varphi$, for any $i \in \{0, 1, 2\}$;
(A13) $J_i\varphi \equiv \neg(J_j\varphi \vee J_k\varphi)$, with $i \neq j \neq k \neq i$;
(A14) $(J_i\varphi \vee \neg J_i\varphi) \equiv 1$, with $i \in \{0, 1, 2\}$;
(A15) $((J_i\varphi \vee J_k\psi) \wedge J_i\varphi) \equiv J_i\varphi$, with $i, k \in \{0, 1, 2\}$;
(A16) $(\varphi \vee J_i\varphi) \equiv \varphi$, with $i \in \{1, 2\}$;
(A17) $J_0(\varphi \vee \psi) \equiv J_0\varphi \wedge J_0\psi$;
(A18) $J_2(\varphi \vee \psi) \equiv (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(A19) $\alpha \to (\beta \to \alpha)$;
(A20) $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$;
(A21) $\alpha \wedge \beta \to \alpha$;
(A22) $\alpha \wedge \beta \to \beta$;
(A23) $(\alpha \to \beta) \to ((\alpha \to \gamma) \to (\alpha \to \beta \wedge \gamma))$;
(A24) $\alpha \to \alpha \vee \beta$;
(A25) $\beta \to \alpha \vee \beta$;
(A26) $(\alpha \to \gamma) \to ((\beta \to \gamma) \to (\alpha \vee \beta \to \gamma))$;
(A27) $(\alpha \to \beta) \to ((\alpha \to \neg\beta) \to \neg\alpha)$;
(A28) $\alpha \to (\neg\alpha \to \beta)$;
(A29) $\neg\neg\alpha \to \alpha$.

Deductive rule

$$[\text{MP}]\ \frac{\varphi \qquad \varphi \to \psi}{\psi}$$

Observe that the axiomatization contains a set of axioms (A19–A29), which, together with the rule of *modus ponens*, yields a complete axiomatization for classical logic (relative to external formulas).

The fact that B^e coincides with the logic induced by the above introduced Hilbert-style axiomatization has been proved in [6] (Finn and Grigolia [17, theorem 3.4] only proved a weak completeness theorem for B^e). We will henceforth indicate by $(\mathbf{Fm}, \vdash_{B^e})$

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both the consequence relation induced by the matrix $\langle \mathbf{WK}^e, \{1\}\rangle$ and the one induced by the above Hilbert-style axiomatization.

The logic B^e is algebraizable with the quasivariety of Bochvar algebras (BCA)—which will be properly introduced in the next section—as its equivalent algebraic semantics. This means that there exists maps $\tau\colon Fm \to \mathcal{P}(Eq)$, $\rho\colon Eq \to \mathcal{P}(Fm)$ from formulas to sets of equations and from equations to sets of formulas such that

$$\gamma_1, \dots, \gamma_n \vdash_{\mathsf{B}_e} \varphi \iff \tau(\gamma_1), \dots, \tau(\gamma_n) \vDash_{\mathsf{BCA}} \tau(\varphi)$$

and

$$\varphi \approx \psi \dashv\vDash_{\mathsf{BCA}} \tau\rho(\varphi \approx \psi).$$

The above conditions are verified by setting $\tau(\varphi) := \{x \approx 1\}$ and $\rho(\varphi \approx \psi) := \{\varphi \equiv \psi\}$ (see [6] for details). Moreover, Bochvar external logic enjoys the (global) deduction theorem, which we recall here.

THEOREM 2.4 (Deduction Theorem). *$\Gamma, \psi \vdash_{\mathrm{B}^e} \varphi$ if and only if $\Gamma \vdash_{\mathrm{B}^e} J_2\psi \to J_2\varphi$.*

§3. Bochvar algebras and Płonka sums. We assume the reader has some familiarity with universal algebra and abstract algebraic logic (standard references are [3, 10] and [18], respectively). In what follows, given a class of algebras K, the usual class-operator symbols $I(\mathsf{K}), S(\mathsf{K}), H(\mathsf{K}), P(\mathsf{K}), P_u(\mathsf{K})$ denote the closure of K under isomorphic copies, subalgebras, homomorphic images, products, and ultraproducts. A class of similar algebras K is a quasivariety if $\mathsf{K} = ISPP_u(K)$. It is a variety if is also closed under homomorphic images or, equivalently, if $\mathsf{K} = HSP(\mathsf{K})$.

The class of *Bochvar algebras*, BCA for short, is introduced by Finn and Grigolia [17, pp. 233–234] as the algebraic counterpart for B^e.

DEFINITION 3.1. *A Bochvar algebra $\mathbf{A} = \langle A, \vee, \wedge, \neg, J_0, J_1, J_2, 0, 1\rangle$ is an algebra of type $\langle 2, 2, 1, 1, 1, 1, 0, 0\rangle$ satisfying the following identities and quasi-identities:*

(1) $\varphi \vee \varphi \approx \varphi$;
(2) $\varphi \vee \psi \approx \psi \vee \varphi$;
(3) $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
(4) $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
(5) $\neg(\neg\varphi) \approx \varphi$;
(6) $\neg 1 \approx 0$;
(7) $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
(8) $0 \vee \varphi \approx \varphi$;
(9) $J_2 J_k \varphi \approx J_k \varphi$, *for every* $k \in \{0, 1, 2\}$;
(10) $J_0 J_k \varphi \approx \neg J_k \varphi$, *for every* $k \in \{0, 1, 2\}$;
(11) $J_1 J_k \varphi \approx 0$, *for every* $k \in \{0, 1, 2\}$;
(12) $J_k(\neg\varphi) \approx J_{2-k}\varphi$, *for every* $k \in \{0, 1, 2\}$;
(13) $J_i \varphi \approx \neg(J_j \varphi \vee J_k \varphi)$, *for* $i \neq j \neq k \neq i$;
(14) $J_k \varphi \vee \neg J_k \varphi \approx 1$, *for every* $k \in \{0, 1, 2\}$;
(15) $(J_i \varphi \vee J_k \varphi) \wedge J_i \varphi \approx J_i \varphi$, *for* $i, k \in \{0, 1, 2\}$;
(16) $\varphi \vee J_k \varphi \approx \varphi$, *for* $k \in \{1, 2\}$;
(17) $J_0(\varphi \vee \psi) \approx J_0\varphi \wedge J_0\psi$;
(18) $J_2(\varphi \vee \psi) \approx (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(19) $J_0\varphi \approx J_0\psi \;\&\; J_1\varphi \approx J_1\psi \;\&\; J_2\varphi \approx J_2\psi \;\Rightarrow\; \varphi \approx \psi$.

BCA forms a quasivariety which is not a variety [16, 17], and it is generated by $\mathbf{WK}^{e}$, i.e., $\mathsf{BCA} = ISP(\mathbf{WK}^{e})$. This is true in virtue of [13, theorem 3.2.2], upon noticing that BCA algebraizes the logic B^{e}, which is defined by the single matrix $\langle \mathbf{WK}^{e}, \{1\} \rangle$. The fact that $\mathbf{WK}^{e}$ generates BCA was firstly stated by Finn and Grigolia [16, 17]. Familiar examples of Bochvar algebras can be obtained by appropriately computing the external functions over a Boolean algebra, as indicates the following example.

EXAMPLE 3.2. *Let* $\mathbf{A}$ *be a (non-trivial) Boolean algebra. Setting the functions* $J_k : A \to A$, *with* $k \in \{0, 1, 2\}$ *as* $J_2 = id$, $J_1 = 0$ *(the constant function onto* 0*) and* $J_0 = \neg$, *then* $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1, J_2, J_1, J_0 \rangle$ *is a Bochvar algebra.*

For this reason, by $\mathbf{B}_n$ we will safely denote both the n-elements Boolean algebra and its BCA expansion obtained according to Example 3.2. Since $\mathbf{WK}^{e}$ generates BCA, a quasi-equation holds in $\mathbf{WK}^{e}$ if and only if it holds in every $\mathbf{A} \in \mathsf{BCA}$.

The original quasi-equational basis for BCA, as provided in Definition 3.1, can be significantly enhanced by reducing the number of axioms and improving their intelligibility.[2] It is known that the operations J_0, J_1 can be defined as $J_2\neg\varphi$ and $\neg(J_2\varphi \vee J_2\neg\varphi)$, respectively. Thus, Bochvar algebras can be equivalently presented in the restricted language $\langle \vee, \wedge, \neg, J_2, 0, 1 \rangle$, and this is particularly convenient for our next goal, namely to provide a new, simpler quasi-equational basis for BCA. This is accomplished in the next theorem.

THEOREM 3.3. *The following is a quasi-equational basis for* BCA.

(1) $\varphi \vee \varphi \approx \varphi$;
(2) $\varphi \vee \psi \approx \psi \vee \varphi$;
(3) $(\varphi \vee \psi) \vee \delta \approx \varphi \vee (\psi \vee \delta)$;
(4) $\varphi \wedge (\psi \vee \delta) \approx (\varphi \wedge \psi) \vee (\varphi \wedge \delta)$;
(5) $\neg(\neg\varphi) \approx \varphi$;
(6) $\neg 1 \approx 0$;
(7) $\neg(\varphi \vee \psi) \approx \neg\varphi \wedge \neg\psi$;
(8) $0 \vee \varphi \approx \varphi$;
(9) $J_0 J_2 \varphi \approx \neg J_2 \varphi$;
(10) $J_2\varphi \approx \neg(J_0\varphi \vee J_1\varphi)$;
(11) $J_2\varphi \vee \neg J_2\varphi \approx 1$;
(12) $J_2(\varphi \vee \psi) \approx (J_2\varphi \wedge J_2\psi) \vee (J_2\varphi \wedge J_2\neg\psi) \vee (J_2\neg\varphi \wedge J_2\psi)$;
(13) $J_0\varphi \approx J_0\psi \;\&\; J_2\varphi \approx J_2\psi \;\Rightarrow\; \varphi \approx \psi$,

where $J_0\varphi \approx J_2\neg\varphi$ *and* $J_1\varphi \approx \neg(J_2\varphi \vee J_0\varphi)$.

The proof of the above Theorem 3.3 requires a significant amount of computations, which are included in the Appendix. Notice that, although J_0, J_1 are definable from the remaining operations of BCA, a detailed investigation of the semantic properties of the full language significantly improve the logical and algebraic understanding of BCA: this is why in several subsequent parts of the paper we will explicitly refer to the full stock of operations.

We now introduce the variety of involutive bisemilattices, which plays a key role to understand the structure theory of Bochvar algebras.

[2] We thank an anonymous referee for pointing this out.

DEFINITION 3.4. *An* involutive bisemilattice *is an algebra* $\mathbf{B} = \langle B, \wedge, \vee, \neg, 0, 1\rangle$ *of type* $(2, 2, 1, 0, 0)$ *satisfying*:

I1. $\varphi \vee \varphi \approx \varphi$;
I2. $\varphi \vee \psi \approx \psi \vee \varphi$;
I3. $\varphi \vee (\psi \vee \delta) \approx (\varphi \vee \psi) \vee \delta$;
I4. $\neg\neg\varphi \approx \varphi$;
I5. $\varphi \wedge \psi \approx \neg(\neg\varphi \vee \neg\psi)$;
I6. $\varphi \wedge (\neg\varphi \vee \psi) \approx \varphi \wedge \psi$;
I7. $0 \vee \varphi \approx \varphi$;
I8. $1 \approx \neg 0$.

The class of involutive bisemilattices forms a variety, which we denote by IBSL. IBSL is the so-called *regularization* of the variety of Boolean algebras: this is the variety satisfying all and only the regular identities that hold in Boolean algebras, namely those identities where exactly the same variables occur on both sides of the equality symbol. The variety IBSL is generated by the three element algebra **WK**, i.e., IBSL $= HSP(\mathbf{WK})$ (see [9, chap. 2], [7, 29]). It is not a direct consequence of Definition 3.1 (nor of Theorem 3.3) that the J_k-free reduct of any Bochvar algebra is an involutive bisemilattice: indeed Definition 3.1 implies that such reduct is a De Morgan bisemilattice (the regularization of de Morgan algebras). However, it is immediate to check that the identity **I6** in Definition 3.4 holds in any Bochvar algebra, as it does in $\mathbf{WK}^{\mathrm{e}}$. Although being the variety generated by the matrix that defines B, IBSL is not its algebraic counterpart, which rather consists of the proper quasivariety generated by **WK**. This quasivariety is called *single-fixpoint involutive bisemilattices*, SIBSL for short,[3] as its members are precisely the involutive bisemilattices with at most one fixpoint for negation, namely those containing at most one element a such that $a = \neg a$.

