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## An 18 Moments Model for Dense Gases: Entropy and Galilean Relativity Principles without Expansions

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**Abstract:** The 14 moments model for dense gases, introduced in the last few years by Arima, Taniguchi, Ruggeri and Sugiyama, is here extended up to 18 moments. They have found the closure of the balance equations up to a finite order with respect to equilibrium; it is also possible to impose for that model the entropy and Galilean relativity principles up to whatever order with respect to equilibrium, but by using Taylor's expansion. Here, the exact solution is found, without expansions, but a bigger number of moments has to be considered and reasons will be shown suggesting that this number is at least 18.

**Keywords:** extended thermodynamics; dense gas; Galilean relativity

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### 1. Introduction

A new model for dense gases is here considered in the framework of extended thermodynamics. Some of the original papers on this subject are [1,2] (see, also, the book [3] for a complete description), while more recent papers are [4–18], and the theory has the advantage of furnishing hyperbolic field equations, with finite speeds of propagation of shock waves and very interesting analytical properties.

It starts from a given set of balance equations, where some arbitrary functions appear; restrictions on this arbitrariness are obtained by imposing the entropy principle and the relativity principle.

This model was applied also to dense gases, for example, by Kremer [19–26], by imposing, up to a finite order with respect to equilibrium, the conditions that come from the above principles; subsequently, in [27,28], an exact non approximated solution of these same conditions has been found for the 14 moments model. A model with an arbitrary number of moments has been considered in [11], where the form of the closure was suggested by the non-relativistic limit of the corresponding relativistic model, and also in this case, an exact non-approximated solution of these same conditions has been found. An interesting aspect of that solution is that, if we try a transition to subsystems, following the procedure of [29], then the result will be different from that obtained by starting from the beginning in that subsystem. To be more precise, for every possible choice of two integer numbers,  $N$  and  $M$ , the article [11] describes a possible model; for a particular choice of  $M$  and  $N$ , the resulting model is the 14 moments one, but this does not mean that it is a subsystem. It would be a subsystem only if, starting from greater values of  $M$  and  $N$  and, then, putting to zero the last Lagrange multipliers, we could obtain the 14 moments one.

In [30], it was supposed that this was a consequence of a bad choice of the hierarchy structure of the balance equations, and another hierarchy was formulated. In the present article, we adopt the same hierarchy of [30], but obtain the same consequence, that is no continuity in the transition to subsystems. However, we think that this is not a defect of the theory, but only a property of the general solution that we find. It may be interpreted as a resistance of nature to the transition to subsystems, in the sense that if we start from a particular subsystem, then it is necessary to remain in that subsystem; but this may seem metaphysical, so we prefer to refrain from expressing this conviction.

We only want to remark that this is a property of the present non-approximated solution and not of the model; in fact, if we look for the Taylor’s expansion of the solution around equilibrium, then the passage can be obtained by the Taylor’s expansion of the closure for subsystems.

Obviously, it is now necessary to speak about the new model [30], which inspired many other subsequent articles, a restricted part of which are [31–41].

In the previous formulation of the theory, the restrictions imposed by the entropy principle and the Galilean relativity principle were so strong, to allow only particular state functions; for example, the function  $p = p(\rho, T)$  relating the pressure  $p$  with the mass density  $\rho$  and the absolute temperature  $T$  was determined, except for a single variable function, so that it was adapted to describe only particular gases or a continuum.

This drawback has been overcome in [30] by considering two blocks of balance equations; for example, in the 18 moments case, which we treat here, they are:

$$\partial_t F^N + \partial_k \tilde{F}^{kN} = P^N, \tag{1}$$

$$\partial_t G^E + \partial_k \tilde{G}^{kE} = Q^E \tag{2}$$

where  $F^N = (F, F^i, F^{ij}, F^{ill})$ ,  $G^E = (G, G^i, G^{ll})$ ,  $\tilde{F}^{kN} = (F^k, F^{ki}, F^{kij}, F^{kill})$ ,  $\tilde{G}^{kE} = (G^k, G^{ki}, G^{kll})$ ,  $P^N = (0, 0, P^{ij}, P^{ill})$ ,  $Q^E = (0, Q^i, Q^{ll})$ .

The first two components of  $P^N$  are zero, because the first two components of Equation (1) are the conservation laws of mass and momentum; the first component of  $Q^E$  is zero, because the first component of Equation (2) is the conservation law of energy. The whole block (2) can be considered an “energy block”, and it was not considered in the previous version of extended thermodynamics.

The difference between the two blocks,  $F$  and  $G$ , can be better understood by considering their counterparts in the kinetic theory of polyatomic gases, as described in [31]. It can be synthesized as follows: In the previous models, the distribution function was considered to depend on position, time and velocity; the moments were obtained by multiplying it by polynomial functions of the velocity and then integrating with respect to velocity in the phase space. Here, it depends also on the internal energy  $I$  of a molecule, and integration is necessary also over  $I$  for  $I \in ] - \infty, +\infty[$ . By multiplying it by polynomial functions of the velocity and then integrating it in that four-dimensional space, we obtain the block (1) for the variables  $F$ . By multiplying it by  $2I$  and the power of the velocity and then integrating, we obtain new variables  $V^E$ ; after that, the block (2) is constituted by  $G^E = V^E + F^{llE}$  (the necessity of this sum lies in the fact that, for example, we have the conservation of total energy, that is internal energy and kinetic energy).

The importance of many moments lies in the fact that the moments theory truncated at a given number of moments can be considered as an approximation of the Bhatnagar–Gross–Krook (BGK) equation (which is similar to the Boltzmann equation); consequently, extending the number of moments increases the goodness of the approximation.

The 14 moments model of [30] considers only a part of Equations (1,2), avoiding considering those for  $F^{ill}$  and  $G^{ll}$ . However, if we want to extend to this model the methods of [11], we have to consider as the last equation a scalar one, such as that for  $G^{ll}$ . On the other hand, the closure of the balance equations cannot be chosen arbitrarily, but according to the Galilean relativity principle, in the manner suggested in Section 4 of [31]. A consequence of this fact is that we cannot consider an equation for  $G^{ll}$  without considering an equation also for  $F^{ill}$ . In effect, the principle would not be violated if we consider also an equation for  $F^{ppll}$ , but in this case, we would obtain a 19 moments model. Therefore, if we want to put together the necessities of the method used in [11] with the restrictions of [31], we have to consider a number of moments that is at least 18. Furthermore, with six moments, we can obtain an exact solution, but this is already known in the literature.

At this moment, it is convenient to clarify the differences between some of the above mentioned models. The 14 moments system of [11] is the same as that introduced by Kremer [19–22], except that in [11], the solution of the conditions is an exact solution not using Taylor’s expansions. All of them belong to the time when the two-block hierarchy of [30] was not introduced or was not fully known. Therefore, they are completely different from the 14-moment system of [30].

Now, for the sequel, it is useful to rewrite Equations (1,2) in a more compact form as:

$$\partial_t F^A + \partial_k \tilde{F}^{kA} = P^A, \tag{3}$$

where  $F^A = (F^N, G^E)$ ,  $\tilde{F}^{kA} = (\tilde{F}^{kN}, \tilde{G}^{kE})$ ,  $P^A = (P^N, Q^E)$ .

In the whole set (3),  $F^A$  are the independent variables, while  $\tilde{F}^{kA}$ ,  $P^A$  are constitutive functions. Restrictions on their generalities are obtained by imposing:

- (1) The entropy principle, which guarantees the existence of an entropy density  $h$  and an entropy flux  $h^k$ , such that the equation:

$$\partial_t h + \partial_k h^k = \sigma \geq 0, \tag{4}$$

holds whatever is the solution of Equation (3).

Thanks to Liu’s theorem [42], this is equivalent to assuming the existence of Lagrange multipliers  $\mu_A$ , such that:

$$dh = \mu_A dF^A, \tag{5}$$

$$dh^k = \mu_A d\tilde{F}^{kA} \tag{6}$$

$$\sigma = \mu_A P^A \geq 0. \tag{7}$$

An idea conceived by Ruggeri is to define the four-potentials  $h'$ ,  $h'^k$  as:

$$h' = \mu_A F^A - h, \quad h'^k = \mu_A \tilde{F}^{kA} - h^k, \tag{8}$$

so that Equations (5, 6) become:

$$dh' = F^A d\mu_A, \quad dh'^k = \tilde{F}^{kA} d\mu_A, \tag{9}$$

which are equivalent to:

$$F^A = \frac{\partial h'}{\partial \mu_A}, \tag{9}$$

$$\tilde{F}^{kA} = \frac{\partial h'^k}{\partial \mu_A}, \tag{10}$$

if the Lagrange multipliers are taken as independent variables. A nice consequence of Equations (9, 10) is that the field equations assume the symmetric form.

We note that (7) is the only condition that we have for the production terms  $P^A$  in the framework of a macroscopic approach, apart for the fact that they are zero at equilibrium. For example, we may write their expressions at the first order with respect to equilibrium; after that, (7) will give the sign of the coefficients.

In the framework of kinetic theory, they can be obtained with integrations involving the distribution function; to this regard, we refer to [30–33], because we have nothing more to say than those results, in this regard.

Other restrictions are given by

- (2) The symmetry conditions, that is the second component of  $F^N$  is equal to the first component of  $\tilde{F}^{kN}$  and the second component of  $\tilde{F}^{kN}$  is a symmetric tensor.

More restrictive conditions may be to impose that the flux in an equation becomes a density in the next equation, but this will lead to less general results, so we prefer not to impose it, as also other authors have done so in the past. In any case, they can be imposed subsequently if requested by the particular physical application under consideration. Similarly, we may impose also the symmetry of the tensors  $\tilde{F}^{kij}$  and  $\tilde{G}^{kij}$ , but this has not been imposed in [30] in order to obtain more general results; so, we do the same thing.

Thanks to Equations (9,10), the above-mentioned symmetry conditions, which we want to impose, assume the form:

$$\frac{\partial h'}{\partial \mu_i} = \frac{\partial h'^i}{\partial \mu} \tag{11}$$

$$\frac{\partial h'^{[i}}{\partial \mu_{j]}} = 0, \tag{12}$$

where we have assumed the decomposition  $\mu_A = (\mu, \mu_i, \mu_{ij}, \mu_{ill}, \lambda, \lambda_i, \lambda_{ll})$  for the Lagrange multipliers. Moreover,  $\mu_{ij}$  is a symmetric tensor.

The next conditions come from

(3) The Galilean relativity principle.

We prefer to devote an entire section, the next one, to describe how to impose this principle. We only remark here that we impose it, combined with the entropy principle, not in the common way [3], but using the methods described in [43]. Therefore, the present article proves also that those methods can be naturally extended to the new two-block set of balance equations.

The result will be that the four-potentials  $h'$  and  $h'^k$  must satisfy the following set of linear partial differential equations:

$$\frac{\partial h'}{\partial \mu} \mu_i + 2 \frac{\partial h'}{\partial \mu_a} (\mu_{ai} + \delta_{ai} \lambda) + \frac{\partial h'}{\partial \mu_{ab}} (2\delta_{i(a}\mu_{b)ll} + \mu_{ill}\delta_{ab} + 2\lambda_{(a}\delta_{b)i}) + 2 \frac{\partial h'}{\partial \mu_{ill}} \lambda_{ll} + \frac{\partial h'}{\partial \lambda} \lambda_i + 2 \frac{\partial h'}{\partial \lambda_i} \lambda_{ll} = 0. \tag{13}$$

$$\frac{\partial h'^k}{\partial \mu} \mu_i + 2 \frac{\partial h'^k}{\partial \mu_a} (\mu_{ai} + \delta_{ai} \lambda) + \frac{\partial h'^k}{\partial \mu_{ab}} (2\delta_{i(a}\mu_{b)ll} + \mu_{ill}\delta_{ab} + 2\lambda_{(a}\delta_{b)i}) + 2 \frac{\partial h'^k}{\partial \mu_{ill}} \lambda_{ll} + \frac{\partial h'^k}{\partial \lambda} \lambda_i + 2 \frac{\partial h'^k}{\partial \lambda_i} \lambda_{ll} + h' \delta^{ik} = 0. \tag{14}$$

In Section 3 of the present article, we find the general solution of the condition (13,14).

In Section 4, we will see the general solution of the condition (11,12).