The examples of Bochvar algebras we have made so far ($\mathbf{WK}^{\mathrm{e}}$ and any Boolean algebra) consist of algebras having an SIBSL-reduct; this is actually true for any Bochvar algebra.

PROPOSITION 3.5. *Every Bochvar algebra has a* SIBSL*-reduct.*

Proof. It suffices to check that every valid SIBSL-quasi-equation is also valid in BCA. To see this, let χ be a quasi-equation in the language of **WK**. We have that

$$\begin{aligned} \mathsf{SIBSL} \vDash \chi &\iff \\ \mathbf{WK} \vDash \chi &\iff \\ \mathbf{WK}^{\mathrm{e}} \vDash \chi &\iff \\ \mathsf{BCA} \vDash \chi&. \end{aligned}$$

The first equivalence holds because **WK** generates SIBSL, the second one because **WK** is the $\langle J_0, J_1, J_2\rangle$-free reduct of $\mathbf{WK}^{\mathrm{e}}$ and the last one because $\mathbf{WK}^{\mathrm{e}}$ generates BCA. □

Clearly, since SIBSL $\subset$ IBSL, every Bochvar algebra has an IBSL-reduct. Although this fact, it is not the case that any (single-fixpoint) involutive bisemilattice can be turned into a Bochvar algebra: the reasons will be clear in the last part of the section.

[3] This quasivariety is firstly investigated in [7], and [26] contains information on its constant-free formulation.

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The fact that the J_k-free reduct of every Bochvar algebra is an involutive bisemilattice (Proposition 3.5) carries to the relevant observation that any such reduct can be represented as a Płonka sum of Boolean algebras [7]. Płonka sums are general constructions introduced by the polish mathematician J. Płonka [27, 28, 30] (more comprehensive expositions are [9, chap. 2], [31, 33])—and now going under his name. In brief, the construction consists of "summing up" similar algebras, organized into a semilattice direct system and connected via homomorphisms, into a new algebra. In more details, the (semilattice direct) system is formed by a family of similar algebras $\{\mathbf{A}_i\}_{i\in I}$ with disjoint universes, such that the index set I forms a lower-bounded semilattice $(I, \vee, i_0)$—we denote by $\leq$ the induced partial order—and, moreover, is made of a family $\{p_{ij}\}_{i\leq j}$ of homomorphisms $p_{ij} : \mathbf{A}_i \to \mathbf{A}_j$, defined from the algebra $\mathbf{A}_i$ to the algebra $\mathbf{A}_j$, whenever $i \leq j$, for $i, j \in I$. Such homomorphisms satisfy a further *compatibility property*: p_{ii} is the identity, for every $i \in I$ and $p_{jk} \circ p_{ij} = p_{ik}$, for every $i \leq j \leq k$.

Given a semilattice direct system of algebras $\langle\{\mathbf{A}_i\}_{i\in I}, (I, \vee, i_0), \{p_{ij}\}_{i\leq j}\rangle$, the Płonka sum over it is the new algebra $\mathbf{A} = \mathcal{P}_l(\mathbf{A}_i)_{i\in I}$ (of the same similarity type of the algebras $\{\mathbf{A}_i\}_{i\in I}$) whose universe is the union $A = \bigcup\limits_{i\in I} A_i$ and whose generic n-ary operation g is defined as

$$g^{\mathbf{A}}(a_1, \dots, a_n) := g^{\mathbf{A}_k}(p_{i_1k}(a_1), \dots p_{i_nk}(a_n)), \tag{1}$$

where $k = i_1 \vee \cdots \vee i_n$ and $a_1 \in A_{i_1}, \dots, a_n \in A_{i_n}$. If the similarity type contains any constant operation e, then $e^{\mathbf{A}} = e^{\mathbf{A}_{i_0}}$. The algebras $\{\mathbf{A}_i\}_{i\in I}$ are called the *fibers* of the Płonka sum. A fiber $\mathbf{A}_i$ is *trivial* if its universe is a singleton. As already stated above, an element a is a *fixpoint* when $a = \neg a$. Equivalently, using the Płonka sum representation, a fixpoint can be understood as the universe of a trivial fiber.

With this terminology at hand, a remarkable result states that any member of the variety IBSL of involutive bisemilattices is isomorphic to the Płonka sum over a semilattice direct system of Boolean algebras (see [7, 9]). In a Płonka sum of Boolean algebras $\mathbf{A} = \mathcal{P}_l(\mathbf{A}_i)_{i\in I}$, a key role is played by the absorption function $\varphi \wedge (\varphi \vee \psi)$, in the following sense. Given $a, b \in A$, it is possible to check that $a, b \in A_i$ for some $i \in I$ if and only if $a \wedge (a \vee b) = a$ and $b \wedge (b \vee a) = b$. Moreover, if $a \in A_i$ and $i \leq j$, it holds $p_{ij}(a) = a \wedge (a \vee b)$, for any $b \in A_j$. In the light of the above observations, given a Bochvar algebra $\mathbf{A}$ we will denote by $\mathcal{P}_l(\mathbf{A}_i)_{i\in I}$ the Płonka decomposition of its IBSL-reduct.

It shall be clear to the reader that a Bochvar algebra can not be represented as a Płonka sum (of some class of algebras) in the usual sense (recalled above). The reason is that the operations J_k, for any $k \in \{0, 1, 2\}$, are not computed according to condition (1). Indeed, if $\mathcal{P}_l(\mathbf{A}_i)_{i\in I}$ is a Płonka sum and $a \in A_i$, Axiom (11) entails that $J_k a \in A_{i_0}$ (with i_0 the least element of the index set I), while condition (1) requires that $J_k a \in A_i$. Nonetheless, the fact that any Bochvar algebra has an IBSL-reduct (Proposition 3) suggests that Płonka sums are good candidates to provide a representation theorem. Indeed we will rely on the Płonka sum representation of the IBSL-reduct of a Bochvar algebra to "reconstruct" the additional operations J_k and provide a (unique) Płonka sum decomposition for any Bochvar algebra. We begin by studying the behavior of the maps J_k with respect to the IBSL-reduct of a Bochvar algebra.

LEMMA 3.6. *Let* $\mathbf{A}$ *be a (non-trivial) Bochvar algebra and* $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ *be the Płonka decomposition of the* IBSL*-reduct of* $\mathbf{A}$, *having* i_0 *as the least element in* I. *Then*:

(1) $J_1^{\mathbf{A}_{i_0}}$ *is the constant map onto* 0;
(2) $J_2^{\mathbf{A}_{i_0}} = id$;
(3) $J_0^{\mathbf{A}_{i_0}} a = \neg a$, *for any* $a \in A_{i_0}$.

Proof. It suffices to notice that (1) holds if and only if

$$\mathsf{BCA} \vDash \varphi \wedge 0 \approx 0 \Rightarrow J_1\varphi \approx 0$$

and this quasi-equation clearly holds in $\mathbf{WK}^{\mathrm{e}}$. The same applies to (2), (3) with respect to the quasi-equations $\varphi \wedge 0 \approx 0 \Rightarrow J_2\varphi \approx \varphi$, $\varphi \wedge 0 \approx 0 \Rightarrow J_1\varphi \approx \neg\varphi$. □

Observe that, when one takes into account the whole algebra $\mathbf{A}$, J_1 in general does not coincide with the constant function 0, as in $\mathbf{WK}^{\mathrm{e}}$ it holds $J_1 1/2 = 1$.
The next result summarizes the key features of the operation J_1.

LEMMA 3.7. *The following hold for every* $\mathbf{A} \in \mathsf{BCA}$ (*with* $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ *the Płonka decomposition of its* IBSL*-reduct*).

(1) *if* $a, b \in A_i$, *for some* $i \in I$, *then* $J_1 a = J_1 b$;
(2) *if* $a \in A_{i_0}$ *then* $J_1 a = 0$;
(3) *if* $a = \neg a$ *then* $J_1 a = 1$;
(4) $p_{i_0 i}(J_1 a) = a \wedge \neg a$, *for every* $a \in A_i$ *and* $i \in I$.

Proof. (1). For every $a, b \in A_i$, $a \wedge (a \vee b) = a$ and $b \wedge (b \vee a) = a$. Since

$$\mathbf{WK}^{\mathrm{e}} \vDash \varphi \wedge (\varphi \vee \psi) \approx \varphi \;\&\; \psi \wedge (\psi \vee \varphi) \approx \psi \Rightarrow J_1\varphi \approx J_1\psi,$$

it follows $J_1 a = J_1 b$.
(2). Follows from (1) in Lemma 3.6.
(3). Holds because $\mathbf{WK}^{\mathrm{e}} \vDash \varphi \approx \neg\varphi \Rightarrow J_1\varphi \approx 1$.
(4). For $a \in A_i$, it holds $p_{i_0 i}(J_1 a) = J_1 a \wedge (J_1 a \vee a)$ and observe that

$$\mathbf{WK}^{\mathrm{e}} \vDash J_1\varphi \wedge (J_1\varphi \vee \varphi) = \varphi \wedge \neg\varphi,$$

so $p_{i_0 i}(J_1 a) = J_1 a \wedge (J_1 a \vee a) = a \wedge \neg a$. □

REMARK 3.8. *It follows from Lemma 3.7 that, for every* $a, b \in A_i$ (*for some* $i \in I$), *i.e.,* a, b *are elements in the same fiber of the Płonka sum,* $J_1 a = J_1 b$, *thus, in particular,* $J_1(a \vee b) = J_1 a \vee J_1 b = J_1 a \wedge J_1 b = J_1(a \wedge b)$.

As a notational convention, let us denote by 1_i and 0_i the top and the bottom elements, respectively, of a generic fiber $\mathbf{A}_i$ in a Płonka sum of Boolean algebras.

LEMMA 3.9. *Let* $\mathbf{A}$ *be a Bochvar algebra with* $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ *the Płonka decomposition of its* IBSL*-reduct. Then*:

(1) $\mathcal{P}_{\nmid}(\mathbf{A}_i)_{i\in I}$ *has surjective homomorphisms*;
(2) *for every* $i \neq i_0$, $p_{i_0 i}$ *is not injective*;
(3) *for every* $a \in A$ *and every* $i \in I$ (*with* $a \in A_i$), $J_2 a \in p_{i_0 i}^{-1}(a)$ *and* $J_0 a \in p_{i_0 i}^{-1}(\neg a)$;
(4) *for every* $a, b \in A_i$, $J_2(a \vee b) = J_2 a \vee J_2 b$.

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Proof. Clearly $\mathbf{WK}^{\mathrm{e}} \vDash J_2\varphi \wedge (J_2\varphi \vee \varphi) \approx \varphi$, and $\mathbf{WK}^{\mathrm{e}} \vDash J_0\varphi \wedge (J_0\varphi \vee \varphi) \approx \neg\varphi$, so, for $a \in A_i$, $p_{i_0 i}(J_2 a) = J_2 a \wedge (J_2 a \vee a) = a$ and $p_{i_0 i}(J_0 a) = \neg a$. This proves (3) and that $p_{i_0 i}$ is surjective, for every $i \in I$. Let now $i \leq j$. Since $p_{i_0 j} = p_{ij} \circ p_{i_0 i}$ is surjective, also p_{ij} is surjective. This shows that $\mathcal{P}_l(\mathbf{A}_i)_{i\in I}$ has surjective homomorphisms (1). (4) holds as it is equivalent to the quasi-equation

$$\varphi \wedge (\varphi \vee \psi) \approx \varphi \;\&\; \psi \wedge (\psi \vee \varphi) \approx \psi \Rightarrow J_2(\varphi \vee \psi) \approx J_2\varphi \vee J_2\psi,$$

which is true in $\mathbf{WK}^{\mathrm{e}}$.