### 2. The Galilean Relativity Principle

There are two ways to impose this principle. One of these is to decompose the variables  $F^A, \tilde{F}^{kA}, P^A, \mu_A$  in their corresponding non-convective parts  $\hat{F}^A, \hat{\tilde{F}}^{kA}, \hat{P}^A, \hat{\mu}_A$  and in velocity-dependent parts, where the velocity is defined, in terms of the field  $F^A$ , by:

$$v^i = F^{-1} F^i. \tag{15}$$

This decomposition can be written as:

$$F^A = X^A_B(\vec{v}) \hat{F}^B, \quad \tilde{F}^{kA} - v^k F^A = X^A_B(\vec{v}) \hat{\tilde{F}}^{kB}, \quad P^A = X^A_B(\vec{v}) \hat{P}^B, \tag{16}$$

$$h' = \hat{h}', \quad h'^k - v^k h' = \hat{h}'^k, \quad \hat{\mu}_A = \mu_B X^B_A(\vec{v}),$$

where:

$$X^A_B(\vec{v}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ v^i & \delta_a^i & 0 & 0 & 0 & 0 & 0 \\ v^i v^j & 2v^i \delta_a^j & \delta_{(a}^i \delta_{b)}^j & 0 & 0 & 0 & 0 \\ v^2 v^i & v^2 \delta_a^i + 2v^i v_a & 2\delta_{(a}^i v_{b)} + v^i \delta_{ab} & \delta_a^i & 0 & 0 & 0 \\ v^2 & 2v_a & 0 & 0 & 1 & 0 & 0 \\ v^2 v^i & v^2 \delta_a^i + 2v^i v_a & 2\delta_{(a}^i v_{b)} & 0 & v^i & \delta_a^i & 0 \\ v^4 & 4v^2 v_a & 4v_a v_b + v^2 \delta_{ab} & 2v_a & v^2 & 2v_a & 1 \end{pmatrix}. \tag{17}$$

After that, all of the conditions are expressed in terms of the non-convective parts of the variables.

This procedure is described in [1,3] for the case considering only the block (1) and is followed in [30] for the whole set (1,2).

Another way to impose this principle leads to easier calculations; it is described in [43] for the case considering only the the block (1) (but, it was already used in [44–46]), and here, we show how it is adapted also for the whole set (1,2).

First of all, we need to know the transformation law of the variables between two reference frames moving with respect to each other with a translational motion with constant translational velocity  $\vec{v}_\tau$ . To know it, we may rewrite (16) in both frames, that is:

$$F_a^A = X^A_B(\vec{v}_a)\hat{F}^B, \tag{18}$$

$$\tilde{F}_a^{kA} - v_a^k F_a^A = X^A_B(\vec{v}_a)\hat{F}^{kB}, \tag{19}$$

$$h'_a = \hat{h}', \tag{20}$$

$$h_a^{ik} - v_a^k h'_a = \hat{h}'^k, \tag{21}$$

$$\hat{\mu}_A = \mu_B^a X^B_A(\vec{v}_a), \tag{22}$$

$$F_r^A = X^A_B(\vec{v}_r)\hat{F}^B, \tag{23}$$

$$\tilde{F}_r^{kA} - v_r^k F_r^A = X^A_B(\vec{v}_r)\hat{F}^{kB}, \tag{24}$$

$$h'_r = \hat{h}' \tag{25}$$

$$h_r^{ik} - v_r^k h'_r = \hat{h}'^k, \tag{26}$$

$$\hat{\mu}_A = \mu_B^r X^B_A(\vec{v}_r), \tag{27}$$

where index  $a$  denotes quantities in the absolute reference frame and index  $r$  denotes quantities in the relative one;  $\hat{F}^B, \hat{F}^{kB}, \hat{h}', \hat{h}'^k, \hat{\mu}_B$  do not have the index  $a$ , nor the index  $r$ , because they are independent of the reference frame.

Now, we can use a property of the matrix  $X^A_B(\vec{v})$ , which is a consequence of its definition (17) and reads:

$$X^C_A(-\vec{v})X^A_B(\vec{v}) = \delta^C_B. \tag{28}$$

Therefore, we may contract (23,24) with  $X^C_A(-\vec{v}_r)$ , so obtaining:

$$\hat{F}^C = X^C_A(-\vec{v}_r)F_r^A, \quad \hat{F}^{kC} = X^C_A(-\vec{v}_r)(\tilde{F}_r^{kA} - v_r^k F_r^A)$$

which can be substituted in (18,19). The result is:

$$F_a^A = X^A_B(\vec{v}_a)X^B_C(-\vec{v}_r)F_r^C, \quad \tilde{F}_a^{kA} - v_a^k F_a^A = X^A_B(\vec{v}_a)X^B_C(-\vec{v}_r)(\tilde{F}_r^{kC} - v_r^k F_r^C). \tag{29}$$

Now, we use another property of the matrix  $X^A_B(\vec{v})$ , which is a consequence of its definition (17) and reads:

$$X^A_B(\vec{u})X^B_C(\vec{w}) = X^A_C(\vec{u} + \vec{w}). \tag{30}$$

Moreover, we use the well-known property:

$$\vec{v}_a = \vec{v}_r + \vec{v}_\tau. \tag{31}$$

In this way, Equation(29) becomes:

$$F_a^A = X^A_C(\vec{v}_\tau)F_r^C, \tag{32}$$

$$\tilde{F}_a^{kA} - v_a^k F_a^A = X^A_C(\vec{v}_\tau)\tilde{F}_r^{kC} - v_r^k X^A_C(\vec{v}_\tau)F_r^C \tag{33}$$

In Equation (33), we can substitute  $X^A_C(\vec{v}_\tau)F_r^C$  from Equation (32), so that it becomes:

$$\tilde{F}_a^{kA} - v_r^k F_a^A = X^A_C(\vec{v}_\tau)\tilde{F}_r^{kC}. \tag{34}$$

Finally, we deduce  $\hat{h}'$ ,  $\hat{h}'^k$  and  $\hat{\mu}_A$  from (25,26,27) and substitute them in (20,21,22), so obtaining:

$$h'_a = h'_r, \tag{35}$$

$$h_a'^k - v_r^k h'_a = h_r'^k, \tag{36}$$

$$\mu_C^r = \mu_B^a X^B_C(\vec{v}_\tau), \tag{37}$$

where, for the last one, we have also used a contraction with  $X^A_C(-\vec{v}_\tau)$ .

Equations (32), (34), (35,36,37) give the requested transformation law between the two reference frames, and it is very interesting that it looks like Equation (16).