(2) Suppose, by contradiction, that there is $j \in I$ such that $j \neq i_0$ and $p_{i_0 j}$ is an injective homomorphism. Thus, by Lemma 3.7 and the fact that $J_2(\varphi \wedge \neg\varphi) \approx 0$ is true in $\mathbf{WK}^{\mathrm{e}}$, it holds $J_1 0_j = 0$ and $J_2 0_j = 0$. Moreover, by (12), $J_0 0_j = 1$, hence $J_i 0 = J_i 0_j$, for every $i \in \{0, 1, 2\}$. Therefore, by the quasi-equation (13), $0 = 0_j$, a contradiction. □

Remark 3.10. *It follows from Lemma 3.9 that, for any $i \in I$, $p_{i_0 i} \circ J_2^{\mathbf{A}_i} = id_{\mathbf{A}_i}$, namely that J_2 (restricted on $\mathbf{A}_i$) is the right inverse of the surjective homomorphism $p_{i_0 i}$.*

Remark 3.11. *Note that, in general, it does not hold that $J_2(\varphi \vee \psi) \approx J_2\varphi \vee J_2\psi$ (the identity is falsified in $\mathbf{WK}^{\mathrm{e}}$).*

Recall that, for any non-trivial Boolean algebra $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1\rangle$ and any $a \in A$, one can turn the interval $[0, a] = \{x \in A \mid x \leq a\}$ into a Boolean algebra $[\mathbf{0}, \mathbf{a}] = \langle [0, a], \wedge, \vee, ^*, 0, a\rangle$, where $x^* = \neg x \wedge a$. We will refer to such an algebra as an *interval Boolean algebra*. In the following result we show that any Boolean algebra in the Płonka sum representation of the IBSL-reduct of a Bochvar algebra is isomorphic to a specific interval Boolean algebra in the lowest fiber.

Proposition 3.12. *Let $\mathbf{A}$ be a Bochvar algebra with $\mathcal{P}_{\mathfrak{l}}(\mathbf{A}_i)_{i\in I}$ the Płonka decomposition of its IBSL-reduct. Then, for every $i \in I$, $J_2 \colon \mathbf{A}_i \to [\mathbf{0}, \top]$ is an isomorphism onto the interval Boolean algebra $[\mathbf{0}, \top] = \langle [0, \top], \wedge, \vee, ^*, 0, \top\rangle$, where $\top = J_2 1_i$, for every $i \neq i_0$.*

Proof. $\mathbf{WK}^{\mathrm{e}} \models J_2(\varphi \wedge \psi) \approx J_2\varphi \wedge J_2\psi$, thus J_2 preserves the $\wedge$ operation. By Lemma 3.9(4) it also preserves $\vee$ when the arguments belong to the same fiber. This implies that $J_2(\mathbf{A}_i)$ is a lattice. To see that is bounded, recall that $J_2 0_i = 0$ (with 0_i the bottom element of $\mathbf{A}_i$); moreover, let $b \in J_2(A_i)$, thus $b = J_2 a$, for some $a \in A_i$, then $J_2 a \vee J_2 1_i = J_2(a \vee 1_i) = J_2 1_i$, i.e., $J_2 a \leq J_2 1_i$ thus $\top = J_2 1_i$ is the top element of the lattice $J_2(\mathbf{A}_i)$. Finally, observe that, for any $a \in A_i$, $J_2\neg a = J_0 a = \neg J_2 a \wedge \neg J_1 a = \neg J_2 a \wedge (J_2 a \vee J_0 a) = \neg J_2 a \wedge (J_2 a \vee J_2 \neg a) = \neg J_2 a \wedge (J_2(a \vee \neg a)) = \neg J_2 a \wedge J_2 1_i = (J_2 a)^*$, i.e., J_2 preserves also the complementation of the interval algebra $[0, \top]$. We have so shown that J_2 is a boolean homomorphism. Moreover, J_2 is injective as $p_{i_0 i}$ is its left-inverse (see Remark 3.10). To see that J_2 is also surjective (onto $[0, \top]$), let $a \in [0, \top]$, i.e., $a \leq J_2 1_i$. By Lemma 3.6, $J_2 a = a \leq J_2 1_i$ hence $a = J_2 a \wedge J_2 1_i = J_2(a \wedge 1_i) = J_2(p_{i_0 i}(a) \wedge 1_i) = J_2(p_{i_0 i}(a))$. This concludes the proof, because $p_{i_0 i}(a) \in A_i$. □

Remark 3.13. *Since $p_{i_0 i}$ is a surjective (but not injective) homomorphism, for any $i \in I$ (and $i \neq i_0$), then $\mathbf{A}_{i_0}/Ker(p_{i_0 i}) \cong \mathbf{A}_i$ (with $Ker(p_{i_0 i} \neq \Delta^{\mathbf{A}_{i_0}}$ for $i \neq i_0$), via the isomorphism f mapping $[x]_{Ker(p_{i_0 i})} \mapsto p_{i_0 i}(x)$. The proof of Proposition 3.12 shows that J_2 is in fact the inverse of f.*

Combining Proposition 3.12 and Remark 3.13 we get that, for any $i \in I[\mathbf{0}, \top] \cong \mathbf{A}_{i_0}/Ker(p_{i_0 i})$. In the following we use the interval characterization proved in Proposition 3.12 to establish some properties that will be used in the subsequent sections.

LEMMA 3.14. *Let* $\mathbf{A}$ *be a Bochvar algebra with* $\mathcal{P}_{ł}(\mathbf{A}_i)_{i\in I}$ *the Płonka decomposition of its* IBSL*-reduct. Then*:

(1) *If* $i < j$, *then* $J_2 1_j < J_2 1_i$. *In particular,* $J_2(\mathbf{A}_j) = [0, J_2 1_j] \subset [0, J_2 1_i] = J_2(\mathbf{A}_i)$.
(2) $J_2(p_{ij}(a)) \leq J_2 a$, *for every* $i \leq j$ *and* $a \in A_i$.

Proof. (1). If $1_i < 1_j$ then $1_i \wedge 1_j = 1_j \neq 1_i$. The following two quasi-equations hold in $\mathbf{WK}^{\mathrm{e}}$:

$$\varphi \vee \neg\varphi \leq \psi \vee \neg\psi \Rightarrow J_2(\psi \vee \neg\psi) \leq J_2(\varphi \vee \neg\varphi),$$
$$J_2(\varphi \vee \neg\varphi) \approx J_2(\psi \vee \neg\psi) \Rightarrow \varphi \vee \neg\varphi \approx \psi \vee \neg\psi.$$

From the former we have that $J_2 1_j \leq J_2 1_i$ and, since $1_i \neq 1_j$, from the latter we conclude $J_2 1_j < J_2 1_i$. This entails $J_2(\mathbf{A}_j) = [0, J_2 1_j] \subset [0, J_2 1_i] = J_2(\mathbf{A}_i)$.
(2) is equivalent to the equation $J_2(\varphi \wedge (\varphi \vee \psi)) \leq J_2\varphi$, which is true in $\mathbf{WK}^{\mathrm{e}}$. □

The following result provides necessary conditions for an SIBSL to be the reduct of a Bochvar algebra.

THEOREM 3.15 (Płonka sum decomposition). *Let* $\mathbf{A}$ *be a Bochvar algebra with* $\mathcal{P}_{ł}(\mathbf{A}_i)_{i\in I}$ *the Płonka sum representation of its* IBSL*-reduct. Then*:

(1) *All the homomorphisms* $\{p_{ij}\}_{i\leq j}$ *are surjective and* $p_{i_0 i}$ *is not injective for every* $i \neq i_0$.
(2) *For every* $i \in I$, *there exists an element* $a_i \in A_{i_0}$ *such that the restriction* $p_{i_0 i}$ *on* $[\mathbf{0}, \mathbf{a_i}]$ *is an isomorphism (with inverse* J_2*) onto* $\mathbf{A}_i$, *with* $a_i \neq a_j$ *for every* $i \neq j$; *in particular, if* $i < j$ *then* $a_j < a_i$.

Moreover, the decomposition is unique up to isomorphism.

Proof. Let $\mathbf{A} \in$ BCA. By Proposition 3, the J_k-free reduct (with $k \in \{0, 1, 2\}$) of $\mathbf{A}$ is a single-fixpoint involutive bisemilattice, thus it is isomorphic to a Płonka sum over a semilattice direct systems $\langle\{\mathbf{A}_i\}_{i\in I}, (I, \leq), p_{ij}\rangle$ of Boolean algebras (see [7, 9]), whose homomorphisms are surjective and $p_{i_0 i}$ is not injective (by Lemma 3.9). Moreover, for every $i \in I$, $J_2 1_i$ is the element in $\mathbf{A}_{i_0}$ such that $\mathbf{A}_i \cong [\mathbf{0}, \mathbf{J_2 1_i}]$, by Proposition 3.12 (which also ensures the isomorphisms are given by J_2 and $p_{i_0 i}$). Lemma 3.14 ensures that, if $i < j$ then $J_2 1_j < J_2 1_i$. Finally, notice that $J_2(\varphi \vee \neg\varphi) \approx J_2(\psi \vee \neg\psi) \Rightarrow \varphi \vee \neg\varphi \approx \psi \vee \neg\psi$ holds in $\mathbf{WK}^{\mathrm{e}}$. Therefore, since $i \neq j$ entails $1_i \neq 1_j$, we conclude $a_i = J_2 1_i \neq J_2 1_j = a_j$.

We now show that the decomposition is unique up to isomorphism, namely that different choices of the element $J_2 a \in p_{i_0 i}^{-1}(a)$ (for any $a \in A_i$) on isomorphic IBSL-reducts lead to isomorphic Bochvar algebras. Observe that the decomposition of the IBSL-reduct of $\mathbf{A}$ is unique up to isomorphism (see [7, 9]). So, suppose that $\mathbf{A}$ and $\mathbf{B}$ are Bochvar algebras whose IBSL-reducts $\mathbf{A}'$, $\mathbf{B}'$ are isomorphic via an isomorphism $\Phi\colon \mathbf{A}' \to \mathbf{B}'$. We claim that $\mathbf{A} \cong \mathbf{B}$ via Φ. To this end, we want to show that Φ preserves also the operations J_k, for any $k \in \{0, 1, 2\}$. Recall from the theory of Płonka sums that Φ preserves the fibers (details can be found in [5, 8]) in the following sense: for any fiber $\mathbf{A}_i$ in the Płonka sum decomposition of $\mathbf{A}'$, $\Phi(\mathbf{A}_i) \cong \mathbf{B}_{\varphi(i)}$, where φ is the

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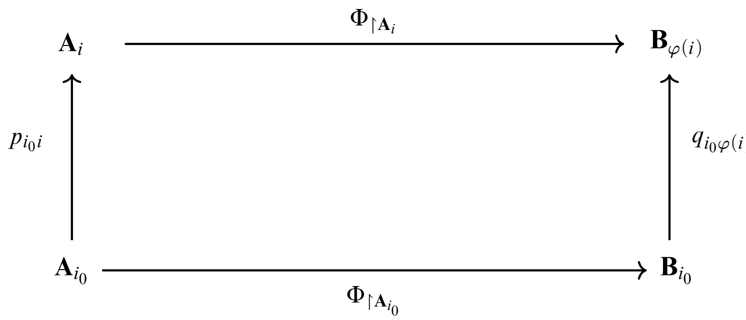
isomorphism induced by Φ on the semilattice of indexes and $\mathbf{B}_{\varphi(i)}$ is a fiber in the Płonka sum decomposition of $\mathbf{B}'$. In particular, the following diagram commutes, for any $i \in I$ (with a slight abuse of notation we indicate $\varphi(i_0)$ with i_0 as it is still a lower bound in $\varphi(I)$).



We claim that $\mathbf{A} \cong \mathbf{B}$ via Φ. Let $a \in A$; in particular, $a \in A_i$, for some $i \in I$. By Lemma 3.9, $J_2^{\mathbf{A}} a \in p_{i_0 i}^{-1}(a)$ and $J_2^{\mathbf{B}} \Phi(a) \in q_{i_0 \varphi(i)}^{-1}(\Phi(a))$, hence, by the commutativity of the above diagram, the fact that J_2 is the isomorphism between $\mathbf{A}_i$ and $\mathbf{A}_{i_0}/Ker(p_{i_0 i})$ (Remark 3.13) and that $\Phi_{\restriction \mathbf{A}_i}$ and $\Phi_{\restriction \mathbf{A}_{i_0}}$ are isomorphisms) follows that $\Phi(J_2^{\mathbf{A}} a) = J_2^{\mathbf{B}} \Phi(a)$. Therefore, we have that Φ is a homomorphism with respect to J_2 (and hence with respect of J_1 and J_0, which can be defined in term of J_2) and this shows that $\mathbf{A} \cong \mathbf{B}$, namely the Płonka sum decomposition is unique up to isomorphism. □

REMARK 3.16. *Observe that, as a consequence of Theorem 3.15, the Płonka sum decomposition of a Bochvar algebra* $\mathbf{A}$ *admits no injective homomorphism (excluding the identical homomorphisms* p_{ii}*). To see this, suppose* p_{ij} *is injective, for some* $i < j$*. Then* p_{ij} *is an isomorphism, as it is also a surjective map. Observe that* $a_j = J_2 1_j < a_i = J_2 1_i$ *and* $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a_i}] \to \mathbf{A}_i$ *is also an isomorphism. Therefore,* $p_{i_0 i}(a_j) \neq 1_i = p_{i_0 i}(a_i)$ *and, by the injectivity of* p_{ij}, $p_{ij} \circ p_{i_0 i}(a_j) = p_{i_0 j}(a_j) \neq 1_j$*, a contradiction.*

We now show that the conditions displayed in the above theorem are also sufficient to equip any SIBSL with a BCA-structure.

THEOREM 3.17. *Let* $\mathbf{A} = \langle A, \wedge, \vee, \neg, 0, 1 \rangle$ *be an involutive bisemilattice whose Płonka sum representation is such that*:

(1) *all homomorphisms are surjective and* $p_{i_0 i}$ *is not injective for every* $i_0 \neq i \in I$;
(2) *for each* $i \in I$ *there exists an element* $a_i \in A_{i_0}$ *such that* $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ *is an isomorphism, with* $a_i \neq a_j$ *for* $i \neq j$ *and, in particular,* $a_j < a_i$ *for each* $i < j$.

Define, for every $a \in A_i$ *and* $i \in I$:

- $J_2(a) = p_{i_0 i}^{-1}(a) \in [0, a_i]$;
- $J_0 a := J_2(\neg a)$;
- $J_1 a := \neg(J_2 a \vee J_2(\neg a))$.

Then $\mathbf{B} = \langle A, \wedge, \vee, \neg, 0, 1, J_2, J_1, J_0 \rangle$ *is a Bochvar algebra.*

Proof. It is immediate to check that assumption (1) implies that $\mathbf{A} \in$ SIBSL. Since $p_{i_0 i} \colon [0, \mathbf{a}_i] \to \mathbf{A}_i$ is an isomorphism (with inverse $p_{i_0 i}^{-1}$) the maps J_2, J_1 and J_0 are well defined and naturally extend to the whole algebra $\mathbf{A}$. It can be mechanically checked that $\mathbf{B}$ satisfies all the quasi-equations in Definition 3.1. □

The first part of condition (2) in Theorems 3.15 and 3.17 can be replaced by the assumption that $1/\operatorname{Ker}_{p_{i_0 i}}$ is a principal filter, for every $i \in I$. Therefore, we obtain the following.

COROLLARY 3.18. *Let* $\mathbf{A} \in \mathsf{SIBSL}$ *be with surjective and non-injective homomorphisms. The following are equivalent*:

(1) $\mathbf{A}$ *is the reduct of a Bochvar algebra*;
(2) *for each* $i \in I$, $1/\operatorname{Ker}_{p_{i_0 i}}$ *is a principal filter, with generator* $a_i \in A_{i_0}$. *Moreover, if* $i \neq j$ *then* $a_i \neq a_j$ *and* $a_j < a_i$ *for each* $i < j$;
(3) *for each* $i \in I$ *there exists an element* $a_i \in A_{i_0}$ *such that* $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ *is an isomorphism. Moreover, if* $i \neq j$ *then* $a_i \neq a_j$ *and* $a_j < a_i$ *for each* $i < j$.

Proof. We just show (2) $\Leftrightarrow$ (3). Let $1/\operatorname{Ker}_{p_{i_0 i}}$ be the filter generated by a_i, for some $a_i \in A_{i_0}$. It is routine to check that $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ is an isomorphism. Conversely, assume that there is an element $a_i \in A_{i_0}$ such that $p_{i_0 i} \colon [\mathbf{0}, \mathbf{a}_i] \to \mathbf{A}_i$ is an isomorphism. Suppose, by contradiction, that $1/\operatorname{Ker}_{p_{i_0 i}}$ is not principal, i.e., there is an element $b \in A_{i_0}$ such that $a_i \nleq b$ and $p_{i_0 i}(b) = 1_i$. Then $c = b \wedge a_i$ is an element in $[0, a_i]$ such that $p_{i_0 i}(c) = 1_i$, in contradiction with the fact that $p_{i_0 i}$ is an isomorphism. □

We conclude this section with an example which empathizes the role of condition (2) in Corollary 3.18 and shows an SIBSL that can not be turned into a Bochvar algebra.

EXAMPLE 3.19. *Let* $\mathcal{P}(\mathbb{Z})$ *be the power set Boolean algebra over the integers. This algebra is uncountable and atomic, with* $\mathbb{Z}$ *as top element. Consider now the non-principal ideal* I *containing all the finite subsets of* $\mathbb{Z}$. *Observe that* $\mathcal{P}(\mathbb{Z})/I$ *is an atomless Boolean algebra. Let* $\mathbf{A}$ *be the Płonka sum built over the two fibers* $\mathcal{P}(\mathbb{Z}), \mathcal{P}(\mathbb{Z})/I$ *and with (the unique non-trivial) homomorphism the canonical map* $p \colon \mathcal{P}(\mathbb{Z}) \to \mathcal{P}(\mathbb{Z})/I$. *Clearly* $\mathbf{A}$ *satisfies condition* (1) *of Theorem* 3.17 (*the homomorphism* p *is both surjective and non-injective hence* $\mathbf{A} \in \mathsf{SIBSL}$). *However the filter* $\mathbb{Z}/\operatorname{Ker}_p$ (*corresponding to the ideal* I) *is not principal, as* I *is not. In the light of condition* (2) *of Corollary* 3.18, $\mathbf{A}$ *is a* SIBSL *which is not the reduct of any Bochvar algebra.*

The results included in the subsequent sections will strongly make use of the Płonka sum decomposition of a Bochvar algebra provided in Theorem 3.15. In the exposition of many results we will take for granted some of the details introduced in this section.

§4. On quasivarieties of Bochvar algebras. The logic B^{e} is algebraizable with respect to the quasivariety of Bochvar algebras and it is well known that there exists a dual isomorphism between the lattice of extensions of B^{e} and the lattice of subquasivarieties of BCA. In this section we characterize such lattices, proving that they consist of the three-elements chain. In order to do so, we take advantage of the recent results in [25], therefore applying general properties of passive structurally complete (PSC) quasivarieties. PSC is a weakened variant of structural completeness (see, among others, [11, 14, 32, 34]), a notion defined, in its algebraic version, as follows.

DEFINITION 4.1. *A quasivariety* K *is structurally complete* (*SC*) *if, for every quasivariety* K':

$$\mathsf{K}' \subsetneq \mathsf{K} \;\Rightarrow\; \mathbb{V}(\mathsf{K}') \subsetneq \mathbb{V}(\mathsf{K}),$$

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where $\mathbb{V}(\mathsf{K}) = HSP(\mathsf{K})$ is the variety generated by K. Recall that a quasi-identity $\varphi_1 \approx \psi_1 \,\&\ldots\&\, \varphi_n \approx \psi_n \Rightarrow \varphi \approx \psi$ is passive in a quasivariety K if, for every substitution $h\colon \mathbf{Fm} \to \mathbf{Fm}$, there exists an algebra $\mathbf{A} \in \mathsf{K}$ such that $\mathbf{A} \nvDash h(\varphi_i) \approx h(\psi_i)$ for some $1 \leq i \leq n$. Put differently, a quasi-identity is passive if it is equivalent to a quasi-identity whose conclusion is $x \approx y$, where x, y are variables that do not appear in the antecedent (see [1, sec. 4.3]) for details.[4] We take the following as a definition of *passive structural completeness* (PSC).

Definition 4.2 [38]. *A quasivariety* K *is PSC if every passive quasi-identity over* K *is valid in* K.

For a quasivariety K, one of the consequences of being PSC amounts to be generated by a single algebra, i.e., to be *singly generated* (see [25, sec. 7]). Recall that an algebra $\mathbf{A}$ is a retract of $\mathbf{B}$ if there exist two homomorphisms $\imath\colon \mathbf{A} \to \mathbf{B}$ and $r\colon \mathbf{B} \to \mathbf{A}$ such that $r \circ \imath$ is the identity map on $\mathbf{A}$. This forces r to be surjective and $\imath$ to be injective: we will call r a *retraction*. $\mathbf{A}$ is a *common retract* of a quasivariety K if it is a retract of every non-trivial member of K. We denote by $Ret(\mathsf{K}, \mathbf{A}) = \{\mathbf{B} \in \mathsf{K} : \mathbf{A} \text{ is a retract of } \mathbf{B}\}$ the members of K having $\mathbf{A}$ as a retract.

The main tool that will be instrumental for our purposes is the following.

Theorem 4.3 [25, theorem 7.11]. *Let* K *be a quasivariety of finite type and* $\mathbf{A} \in \mathsf{K}$ *a finite* 0*-generated algebra. Then* $Ret(\mathsf{K}, \mathbf{A})$ *is a maximal PSC subquasivariety of* K.[5]

The following summarizes the properties of the quasivariety BCA with respect to the above introduced notions.

Proposition 4.4. *The following hold*:

(i) BCA *is not PSC*;
(ii) $\mathbf{B}_2$ *is the* 0*-generated algebra in* BCA;
(iii) $Ret(\mathsf{BCA}, \mathbf{B}_2)$ *is a maximal PSC subquasivariety of* BCA.

Proof. (i). The quasi-identity

$$J_1\varphi \approx 1 \Rightarrow \psi \approx 1 \qquad (NF)$$

is passive in BCA. Indeed, for each substitution, the antecedent $J_1\varphi \approx 1$ is falsified in every Bochvar algebra containing no trivial algebra in its Płonka sum decomposition. On the other hand, it is immediate to check that $\mathsf{BCA} \nvDash (NF)$.

(ii) Is straightforward, as every Boolean algebra is also a Bochvar algebra (see Example 3.2), and $\mathbf{B}_2$ is the 0-generated Boolean algebra.

(iii) Follows from (ii) and Theorem 4.3. □

It is easy to check that, in BCA, (NF) is equivalent to the quasi-identity

$$\varphi \approx \neg\varphi \;\Rightarrow\; \psi \approx \delta.$$

Since (NF) is valid in the quasivariety $Ret(\mathsf{BCA}, \mathbf{B}_2)$, it demands that the underlying involutive bisemilattice of any of its non-trivial members lacks trivial fibers. A stronger fact is established in the following lemma.

---

4 Passive rules were originally introduced under the name of overflow rules by Wroński in [38].

5 K' is a maximal PSC subquasivariety of K when for every PSC quasivariety K'', if $\mathsf{K}' \subseteq \mathsf{K}'' \subseteq \mathsf{K}$ then $\mathsf{K}'' = \mathsf{K}'$.

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LEMMA 4.5. *Let* $\mathbf{A} \in \mathsf{BCA}$ *be non-trivial. Then* $\mathbf{A} \in Ret(\mathsf{BCA}, \mathbf{B}_2)$ *if and only if* $\mathbf{A} \models (NF)$.