Now, if the Lagrange multipliers are taken as independent variables, Equation (37) is only a change of independent variables from  $\mu_B^a$  to  $\mu_C^r$ , while (32), (34), (35,36) are conditions, because they involve constitutive functions:

$$F_a^A = F^A(\mu_B^a), \tag{38}$$

$$\tilde{F}_a^{kA} = \tilde{F}^{kA}(\mu_B^a), \tag{39}$$

$$h'_a = h'(\mu_B^a), \tag{40}$$

$$h_a'^k = h'^k(\mu_B^a), \tag{41}$$

$$F_r^A = F^A(\mu_B^r), \tag{42}$$

$$\tilde{F}_r^{kA} = \tilde{F}^{kA}(\mu_B^r), \tag{43}$$

$$h'_r = h'(\mu_B^r) \tag{44}$$

$$h_r'^k = h'^k(\mu_B^r), \tag{45}$$

where the form of the functions  $F^A$ ,  $\tilde{F}^{kA}$ ,  $h'$ ,  $h'^k$  does not depend on the reference frame for the Galilean relativity principle. If we substitute  $\mu_B^a$  from Equation (37) in (38,39,40,41) and then substitute the result in (32), (34), (35,36), we obtain:

$$F^A(\mu_C^r X^C_B(-\vec{v}_\tau)) = X^A_C(\vec{v}_\tau)F_r^C, \tag{46}$$

$$\tilde{F}^{kA}(\mu_C^r X^C_B(-\vec{v}_\tau)) - v_r^k X^A_C(\vec{v}_\tau)F_r^C = X^A_C(\vec{v}_\tau)\tilde{F}_r^{kC} \tag{47}$$

$$h'(\mu_C^r X^C_B(-\vec{v}_\tau)) = h'_r \tag{48}$$

$$h'^k(\mu_C^r X^C_B(-\vec{v}_\tau)) - h'_r v_r^k = h_r'^k. \tag{49}$$

These expressions calculated in  $v_\tau^i = 0$  are nothing more than Equations (42,43,44,45), as we expected. However, for the Galilean relativity principle, they must be coincident for whatever value of  $v_\tau^i$ ; this amounts to saying that the derivatives of (46,47,48,49) with respect to  $v_\tau^i$  must hold.

This constraint can be written explicitly more easily if we take into account that  $\mu_C^r X^C_B(-\vec{v}_\tau) = \mu_B^a$ , which can be written explicitly by use of (17) and reads:

$$\begin{aligned} \mu^a &= \mu^r - \mu_i^r v_\tau^i + \mu_{ij}^r v_\tau^i v_\tau^j - v_\tau^2 v_\tau^i \mu_{ill}^r + \lambda^r v_\tau^2 - \lambda_i^r v_\tau^i v_\tau^2 + \lambda_{ll}^r v_\tau^4, \\ \mu_h^a &= \mu_h^r - 2\mu_{ih}^r v_\tau^i + (v_\tau^2 \delta_h^i + 2v_\tau^i v_\tau^h) \mu_{ill}^r - 2\lambda^r v_{\tau h} + \lambda_i^r (v_\tau^2 \delta_h^i + 2v_\tau^i v_{\tau h}) - 4v_\tau^2 v_h^\tau \lambda_{ll}^r, \\ \mu_{hk}^a &= \mu_{hk}^r - (2\delta_{(h}^i v_k^\tau) + v_\tau^i \delta_{hk}) \mu_{ill}^r - 2\lambda_i^r v_{\tau(h} \delta_{k)}^i + (4v_h^\tau v_k^\tau + v_\tau^2 \delta_{hk}) \lambda_{ll}^r, \\ \mu_{hll}^a &= \mu_{hll}^r - 2v_h^\tau \lambda_{ll}^r, \\ \lambda^a &= \lambda^r - \lambda_i^r v_\tau^i + v_\tau^2 \lambda_{ll}^r, \\ \lambda_h^a &= \lambda_h^r - 2v_h^\tau \lambda_{ll}^r, \\ \lambda_{ll}^a &= \lambda_{ll}^r, \end{aligned} \tag{50}$$

from which:

$$\begin{aligned} \frac{\partial \mu^a}{\partial v_\tau^i} &= -\mu_i^a, & \frac{\partial \mu_h^a}{\partial v_\tau^i} &= -2\mu_{ih}^a - 2\lambda^a \delta_{hi}, & \frac{\partial \mu_{hk}^a}{\partial v_\tau^i} &= -2\lambda_{(h}^a \delta_{k)i} - 2\delta_{i(h} \mu_{k)ll}^a - \mu_{ill}^a \delta_{hk}, \\ \frac{\partial \mu_{hll}^a}{\partial v_\tau^i} &= -2\delta_{ih} \lambda_{ll}^a, & \frac{\partial \lambda^a}{\partial v_\tau^i} &= -\lambda_i^a, & \frac{\partial \lambda_h^a}{\partial v_\tau^i} &= -2\lambda_{ll}^a \delta_{hi}, & \frac{\partial \lambda_{ll}^a}{\partial v_\tau^i} &= 0. \end{aligned} \tag{51}$$

Consequently, the derivatives of (48,49) with respect to  $v_\tau^i$  become (13,14), where we have omitted index  $a$  denoting variables in the absolute reference frame, because they remain unchanged if we change  $v_\tau^i$  with  $-v_\tau^i$ , that is if we exchange the absolute and the relative reference frames.

It is not necessary to impose the derivatives of (46,47) with respect to  $v_\tau^i$ , because they are consequences of (13,14) and (9,10). Consequently, the Galilean relativity principle amounts simply to the two equations, (13,14).

Therefore, we have to find the most general functions satisfying them. After that, we have to use Equation (9) to obtain the Lagrange multipliers in terms of the variables  $F^A$ . By substituting them in (10) and in  $h' = h'(\mu_A)$ ,  $h'^k = h'^k(\mu_A)$ , we obtain the constitutive functions in terms of the variables  $F^A$ . If we want the non-convective parts of our expressions, it suffices to calculate the left-hand side of Equation (9) in  $\vec{v} = \vec{0}$ , so that they become:

$$\hat{F}^A = \frac{\partial h'}{\partial \mu_A}. \tag{52}$$

>From this equation, we obtain the Lagrange multipliers in terms of  $\hat{F}^A$  (obviously, they will be  $\hat{\mu}_A$ ) and, after that, substitute them in  $h' = h'(\mu_A)$ ,  $h'^k = h'^k(\mu_A)$  (the last of which will in effect be  $\hat{h}'^k$ ) and into  $\hat{F}^{kA} = \frac{\partial h'^k}{\partial \mu_A}$ , that is Equation (10) calculated in  $\vec{v} = \vec{0}$ .