Proof. $(\Rightarrow)$. By Proposition 4.4, $Ret(\mathsf{BCA}, \mathbf{B}_2)$ is a (maximal) PSC quasivariety and (NF) is a passive quasi-identity, therefore $Ret(\mathsf{BCA}, \mathbf{B}_2) \models (NF)$. $(\Leftarrow)$. Suppose $\mathbf{A} \models (NF)$, therefore its IBSL-reduct lacks trivial fibers. If $\mathbf{A}$ is a Boolean algebra, the conclusion immediately follows. Differently, $|I| \geq 2$. Let $i > i_0$ and consider the set $X = \{a \in A_{i_0} \mid p_{i_0 i}(a) = 1_i\}$; let F_0 be an ultrafilter on $\mathbf{A}_0$ containing X (it exists since the filter generated by X is proper, as the Płonka decomposition lacks trivial algebras). Since all the homomorphisms between fibers of $\mathbf{A}$ are surjective and $\mathbf{A}$ lacks trivial fibers, $F_j = p_{i_0 j}[F_0]$ is an ultrafilter on $\mathbf{A}_j$, for each $j \in I$ (this readily follows from the fact that F_0 is an ultrafilter extending X). Set $F = \bigcup_{k \in I} p_{i_0 k}[F_0]$ and let $r \colon \mathbf{A} \to \mathbf{B}_2$ be defined, for each $a \in A$, as

$$r(a) = \begin{cases} 1, & \text{if } a \in F, \\ 0, & \text{otherwise.} \end{cases}$$

We want to show that r is a retraction. We show that r is compatible with $\wedge$, $\neg$ and J_2. Let $a \in A_i$, $b \in A_j$ and set $k = i \vee j$: we have $r(a \wedge b) = 1 \iff a \wedge b \in F_k \iff p_{ik}(a), p_{jk}(b) \in F_k$. Suppose, by contradiction, $a \notin F_i$. Then $p^{-1}_{i_0 i}(a) \notin F_0$ which entails $p_{i_0 k}(p^{-1}_{i_0 i}(a)) \notin F_k$. However, by the composition property of Płonka homomorphisms, $p_{i_0 k}(p^{-1}_{i_0 i}(a)) = p_{ik}(p_{i_0 i}(p^{-1}_{i_0 i}(a)) = p_{ik}(a) \in F_k$, a contradiction. So $a \in F_i$. The same argument applies to b, and we conclude $b \in F_j$. Therefore $r(a) \wedge r(b) = 1 = r(a \wedge b)$.

Let $r(a) = 1$, for some $a \in A$, then $a \in F_j$, for some $j \in I$ and, since F_j is an ultrafilter on $\mathbf{A}_j$, $\neg a \notin F_j$, thus $r(\neg a) = 0 = \neg r(a)$.

Finally, we show the compatibility of r with J_2. For any $a \in A_j$, we have $r(J_2 a) = 1 \iff J_2 a \in F_0 \iff p_{i_0 i}(J_2 a) = a \in F_i \iff J_2 r(a) = 1$. So r is a surjective homomorphism. Setting $\iota \colon \mathbf{B}_2 \to \mathbf{A}$ such that $\iota(1) = 1, \iota(0) = 0$, we have that r is a retraction. This shows $\mathbf{B}_2$ is a retract of $\mathbf{A}$, so $\mathbf{A} \in Ret(\mathsf{BCA}, \mathbf{B}_2)$. □

COROLLARY 4.6. $Ret(\mathsf{BCA}, \mathbf{B}_2)$ *is the quasivariety axiomatized by adding* (NF) *to the quasi-equational theory of* BCA. *Moreover, a Bochvar algebra belongs to* $Ret(\mathsf{BCA}, \mathbf{B}_2)$ *if and only if its Płonka sum decomposition lacks trivial fibers.*

In analogy with the terminology introduced in [26], we call the quasivariety $Ret(\mathsf{BCA}, \mathbf{B}_2)$: *nonparaconsistent Bochvar algebras*, NBCA in brief. Since NBCA is a (maximal) PSC subquasivariety of BCA, we know that NBCA, as a quasivariety, is generated by a single algebra $\mathbf{A}$, namely $\mathsf{NBCA} = ISPP_u(\mathbf{A}) = ISP(\mathbf{A})$ (see [25, theorem 4.3]). We now introduce an example of a Bochvar algebra which will play an important role.

EXAMPLE 4.7 ($\mathbf{B}_4 \oplus \mathbf{B}_2$). *Let* $\mathbf{B}_4 \oplus \mathbf{B}_2$ *denote the involutive bisemilattice whose Płonka sum consists of a system made of the four-element Boolean algebra* $\mathbf{B}_4$, *the two-element one* $\mathbf{B}_2$, *the two-element lattice as index set as in the following diagram (where the arrows stand for the homomorphism* $p_{i_0 i} \colon \mathbf{B}_4 \to \mathbf{B}_2$).

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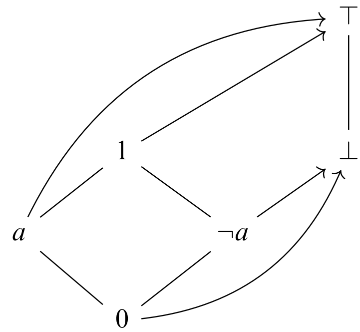

It follows from the structure theory developed in Section 3 that the unique way to turn $\mathbf{B}_4 \oplus \mathbf{B}_2$ *into a Bochvar algebra is by defining* $J_2 \top = a$, $J_2 \perp = 0$, $J_1 \top = J_1 \perp = \neg a$ *and* $J_0 \top = 0$, $J_0 \perp = a$ *(recall that* J_2 *is the identity on* $\mathbf{B}_4$, J_1 *is the constant onto* 0 *and* J_0 *is negation). With a slight abuse of notation we will indicate this unique Bochvar algebra by* $\mathbf{B}_4 \oplus \mathbf{B}_2$, *as its* IBSL*-reduct.*

THEOREM 4.8. *The quasivariety* NBCA *is generated by* $\mathbf{B}_4 \oplus \mathbf{B}_2$.

Proof. We show that NBCA $= ISP(\mathbf{B}_4 \oplus \mathbf{B}_2)$. The right to left inclusion is obvious, as subalgebras of direct products of an involutive bisemilattice without trivial fibers preserve the property of lacking trivial fibers.

For the converse, the proof is an adaption of [26, theorem 7], and we only sketch its main ingredients (leaving the details to the reader). Preliminarily recall that for quasivarieties K, K′, K $\subseteq$ K′ if and only if every finitely generated member of K belongs to K′. Moreover, for algebras $\mathbf{A}, \mathbf{B}$, $\mathbf{A} \in ISP(\mathbf{B})$ if and only if that there exists a family $H \subseteq Hom(\mathbf{A}, \mathbf{B})$ such that $\bigcap_{h \in H} Ker(h) = \Delta^{\mathbf{A}}$. It is possible to show that, for a finitely generated $\mathbf{A} \in$ NBCA, there exists a family of homomorphisms $H \subseteq Hom(\mathbf{A}, \mathbf{B}_4 \oplus \mathbf{B}_2)$ such that $\bigcap_{h \in H} Ker(h) = \Delta^{\mathbf{A}}$. Indeed, $\mathbf{B}_4 \oplus \mathbf{B}_2 \cong \mathbf{WK}^{\mathrm{e}} \times \mathbf{B}_2$ and $\mathbf{WK}^{\mathrm{e}}$ generates BCA, so there exists $H \subseteq Hom(\mathbf{A}, \mathbf{WK})$ such that $\bigcap_{h \in H} Ker(h) = \Delta^{\mathbf{A}}$. Moreover, by Lemma 4.5, $\mathbf{B}_2$ is a retract of $\mathbf{A}$, so there exists a retraction $r \colon \mathbf{A} \to \mathbf{B}_2$. Now, the family $H \times \{r\}$ is a family of homomorphisms from $\mathbf{A}$ to $\mathbf{B}_4 \oplus \mathbf{B}_2$, defined for each $h \in H$ and $a \in A$ by $a \mapsto \langle h(a), g(a) \rangle$. It is easy to check that $\bigcap_{h \in H} Ker\langle h, g \rangle = \Delta^{\mathbf{A}}$, so $\mathbf{A} \in ISP(\mathbf{B}_4 \oplus \mathbf{B}_2)$, as desired. □

Upon noticing that every non-trivial NBCA lacks trivial subalgebras (this amounts to say that NBCA is *Kollár*), [25, corollary 7.8] ensures that $\mathbf{B}_2$ is the unique relatively simple member of NBCA, namely it is the unique algebra in the quasivariety whose lattice of relative congruences is a two-element chain. Moreover, $\mathbf{B}_4 \oplus \mathbf{B}_2$ is a relatively subdirectly irreducible member of NBCA which is therefore not simple. In other words, NBCA is not relatively semisimple, unlike BCA.

Observe that any Bochvar algebra satisfies the absorption law $\varphi \approx \varphi \wedge (\varphi \vee \psi)$ if and only if its involutive bisemilattice reduct is a Boolean algebra. We call JBA the quasivariety axiomatized by adding the absorption law to the quasi-equational theory of BCA. It is immediate to verify that JBA and BA are term equivalent by interpreting

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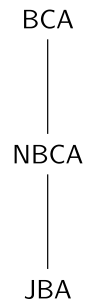
the operation J_2 as the identity map. The next theorem characterizes the structure of the lattice of non-trivial subquasivarieties of BCA, proving that it consists of the following three-elements chain.



THEOREM 4.9. *The only non-trivial subquasivarieties of* BCA *are* NBCA *and* JBA. *They form a three element chain* JBA $\subset$ NBCA $\subset$ BCA.

Proof. We already proved JBA $\subset$ NBCA $\subset$ BCA. We only have to show that these are the only non-trivial ones. Suppose that K $\subseteq$ BCA and K $\nsubseteq$ NBCA. Therefore there exists $\mathbf{A} \in$ K and $\mathbf{A} \notin$ NBCA. This entails $\mathbf{A} \in$ BCA and $\mathbf{A}$ has a unique trivial fiber with universe $\{a\}$. Clearly $g\colon \mathbf{WK}^{\mathrm{e}} \to \mathbf{A}$ mapping $1^{\mathbf{WK}^{\mathrm{e}}} \to 1^{\mathbf{A}}, 0^{\mathbf{WK}^{\mathrm{e}}} \to 0^{\mathbf{A}}, 1/2 \to a$ is an embedding. Therefore $\mathbf{WK}^{\mathrm{e}} \in S(\mathbf{A})$, whence $\mathbf{WK}^{\mathrm{e}} \in$ K. Since $\mathbf{WK}^{\mathrm{e}}$ generates BCA, K = BCA.

Suppose now K $\subseteq$ NBCA and K $\nsubseteq$ JBA and let $\mathbf{A} \in$ K, $\mathbf{A} \notin$ JBA. This entails that the Płonka sum decomposition of $\mathbf{A}$ has at least two fibers $\mathbf{A}_{i_0}, \mathbf{A}_i\,(i_0 < i)$ and no fiber is trivial. Moreover, by Lemma 3.9, $\mathbf{A}_{i_0}$ has cardinality ≥ 4 (for otherwise $\mathbf{A}_i$ would be trivial). Let $h\colon \mathbf{B}_4 \oplus \mathbf{B}_2 \to \mathbf{A}$ mapping $1^{\mathbf{B}_4 \oplus \mathbf{B}_2} \to 1^{\mathbf{A}}, 0^{\mathbf{B}_4 \oplus \mathbf{B}_2} \to 0^{\mathbf{A}}$, $1^{\mathbf{B}_2} \to 1_i, 0^{\mathbf{B}_2} \to 0_i$, $a \to J_2(1_i), \neg a \to \neg J_2 1_i$. Clearly h is an embedding, so $\mathbf{B}_4 \oplus \mathbf{B}_2 \in S(\mathbf{A})$, which entails K = NBCA. ☐

COROLLARY 4.10. *The quasivariety* NBCA *is structurally complete.*

Proof. The only proper subquasivariety of NBCA is the variety JBA, so $\mathbb{V}$(JBA) = JBA $\subsetneq$ NBCA $\subsetneq$ $\mathbb{V}$(NBCA). ☐

Moreover, from the fact that BCA is not SC we can infer the following.

COROLLARY 4.11. $\mathbb{V}$(NBCA) = $\mathbb{V}$(BCA).

Let now switch our attention to the logical setting, relying on the bridge results connecting an algebraizable logic (and its extensions) with its algebraic counterpart(s). Let NB^{e} be the logic obtained by adding to B^{e} the rule

$$J_1\varphi \vdash \psi. \qquad (EFJ)$$

This logic is a proper extension of B^{e}, as (EFJ) is the logical pre-image of (NF) via the transformer formula-equations transformer τ and (NF) is not valid in B^{e} (a counterexample is easily found in $\mathbf{WK}^{\mathrm{e}}$). In the light of Theorem 4.9 we obtain the following.

COROLLARY 4.12. NB^{e} *is complete with respect to the matrix* $\langle \mathbf{B}_4 \oplus \mathbf{B}_2, \{1\}\rangle$. *Moreover, the only non-trivial extensions of* B^{e} *are* NB^{e} *and* CL.

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In a logical perspective, the notions of PSC and SC have been investigated in several contributions, such as [2, 32, 34, 38]. For a logic $\vdash$, being SC amounts to the fact that each admissible rule is derivable in $\vdash$. In other words, $\vdash$ is SC if for every rule (R) of the form $\langle \Gamma, \psi \rangle$:

$$(\vdash \varphi \iff \vdash_R \varphi) \Rightarrow \Gamma \vdash \psi,$$

where $\vdash_R$ is the extension of $\vdash$ obtained by adding (R) to $\vdash$. Clearly, the converse implication in the above display is always true. A *passive* rule is of the form $\langle \Gamma, y \rangle$, where no member of Γ contains occurrences of y, namely $y \notin Var(\Gamma)$. Accordingly, we say that a logic $\vdash$ is PSC if every passive, admissible rule is derivable.[6]

The following corollary emphasizes the logical meaning of the previous results on the subquasivarieties of BCA.

Corollary 4.13. $\mathrm{B^e}$ *is not PSC, while* $\mathrm{NB^e}$ *is SC.*

Proof. In order to prove the first statement we show (EFJ) is passive, admissible and non-derivable in $\mathrm{B^e}$. That (EFJ) is passive and non-derivable in $\mathrm{B^e}$ is clear. Let now φ be a theorem of $\mathrm{NB^e}$, and remind it is the logic obtained adding (EFJ) to $\mathrm{B^e}$. Suppose φ is not a theorem of $\mathrm{B^e}$. Then $\mathsf{BCA} \not\models \varphi \approx 1$, and $\mathsf{NBCA} \models \varphi \approx 1$. However, this contradicts Corollary 4.11. So, $\mathrm{B^e}$ is not PSC.

That $\mathrm{NB^e}$ is SC follows straightforwardly upon noticing it has CL as unique proper (non-trivial) extension, and that $\varphi \vee \neg\varphi$ is not a theorem of $\mathrm{NB^e}$. □

§5. Amalgamation in quasivarieties of Bochvar algebras. In the context of algebraizable logics, several logical properties can be established by means of the so-called bridge theorems, whose general form is

$$\text{a logic L has the property } P \iff \mathsf{K} \text{ has the property } Q,$$

where the quasivariety K is the equivalent algebraic semantics of L. A valid instance of the above equivalence can be obtained by replacing P with "Deduction theorem" and Q with "Equationally definable principal relative congruences" (see [13, theorem Q.9.3]).

In the light of the results of Section 4, BCA and NBCA are the only interesting quasivarieties of Bochvar algebras. In this section we show that they enjoy the amalgamation property (AP) or, equivalently, that their associated logics enjoy the Craig interpolation property. The strategy for proving (AP) for BCA consists in providing a sufficient condition implying (AP), established in [24, theorem 9] for varieties, and that naturally extends to quasivarieties (Theorem 5.2).

Recall that a *V-formation* (see Figure 2) is a 5-tuple $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ such that $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are similar algebras, and $i\colon \mathbf{A} \to \mathbf{B}$, $j\colon \mathbf{A} \to \mathbf{C}$ are embeddings. A class K of similar algebras is said to have the *amalgamation property* if for every V-formation with $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathsf{K}$ there exists an algebra $\mathbf{D} \in \mathsf{K}$ and embeddings $h\colon \mathbf{B} \to \mathbf{D}$, $k\colon \mathbf{C} \to \mathbf{D}$ such that $k \circ j = h \circ i$. In such a case, we also say that $(\mathbf{D}, h, k)$ is an *amalgam* of the V-formation $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$.

The following lemma is originally due to Grätzer [20] (it can be also found in [24]), while the subsequent theorem is the obvious adaptation to quasivarieties of a theorem

[6] In the context of finitary algebraizable logics, the fact that the logical definition of PSC is the "right" translation of the algebraic one can be inferred by comparing [25, corollary 3.3] and [32, theorem 7.5].

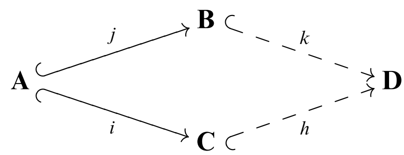

Figure 2. A generic amalgamation schema.

by Metcalfe, Montagna, and Tsinakis [24, theorem 9]. We insert the proofs for the completeness of the exposition.

LEMMA 5.1 [20]. *Let* Q *be a quasivariety. The following are equivalent*:

(1) Q *has* (AP);
(2) *for every* V*-formation* $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ *and elements* $x \neq y \in B$ $(x \neq y \in C$, *respectively*) *there exists* $\mathbf{D}_{xy} \in \mathsf{Q}$ *and homomorphisms* $h_{xy} \colon \mathbf{B} \to \mathbf{D}_{xy}$ *and* $k_{xy} \colon \mathbf{C} \to \mathbf{D}_{xy}$ *such that* $h_{xy}(x) \neq h_{xy}(y)$ $(k_{xy}(x) \neq k_{xy}(y)$, *respectively*) *and* $h \circ i = k \circ j$.

Proof. (1) $\Rightarrow$ (2) is obvious.
(2) $\Rightarrow$ (1). Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in Q. Define $\mathbf{D} = \prod_{x \neq y \in B} \mathbf{D}_{xy}$. By assumption, for every $x \neq y \in B$ there exist $h_{xy} \colon \mathbf{B} \to \mathbf{D}_{xy}$ and $k_{xy} \colon \mathbf{C} \to \mathbf{D}_{xy}$ s. t. $h(x) \neq h(y)$ and $h \circ i = k \circ j$. By the universal property of the product, $\mathbf{D}$ and the homomorphisms h and k, where $\pi_{xy} \circ h = h_{xy}$ and $\pi_{xy} \circ k = k_{xy}$ (with $\pi \colon \mathbf{D} \to \mathbf{D}_{xy}$ the projection) is the amalgam. □

The following provides a sufficient condition for a quasivariety to have the (AP) and it reduces somehow the search for an amalgam to a subclass of a quasivariety. As a notational convention, by $Co^{\mathbf{A}}_{\mathsf{K}}$ we denote the lattice of K-congruences on an algebra $\mathbf{A}$, namely the congruences θ on $\mathbf{A}$ such that $\mathbf{A}/\theta \in \mathsf{K}$. Let $\{\theta_i\}_{i \in I}$ be a family of K-congruences on an algebra $\mathbf{A} \in \mathsf{K}$. We say that $\mathbf{A}$ is subdirectly irreducible relative to K, or just relatively subdirectly irreducible, when $\bigwedge_{i \in I} \theta_i = \Delta^{\mathbf{A}}$ entails $\theta_i = \Delta^{\mathbf{A}}$ for some $i \in I$. Moreover, given a quasivariety K, by K_{RSI} we indicate the class of relatively subdirectly irreducible members of K. If K is a variety, we simply write K_{SI}.

THEOREM 5.2 (essentially [24]). *Let* K *be a subclass of a quasivariety* Q *satisfying the following properties*:

(1) $\mathsf{Q}_{RSI} \subseteq \mathsf{K}$;
(2) K *is closed under* I *and* S;
(3) *for every algebras* $\mathbf{A}, \mathbf{B} \in \mathsf{Q}$ *such that* $\mathbf{A} \leq \mathbf{B}$ *and every* $\theta \in Co^{\mathbf{A}}_{\mathsf{K}}$ *such that* $\mathbf{A}/\theta \in \mathsf{K}$ *there exists* $\Phi \in Co^{\mathbf{B}}_{\mathsf{K}}$ *extending* θ *with respect to* K, *i.e.*, $\mathcal{B}/\Phi \in \mathsf{K}$ *and* $\Phi \cap A^2 = \theta$;
(4) *every* V*-formation of algebras in* K *has an amalgam in* Q.

Then Q *has the* (AP).

Proof. We show that Q satisfies the condition (2) in Lemma 5.1. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, i, j)$ be a V-formation in Q and $x \neq y \in B$. By Zorn lemma, it is possible to find a relative

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congruence Ψ of $\mathbf{B}$ maximal with respect to the property $(x, y) \notin \Psi$. Let $\theta = \Psi \cap A^2$, and define the map $f : \mathbf{A}/\theta \to \mathbf{B}/\Psi$, $[a]_\theta \mapsto f([a]_\theta) := [a]_\Psi$. Observe that f is an injective homomorphism. Indeed, for $[a]_\theta \neq [b]_\theta$, i.e., $(a, b) \notin \theta$, hence $(a, b) \notin \Psi$ (Ψ is maximal with respect to this property), i.e., $[a]_\Psi \neq [b]_\Psi$. $\mathbf{B}/\Psi \in \mathsf{Q}_{RSI}$ (since Ψ is completely meet-irreducible), thus, by hypothesis (1), $\mathbf{B}/\Psi \in \mathsf{K}$; $\mathbf{A}/\theta \leq \mathbf{B}/\Psi$, hence $\mathbf{A}/\theta \in \mathsf{K}$ (by hyp. (2)). Since $\mathbf{A} \leq \mathbf{C}$ (upon identifying $\mathbf{A}$ with $j(\mathbf{A})$) and $\theta \in Co_{\mathsf{K}}^{\mathbf{A}}$, by hyp. (3), there exists $\Phi \in Co_{\mathsf{K}}^{\mathbf{C}}$ s.t. $\Phi \cap A^2 = \theta$ and $\mathbf{C}/\Phi \in \mathsf{K}$. The map $g : \mathbf{A}/\theta \to \mathbf{C}/\Phi$ defined as $[a]_\theta \mapsto g([a]_\theta) := [a]_\Phi$ is an injective homomorphism. Therefore $(\mathbf{A}/\theta, \mathbf{B}/\Psi, \mathbf{C}/\Phi, f, g)$ is a V-formation of algebras in K. By hyp. (4), there exists an amalgam $(h, k, \mathbf{D})$ in Q. Define the homomorphisms $h' : \mathbf{B} \to \mathbf{D}$ and $k' : \mathbf{C} \to \mathbf{D}$ as $h' = h \circ \pi_\Psi$ and $k' = k \circ \pi_\Phi$ (π_Ψ and π_Φ the projections onto the quotients $\mathbf{B}/\Psi$ and $\mathbf{C}/\Phi$, resp.). Observe that $h'(x) \neq h'(y)$ and $h' \circ i = k' \circ j$. Indeed, $h'(x) = h(\pi_\Psi(x)) = h([x]_\Psi) \neq h([y]_\Psi) = h(\pi_\Psi(y)) = h'(y)$ (where we have used the injectivity of h and the fact that $[x]_\Psi \neq [y]_\Psi$). As for the latter, let $a \in A$, $h' \circ i(a) = h(\pi_\Psi(i(a))) = h([i(a)]_\Psi) = k([j(a)]_\Phi) = k(\pi_\Phi(j(a))) = k'(j(a)) = k' \circ j(a)$ (where we have used the fact that $(\mathbf{D}, f, g)$ is an amalgam). Finally, by Lemma 5.1, we conclude that Q has the (AP). □

Remark 5.3. B^{e} *is a finitary logic with a Deduction Theorem* (*Theorem* 2.4): *this is a stronger property than the local deduction, which implies that the logic enjoys the filter extension property* (*see* [13, *theorem* 2.3.5]). *This translates into the relative congruence extension property* (*by the algebraizability of* B^{e}, *the lattice of logical filters is dually isomorphic to that of the relative congruences*).