It has to be noted that from (15) it follows  $\hat{F}^i = 0$ , so that one of the equations (52) is  $0 = \frac{\partial h'}{\partial \mu_i}$ ; this does not mean that  $h'$  does not depend on  $\mu_i$ , but this is simply an implicit function defining jointly with the other equations (52) the quantities  $\hat{\mu}_A$  in terms of  $\hat{F}^A$ .

By using a procedure similar to that of the paper [43], we can prove that we obtain the same results of the firstly described approach.



### 3. The General Solution of the Conditions Translating the Galilean Relativity Principle

Now, we search the solution of the conditions (13,14), which are equivalent to the Galilean relativity principle for our model.

By extending to our model the method used in [11] in a different context, we firstly define the following functions of the Lagrange multipliers and of a generic vector  $w^i$ :

$$\begin{aligned} \eta &= \mu + \mu_i w^i + \mu_{ij} w^i w^j + \mu_{ill} w^i w^2 + \lambda w^2 + \lambda_i w^i w^2 + \lambda_{ll} w^4, \\ \eta_a &= \mu_a + 2\mu_{ia} w^i + \mu_{ill}(w^2 \delta_a^i + 2w^i w_a) + 2\lambda w_a + \lambda_i (w^2 \delta_a^i + 2w^i w_a) + 4\lambda_{ll} w^2 w_a, \\ \eta_{ab} &= \mu_{ab} + \mu_{ill}(2\delta_{(a}^i w_{b)}) + w^i \delta_{ab}) + 2\lambda_{(a} w_{b)} + \lambda_{ll}(w^2 \delta_{ab} + 4w_a w_b), \\ \eta_{all} &= \mu_{all} + 2\lambda_{ll} w_a, \\ \mathcal{L} &= \lambda + \lambda_i w^i + \lambda_{ll} w^2 \quad ; \end{aligned} \tag{53}$$

It is clear that this definition is built by taking a linear combination of the Lagrange multipliers through the first five columns of the matrix  $X^A_B$  introduced in (17), but with  $(\vec{w})$  instead of  $(\vec{v})$ .

As a consequence, from (53), we find the counterpart of (51), that is:

$$\begin{aligned} \frac{\partial \eta}{\partial w^r} = \eta_r, \quad \frac{\partial \eta_a}{\partial w^r} = 2\eta_{ar} + 2\mathcal{L} \delta_{ar}, \quad \frac{\partial \eta_{ab}}{\partial w^r} = 2\delta_{r(a} \eta_{b)l} + \eta_{rl} \delta_{ab} + 2\lambda_{(a} \delta_{b)r} + 4\lambda_{ll} w_{(a} \delta_{b)r}, \\ \frac{\partial \eta_{all}}{\partial w^r} = 2\delta_{ar} \lambda_{ll}, \quad \frac{\partial \mathcal{L}}{\partial w^r} = \lambda_r + 2w_r \lambda_{ll}. \end{aligned} \tag{54}$$

Let us now insert in (53):

$$w_j = -\frac{\lambda_j}{2\lambda_{ll}} \tag{55}$$

(we will discuss at the end of this section the presence of  $\lambda_{ll}$  in the denominator), and let us consider the result as a change of independent variables from  $\mu, \mu_a, \mu_{ab}, \mu_{all}, \lambda, \lambda_a, \lambda_{ll}$  to  $\eta, \eta_a, \eta_{ab}, \eta_{all}, \mathcal{L}, \lambda_a, \lambda_{ll}$ . Consequently,  $h'$  will be a composite function  $h' = H(\vec{\eta}(\mu_A), \lambda_a, \lambda_{ll})$  from which, for the derivation rule of composite functions, we obtain:

$$\frac{\partial h'}{\partial \lambda_r} = \frac{\partial H}{\partial \eta} \frac{\partial \eta}{\partial \lambda_r} + \frac{\partial H}{\partial \eta_a} \frac{\partial \eta_a}{\partial \lambda_r} + \frac{\partial H}{\partial \eta_{ab}} \frac{\partial \eta_{ab}}{\partial \lambda_r} + \frac{\partial H}{\partial \eta_{all}} \frac{\partial \eta_{all}}{\partial \lambda_r} + \frac{\partial H}{\partial \mathcal{L}} \frac{\partial \mathcal{L}}{\partial \lambda_r} + \frac{\partial H}{\partial \lambda_r}.$$

This expression can be rewritten by using (54) and remembering that  $\vec{\eta}$  depends on  $\lambda_r$ , whether explicitly or by means of  $w_j$  given by (55); so, it becomes:

$$\begin{aligned} \frac{\partial h'}{\partial \lambda_r} &= \frac{\partial H}{\partial \eta} \left( w^r w^2 + \eta^r \frac{-1}{2\lambda_{ll}} \right) + \frac{\partial H}{\partial \eta_a} \left[ w^2 \delta_a^r + 2w^r w_a + (2\eta_{ar} + 2\delta_{ar} \mathcal{L}) \frac{-1}{2\lambda_{ll}} \right] + \\ &+ \frac{\partial H}{\partial \eta_{ab}} \left[ 2\delta_{r(a} w_{b)} - \frac{1}{2\lambda_{ll}} (2\delta_{r(a} \eta_{b)l} + \eta_{rl} \delta_{ab} + 2\delta_{r(a} \lambda_{b)}) + 4\lambda_{ll} \delta_{r(a} w_{b)} \right] + \\ &+ \frac{\partial H}{\partial \eta_{all}} 2\lambda_{ll} \delta_{ar} \frac{-1}{2\lambda_{ll}} + \frac{\partial H}{\partial \mathcal{L}} \left[ w^r - \frac{1}{2\lambda_{ll}} (\lambda^r + 2w^r \lambda_{ll}) \right] + \frac{\partial H}{\partial \lambda_r}, \end{aligned} \tag{56}$$

where overlined terms cancel each other. Similarly, Equation (13) becomes:

$$\begin{aligned}
 0 = & \mu_r \frac{\partial H}{\partial \eta} + 2(\mu_{air} + \delta_{ar}\lambda) \left( \frac{\partial H}{\partial \eta} w_a + \frac{\partial H}{\partial \eta_a} \right) + \\
 & +(2\delta_{r(a\mu_b)ll} + \mu_{rll}\delta_{ab} + 2\lambda_{(a\delta_b)r}) \left( \frac{\partial H}{\partial \eta} w_a w_b + 2\frac{\partial H}{\partial \eta_a} w_b + \frac{\partial H}{\partial \eta_{ab}} \right) + \\
 & +2\lambda_{ll} \left( \frac{\partial H}{\partial \eta} w^a w_a w_r + \frac{\partial H}{\partial \eta_a} (w^b w_b \delta_a^r + 2w^r w_a) + \frac{\partial H}{\partial \eta_{ab}} (2\delta_a^r w_b + \delta_{ab} w^r) + \frac{\partial H}{\partial \eta_{rll}} \right) + \\
 & +\lambda_r \left( \frac{\partial H}{\partial \eta} w^a w_a + 2\frac{\partial H}{\partial \eta_a} w^a + \frac{\partial H}{\partial \mathcal{L}} \right) + 2\lambda_{ll} \frac{\partial h'}{\partial \lambda_r},
 \end{aligned}$$

with  $\frac{\partial h'}{\partial \lambda_r}$  given by (56). By using (53), we observe now that the coefficients of  $\frac{\partial H}{\partial \eta}, \frac{\partial H}{\partial \eta_a}, \frac{\partial H}{\partial \eta_{ab}}, \frac{\partial H}{\partial \eta_{all}}, \frac{\partial H}{\partial \mathcal{L}}$  are all zero. Therefore, from the above equations, there remains only  $\frac{\partial H}{\partial \lambda_r} = 0$ . We may conclude that the general solution of (13) is:

$$h' = H(\vec{\eta}(\mu_A), \lambda_{ll}). \tag{57}$$

Let us now consider Equation (14).

By defining  $H^k$  from:

$$h'^k = H^k + \frac{\lambda^k}{-2\lambda_{ll}} h', \tag{58}$$

it becomes:

$$\begin{aligned}
 \frac{\partial H^k}{\partial \mu} \mu_r + 2\frac{\partial H^k}{\partial \mu_a} (\mu_{ar} + \delta_{ar}\lambda) + \frac{\partial H^k}{\partial \mu_{ab}} (2\delta_{r(a\mu_b)ll} + \mu_{rll}\delta_{ab} + 2\lambda_{(a\delta_b)r}) + \\
 +2\frac{\partial H^k}{\partial \mu_{rll}} \lambda_{ll} + \frac{\partial H^k}{\partial \lambda} \lambda_r + 2\frac{\partial H^k}{\partial \lambda_r} \lambda_{ll} = 0
 \end{aligned} \tag{59}$$

(where we have taken into account that  $h'$  satisfies (13)), which is like (13), but with  $H^k$  instead of  $h'$ . Therefore, with a similar method, we obtain that the general solution of (14) is:

$$h'^k = \frac{\lambda^k}{-2\lambda_{ll}} H(\vec{\eta}(\mu_A), \lambda_{ll}) + H^k(\vec{\eta}(\mu_A), \lambda_{ll}). \tag{60}$$

Now, it is clear that the present general solution can be used only in the sub-manifold with  $\lambda_{ll} \neq 0$ .

In order to avoid confusion in the terminology, we remark that this  $\lambda_{ll}$  is different from that in [29]. In fact, in that article, only the block (1) was considered and  $\lambda_{ll}$  was the trace of the Lagrange multiplier with  $N = ij$ , and it was different from zero, because it was the Lagrange multiplier of the energy conservation law. Now, we have two blocks, so that this conservation law is (2) with  $E = 0$ ; in other words, the variable  $\lambda_{ll}$  of [29] corresponds to the variable  $\lambda$  of the present article.

It is not possible to choose  $\lambda_{ll} \neq 0$ , which derives, as a subsystem, the 14-moment system [30]; in fact, Equations (13, 14) calculated in the subsystem with  $\mu_{iil} = 0$  become:

$$\begin{aligned}
 \frac{\partial h'_S}{\partial \mu} \mu_i + 2\frac{\partial h'_S}{\partial \mu_a} (\mu_{ai} + \delta_{ai}\lambda) + 2\frac{\partial h'_S}{\partial \mu_{ai}} \lambda_a + \frac{\partial h'_S}{\partial \lambda} \lambda_i + 2\left( \frac{\partial h'}{\partial \mu_{iil}} + \frac{\partial h'}{\partial \lambda_i} \right)_S \lambda_{ll} = 0. \\
 \frac{\partial h'^k_S}{\partial \mu} \mu_i + 2\frac{\partial h'^k_S}{\partial \mu_a} (\mu_{ai} + \delta_{ai}\lambda) + 2\frac{\partial h'^k_S}{\partial \mu_{ai}} \lambda_a + \frac{\partial h'^k_S}{\partial \lambda} \lambda_i + h'_S \delta^{ik} + 2\left( \frac{\partial h'^k}{\partial \mu_{iil}} + \frac{\partial h'^k}{\partial \lambda_i} \right)_S \lambda_{ll} = 0,
 \end{aligned}$$

where the index  $S$  denotes a quantity calculated in the subsystem. The last term in both equations is not present if we start from the beginning in the subsystem. Therefore, if  $\lambda_{ll} \neq 0$ , the linear expression of  $h'$  and  $h'^k$  with respect to  $S$  plays an additional role.

Obviously, this problem does not exist in the case of the near equilibrium state of ET with more moments. For the same reason ( $\lambda_{ll} \neq 0$ ), we can say that the six-moment model has an exact solution, but it cannot be deduced with the subsystems methodology from the present exact solution of the 18-moment model.

Now, the condition  $\lambda_{ll} \neq 0$  is a problem if we try a transition to the subsystem  $\lambda_{ll} = 0$ ,  $\mu_{ill} = 0$  in order to obtain the model in [30]. However, we think that this is not a defect of the theory, but only a property of the general solution that we find. It exhibits a sort of resistance to be extended also to a model with less field equations. Another possibility is that subsystems have to be defined in another way and not simply by putting equal to zero the exceeding Lagrange multipliers; in fact, when people drop the hierarchy of the balance equations, this does not mean that the subsequent equations are identically satisfied, but simply that we are doing a necessary approximation. From this point of view, it may be possible to recover a subsystem by considering the exceeding Lagrange multipliers as functions of the remainder. However, this is an argument for future research, because it goes far beyond the scope of the present article.