Theorem 5.4. BCA *has the Amalgamation Property* (*AP*).

Proof. We show that $\mathsf{K} = \mathsf{BCA}_{RSI} = \{\mathbf{WK}^e, \mathbf{B}_2\}$ satisfies the assumptions (1)–(4) of Theorem 5.2. (1), (2) and (4) are immediate. As concerns (3): suppose that $\mathbf{A}, \mathbf{B} \in \mathsf{BCA}$ with $\mathbf{A} \leq \mathbf{B}$, $\theta \in Co_{\mathsf{K}}^{\mathbf{A}}$ and $\mathbf{A}_{/\theta} \in \mathsf{K}$. $\mathbf{B}$ decomposes into a Płonka sum $\mathcal{P}_l(\mathbf{B}_i)_{i \in I}$ and, since $\mathbf{A} \leq \mathbf{B}$, and $\mathsf{S}(\mathcal{P}_l(\mathbf{B}_i)) \subseteq \mathcal{P}_l(\mathsf{S}(\mathbf{B}_i))$, then $\mathbf{A}$ decomposes into a Płonka sum $\mathcal{P}_l(\mathbf{A}_j)$ of subalgebras of $\mathbf{B}_i$, over a semilattice of indexes $J \leq I$, thus, in particular, $i_0 \in J$. Observe that, for every $i \in I$, $\theta_i = \theta \cap A_i^2$ is a (Boolean) congruence on $\mathbf{A}_i$. The hypothesis that $\mathbf{A}_{/\theta} \in \mathsf{K}$ implies that θ_{i_0} is a maximal congruence on $\mathbf{A}_{i_0}$ (a congruence corresponding to a maximal ideal). Since BCA has the relative congruence extension property and $\mathbf{A} \leq \mathbf{B}$ then there exists a relative congruence Ψ on $\mathbf{B}$ extending θ ($\Psi \cap A^2 = \theta$). $\Psi_{i_0} = \Psi \cap B_{i_0}$ is a (Boolean) congruence on $\mathbf{B}_{i_0}$; let Φ_{i_0} (one of) its maximal extension on $\mathbf{B}_{i_0}$ and Φ the congruence on $\mathbf{B}$ defined as follows: $(x, y) \in \Phi$ iff $(J_2 x, J_2 y) \in \Phi_{i_0}$. It is immediate to check that $\Phi \in Co_{\mathsf{K}}^{\mathbf{B}}$ and $\mathbf{B}/\Phi \in \mathsf{K}$. Finally, it also holds that $\Phi \cap A^2 = \theta$: $\theta \subseteq \Phi \cap A^2$ follows by construction. On the other hand, let $a, b \in A$ (with $a \in A_i$ and $b \in B_j$) and $(a, b) \in \Phi$, i.e., $(J_2 a, J_2 b) \in \Phi_{i_0}$, hence $(J_2 a, J_2 b) \in \theta_{i_0}$ (by construction), thus $J_2 a, J_2 b \in [1]_\theta$ or $J_2 a, J_2 b \in [0]_\theta$. Suppose $J_2 a, J_2 b \in [1]_\theta$ (the other case is analogous), therefore $a = p_{i_0 i}(J_2 a) \in [1_i]_\theta$ and $b = p_{i_0 j}(J_2 b) \in [1_j]_\theta$. The assumption that $\mathbf{A}/\theta \in \mathsf{K}$ implies that $[1_i]_\theta = [1_j]_\theta$, from which $(a, b) \in \theta$. □

We conclude this section by proving that also the quasivariety NBCA has the (AP). Before proceeding further, it is worth noticing that we cannot apply the same strategy used in the case of BCA. This is a consequence of the following remark, which also proves that the logic NB_e does not have a local deduction theorem.

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REMARK 5.5. *Observe that* $\mathsf{NBCA}_{RSI} \subseteq IS(\mathbf{B}_4 \oplus \mathbf{B}_2)$, *because* $ISP(\mathbf{B}_4 \oplus \mathbf{B}_2) = IP_S S(\mathbf{B}_4 \oplus \mathbf{B}_2)$. *So, the only relatively subdirectly irreducible members of* NBCA *are* $\mathbf{B}_4 \oplus \mathbf{B}_2$ *and* $\mathbf{B}_2$. *The class* $K = \{\mathbf{B}_4 \oplus \mathbf{B}_2, \mathbf{B}_2\}$ *does not satisfy condition* (3) *of Theorem* 5.2, *consider the algebra* $\mathbf{B}_4 \oplus \mathbf{B}_2$ *depicted in* 4.7, *where* $J_1(\bot) = \neg a$. *Observe that* $\mathbf{B}_4 \leq \mathbf{B}_4 \oplus \mathbf{B}_2$ *and consider* $\theta = Cg^{\mathbf{B}_4}_{\mathsf{NBCA}}(1, \neg a)$. *Clearly* $\mathbf{B}_4/\theta \cong \mathbf{B}_2$ *but, for each* NBCA-*congruence* Φ *on* $\mathbf{B}_4 \oplus \mathbf{B}_2$, *if* $(\neg a, 1) \in \Phi$ *then* $\Phi = \nabla \neq \theta \cap \mathbf{B}_4^2$. *This shows that* NBCA *fails the relative congruence extension property or, equivalently, that* NB_e *fails to have a local deduction theorem.*

Nonetheless, (AP) holds for NBCA, as shown in the following.

THEOREM 5.6. NBCA *has the amalgamation property.*

Proof. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, f, g)$ be a V-formation in NBCA. By Theorem 5.4, there exists an amalgam $(h, k, \mathbf{D})$ with $\mathbf{D} \in \mathsf{BCA}$. If $\mathbf{D}$ has no trivial fibers, then $(h, k, \mathbf{D})$ is also an amalgam in NBCA. Otherwise, let $u \in I$ be the index of the trivial fiber $\mathbf{D}_u$ with universe $\{u\}$, where I is the underlying semilattice of $\mathbf{D}$ and homomorphisms p_{ij} for every $i \leq j$. Observe that $k(b) \neq u$ and $h(c) \neq u$, for each $b \in B$, $c \in C$. Consider an ultrafilter F_0 over $\mathbf{D}_{i_0}$ (the lowest fiber in $\mathbf{D}$) and $F_i = p_{i_0 i}[F_0]$, for each $i_0 \leq i$ and set $F = \bigcup_{i<u} F_i$. Each F_i is an ultrafilter over the algebra $\mathbf{D}_i$ (since homomorphisms are surjective). Consider the algebra $\mathbf{D}' = \mathbf{D} \times \mathbf{B}_2$, which shares with $\mathbf{D}$ the semilattice structure I and whose homomorphisms are denoted by q_{ij} for $i \leq j$. Observe that this algebra does not contain trivial fibers as u is the top element of I and $\mathbf{D}'_u$ is the two-elements Boolean algebra with universe $\{\langle u, \top\rangle, \langle u, \bot\rangle\}$. This entails that $\mathbf{D}' \in \mathsf{NBCA}$. Define the map $k' \colon \mathbf{B} \to \mathbf{D}'$ such that, for any $b \in B$:

$$k'(b) = \begin{cases} \langle k(b), \top\rangle, \text{ if } k(b) \in F, \\ \langle k(b), \bot\rangle, \text{ otherwise.} \end{cases}$$

Similarly, consider $h' \colon \mathbf{C} \to \mathbf{D}'$ defined by the same rule when applied to elements in C. We show that k' is an embedding. Clearly the map in injective, as k is. It is also clear that k' preserves the Boolean operations. Moreover, for $b \in B$,

$$\begin{aligned} k'(J_2 b) = \langle k(J_2 b), \top\rangle &\iff \\ k(J_2 b) = J_2 k(b) \in F &\iff \\ k(b) \in F &\iff \\ J_2 k'(b) = J_2\langle(k(b), \top\rangle = \langle J_2 k(b), J_2 \top\rangle &= \langle k(J_2 b), \top\rangle, \end{aligned}$$

where the second equivalence is justified because, for any $i \in I$ and $d_i \in D_i$, $J_2 d_i \in p_{0i}^{-1}(d_i)$. This, together with an analogous argument applied to h', show that k', h' are embeddings. In order to conclude the proof, recall that for $a \in A$, $k \circ f(a) = h \circ g(a)$, so the following equivalences hold:

$$\begin{aligned} k' \circ f(a) = \langle k \circ f(a), \top\rangle &\iff \\ k \circ f(a) \in F &\iff \\ g'' \circ g(a) = \langle g' \circ g(a), \top\rangle &= \langle k \circ f(a), \top\rangle. \end{aligned}$$

This proves $k' \circ f(a) = h' \circ g(a)$, as desired. □

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§A. Appendix. This appendix is devoted to the proof of Theorem 3.3. Let us denote by BCA_2 the quasivariety axiomatized by (1)–(13) in Theorem 3.3.

LEMMA A.1. *The following identities hold in* BCA_2.

(1) $1 \wedge \varphi \approx \varphi$.
(2) $J_k\varphi \vee \neg J_k\varphi \approx 1, \forall k \in \{0,1,2\}$.
(3) $J_k\varphi \wedge \neg J_k\varphi \approx 0, \forall k \in \{0,1,2\}$.
(4) $J_2(\varphi \vee \neg\varphi) \approx J_2\varphi \vee J_2\neg\varphi$.
(5) $J_2J_k\varphi \approx J_k\varphi$, *for every* $k \in \{0,1,2\}$.
(6) $J_0J_k\varphi \approx \neg J_k\varphi$, *for every* $k \in \{0,1,2\}$.
(7) $J_i\varphi \approx \neg(J_j\varphi \vee J_k\varphi)$, *for* $i \neq j \neq k \neq i$.
(8) $((J_i\varphi \vee J_k\varphi) \wedge J_i\varphi) \approx J_i\varphi$, *for* $i, k \in \{0,1,2\}$.

Proof. Let $\mathbf{A} \in \mathsf{BCA}_2$ and $a, b \in A$.

(1) $1 \wedge a = \neg(\neg 1 \vee \neg a) = \neg(0 \vee \neg a) = \neg(\neg a) = a$.

(2) The case $k = 0$ follows immediately by the case $k = 2$ (which holds by Definition 3.3). For $k = 1$: $J_1a \vee \neg J_1a = \neg(J_2a \vee J_0a) \vee (J_2a \vee J_0a) = (\neg J_2a \wedge \neg J_0a) \vee (J_2a \vee J_0a) = (\neg J_2a \vee J_2a \vee J_0a) \wedge (\neg J_0a \vee J_2a \vee J_0a) = (1 \vee J_0a) \wedge (1 \vee J_2a) = (J_0a \vee \neg J_0a \vee J_0a) \wedge (J_2a \vee \neg J_2a \vee J_2a) = (J_0a \vee \neg J_0a) \wedge (J_2a \vee \neg J_2a) = 1 \wedge 1 = 1$.

(3) Follows from (2) (and De Morgan laws).

(4) $J_2(a \vee \neg a) = (J_2a \wedge J_2\neg a) \vee (J_2\neg a \wedge J_2\neg a) \vee (J_2a \wedge J_2a) = ((J_2a \wedge J_2\neg a) \vee J_2\neg a) \vee J_2a = ((J_2a \vee J_2\neg a) \wedge (J_2\neg a \wedge J_2\neg a)) \vee J_2a = ((J_2a \vee J_2\neg a) \wedge J_2\neg a) \vee J_2a = (J_2a \vee J_2\neg a \vee J_2a) \wedge (J_2\neg a \vee J_2a) = (J_2a \vee J_2\neg a) \wedge (J_2\neg a \vee J_2a) = J_2a \vee J_2\neg a$, where we have used (12) and distributivity.

(5) $k = 2$: it follows directly from $\neg J_2a = J_2\neg J_2a$, which holds as $\neg J_2a = J_0J_2a = J_2\neg J_2a$ (where we have used (9) and the definition of J_0). $k = 0$: $J_2J_0a = J_2J_2\neg a = J_2\neg a = J_0a$. $k = 1$: $J_2J_1a = J_2\neg(J_2a \vee J_0a) = J_0(J_2a \vee J_0a) = J_0(J_2a \vee J_2\neg a) = J_0J_2(a \vee \neg a) = \neg J_2(a \vee \neg a) = \neg(J_2a \vee J_2\neg a) = \neg(J_2a \vee J_0a) = J_1a$, where we have use the previous (4).

(6) $k = 2$ is included in Definition 3.3, $k = 0$ follows immediately by (9). For $k = 1$: $J_0J_1a = J_2\neg J_1a = J_2(J_2a \vee J_0a) = (J_2J_2a \wedge J_2J_0a) \vee (J_2\neg J_2a \wedge J_2J_0a) \vee (J_2J_2a \wedge J_2\neg J_0a) = (J_2a \wedge J_0a) \vee (J_0J_2a \wedge J_0a) \vee (J_2a \wedge J_0J_0a) = (J_0a \wedge (J_2a \vee \neg J_2a)) \vee (J_2a \wedge \neg J_0a) = (J_0a \wedge 1) \vee (J_2a \wedge \neg J_0a) = J_0a \vee (J_2a \wedge \neg J_0a) = (J_0a \vee J_2) \wedge (J_0a \vee \neg J_0a) = (J_0a \vee J_2) \wedge 1 = J_0a \vee J_2 = \neg J_1a$.

(7) We only have to show the case $J_0\varphi \approx \neg(J_2\varphi \vee J_1\varphi)$ (as the others hold by Definition 3.3 and by the definition of J_1). $J_0a = J_2\neg a = \neg(J_0\neg a \vee \neg J_1\neg a) = \neg(J_2a \vee J_1a)$.

(8) We just show the case $i = 2$, $k = 0$ (as the others are analogous). $J_2a \wedge (J_2a \vee J_0a) = J_2a \wedge \neg J_1a = \neg(J_0a \vee J_1a) \wedge \neg J_1a = \neg J_0a \wedge \neg J_1a \wedge \neg J_1a = \neg J_0a \wedge \neg J_1a = \neg(J_0a \vee J_1a) = J_2a$. □

Observe that, by Lemma A.1 (in particular, (8) and (9)), it follows that the image $J_2(\mathbf{A})$ (and hence of J_0 and J_1) of a Bochvar algebra forms the universe of a Boolean algebra: a fact that we will use several times (in the proofs) of the next lemma, where we will indicate with $\leq$ the order of the mentioned Boolean algebra.

LEMMA A.2. *The following identities and quasi-identities hold in* BCA_2.

(1) $J_21 \approx 1$, $J_00 \approx 1$, $J_20 \approx 0$, $J_01 \approx 0$.
(2) $J_2\varphi \vee J_0\varphi \approx J_2(1 \vee \varphi)$.

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(3) $J_2(1 \vee \varphi) \approx J_2(1 \vee \neg\varphi)$.
(4) $J_2(1 \vee \varphi) \approx J_2(1 \vee (0 \wedge \varphi))$.
(5) $J_i\varphi \leq \neg J_k\varphi$, *for every* $i \neq k \in \{0, 1, 2\}$.
(6) $J_2(\varphi \wedge 0) \approx 0$.
(7) $J_2(1 \vee (\varphi \wedge \psi)) \approx J_2(1 \vee \varphi) \wedge J_2(1 \vee \psi)$.
(8) $J_1(\varphi \wedge \psi) \approx J_1\varphi \vee J_1\psi$.
(9) $J_0(\varphi \wedge \psi) \approx (J_2\varphi \wedge J_0\psi) \vee (J_0\varphi \wedge \neg J_1\psi)$.
(10) $J_2(\varphi \wedge \psi) \approx \neg(J_2\varphi \wedge J_0\psi) \wedge J_2\varphi \wedge \neg J_1\psi$.
(11) $J_2(\varphi \wedge \psi) \approx J_2\varphi \wedge J_2\psi$.
(12) $J_0(\varphi \vee \psi) \approx J_0\varphi \wedge J_0\psi$.
(13) $\varphi \vee J_k\varphi \approx \varphi$, *for* $k \in \{1, 2\}$.
(14) $J_0\varphi \approx J_0\psi \;\&\; J_1\varphi \approx J_1\psi \;\&\; J_2\varphi \approx J_2\psi \;\Rightarrow\; \varphi \approx \psi$.

Proof. Let $\mathbf{A} \in \mathsf{BCA}_2$ and $a, b \in A$.

(1) $J_21 = J_2(J_2a \vee \neg J_2a) = (J_2J_2a \vee J_2\neg J_2a) \wedge (J_2\neg J_2a \vee J_2\neg J_2a) \vee (J_2J_2a \vee J_2J_2a) = (J_2a \wedge J_0J_2a) \vee J_0J_2a \vee J_2a = (J_2a \wedge \neg J_2a) \vee \neg J_2a \vee J_2a = 0 \vee \neg J_2a \vee J_2a = 0 \vee 1 = 1$.
$J_20 = \neg(J_00 \vee J_10) = \neg(1 \vee J_11) = \neg 1 = 0$. The last equality follows from this one.
(2) $J_2(1 \vee a) = (J_21 \wedge J_2a) \vee (J_21 \wedge J_0a) \vee (J_20 \wedge J_2a) = (1 \wedge J_2a) \vee (1 \vee J_0a) \vee (0 \wedge J_2a) = J_2a \vee J_0a \vee 0 = J_2a \vee J_0a$.
(3) It follows directly from the previous point, upon observing that $J_2\varphi \approx J_0\neg\varphi$.
(4) Observe that $1 \vee (0 \wedge a) = (1 \vee 0) \wedge (1 \vee a) = 1 \wedge (1 \vee a) = 1 \vee a$ (by Lemma A.1(1)).
(5) Immediate from Lemma A.1(7).
(6) Observe that, by the previous point, $J_2(0 \wedge a) \leq \neg J_0(0 \wedge a) = \neg J_2(1 \vee \neg a) = \neg J_2(1 \vee a)$. Therefore $J_2(0 \wedge a) \leq \neg J_2(1 \vee a) = \neg J_2(1 \vee (0 \wedge a)) = \neg(J_2(0 \wedge a) \vee J_0(0 \wedge a)) = \neg J_2(0 \wedge a) \wedge \neg J_0(0 \wedge a) \leq \neg J_2(0 \wedge a)$, hence $J_2(0 \wedge a) = 0$.
(7) Applying De Morgan laws and 12, we have

$$\begin{aligned}
J_2(1 \vee (a \wedge b)) &= J_2(1 \vee \neg a \vee \neg b) \\
&= J_2((1 \vee \neg a) \vee \neg b) \\
&= (J_2(1 \vee \neg a) \wedge J_0b) \vee (J_2(1 \vee \neg a) \wedge J_2a) \vee (J_0(1 \vee \neg a) \wedge J_0b) \\
&= (J_2(1 \vee a) \wedge J_0b) \vee (J_2(1 \vee a) \wedge J_2a) \vee (J_2(0 \wedge a) \wedge J_0b) \\
&= (J_2(1 \vee a) \wedge J_0b) \vee (J_2(1 \vee a) \wedge J_2a) \vee 0 \\
&= (J_2(1 \vee a) \wedge J_0b) \vee (J_2(1 \vee a) \wedge J_2a) \\
&= J_2(1 \vee a) \wedge (J_0b \vee J_2b) \\
&= J_2(1 \vee a) \wedge J_2(1 \vee b).
\end{aligned}$$

(8) The claim is equivalent to (7). Indeed $J_1(\varphi \wedge \psi) \approx J_1\varphi \vee J_1\psi$ iff $\neg J_1(\varphi \wedge \psi) \approx \neg J_1\varphi \wedge \neg J_1\psi$ iff $J_2(\varphi \wedge \psi) \vee J_0(\varphi \wedge \psi) \approx (J_2\varphi \vee J_0\varphi) \wedge (J_2\psi \vee J_0\psi)$ iff $J_2(1 \vee (\varphi \wedge \psi)) \approx J_2(1 \vee \varphi) \wedge J_2(1 \vee \psi)$.
(9) Easy calculation using (12), De Morgan laws, distributivity and Lemma A.1(6).

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(10) By Lemma A.1(6), we have

$$\begin{aligned}\neg J_2(a\wedge b) &= J_0(a\wedge b)\vee J_1(a\wedge b)\\ &= (J_2a\wedge J_0b)\vee(J_0a\wedge\neg J_1b)\vee J_1(a\wedge b) && (9)\\ &= (J_2a\wedge J_0b)\vee(J_0a\wedge\neg J_1b)\vee J_1b\vee J_1a && (8)\\ &= (J_2a\wedge J_0b)\vee((J_0a\vee J_1b)\wedge(\neg J_1b\vee J_1b))\vee J_1a\\ &= (J_2a\wedge J_0b)\vee((J_0a\vee J_1b)\wedge 1)\vee J_1a\\ &= (J_2a\wedge J_0b)\vee J_0a\vee J_1b\vee J_1a\\ &= (J_2a\wedge J_0b)\vee\neg J_2a\vee J_1b,\end{aligned}$$

thus the conclusion follows by De Morgan laws.

(11) By the previous point, we have

$$\begin{aligned}J_2(a\wedge b) &= \neg(J_2a\wedge J_0b)\wedge J_2a\wedge\neg J_1b\\ &= (\neg J_2a\vee\neg J_0b)\wedge J_2a\wedge\neg J_1b\\ &= ((\neg J_2a\wedge J_2a)\vee(\neg J_0b\wedge J_2a))\wedge\neg J_1b\\ &= (0\vee(\neg J_0b\wedge J_2a))\wedge\neg J_1b\\ &= \neg J_0b\wedge J_2a\wedge\neg J_1b\\ &= J_2a\wedge J_2b.\end{aligned}$$

(12) $J_0(a\vee b)=J_2(\neg a\wedge\neg b)=J_2\neg a\wedge J_2\neg b=J_0a\wedge J_0b$.

(13) We show that the antecedent of the quasi-identity (13) is satisfied, so is the consequent. $J_2(a\vee J_2a)=(J_2a\wedge J_2J_2a)\vee(J_2\neg a\wedge J_2J_2a)\vee(J_2a\wedge J_2\neg J_2a)=(J_2a\wedge J_2a)\vee(J_0a\wedge J_2a)\vee(J_2a\wedge J_2J_0a)=J_2a\vee(J_0a\wedge J_2a)\vee(J_0a\wedge J_2a)=J_2a\vee(J_0a\wedge J_2a)=J_2a$, where in the last passage we have used the dual version of (9).

$J_0(a\vee J_2a)=J_0a\wedge J_0J_2a=J_0a\wedge\neg J_2a=J_0a\wedge(J_0a\vee J_1a)=J_0a$.
Thus, by the quasi-identity (13) we have the conclusion.

The case of $k=1$ is proved analogously.

(14) We just have to show that $J_0\varphi\approx J_0\psi\ \&\ J_2\varphi\approx J_2\psi$ implies $J_0\varphi\approx J_0\psi\ \&\ J_1\varphi\approx J_1\psi\ \&\ J_2\varphi\approx J_2\psi$. Suppose $J_0a=J_0b$ and $J_2a=J_2b$. Then $J_1a=\neg(J_2a\vee J_0a)=\neg(J_2b\vee J_0b)=J_1b$. □

Proof of Theorem 3.3. The original axiomatization of BCA (Definition 3.1) includes all the identities (1)–(13); all the remaining identities (and quasi-identities) appearing in Definition 3.1 but not in Theorem 3.3 have been shown to follow, from the axiomatization provided in Theorem 3.3 in Lemmas A.1 and A.2. □

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