Another aspect to note is that the above-mentioned sub-manifold does not contain equilibrium if we define equilibrium as the state with  $\mu_{ij} = 0$ ,  $\mu_{ill} = 0$ ,  $\lambda_i = 0$ ,  $\lambda_{ll} = 0$ . However, there is no necessity to have  $\lambda_{ll} = 0$  at equilibrium! To this end, let us recall the definition of equilibrium in terms of the Lagrange multipliers as that exposed in [2,3]. It starts from the entropy production  $\sigma$  in (7); now, the productions  $P^A$  are zero at equilibrium, so that  $\sigma$  has a minimum at equilibrium for Equation (4). If we assume that  $P^{ij}$ ,  $P^{ill}$ ,  $Q^i$ ,  $Q^{ll}$  can be taken as part of the independent variables, then the derivatives of  $\sigma$  with respect to them must be zero at equilibrium, and this implies that  $\mu_{ij} = 0$ ,  $\mu_{ill} = 0$ ,  $\lambda_i = 0$ ,  $\lambda_{ll} = 0$ . However, it is this last assumption that may be wrong; for example, we may have that  $P^{ll}$  and  $Q^{ll}$  are proportional at first order with respect to the equilibrium, so that they are not independent variables! This would imply that  $\lambda_{ll}$  and  $\mu_{ll}$  are not necessarily zero at equilibrium, but only a linear combination of them. Instead of this, the non-convective parts of  $\mu_{<ij>}$ ,  $\mu_{ill}$ ,  $\lambda_i$  are zero at equilibrium, because this state can be defined also in terms of the moments, and in this case, there are only two scalar independent variables (besides the velocity); and, for the representation theorems (see [47–55]), every vectorial function depending on them must be zero, and every second order tensorial function must be proportional to the identical matrix.

From this point of view, we find further grounds to appreciate the new idea of the two blocks of equations, exposed in [30], because this allows the presence of two scalar Lagrange multipliers,  $\lambda_{ll}$  and  $\mu_{ll}$ . Obviously, there is much more to investigate in this aspect, and we leave it also for future considerations.

#### 4. The General Solution of the Symmetry Conditions

Now, we search the general solution of the symmetry conditions (11,12). By using (57) and (60), Equation (11) becomes:

$$\frac{\partial H}{\partial \eta_B} \frac{\partial \eta_B}{\partial \mu_i} = \frac{\lambda^i}{-2\lambda_{ll}} \frac{\partial H}{\partial \eta_B} \frac{\partial \eta_B}{\partial \mu} + \frac{\partial H^i}{\partial \eta_B} \frac{\partial \eta_B}{\partial \mu},$$

which, by using (53), becomes:

$$\frac{\partial H}{\partial \eta} w^i + \frac{\partial H}{\partial \eta_i} = \frac{\lambda^i}{-2\lambda_{ll}} \frac{\partial H}{\partial \eta} + \frac{\partial H^i}{\partial \eta},$$

or  $\frac{\partial H}{\partial \eta_i} = \frac{\partial H^i}{\partial \eta}$ , because of Equation (55).

Similarly, by using (60), Equation (12) becomes:

$$\frac{\partial H}{\partial \eta_B} \frac{\lambda^{[k}}{-2\lambda_{ll}} \frac{\partial \eta_B}{\partial \mu_{i]}} + \frac{\partial H^{[k}}{\partial \eta_B} \frac{\partial \eta_B}{\partial \mu_{i]}} = 0,$$

which, by using (53), becomes:

$$\frac{\partial H}{\partial \eta} \frac{\lambda^{[k}}{-2\lambda_{ll}} w^{i]} + \frac{\partial H^{[k}}{\partial \eta} w^{i]} + \frac{\lambda^{[k}}{-2\lambda_{ll}} \frac{\partial H}{\partial \eta_{i]}} + \frac{\partial H^{[k}}{\partial \eta_{i]}} = 0,$$

or  $\frac{\partial H^{[k}}{\partial \eta_{i]}} = 0$ , because of Equation (55).

Therefore, we have obtained that the symmetry conditions (11,12) expressed in terms of the new variables and of the new functions  $H$  and  $H^k$  become:

$$\frac{\partial H}{\partial \eta_i} = \frac{\partial H^i}{\partial \eta}, \quad \frac{\partial H^{[i}}{\partial \eta_{j]}} = 0, \tag{61}$$

which are very similar to (11,12). Similarly, if we would impose as a further symmetry condition that the flux  $\tilde{F}^{ij}$  is equal to the density  $F^{ij}$  of the subsequent equation, for Equations (9,10), this condition amounts to:

$$\frac{\partial h'}{\partial \mu_{ij}} = \frac{\partial h'^i}{\partial \mu_j}, \tag{62}$$

which can be expressed in the new functions and new variables as:

$$\frac{\partial H}{\partial \eta_{ij}} = \frac{\partial H^i}{\partial \eta_j}, \tag{63}$$

where (61) has been used. Furthermore, in this case, Equations (62) and (63) are very similar. This similarities were not foregone. In fact, if we would impose as a further symmetry condition that the flux  $\tilde{G}^i$  is equal to the density  $G^i$  of the subsequent equation, for Equations (9,10), this condition amounts to:

$$\frac{\partial h'}{\partial \lambda_i} = \frac{\partial h'^i}{\partial \lambda}, \tag{64}$$

which, by using (57) and (60), becomes:

$$\frac{\partial H}{\partial \eta_B} \frac{\partial \eta_B}{\partial \lambda_i} = \frac{\lambda^i}{-2\lambda_{ll}} \frac{\partial H}{\partial \eta_B} \frac{\partial \eta_B}{\partial \lambda} + \frac{\partial H^i}{\partial \eta_B} \frac{\partial \eta_B}{\partial \lambda}; \tag{65}$$

but  $\eta_B$  depends on  $\lambda_i$  both directly, rather than by means of  $w^j$ ; so, by using (53) and (54), we have:

$$\begin{aligned} \frac{\partial \eta}{\partial \lambda_i} &= w^i w^2 + \eta_i \frac{-1}{2\lambda_{ll}} \quad , \quad \frac{\partial \eta_a}{\partial \lambda_i} = w^2 \delta_a^i + 2w^i w_a + (2\eta_{ai} + 2\mathcal{L} \delta_{ai}) \frac{-1}{2\lambda_{ll}} , \\ \frac{\partial \eta_{ab}}{\partial \lambda_i} &= 2\delta_{i(a} w_{b)} + [2\delta_{i(a} \eta_{b)ll} + \eta_{ill} \delta_{ab} + 2\lambda_{(a} \delta_{b)i} + 4\lambda_{ll} w_{(a} \delta_{b)i}] \frac{-1}{2\lambda_{ll}} , \\ \frac{\partial \eta_{all}}{\partial \lambda_i} &= 2\delta_{ai} \lambda_{ll} \frac{-1}{2\lambda_{ll}} \quad , \quad \frac{\partial \mathcal{L}}{\partial \lambda_i} = w^i + (\lambda_i + 2w_i \lambda_{ll}) \frac{-1}{2\lambda_{ll}} . \end{aligned}$$

By substituting these in the left-hand side of Equations (65) and (53), (61) and (63) in its right-hand side (and by using also (55)), it becomes:

$$\frac{\partial H^i}{\partial \mathcal{L}} = \frac{-1}{2\lambda_{ll}} \left\{ \frac{\partial H}{\partial \eta} \eta_i + \frac{\partial H}{\partial \eta_a} (2\eta_{ai} + 2\delta_{ai} \mathcal{L}) + \frac{\partial H}{\partial \eta_{ab}} (2\delta_{i(a} \eta_{b)ll} + \eta_{ill} \delta_{ab}) + 2 \frac{\partial H}{\partial \eta_{ill}} \lambda_{ll} \right\} , \quad (66)$$

which is quite different from (64). Other possible symmetry conditions are:

$$\tilde{F}^{ill} = F^{ill} \quad , \quad \text{that is,} \quad \frac{\partial h'}{\partial \mu_{ill}} = \frac{\partial h'^i}{\partial \mu_{rs}} \delta_{rs} , \quad (67)$$

$$\tilde{G}^{ll} = G^{ll} \quad , \quad \text{that is,} \quad \frac{\partial h'}{\partial \lambda_{ll}} = \frac{\partial h'^i}{\partial \lambda_j} \delta_{ij} , \quad (68)$$

$$\text{Symmetry of } \tilde{F}^{kij} \text{, } \tilde{F}^{kill} \text{ and } \tilde{G}^{ki} \text{, that is,} \quad \frac{\partial h'^{[k}}{\partial \mu_{i]j}} = 0 , \quad (69)$$

$$\frac{\partial h'^{[k}}{\partial \mu_{i]l}} = 0 , \quad (70)$$

$$\frac{\partial h'^{[k}}{\partial \lambda_{i]}} = 0 . \quad (71)$$

With the usual passages, we obtain that (67, 69, 70) expressed in terms of the new variables and of the new functions  $H$  and  $H^k$  become:

$$\frac{\partial H}{\partial \eta_{ill}} = \frac{\partial H^i}{\partial \eta_{rs}} \delta_{rs} , \quad (72)$$

$$\frac{\partial H^{[k}}{\partial \eta_{i]j}} = 0, \quad \frac{\partial H^{[k}}{\partial \eta_{i]l}} = 0, \quad (73)$$

while (71, 68) (by using also (61), (63), (66), (72,73)) become:

$$\begin{aligned} \frac{\partial H}{\partial \eta_{[k}} \eta_{i]} + 2 \frac{\partial H}{\partial \eta_{a[k}} \eta_{i]a} + \frac{\partial H}{\partial \eta_{ll[k}} \eta_{i]ll} &= 0 \quad , \quad (74) \\ \frac{\partial H}{\partial \eta_a} \eta_a + 2 \frac{\partial H}{\partial \eta_{ab}} (\eta_{ab} + \mathcal{L} \delta_{ab}) + \frac{\partial H}{\partial \eta_{all}} \eta_{all} + 2 \frac{\partial H}{\partial \lambda_{ll}} \lambda_{ll} + 3H &+ + 2 \frac{\partial H^a}{\partial \eta_{all}} \lambda_{ll} + 2 \frac{\partial H^a}{\partial \eta_{ab}} \eta_{bll} = 0, \end{aligned}$$

where, for the last one, we have used  $\frac{\partial H^k}{\partial \eta_{kb}} = \frac{\partial H}{\partial \eta_{bl}}$ , which is a consequence of (72, 73).

We recall that the functions  $H$  and  $H^a$  do not depend on  $\lambda^i$ ; it is interesting that also in their coefficients, the contribution of  $\lambda^i$  to (61), (63), (66), (72,73), (74) disappeared automatically.

Now, we have enclosed the conditions (63), (66), (72,73), (74) for the sake of completeness, but as said from the beginning, we avoid imposing them in order to not lose generality. Instead of this, the condition

(61) is physically important; so we end this consideration by finding the solution, and it is easy to see that it is:

$$H = \frac{\partial \psi}{\partial \eta}, \quad H^k = \frac{\partial \psi}{\partial \eta_k}, \quad (75)$$

with  $\psi(\vec{\eta}, \lambda_{ll})$ , an arbitrary function; in fact, (61) is nothing more than the integrability conditions to obtain  $\psi$  from (75).

## 5. Conclusions

We have considered a model following the new guidelines of two blocks of equations, obtaining a non-linear and exact solution of the conditions arising from the entropy principle, the Galilean relativity principle and the symmetry conditions, which consist of the angular momentum conservation and in the fact that the flux in the conservation law of mass is the density in the conservation law of momentum. Differently from [11], the new model based on the two blocks of equations introduced in [30] is defined also in a neighborhood of equilibrium. There remains the resistance of the model to be confined in a subsystem; but this is not a defect. In fact, in [3], it was shown that the number of moments to include in a model is determined by the particular physical application under consideration: If a model with a determined number of moments fits the experimental results better, then that is the correct number of moments to be considered. From this point of view, it is not reasonable to force it to be described also with one of the subsystems. Obviously, this may be the starting point of many other considerations and research projects.

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## Author Contributions

M.C. Carrisi and S. Pennisi conceived the idea and performed the calculations with a mutual interaction in all parts of the article. Both of them wrote the paper.

## Conflicts of Interest

The authors declare no conflict of interest.

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